Lesson 3-9 Step Functions

Vocabulary

step function
floor symbol ⌊ ⌋
ceiling symbol ⌈ ⌉
floor function, greatest-integer function, rounding-down function, int function
ceiling function, rounding-up function

BIG IDEA *Step functions* have applications in many situations that involve rounding.

In 2007, the U.S. postage rate for first class flats (certain large envelopes) was $0.70 for the first ounce plus $0.17 for each additional ounce or part of an ounce. First-class mail rates for flats up to 13 ounces are given in the table below. Notice that the phrase "up to and including the given weight" means that the weight is *rounded up* to the nearest ounce. For instance, an envelope weighing 4.4 ounces is charged at the 5-ounce rate.

2007 First-Class Mail Rates for Flats*

Weight (oz)	Rate (dollars)	Weight (oz)	Rate (dollars)
1	0.70	8	1.89
2	0.87	9	2.06
3	1.04	10	2.23
4	1.21	11	2.40
5	1.38	12	2.57
6	1.55	13	2.74
7	1.72		

*Rate is for a flat up to and including the given weight.

Mental Math

Each sequence below is either arithmetic or consists of consecutive powers of a number. Give the next two terms in the sequence.

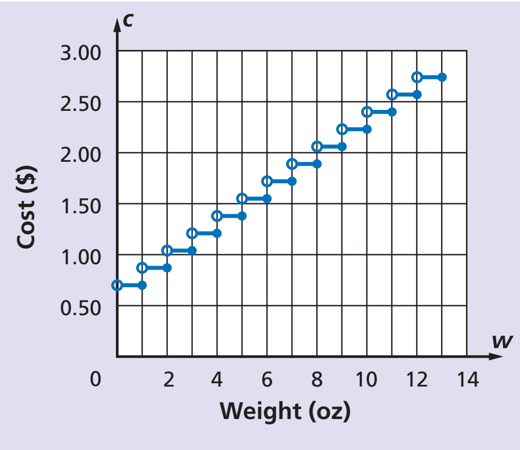
a. 2, 4, 6, 8, ...

b. 23, 17, 11, 5, ...

c. 1, –1, 1, –1, ...

The graph at the right shows the cost of mailing a first-class flat for weights up to 13 ounces. Because the cost is rounded up, the left end of each segment is not included on the graph and the right end of each segment is included. Because no single weight has two costs, this graph pictures a function.

The domain is the set of possible weights of a flat in ounces between 0 and 13 ounces, and the range is the set of costs {$0.70, $0.87, $1.04, … , $2.74}.



The mail-rates function is not a linear function, but it is a piecewise linear function. Because its graph looks like a series of steps, it is called a **step function**. Each step is part of a horizontal line. Two step functions commonly used are the *floor function* and the *ceiling function*.

The Floor and Ceiling Functions

The **floor symbol** $\lfloor\ \rfloor$ and the **ceiling symbol** $\lceil\ \rceil$ are defined as follows.

Definition of Greatest Integer/Least Integer

$\lfloor x \rfloor$ = the greatest integer less than or equal to x, and

$\lceil x \rceil$ = the least integer greater than or equal to x.

The **floor function** is the function f with $f(x) = \lfloor x \rfloor$, for all real numbers x. It is also called the **greatest-integer function**, or the **rounding-down function**. On some calculators and in some computer languages it is called the **int function**. Another notation you may see for the floor function is $f(x) = [\![x]\!]$.

The **ceiling function** is the function f with $f(x) = \lceil x \rceil$, for all real numbers x. It is also called the **rounding-up function**.

STOP QY1

▶ QY1

What names does your calculator use for the floor and ceiling functions?

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Example 1

Evaluate each of the following.

a. $\left\lfloor 5\frac{7}{8} \right\rfloor$ b. $\lfloor -4.2 \rfloor$ c. $\lceil \pi \rceil$ d. $\lceil 13 \rceil$

Solution

a. $\left\lfloor 5\frac{7}{8} \right\rfloor$ is the greatest integer less than or equal to $5\frac{7}{8}$. So, $\left\lfloor 5\frac{7}{8} \right\rfloor = \underline{\ ?\ }$.

b. $\lfloor -4.2 \rfloor$ is the $\underline{\quad ?\quad}$ less than or equal to -4.2. So, $\lfloor -4.2 \rfloor = \underline{\ ?\ }$.

c. $\lceil \pi \rceil$ is the least integer greater than or equal to $\pi \approx \underline{\ ?\ }$.
So, $\lceil \underline{\ ?\ } \rceil = \underline{\ ?\ }$

d. $\lceil 13 \rceil$ is the $\underline{\quad ?\quad}$ greater than or equal to 13. So, $\lceil \underline{\ ?\ } \rceil = \underline{\ ?\ }$.

The Graph of the Floor Function

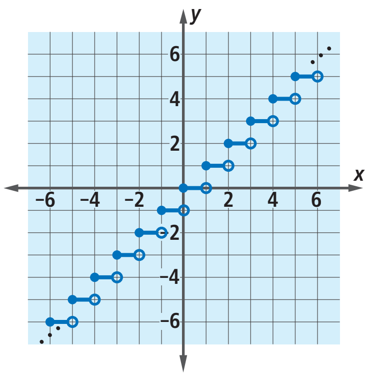
One way to sketch the graph of a step function is to make a table of values so you can see the pattern.

Example 2

Graph the function f defined by $f(x) = \lfloor x \rfloor$.

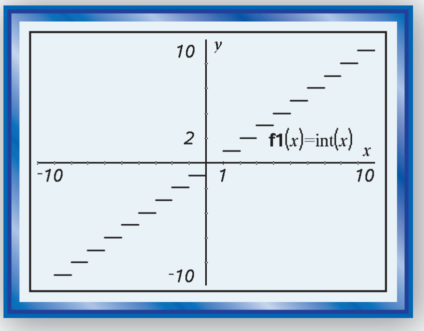
Solution Make a table of values. For all x greater than or equal to 0 but less than 1, the greatest integer less than or equal to x is 0. For all x greater than or equal to 1 but less than 2, the greatest integer less than or equal to x is 1. In a similar manner, you can get the other values in the table below. The graph is at the right below.

x	$f(x) = \lfloor x \rfloor$
$-3 \le x < -2$	-3
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2
$3 \le x < 4$	3



In the graph in Example 2, the open circles at (1, 0), (2, 1), (3, 2), and so on, indicate that these points do not lie on the graph of $f(x) = \lfloor x \rfloor$. At these points, the function value jumps to the next step. The solid circles indicate that the points (1, 1), (2, 2), (3, 3), and so forth, do lie on the graph. Notice that the domain of the greatest-integer function is the set of real numbers, but the range is the set of integers.

If your graphing utility has the `int`, or `floor` function, it will graph the greatest-integer function for you. The graph from one graphing utility is shown at the right. By default, some graphing utilities connect successive pixels, so they may join successive steps. This makes it appear as if the graph does not represent a function. On these graphing utilities, you can get the correct graph by switching from connected mode to dot mode.



Applications of Step Functions

The floor or ceiling function is appropriate when function values must be integers and other formulas would give noninteger values.

Example 3

In March 2008, New York City taxi rates were an initial fee of \$2.50 plus \$0.40 for each full $\frac{1}{5}$-mile traveled.

a. Write a formula for $T(m)$, the charge for a trip of m miles.

b. What is the charge for an 8.75-mile trip in a New York City taxi?

The taxi to resident ratio in New York City is 1:149.

Solution

a. Because there are 5 one-fifths of a mile in each mile, multiply the miles by 5 to determine the number of $\frac{1}{5}$-miles traveled. This number, $5m$, may not be a whole number, so use the greatest-integer function to change it to an integer before multiplying by \$0.40. An equation for this function is $T(m) = 2.50 + 0.40\lfloor 5m \rfloor$.

b. The charge for a trip of 8.75 miles can be computed by substituting $m = 8.75$ into the formula for $T(m)$.

$$\begin{aligned} T(m) &= \$2.50 + \$0.40\lfloor 5 \cdot 8.75 \rfloor \\ &= \$2.50 + \$0.40\lfloor 43.75 \rfloor \\ &= \$2.50 + \$0.40 \cdot 43 \\ &= \$19.70 \end{aligned}$$

STOP QY2

QY2

What is the charge for a 15.3-mile trip in a NYC taxi?

GUIDED

Example 4

Users of pre-paid calling cards are billed in 1-minute increments. This means that customers are billed for a full minute when any part of a minute is used. If the Call-Me-Often Phone Card Company charges \$0.03 per minute with a 1-minute billing increment, what is the charge for a 5-minute, 40-second phone call?

Solution Call-Me-Often's charge is rounded up to the nearest minute, so use a ceiling function.

$0.03\left\lceil 5\frac{40}{60} \right\rceil = 0.03(\underline{\ ?\ }) = \underline{\ ?\ }$

Call-Me-Often charges ___?___ *cents for the call.*

Questions

COVERING THE IDEAS

In 1 and 2, refer to the postage example at the beginning of this lesson.

1. What is the cost to mail a letter weighing 4.3 ounces?
2. What is the domain of the function?
3. In your own words, write the meaning of $\lceil x \rceil$. Why do you think it is also called the ceiling function?

In 4–7, evaluate.

4. $\left\lfloor 4\frac{3}{4} \right\rfloor$
5. $\lfloor 4\pi \rfloor$
6. $\lfloor -5.87 \rfloor$
7. $\lceil 7 - 0.5 \rceil$
8. a. **Fill in the Blanks** The function f defined by $f(x) = \lfloor x \rfloor$ is called the __?__ or __?__ function.
   b. The range of $f: x \rightarrow \lfloor x \rfloor$ is __?__.
   c. Why are there open circles at (1, 0), (2, 1), (3, 2), and so on in the graph of f?
9. Give the domain and range of the function.
   a. $f(x) = \lceil x \rceil$
   b. the function in Example 3
10. Refer to Example 4. A 2-minute billing increment charges for parts of minutes as if they were the next even minute (for example, a 3-minute call is billed for 4 minutes). If Call-Me-Often Phone Card Company charges \$0.03 per minute with a 2-minute billing increment, what does an 18-minute, 10-second phone call cost?

APPLYING THE MATHEMATICS

11. Let $r(x) = \lfloor x + 0.5 \rfloor$.
    a. Find $r(1.2)$.
    b. Find $r(1.7)$.
    c. What kind of rounding does r do?

In 12 and 13, an auditorium used for a high school graduation has 750 seats available for its g graduates.

12. **Multiple Choice** If the tickets are divided evenly among the graduates, which of the following represents the number of tickets each graduate may have?

A $\left\lceil \frac{750}{g} \right\rceil$ B $\left\lfloor \frac{750}{g} \right\rfloor$ C $\left\lceil \frac{g}{750} \right\rceil$ D $\left\lfloor \frac{g}{750} \right\rfloor$

13. Write an expression for the number of tickets left over, if any, after each graduate gets his or her tickets.

14. A used-car salesperson is paid \$350 per week plus a commission of \$100 for each \$1500 in sales during the week.

a. Find the salesperson's salary during a week in which he or she had \$3500 in sales.

b. When the person has d dollars in sales, write an equation that gives the weekly earnings E.

c. Is it possible for the salesperson to earn exactly \$1000 a week? Why or why not?

15. The table at the right shows the typical fees charged by the postal service for its COD (collect on delivery) service as a function of the amount of money to be collected from the recipient (as of 2008).

Amount Collected from Recipient (dollars)	COD Fee (dollars)
0.01 to \$50.00	5.10
50.01 to 100.00	6.25
100.01 to 200.00	7.40
200.01 to 300.00	8.55
300.01 to 400.00	9.70
400.01 to 500.00	10.85
500.01 to 600.00	12.00
600.01 to 700.00	13.15
700.01 to 800.00	14.30
800.01 to 900.00	15.45
900.01 to 1000.00	16.60

a. Can these data be modeled by a step function?

b. **Fill in the Blank** Complete the following piecewise definition of a function that gives the COD fee $F(a)$ (in dollars) as a function of the amount a (in dollars) to be collected.

$$F(a) = \begin{cases} 5.10, \text{ if } a \leq 50 \\ \underline{\ ?\ }, \text{ if } 50 < a \leq 1000 \end{cases}$$

16. The Fine Furniture Factory pays employees a bonus based on their monthly sales. For sales of \$5,000 up to \$25,000 the bonus is \$500. For sales of \$25,000 up to \$40,000, the bonus is \$1,000. For sales of \$40,000 or more, the bonus is \$2,000. Write a piecewise linear function to give the bonus b for monthly sales m.

17. The formula $W = d + 2m + \left\lfloor \frac{3(m+1)}{5} \right\rfloor + y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor + 2$ gives the day of the week based on our current calendar where d = the day of the month of the given date; m = the number of the month in the year with January and February regarded as the 13th and 14th months of the previous year; that is, 2/22/90 is 14/22/89. The other months are numbered 3 to 12 as usual; and y = the year as a 4-digit number. Once W is computed, divide by 7 and the remainder is the day of the week, with Saturday = 0, Sunday = 1, …, Friday = 6. Enter the formula into a spreadsheet to answer the questions.

a. On what day of the week were you born?

b. On what day of the week was the Declaration of Independence adopted?

c. On what day of the week was January 1, 2001, the first day of the current millennium?

REVIEW

18. **Multiple Choice** Which of the following is *not* an arithmetic sequence? (Lesson 3-8)

A $a_n = 3 - 7n$

B $b_n = n + n$

C $\begin{cases} c_1 = 1 \\ c_2 = 5 \\ c_n = c_{n-1} + c_{n-2}, \text{ for } n \geq 3 \end{cases}$

D $\begin{cases} d_1 = 1 \\ d_2 = 5 \\ d_n = d_{n-1} + 4, \text{ for } n \geq 3 \end{cases}$

E $\begin{cases} e_1 = -6 \\ e_n = e_{n-1} + 4, \text{ for } n \geq 2 \end{cases}$

19. Consider the arithmetic sequence $\sqrt{2}, \sqrt{2} + 2\sqrt{3}, \sqrt{2} + 4\sqrt{3}, \sqrt{2} + 6\sqrt{3}, \ldots$. (Lesson 3-8)
   a. Write a recursive definition of the sequence.
   b. Write an explicit formula for the nth term of the sequence.
   c. Find the 101st term of the sequence.

20. The table at the right shows the number of voters (in thousands) that voted in each of the presidential elections in the United States from 1980 to 2004. (Lesson 3-5)
   a. Find an equation for the regression line for these data.
   b. According to the answer in Part a, what would be the predicted voter turnout in 2008?
   c. What is the slope of the line you found in Part a? Name a real-life factor that may influence this slope.

Year	Voters (thousands)
1980	86,515
1984	92,652
1988	91,594
1992	104,405
1996	96,456
2000	105,586
2004	122,294

21. A line passes through the points (2, 2) and (0, -3). (Lesson 3-4)
   a. Find an equation for this line in point-slope form using the point (2, 2).
   b. Find an equation for this line in point-slope form using the point (0, -3).
   c. Verify that your equations from Parts a and b are equivalent.

EXPLORATION

22. a. Solve the equation $\left\lfloor \frac{x}{2} \right\rfloor = \frac{x}{2}$.
   b. Generalize Part a to solve $\left\lfloor \frac{x}{n} \right\rfloor = \frac{x}{n}$.

QY ANSWERS

1. Answers vary. Sample: floor(x); ceiling(x)

2. \$32.90