Lesson 3-8

Formulas for Linear (Arithmetic) Sequences

Vocabulary

linear sequence,
arithmetic sequence

BIG IDEA As with other linear functions, two points on the graph of a *sequence*, or one point on the graph and its constant difference, are enough to determine an explicit formula.

Mental Math

Tell whether the graph of the function is a line, a parabola, a hyperbola, or none of these.

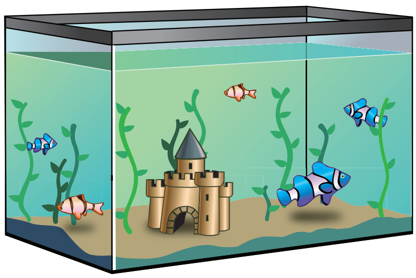
a. $f(x) = 6x^2$

b. $g(x) = -2x$

c. $h(x) = -\frac{3}{x^2}$

d. $j(x) = \frac{17}{x}$

Demetrius is adding water to a fish tank one gallon at a time. The tank weighs 32 pounds when empty and each gallon of water weighs 8.33 pounds. Let $W(g) = 32 + 8.33g$ represent the weight of the tank after g gallons of water are added. Notice that this equation is in slope-intercept form with a slope of 8.33 and a y-intercept of 32. The increase of 8.33 pounds after each gallon is added is the *constant difference* between terms of the sequence. In this case, the constant difference is a constant increase.



A sequence with a *constant difference* between successive terms is called a **linear** or **arithmetic sequence**. (Here the word *arithmetic* is used as an adjective; it is pronounced a-rith-MEH-tik.) Because there is a constant difference between successive terms, a linear sequence is a linear function with the domain of a sequence.

Developing an Explicit Formula for an Arithmetic Sequence

As you learned in Lesson 1-8, an explicit formula for a sequence can be developed by examining the pattern in a table.

Example 1

a. Write an explicit formula for the arithmetic sequence 4, 7, 10, 13, ...

b. Compute the 30th term of the sequence.

Solution

a. To develop an explicit formula, use the constant difference to write each term after the first. Consider the pattern in the table at the right.

n = number of term	term = a_n
1	4
2	$4 + 1 \cdot 3 = 7$
3	$4 + 2 \cdot 3 = 10$
4	$4 + 3 \cdot 3 = 13$
5	$4 + 4 \cdot 3 = 16$
⋮	⋮
n	$4 + (n - 1) \cdot 3 = a_n$

Notice that in term n, the number of 3s added to the initial term is 1 less than n. So, an explicit formula for the sequence is

$a_n = 4 + (n - 1) \cdot 3.$

$a_n = 3n + 1$

b. a_{30} is the 30th term of the sequence. Substitute 30 for n.

$a_{30} = 3\,(30) + 1$

$a_{30} = 91$

Because the sequence in Example 1 is a function, it can be written as a set of ordered pairs: $\{(1, 4), (2, 7), (3, 10), \ldots\}$. These points lie on a line with slope, or constant increase, of 3. Substitute this slope and the point (1, 4) in the point-slope form of the equation of a line.

$y - 4 = 3(x - 1)$

$a_n - 4 = 3(n - 1)$ Substitute.

$a_n = 4 + 3(n - 1) = 3n + 1$ Solve for a_n.

This is the formula found in Part a of Example 1. This suggests the following theorem.

*n*th Term of an Arithmetic Sequence Theorem

The nth term a_n of an arithmetic (linear) sequence with first term a_1 and constant difference d is given by the explicit formula $a_n = a_1 + (n - 1)d$.

Proof Each ordered pair of the arithmetic sequence is of the form $(x, y) = (n, a_n)$. The first ordered pair is $(x_1, y_1) = (1, a_1)$. Because arithmetic sequences represent constant-increase or constant-decrease situations, a graph of the sequence consists of points that lie on a line. The slope of the line is the constant difference d. Substitute these values into the point-slope form of a linear equation.

$y - y_1 = m(x - x_1)$ Point-slope form

$a_n - a_1 = d(n - 1)$ Substitute.

$a_n = a_1 + (n - 1)d$ Solve for a_n.

STOP **QY**

QY

Use the theorem to find an explicit formula for the sequence 6, 10, 14, 18, ...

Recursive Notation for Arithmetic Sequences

A recursive formula for the arithmetic sequence in Example 1 is

$$\begin{cases} a_1 = 4 \\ a_n = a_{n-1} + 3, \text{ for integers } n \geq 2. \end{cases}$$

The second line of this formula can be rewritten as

$$a_n - a_{n-1} = 3.$$

This shows the constant difference between term n and term $(n-1)$. The constant difference 3 is the slope between the points $(n-1, a_{n-1})$ and (n, a_n).

More generally, suppose a sequence is defined recursively as

$$\begin{cases} a_1 \\ a_n = a_{n-1} + d, \text{ for integers } n \geq 2. \end{cases}$$

When $n \geq 2$, we can rewrite the second line as $a_n - a_{n-1} = d$. This means that the difference between consecutive terms is the constant d. By definition, the sequence is arithmetic. This proves the following theorem.

Constant-Difference Sequence Theorem

The sequence defined by the recursive formula

$$\begin{cases} a_1 \\ a_n = a_{n-1} + d, \text{ for integers } n \geq 2 \end{cases}$$

is the arithmetic sequence with first term a_1 and constant difference d.

Example 2

A cell-phone company charges 25¢ per minute for overseas calls along with a 30¢ service charge.

a. Write a recursive formula for the arithmetic sequence that represents the cost of a call lasting n minutes.

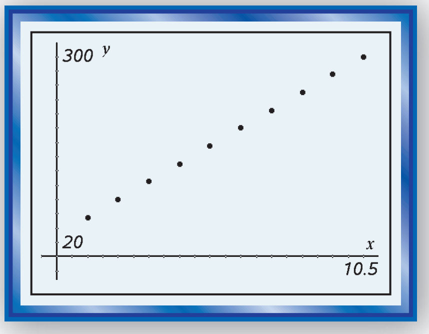
b. Graph the sequence.

Solution

a. When the call begins you are immediately charged 55¢ (25¢ for your first minute plus the 30¢ service charge). Then you pay 25¢ for each additional minute that the call lasts, so the difference between successive terms is $d = 25$. A recursive formula for this arithmetic sequence is

$$\begin{cases} a_1 = 55 \\ a_n = a_{n-1} + 25, \text{ for integers } n \geq 2\,. \end{cases}$$

b. You can graph arithmetic sequences using the methods you used in Lesson 3-7. The graph of the first ten values of the sequence is shown at the right.



You should be able to translate from a recursive formula to an explicit formula for an arithmetic sequence, and vice versa. To go from recursive to explicit form you can use the theorems on the previous two pages. Use the second theorem to find a_1 and d and then substitute the values into the first theorem. To go from explicit to recursive form, you can use the explicit formula to find a_1, and substitute the known values of a_1 and d into the recursive pattern for an arithmetic sequence.

Example 3

Some car dealers offer interest-free loans, provided that you pay back the amount borrowed by a certain date. Suppose a car that costs \$14,736 requires monthly payments of \$245.60. Let a_n be the amount you still owe after n months.

a. Write a recursive formula for this sequence.

b. Write an explicit formula for this sequence.

Solution

a. After 1 month you owe \$14,736 − \$245.60 = \$14,490.40, so $a_1 = 14{,}490.40$. Each month you pay \$245.60, so the difference between successive terms is $d = -245.60$. A recursive formula for this sequence is

$$\begin{cases} a_1 = 14{,}490.40 \\ a_n = a_{n-1} - 245.60, \text{ for integers } n \geq 2. \end{cases}$$

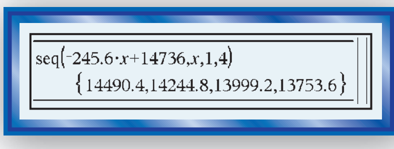
b. You know that $a_1 = 14{,}490.40$ and $d = -245.60$.

So, $a_n = 14{,}490.40 + (n - 1)(-245.60)$

$a_n = -245.60n + 14{,}736$

where n = the number of payments and a_n = the current balance owed.

Many calculators have a sequence command `seq` that lets you generate the terms of a sequence if you know its explicit form. At the right is how one calculator generates the first four terms of the sequence in Example 3. The "x,1,4" tells the machine to start at $x = 1$ and end at $x = 4$.



Questions

COVERING THE IDEAS

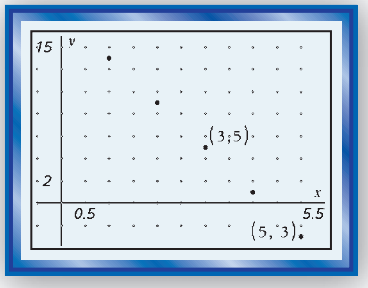
1. Write the 50th term of the sequence from Example 1.
2. What is an arithmetic sequence?
3. What is an explicit formula for the nth term of an arithmetic sequence with first term a_1 and constant difference d?
4. What is the connection between the slope of a linear function and the constant difference of an arithmetic sequence?
5. Consider the arithmetic sequence 12, 16, 20, 24, 28, … .
   a. Write an explicit formula for the nth term of the sequence.
   b. Write a recursive definition of the sequence.
   c. Calculate the 47th term of this sequence.

In 6 and 7, refer to Example 2.

6. What term of the sequence gives the cost of a 20-minute overseas phone call?
7. Rewrite the formula if the service charge is $1.00 and the cost per minute is 30¢.
8. An arithmetic sequence has an initial term of –127 and a constant difference of 42.
   a. Write an explicit formula for the sequence.
   b. Use the `seq` command on a calculator to find the first seven terms.
9. Suppose $t_n = 12 + 7(n - 1)$.
   a. Write a recursive formula for the arithmetic sequence t.
   b. Compute t_{77}.
10. Consider the sequence $\begin{cases} a_1 = 10.5 \\ a_n = a_{n-1} + 4.3, \text{ for integers } n \geq 2. \end{cases}$
    a. Write its first three terms.
    b. Write an explicit formula for the sequence.
11. Write a recursive formula for the linear sequence –70, –47.5, –25, –2.5, … .

APPLYING THE MATHEMATICS

12. In Chapter 1, the sequence of triangular numbers was described by the explicit formula $t(n) = \frac{n(n+1)}{2}$. Is this an example of an arithmetic sequence? Why or why not?



13. At the right is a graph of the first five terms of an arithmetic sequence. Write an explicit formula for the sequence.

14. A formula for the sum of the measures of the interior angles of a convex polygon is $S_n = 180(n - 2)$ for $n \geq 3$, where n is the number of sides of the polygon.
   a. Evaluate S_n for $n = 3, 4, 5, 6, 7$.
   b. Explain why the results of Part a represent the terms of an arithmetic sequence.
   c. Find a recursive formula for S_n.

In 15 and 16, a local radio station holds a weekly contest to give away a cash prize. The announcer calls a number and if the person who answers guesses the correct amount of money in the pot, he or she wins the money. If the person misses, $25 is added to the money pot.

15. On the 12th call, a contestant won $675. How much was in the pot at the beginning?

16. Suppose the pot starts with $140. On what call could the winner receive $1,115?

17. A 16 ounce jar candle is advertised to burn an average of 110 hours. Suppose that the candle burns at a constant rate.
   a. Write the first three terms of the sequence that shows how many ounces of the candle remain after each hour it burns.
   b. Write the explicit formula for r_n, the number of ounces of the candle left after n hours of burning.

REVIEW

18. Graph the first eight terms of the sequence $r_n = \frac{n^3}{n!}$. Describe how the value of r_n changes as n increases. Recall that $n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. **(Lesson 3-7)**

19. Verify that the explicit formula for the nth triangular number $T_n = \frac{n(n+1)}{2}$ satisfies the recursive formula
$$\begin{cases} T_1 = 1 \\ T_n = T_{n-1} + n. \end{cases}$$
(Lessons 3-6, 1-8)

20. Write the first twelve terms of the sequence defined by
$$\begin{cases} p_1 = 1 \\ p_2 = -1 \\ p_n = p_{n-1} \cdot p_{n-2}, \text{ for integers } n \geq 3. \end{cases}$$
(Lesson 3-6)

21. **Multiple Choice** Which of the following are *not* sufficient criteria to determine a (unique) line? (**Lesson 3-4**)

A two distinct points

B a point and a slope

C a slope and a y-intercept

D a slope and an x-intercept

E All of the above uniquely determine a line.

22. Suppose you have two solutions of oxalic acid, one at 0.1 mol/L, the other 0.5 mol/L. (**Lessons 3-2**)

a. If you mix A liters of the first solution with B liters of the second, what will be the concentration of the resulting solution?

b. How many liters of the 0.5 mol/L solution must be added to 4 liters of the 0.1 mol/L solution to get 1 mole of acid in the final mixture?

23. Suppose y varies directly as x, and $y = -24$ when $x = 48$. (**Lesson 2-4, 2-1**)

a. Find the constant of variation k.

b. How is k represented on the graph of the function?

EXPLORATION

24. The following recursively defined sequence generates numbers known as *hailstone numbers*.

$$\begin{cases} a_1 \text{ is any positive integer.} \\ \text{For } n > 1, \text{ If } a_{n-1} \text{ is even, } a_n = \frac{a_{n-1}}{2}. \\ \qquad\qquad\quad \text{If } a_{n-1} \text{ is odd, } a_n = 3a_{n-1} + 1. \end{cases}$$

For example, if $a_1 = 46$, then $a_2 = 23$, $a_3 = 70$, $a_4 = 35$, $a_5 = 106\ldots$.

a. Explore this sequence rule for at least five different values of a_1. Continue generating terms until you can predict what will happen in the long run.

b. Look up hailstone numbers on the Internet to find out how they got their name and what is known about them. Briefly describe what you find.

Hailstones rise and fall within clouds, growing larger and larger until they are so heavy they fall out of the cloud.

QY ANSWER

$a_n = 6 + (n - 1) \cdot 4 = 4n + 2$