

Lesson

3-7

Graphs of Sequences

Vocabulary

Fibonacci sequence

► **BIG IDEA** Sequences are graphed like other functions. The major differences are that the graph of a sequence is discrete and you can obtain some values of sequences using a recursive definition.

As you saw in Lesson 1-8 and in the last lesson, sequences can be described in two ways:

- An *explicit formula* gives an expression for the n th term of a sequence in terms of n . An example is $a_n = 4n - 6$.
- A *recursive formula* gives a first term or first few terms and an expression for the n th term of a sequence in terms of previous terms. An example is

$$\begin{cases} a_1 = -2 \\ a_n = a_{n-1} + 4, \text{ for } n > 1. \end{cases}$$

To graph a sequence, plot each ordered pair (n, a_n) . You can generate the ordered pairs using a written description of a sequence, an explicit formula, or a recursive formula. The next three examples explore these possibilities.

Mental Math

- What is 30% of $70x$?
- What is 70% of $30x$?
- What is 75% of $8x - 24y$?

GUIDED

Example 1

Consider the sequence with recursive formula $\begin{cases} a_1 = -2 \\ a_n = a_{n-1} + 4, \text{ for } n > 1. \end{cases}$

- Make a table of values of the first six terms of this sequence.
- Graph the first six terms of the sequence.

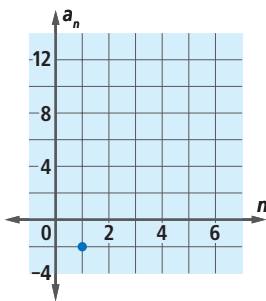
Solution

- Make a table with n in one column and a_n in another column, as shown at the right. From the recursive definition, $a_1 = -2$ and each succeeding term is 4 larger than the previous term.

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n	a_n
1	-2
2	?
3	?
4	?
5	?
6	?

- b. Plot the points from your table on a coordinate grid, with n as the independent variable. The points should be collinear. (The graph at the right is not complete.)



In Example 1, you could calculate each term of the sequence by hand. Using a spreadsheet, you can automatically generate the terms of a sequence if you know an explicit or recursive formula for it. Example 2 shows how to do this, and also points out an important difference between the graph of a line and the graph of a sequence.

Example 2

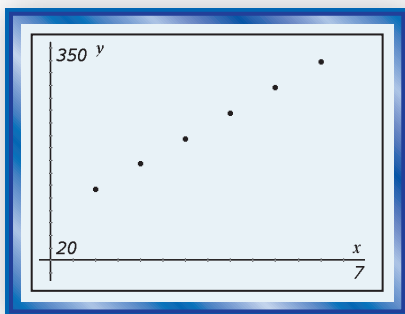
Recall the explicit formula for the sequence of lengths L_n of a train with n boxcars from Lesson 3-6, $L_n = 72 + 41n$.

- Graph the first six terms of L_n using the explicit formula.
- Graph the function $L(n) = 72 + 41n$ using a domain of the set of all reals on the same axes as the graph of the sequence.
- Compare and contrast the graphs.

Solution

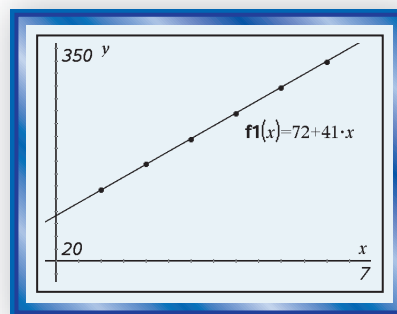
- Enter the index values 1 through 6 into column A of a spreadsheet. Then use the explicit formula to generate values for the train-length sequence. On one calculator, we do this by entering the formula at the top of column B, as shown at the right. On other machines, you might do this by using the fill function.

Create a scatterplot of the sequence, as shown below.



- Graph $L(n) = 72 + 41n$ on your plot from Part a, as shown at the right.

A	B	C	D	E	F
	=72+41*a				
1	113				
2	154				
3	195				
4	236				
5	277				
6	318				
AI	1				



- c. Notice the similarities and differences between the two graphs.
Both graphs have the same constant rate of change of 41 feet per car. The difference is that the line graph is continuous, while the sequence graph is discrete.

Graphing a Sequence Using a Recursive Formula

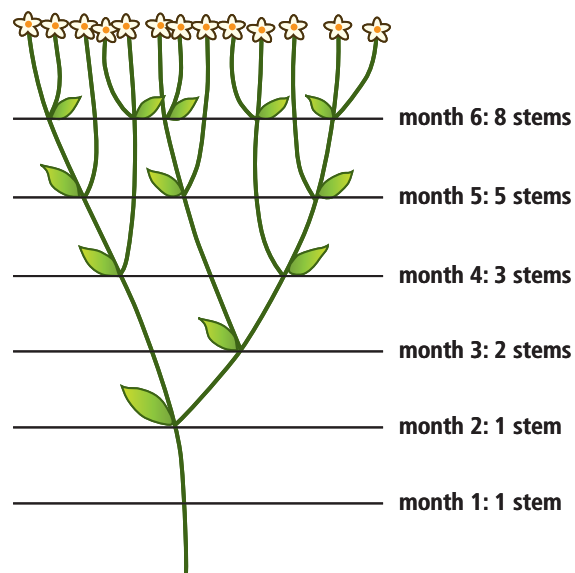
Sometimes, it is easier to write a recursive formula for a sequence than an explicit formula. Example 3 illustrates such a situation. It also shows how spreadsheets can be used to generate terms and graph a recursively defined sequence.

Example 3

The sneezewort plant (also called sneezeweed or *Achillea ptarmica*) starts as a single stem. After two months of growth, the stem sends off a shoot that becomes a new stem, and produces a new shoot every month thereafter. The new shoots must mature for two months before they are strong enough to produce shoots of their own.

Let F_n be the number of stems in month n . The first two terms are equal to 1. Beginning with the third term, each term is found by adding the previous two terms.

- Using F_n to represent the number of stems and shoots in month n , write a recursive formula for the sequence.
- Graph the first ten terms of the sequence.



Solution

- After the first two terms, each term of the sequence is calculated by adding the previous two terms. As you know, when F_n is the n th term, then the previous $(n - 1)$ st term is F_{n-1} . Similarly, the term preceding F_{n-1} is the $(n - 2)$ nd term, or F_{n-2} . So a recursive definition is

$$\begin{cases} F_1 = 1 \\ F_2 = 1 \\ F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 3. \end{cases}$$

(continued on next page)

Alternatively, you could write

$$\begin{cases} F_1 = 1 \\ F_2 = 1 \\ F_{n+1} = F_n + F_{n-1}, \text{ for } n \geq 1. \end{cases}$$

- b. Use a spreadsheet. Enter the index numbers 1 through 10 into column A. The terms of the sequence will be in column B.

The recursive definition from Part a says that $F_1 = F_2 = 1$. So enter 1 in each cell B1 and B2.

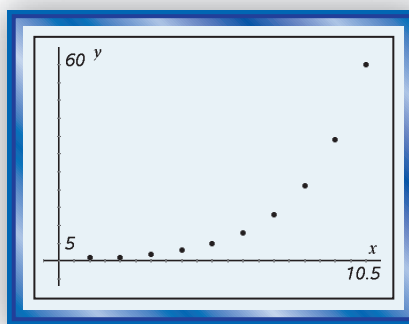
Each of the terms F_3 through F_{10} is defined as the sum of the two previous terms. So, each cell from B3 through B10 needs to be defined as the sum of the previous two cells. Enter $=B1+B2$ in cell B3, as shown at the right. Then copy and paste this formula into cells B4 through B10 to generate the rest of the desired terms of the sequence. When this is done, 55 should appear in cell B10, as shown below at the left.

A	B	C	D	E	F
1	1				
2	1				
3	2				
4					
5					
6					

B3 =b1+b2

A	B	C	D	E	F
5	5				
6	8				
7	13				
8	21				
9	34				
10	55				

B10 =b8+b9



Create a scatter plot of the sequence, as shown above at the right.

The sequence in Example 3 is called the **Fibonacci** (pronounced “Fee-boh-NOTCH-ee”) **sequence**. It is named after Leonardo of Pisa, a 12th century mathematician who wrote under the name Fibonacci. The Fibonacci numbers arise in a wide variety of contexts, and are so mathematically rich that there is an entire publication, the *Fibonacci Quarterly*, devoted to the mathematics arising from them.

STOP QY

You may also have noticed that the points on the graphs in Examples 1 and 2 are collinear, but the points on the graph in Example 3 are not. This is because in the sequences in Examples 1 and 2, there is a constant difference between terms, but in the Fibonacci sequence, the difference between terms is not constant. The sequences in Examples 1 and 2 are examples of *arithmetic sequences*. You will learn more about arithmetic sequences in the next lesson.

QY

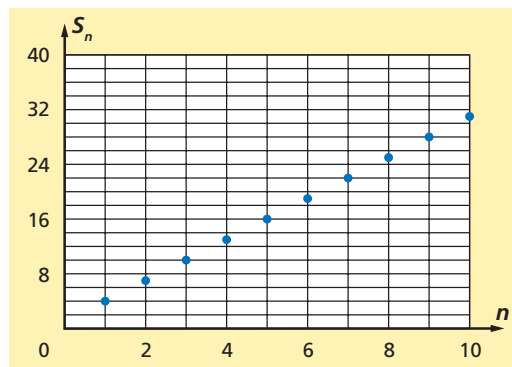
In the sequence F of Example 3, find F_{11} .

Questions

COVERING THE IDEAS

- Refer to Example 1.
 - What is a_{10} ?
 - Write a recursive rule for a_{n+1} .
- Refer to Example 2.
 - Write the recursive formula for L_n .
 - Explain why it does not make sense to let n be any real number.
- Multiple Choice** Which formula below gives the sequence graphed at the right?

A $s_n = n^2$	B $\begin{cases} s_1 = 1 \\ s_n = s_{n-1} + 1, n > 1 \end{cases}$
C $\begin{cases} s_1 = 40 \\ s_n = s_{n-1} - 3, n > 1 \end{cases}$	D $\begin{cases} s_1 = 4 \\ s_n = s_{n-1} + 3, n > 1 \end{cases}$
- Give one reason why you might choose to model a situation with a recursively defined sequence rather than an explicitly defined one.
 - What is the tenth Fibonacci number?
 - How many 3-digit Fibonacci numbers are there?
 - How many 5-digit Fibonacci numbers are there?



APPLYING THE MATHEMATICS

- Graph the first ten terms of the sequence whose explicit formula is $s_n = n^2$.
 - On the same axes, graph the first ten terms of the sequence whose recursive formula is

$$\begin{cases} r_1 = 1 \\ r_n = r_{n-1} + 2n - 1, n > 1. \end{cases}$$
 - What do the graphs suggest about the sequence of numbers 1, 4, 9, 16, ... ?
- Forrest, a crime scene investigator, is called to investigate a missing person's case. On entering the person's apartment, he discovers hundreds of bugs crawling around. Suppose that the number of bugs increases by 20% every day the apartment is left unoccupied, and that on the first day the apartment was left unoccupied, there were about 10 bugs.
 - Write a recursive definition that describes the sequence giving the number b_n of bugs on day n .
 - Graph the first 20 terms of the sequence you found in Part a.
 - A detailed census determines that there are 450 bugs in the apartment. For how long has the apartment been unoccupied?

8. The *tribonacci* numbers R_n are defined as follows:
- (1) The first three tribonacci numbers are 1, 1, and 2.
 - (2) Each later tribonacci number is the sum of the *three* preceding numbers.
 - a. Using the rule, the fourth tribonacci number is $1 + 1 + 2 = 4$. Compute the fifth, sixth, and seventh tribonacci numbers.
 - b. Give a recursive formula for the tribonacci numbers.
 - c. Use a spreadsheet to find the 17th tribonacci number greater than 1000.
9. Kamilah gets a part-time job and saves \$75 each week. At the start of the summer, her bank account has a balance of \$500. Let b_n be the balance at the start of week n (so $b_1 = \$500$).
- a. Make a table of values of b_n for $n = 1, 2, \dots, 6$.
 - b. Write a recursive formula for b_n .
 - c. Make a graph of the first 20 values of b_n .
 - d. Are the points on the graph collinear? Justify your answer.
10. A population of rabbits is counted annually; the number of rabbits in year n is r_n . The values of r_n can be modeled by a recursively defined sequence:
- $$\begin{cases} r_1 = 150 \\ r_{n+1} = 0.008r_n(200 - r_n), \text{ for } n \geq 1. \end{cases}$$
- a. Graph the first 20 terms of this sequence.
 - b. In the long term, what happens to the population of rabbits? Does it increase, decrease, stabilize at a particular value, or follow some other pattern you can describe?
 - c. Try graphing the sequence with several different values of r_1 , for $2 \leq r_1 < 200$. How does the starting value affect the long-term population?
11. a. Graph the first ten terms of the sequence given by the recursive formula $a_n = \frac{1}{2}\left(a_{n-1} + \frac{9}{a_{n-1}}\right)$, $n > 1$, with $a_1 = 20$.
- b. Describe any patterns you see in the graph in words. For example, do the terms get larger or smaller? Do they appear to get closer to a particular number?
- c. Repeat Parts a and b using the recursive formula $a_1 = 20$, $a_n = \frac{1}{2}\left(a_{n-1} + \frac{25}{a_{n-1}}\right)$. How do your answers change?



The largest litter of rabbits on record is 24. This occurred in 1978 and 1999.

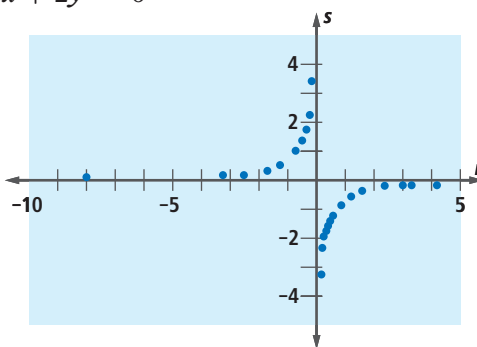
REVIEW

12. Write the first five terms of the sequence defined by
- $$\begin{cases} b_1 = 2 \\ b_{n+1} = 2b_n + 1, \text{ for integers } n \geq 1. \end{cases} \quad (\text{Lesson 3-6})$$

13. Consider the sequence that begins 81, 27, 9, 3, ... (Lesson 3-6)
- Fill in the Blank** From the second term on, each term is _____ the previous term.
 - Write a recursive formula for this sequence.
 - Compute the next four terms in the sequence.
14. The table at the right lists body lengths and weights for several humpback whales. Compute the regression line for these data and use it to estimate the weight of a humpback whale that is 12 meters long. (Lesson 3-5)
15. **Multiple Choice** Which of the following equations describes a line that does not have an x -intercept? (Lesson 3-3)
- A $4x + 3y = 0$ B $2x - 2y = 1$ C $0x + 0y = 3$
 D $-3x + 0y = 0$ E $0x + 2y = -6$

Length (meters)	Weight (metric tons)
12.5	23
13.4	30
13.5	29
14.5	36
13.0	27
14.0	32

16. Two perpendicular lines \overleftrightarrow{AB} and \overleftrightarrow{AC} have been drawn and their slopes measured. If the slope of \overleftrightarrow{AB} is r , then the slope s of \overleftrightarrow{AC} is a function of r . The point (r, s) is plotted as the lines rotate. (Lesson 2-7)



Slope \overleftrightarrow{AB} ; r	Slope \overleftrightarrow{AC} ; s
4.11	-0.24
3.30	-0.30
3.05	-0.33
2.35	-0.42
1.76	-0.57
1.40	-0.71
1.17	-0.86
0.99	-1.01
0.86	-1.16
0.77	-1.29
0.70	-1.44
0.63	-1.58
0.58	-1.72
0.53	-1.87
-18.08	0.06

- Use the graph to determine whether there is a direct or inverse variation relationship between r and s .
- Ana used the table function of her calculator to generate the table of s and r values at the right. Based on her data, determine an equation relating r and s .

EXPLORATION

17. The Fibonacci numbers have many arithmetic properties. Label the Fibonacci numbers $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, and so on. Use a list of the first 15 Fibonacci numbers to explore each of the following.
- Let s_n be the sum of the first n Fibonacci numbers. That is, $s_1 = F_1$, $s_2 = F_1 + F_2$, $s_3 = F_1 + F_2 + F_3$, etc. Make a table of values of s_n for $n = 1, 2, \dots, 14$.
 - Find a pattern in your table that allows you to quickly compute F_n from a list of Fibonacci numbers.
 - Do some research and find some other arithmetical properties of the Fibonacci numbers.

QY ANSWER

$$F_{11} = 89$$