Lesson 3-6

Recursive Formulas for Sequences

Vocabulary

recursive formula

recursive definition

BIG IDEA Some sequences can be defined by giving their first terms and indicating how later terms are related to each other.

Recall from Chapter 1 that a *sequence* is a function whose domain is the set of natural numbers or a subset of the natural numbers less than a given number. If you know an *explicit formula* for a sequence, then you can write the terms of a sequence rather quickly.

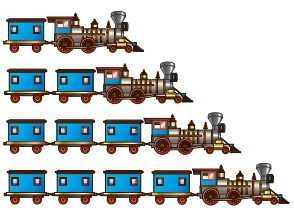
Consider a freight train consisting of one engine and a series of identical boxcars. The engine is 72 feet long and each boxcar is 41 feet long. So the length of a train with n boxcars is defined by the explicit formula $L_n = 72 + 41n$. The first four terms of the sequence are:

$L_1 = 72 + 41 \cdot 1 = 113$

$L_2 = 72 + 41 \cdot 2 = 154$

$L_3 = 72 + 41 \cdot 3 = 195$

$L_4 = 72 + 41 \cdot 4 = 236$



We say the formula $L_n = 72 + 41n$ *generates* the sequence 113, 154, 195, 236, ….

STOP QY1

Mental Math

Consider the sequence $t_n = \frac{n^2}{2} + 7$.

a. Find t_7.

b. Find t_{100}.

c. Find $t_{12} - t_{10}$.

d. What is the domain of the sequence?

QY1

Generate the first five terms of a sequence S when $S_n = 34 - 12(n - 1)$.

Generating Sequences Using a Calculator

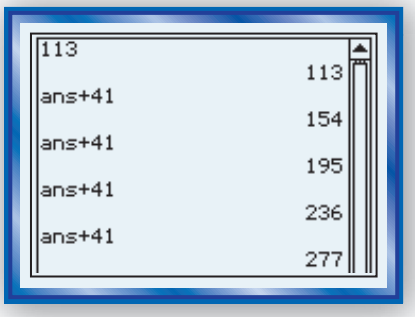
You can use a calculator to generate sequences.

Activity

Use a calculator to generate the sequence of freight train lengths.

Step 1 Store the first value of this sequence, 113. When we enter 113 and press ENTER, this calculator stores it in the variable ans.

Step 2 Add 41 to ans, and repeatedly press ENTER. The first five terms of the sequence are shown at the right. To find additional terms, continue pressing ENTER.



STOP QY2

Any sequence of numbers that is defined by adding, subtracting, multiplying, or dividing by a constant can be generated on your calculator using the approach of the Activity.

▶ QY2

In the Activity, if you pressed ENTER 35 times, what term of the sequence would appear?

What Is a Recursive Formula?

Notice that no explicit formula was used to generate the sequence in the Activity. Instead, each term of the sequence was derived from the preceding one. For instance, to find the 6th term of the sequence, you add 41 to the preceding term (the 5th term, 271). This process generates the sequence *recursively*.

Definition of Recursive Formula

A **recursive formula** or **recursive definition** for a sequence is a set of statements that

a. indicates the first term (or first few terms) of the sequence, and
b. tells how the next term is calculated from the previous term or terms.

Consider the sequence $L = 113, 154, 195, 236, \ldots$ of lengths of a train with n boxcars. You can write this sequence recursively as

$$\begin{cases} L_1 = 113 \\ L_n = \text{previous term} + 41, \text{ for integers } n \geq 2. \end{cases}$$

The brace { indicates that both lines are needed for the recursive definition. You should be able to read and evaluate recursive definitions.

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Example 1

Verify that the recursive definition of L above generates the first four terms of the sequence.

Solution The first term is given,

$L_1 = 113$.

According to the second line of the definition,

$L_2 = \text{previous term} + 41 = \underline{\ ?\ } + 41 = \underline{\ ?\ }$.

To find L_3, use the definition again.

$L_3 = \text{previous term} + 41 = \underline{\ ?\ } + 41 = \underline{\ ?\ }$.

(continued on next page)

L_4 = previous term + 41 = __?__ + 41 = __?__.

The first four terms are 113, __?__, __?__, __?__, which checks according to what was calculated before.

Notation for Recursive Formulas

In Example 1, the sequence is described using the symbol L and the symbol L_n denotes term number n. The term that precedes term n is term number $(n - 1)$, and the term that follows term n is term number $(n + 1)$. So the symbol L_{n-1} can be used in place of the words *previous term*. Thus the sequence of Example 1 can be defined as follows:

$$\begin{cases} L_1 = 113 \\ L_n = L_{n-1} + 41, \text{ for integers } n \geq 2 \end{cases}$$

Or, if you prefer, you could let L_n be the previous term and L_{n+1} be the next term. In this case, the definition would be

$$\begin{cases} L_1 = 113 \\ L_{n+1} = L_n + 41, \text{ for integers } n \geq 1. \end{cases}$$

A recursive definition always includes at least two lines. One or more lines define the initial values of the sequence, and another line defines the relationship between consecutive terms.

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Example 2

Consider the sequence A = 10,000; 9600; 9200; ... of dollar amounts left on a car loan if $400 is repaid each month. Write a recursive definition for the sequence A_n.

Solution The first line of the definition gives the initial term. So, the first line is

A_1 = __?__.

The second line relates each term to the previous term. So, the second line is

$A_{n+1} = A_n -$ __?__, for integers $n \geq$ __?__.

If you can describe a sequence in words, then you can use that description to write a recursive formula for the sequence.

Example 3

The NCAA Women's Basketball Tournament originally consists of 64 teams, which are paired in single-elimination contests. At the end of each round of play, the number of teams proceeding to the next round is half the previous number.

The sequence 64, 32, 16, 8, 4, 2 gives the number of teams in the tournament at the beginning of round n.

a. Use words to write a recursive definition of this sequence.

b. Write a recursive formula for this sequence.

Solution

a. Identify the first term and the rule for generating all following terms. The first term is 64. Each term after the first is found by dividing the previous term by 2.

b. Choose a name for the sequence. We call it b_n. Then write the recursive formula. Note that there are only six terms.

$$\begin{cases} b_1 = 64 \\ b_n = \frac{1}{2}b_{n-1}, \text{ for integers } n = \{2, 3, 4, 5, 6\} \end{cases}$$

The Tennessee Lady Vols defeated Rutgers in the 2007 NCAA Women's Championship game.

CAUTION: The first term of the sequence in Example 3 is 64, the term written in the first line of the recursive definition. Although 32 is the first term given by the rule for b_n in the solution, it is the second term of the sequence. Remember, b_n represents the nth term in this definition, and the formula for b_n is only defined for $n = \{2, 3, 4, 5, 6\}$.

Recursive Definitions and Spreadsheets

A spreadsheet is another efficient tool for generating a sequence from its recursive definition.

Example 4

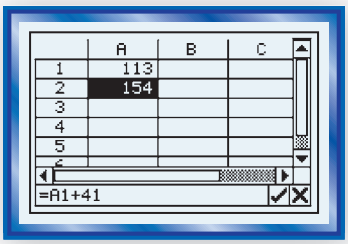
Use a spreadsheet to generate the first five terms of the sequence

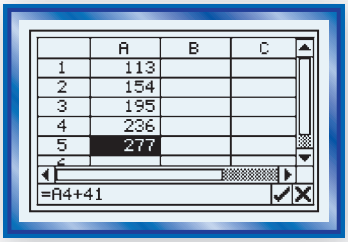
$$\begin{cases} L_1 = 113 \\ L_n = L_{n-1} + 41, \text{ for integers } n \geq 2. \end{cases}$$

Solution Enter 113 in cell A1.

Then in A2, enter =A1+41.

Each cell after A2 can be defined as the value in the previous cell + 41. A shortcut to do this is to copy cell A2 and paste it into cells A3–A5. For example, after the paste, the screen at right shows that the formula in A5 has been updated to refer to cell A4. You could also use the fill down function on the spreadsheet instead of copying and pasting.





So the value in cell A5 $= L_5 = 277$.

The first five terms are 113, 154, 195, 236, and 277.

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Example 5

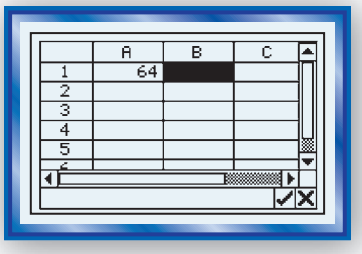
Use a spreadsheet to generate the six terms of the sequence in Example 3.

Solution Enter 64 in cell __?__.

Enter __?__ in cell A2.

To see the six terms of this sequence in your spreadsheet, copy cell __?__ and then paste it into cells __?__ through __?__.

The six terms of the sequence are __?__, __?__, __?__, __?__, __?__, and __?__.



Questions

COVERING THE IDEAS

In 1 and 2, refer to the Activity.

1. What is the meaning of ans on the CAS?
2. How long is the freight train with 12 boxcars?
3. What is a recursive definition for a sequence?
4. Suppose t_n denotes the nth term of a sequence.
   a. What does t_{n-1} denote?
   b. What does t_{n-2} denote?
   c. What does t_{n+1} denote?

In 5 and 6, refer to Example 3.

5. Rewrite the recursive formula if b_n is the previous term instead of the next term.
6. Suppose that the NCAA Women's Basketball Tournament is expanded to include 128 teams.
   a. How many terms are now in the sequence describing the tournament?
   b. Write a recursive formula for this sequence using recursive notation.

In 7 and 8, a description of a sequence is given.

   a. **Write the first five terms of the sequence.**
   b. **Write the recursive formula.**

7. The first term is –2; each term after the first is 8 more than the previous term.
8. The first term is 4; each term after the first is $-\frac{1}{4}$ times the previous term.

9. Explain in your own words why a recursive formula must have at least two parts.

10. Consider the sequence that begins $-19, -26, -33, -40, \ldots$.
    a. What is the first term?
    b. **Fill in the Blank** From the second term on, each term is __?__ the previous term.
    c. Write a recursive formula for the sequence.

11. Consider the decreasing sequence of negative even integers, beginning with $-2, -4, -6, \ldots$. Write a recursive definition for this sequence if t_{n+1} is the next term in the second line.

12. Generate the first eight terms of the sequence below by hand or using a spreadsheet.

$$\begin{cases} S_1 = 12 \\ S_n = 2S_{n-1} + 3, \text{ for integers } n \geq 2 \end{cases}$$

APPLYING THE MATHEMATICS

13. In an auditorium with 25 rows, the first row has 8 seats and each succeeding row has 2 more seats than the row in front of it.
    a. Write a recursive formula for a sequence that gives the number S_n of seats in row n.
    b. Find the number of seats in the 14th row.

14. You are given a penny on the first day of September. On each subsequent day you are given twice as many as the previous day.
    a. Write a recursive formula to represent how much you will receive on the nth day of September.
    b. How much money will you receive on September 30?

15. Let S be a sequence. Is the set of ordered pairs of the form (n, S_n) a function? Why or why not?

16. The table at the right gives the cost per book of printing 100 or more books, based on the number of color pages in the book.
    a. Describe in words the sequence that gives the cost per book for books with n color pages.
    b. Write a recursive formula for the sequence that gives the cost per book for books with n color pages.

Number of Color Pages	Cost ($)
1	5.55
2	5.70
3	5.85
4	6.00
5	6.15
6	6.30
7	6.45
8	6.60

17. Consider the sequence $6.42, 3.69, 0.96, -1.77, -4.5, \ldots$.
    a. Write a description of this sequence in words.
    b. Write a recursive definition of this sequence.
    c. Find the eighth term of the sequence.

18. The first term of a sequence is 1. Suppose each term after that is the sum of its term number and the previous term.
    a. Write a recursive definition for this sequence.
    b. Explain how you would use a spreadsheet to generate this sequence through term 12.
    c. What is the 12th term?

REVIEW

19. The table at the right gives the total attendance and payroll of some of the Major League Baseball teams in 2006. Make a scatterplot of the data and find the line of best fit. Does there appear to be a linear relationship between attendance and payroll? **(Lesson 3-5)**

Team	Attendance (thousands)	Payroll (millions of dollars)
Yankees	4200	198.6
Angels	3406	103.6
White Sox	2957	102.8
Red Sox	2930	120.1
Rockies	2105	41.1
Indians	1998	56.8
Pirates	1861	46.8
Royals	1372	47.3

20. **True or False** The line of best fit for a set of collinear points is the line through those points. **(Lesson 3-5)**

21. Suppose a store sells custom-made baseball caps with a graduated pricing scheme. The first 10 caps cost \$14 each, the next 10 cost \$12 each, and each cap after the 20th costs \$10. **(Lesson 3-4)**
    a. Draw a graph showing how the total cost C in dollars relates to the number of caps n that you order.
    b. Express the relation between C and n as a piecewise function with linear pieces.

22. A formula for the area of a trapezoid with height h and bases b_1 and b_2 is $A = \frac{1}{2}h(b_1 + b_2)$. Solve this formula for h. **(Lesson 1-7)**

23. When you drop an object, its distance fallen in feet is modeled by $d(t) = 4.9t^2$, where t is time in seconds. **(Lesson 1-3)**
    a. What does the sentence $d(3) = 44.1$ mean in English?
    b. Does the sentence $d(-1) = 4.9$ have meaning? Why or why not?

EXPLORATION

24. There is an expression *two steps forward, one step back*.
    a. Write a recursive definition for the sequence of stopping points on a path following these directions. Let the nth term of the sequence be your position after you have followed these directions n times.
    b. What does the phrase mean?

QY ANSWERS

1. 34, 22, 10, –2, –14

2. the 36th term of the sequence, 1548