Lesson 3-4 Finding an Equation of a Line

Vocabulary

point-slope form

piecewise linear function

▶ **BIG IDEA** Postulates and theorems of geometry about lines tell when exactly one line is determined from given information. An equation of that line can often be determined algebraically.

Mental Math

Suppose you pull a chip from a bag. What is the probability of choosing a blue chip if

a. there are 8 blue chips and 4 red chips?

b. there are 3 blue chips, 7 red chips, and 1 white chip?

c. there are *b* blue chips, *r* red chips, and *w* white chips?

In geometry you learned that *there is exactly one line through a given point parallel to a given line.* Since nonvertical lines are parallel if and only if they have the same slope, this means that there is exactly one line through a given point with a given slope. Using algebra you can determine the equation of this line.

Finding a Linear Equation

Example 1

In a physics experiment, a spring is 11 centimeters long with a 13-gram weight attached. Its length increases 0.5 centimeter with each additional gram of weight. This is a constant-increase situation up to the spring's elastic limit. Write a formula relating spring length *L* and weight *W*. Then graph the equation.

Solution The slope is $0.5\frac{\text{cm}}{\text{gram}}$. Because the unit for slope is centimeters per gram, length is the dependent variable and weight is the independent variable. The point (13, 11) is on the line. Substitute these values into the slope formula.

$$\frac{L-11}{W-13}=0.5$$

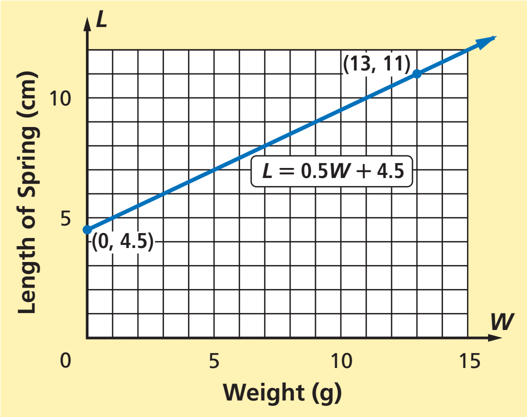
To put this equation in slope-intercept form, multiply both sides by *W* – 13.

$$L-11=0.5(W-13)$$

Then solve for *L*.

$$L-11=0.5W-6.5$$
$$L=0.5W+4.5$$

Now you know the *y*-intercept of this line as well as the point (13, 11). Use this information to graph the equation, as shown at the right.



STOP QY1

> ▶ QY1
> What does the y-intercept in the equation $L = 0.5W + 4.5$ represent in Example 1?

Point-Slope Form of a Line

Each of the equations $L = 0.5W + 4.5$ and $L - 11 = 0.5(W - 13)$ describes the situation of Example 1. The slope-intercept form is useful for computing values of L quickly if you know values of W. The form $L - 11 = 0.5(W - 13)$ shows the slope and a specific point on the graph, and it can be used to determine the slope-intercept form.

Point-Slope Theorem

If a line contains the point (x_1, y_1) and has slope m, then it has the equation $y - y_1 = m(x - x_1)$.

Proof Let ℓ be the line with slope m containing (x_1, y_1). If (x, y) is any other point on ℓ, then using the definition of slope,

$$m = \frac{y - y_1}{x - x_1}.$$

Multiplying both sides by $x - x_1$ gives

$$y - y_1 = m(x - x_1).$$

This is the desired equation of the line.

The equation $y - y_1 = m(x - x_1)$ is called a **point-slope form** of an equation for a line. Solving for y in the point-slope form of a line gives a form that combines aspects of the slope-intercept and point-slope forms.

$$y = m(x - x_1) + y_1$$

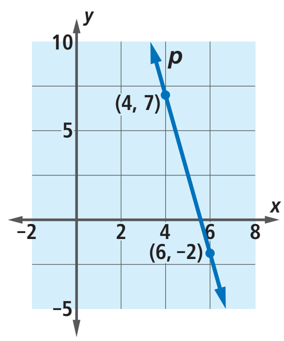
Finding an Equation of a Line through Two Points

Two points determine a line. You use this idea every time you draw a line through two points with a straightedge. It is another postulate from geometry. If you know two points on a line, you can find its equation by first computing its slope. Then use either point in the point-slope form $y - y_1 = m(x - x_1)$ or $y = m(x - x_1) + y_1$.

GUIDED

Example 2

Find an equation of the line p through (4, 7) and (6, –2).



Solution First, compute the slope of the line p.

$m = \underline{\ ?\ }$

Second, because you know two points on the line and neither of them is the y-intercept, use one of the points in the point-slope form.

$$y - \underline{\ ?\ } = \underline{\ ?\ }(x - \underline{\ ?\ })$$

This is one equation for the line p. In slope-intercept form, this equation is $\underline{\quad ?\quad}$.

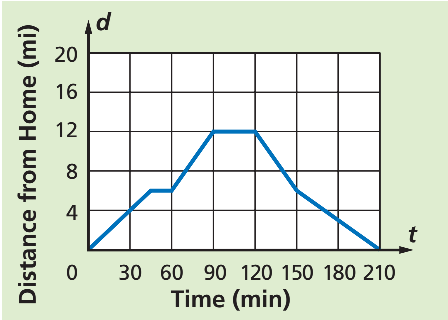
STOP **QY2**

QY2

Check your solution to Example 2 by finding an equation of the line using the point you did not use before and putting the equation in slope-intercept form.

Piecewise Linear Functions and Graphs

In Chapter 1 you saw the graph at the right of Frank's bicycle trip to Chuck's house. You focused on the domain and range of the whole trip and on function representations for different parts of the trip. For instance, at time $t = 60$ minutes, Frank was about 6 miles from home, so $d = f(60) = 6$.



Because this graph does not have a constant rate of change over its domain $\{t|\ 0 \le t \le 210\}$, it cannot be represented by a single linear equation. But you can write an equation for each segment of the graph that does have a constant rate of change. Because the graph is described in pieces, the graph and the function are called **piecewise linear**. The graph is the union of several segments. Each segment can be described by a linear function that has a restricted domain indicating where the segment starts and stops.

Writing a function for a piecewise linear graph requires writing the equation of each line segment.

Example 3

Write a piecewise linear function for the first two segments of Frank's bicycle trip as described in the graph above. Estimate the values of segment endpoints where necessary.

Solution **First piece:** This segment appears to begin when $t = 0$ and end when $t = 45$. The ordered pairs represented by the endpoints are (0, 0) and about (45, 6). Write an equation for the line through these two points.

Find the slope: $m = \frac{6 - 0}{45 - 0} = \frac{6}{45} = \frac{2}{15}$.

Because (0, 0) is on the line, 0 is the d-intercept. So use the slope-intercept form of a line to get

$$d = \frac{2}{15}t + 0 \text{ or } d = \frac{2}{15}t.$$

(continued on next page)

Since this equation describes only the part of the situation for *t*-values from 0 through 45, write

$$d = \frac{2}{15}t \text{ for } 0 \le t \le 45.$$

Second piece: This appears to be a constant function where $d = 6$ between $t = 45$ and $t = 60$. So,

$$d = 6 \text{ for } 45 < t \le 60.$$

In the second piece, we have a choice whether to include the value 45 in the domain. This is because (45, 6) is already included in the first piece and it is not necessary to repeat it.

The piecewise function: Combine the functions for the two pieces and their domains into one formula using a brace.

$$d = f(t) = \begin{cases} \frac{2}{15}t & \text{for } 0 \le t \le 45 \\ 6 & \text{for } 45 < t \le 60 \end{cases}$$

Writing a piecewise formula for the last four segments of the trip is left to you in the Questions.

The piecewise formula can be used to determine Frank's distance from home at any time, even if the distance cannot be easily determined from the graph. For instance, to determine how far Frank was from home after 20 minutes, use the first equation because $t = 20$ is in the domain $0 \le t \le 45$. So, when $t = 20$, $d = f(20) = \frac{2}{15}(20) \approx 2.67$ miles.

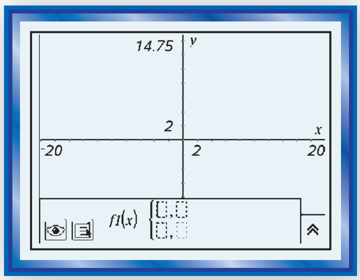
STOP QY3

> **QY3**
> How far from home was Frank after 55 minutes?

If you know the formula for a piecewise function, a CAS or graphing utility can be used to graph it.

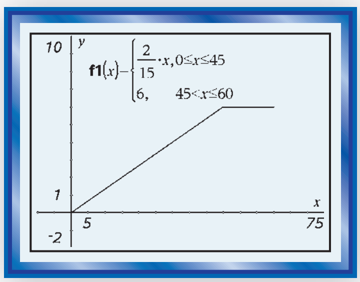
Activity

Step 1 Open a graphing utility and clear any functions that have been entered. Find out how to enter a piecewise function. On the CAS pictured at the right, a template is used to enter a piecewise function.



Step 2 Enter the piecewise formula you found in Example 3. Include the domain restrictions for each line of the formula.

Step 3 Graph the function. Adjust your window as necessary to see the whole graph.



Step 4 Compare your graph to the first part of the graph found in this lesson. Do they look the same?

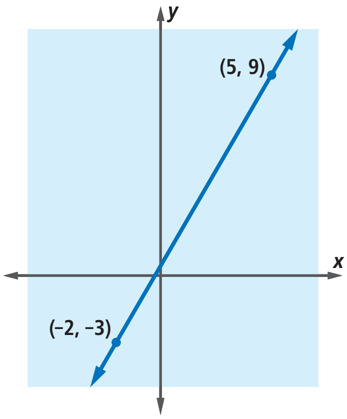
Questions

COVERING THE IDEAS

1. How many points determine a line?
2. Name a point on the line $L = 0.5W + 4.5$ other than the two used in Example 1.
3. **True or False** You can determine the equation of a line knowing only its slope.
4. **Fill in the Blank** The point-slope form of the equation for a line with slope m and passing through point (x_1, y_1) is __?__.
5. Give a strategy for finding an equation for a line when you know two points on the line.
6. A line passes through the points $\left(\frac{1}{3}, \frac{2}{5}\right)$ and $\left(\frac{7}{3}, \frac{9}{10}\right)$.
   a. Compute the slope of the line.
   b. Use $\left(\frac{1}{3}, \frac{2}{5}\right)$ and the slope to determine an equation of the line.
   c. Check that $\left(\frac{7}{3}, \frac{9}{10}\right)$ satisfies the equation.

In 7 and 8, write an equation for the line with the given information.

7. slope 6 and y-intercept $\sqrt{3}$
8. slope $-\frac{3}{2}$ and passing through $(-4, 1)$
9. Find an equation in point-slope form for the line graphed at the right.



10. Refer to Example 3.
    a. Write the equations describing the last four segments of Frank's bicycle trip.
    b. How far from home was Frank after 132 minutes?
    c. Graph the whole piecewise function on a CAS.

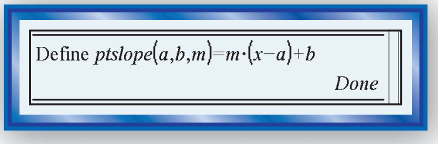
APPLYING THE MATHEMATICS

11. A club will be charged \$247.50 for printing 150 t-shirts and \$501.25 for printing 325 t-shirts. Let c be the cost of printing s shirts.
    a. Write c as a linear function of s.
    b. How much will it cost to print 0 t-shirts? (This is the set-up cost.)
    c. How much will it cost to print 100 t-shirts?
12. Find an equation for the line through $(5, 3)$ and parallel to $y = \frac{4}{7}x - 12$.

13. Refer to the `slope` function defined in Question 14 of Lesson 3-1.

   a. Use `slope` to find the slope of the line through $P = (17.3, 2.4)$ and $Q = (43.9, 22.6)$.

   b. Define a new function `ptslope` that will find the equation of a line with slope m and passing through the point (a, b) in point-slope form. A sample from one CAS is shown at the right.



   c. Use `ptslope` to find an equation of $\overleftrightarrow{PQ}$ from Part a. Use $a = 17.3$, $b = 2.4$, and the slope you found in Part a as m.

   d. You can define one function, call it `line2pt(a,b,c,d)`, that combines the calculations of `slope` and `ptslope`. Define this function on a CAS.

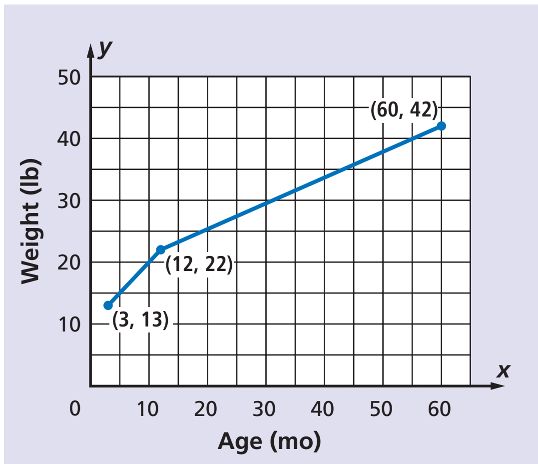
   e. Input the coordinates of points P and Q into `line2pt`. Are your results the same as in Part c?

14. An approximate conversion from degrees C on the Celsius scale to degrees F on the Fahrenheit scale is $F \approx 2C + 32$.

   a. Use the facts that $32°\text{F} = 0°\text{C}$, $100°\text{C} = 212°\text{F}$, and that the two scales are related linearly to write a more accurate formula.

   b. Graph the two formulas to determine when the approximation is within 3° of the actual Fahrenheit temperature.

15. The graph at the right represents the average weight in pounds of a child from age 3 months to age 5 years (60 months). Use the graph to answer the following questions.



   a. Write a piecewise linear function for this situation.

   b. What is the average weight of a 30-month-old child?

16. In the book, *The Lion, the Witch, and the Wardrobe* (C.S. Lewis, 1961), a group of children find a portal to the world of Narnia where time passes at a different rate than on Earth. The children are in Narnia for 15 years while only 5 minutes pass on Earth. Peter, the oldest child in the group, is 13 when he enters Narnia. Write a linear equation that shows Peter's age n in Narnia for each minute m that passes on Earth.

REVIEW

17. Fill in the Blanks The equation $Ax + By = C$ describes a line with an undefined slope whenever A is __?__ and B is __?__. **(Lesson 3-3)**

18. Write an equation of the line with x-intercept -2 and y-intercept 4 in standard form. **(Lesson 3-3)**

19. Multiple Choice Which of the following is not a linear combination? **(Lesson 3-2)**

A $4a + (-3)b$ B $12x + 12y + 12z$ C $T - F$

D $7A + 3C^2$ E $e + e + e + 3f$

20. Makayla finds that the amount she studies varies inversely as the number of phone calls she receives. One day she studied 3.5 chapters with 3 phone calls. How many chapters can she study if she gets 5 phone calls? **(Lesson 2-2)**

21. Consider the sequence r whose first 7 terms are -5, 3, 1700, -65.4, 0.354, 29, and -1327. **(Lesson 1-8)**

a. What is r_3?

b. Write the sentence $r_7 = -1327$ in words as you would read it.

22. The following table of values shows the height of Azra's dog for certain numbers of months after she bought him. **(Lesson 1-5)**

Month	0	1	2	3	4	5	6
Height (in.)	10	14	16	19	22	23	24

a. Graph the values in the table on the coordinate axes, with time on the horizontal axis and height on the vertical axis.

b. Azra thinks the domain of her function is all positive real numbers between 0 and 6, but her brother Marquis thinks it is only the whole numbers from 0 to 6. Who is right and why?

EXPLORATION

23. In 1912, the Olympic record for the men's 100-meter dash was 10.8 seconds. In 1996 it was 9.84 seconds. Assume a linear relationship between the year and the record time.

a. According to the information given, determine a linear equation to model the given data.

b. Using your model, compute the predicted world record in 2006.

c. A new world record of 9.76 seconds for the 100-meter dash was set in 2006. Compare your predicted record with the actual 2006 value.

QY ANSWERS

1. 4.5 cm, the length of the spring with no weights attached

2. Answers vary. Sample: $y + 2 = -4.5(x - 6)$. In slope-intercept form: $y = -4.5x + 25$.

3. 6 miles