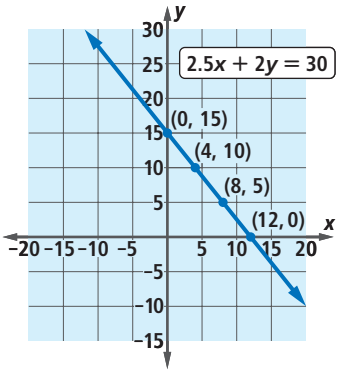
Lesson 3-3

The Graph of $Ax + By = C$

BIG IDEA If a linear combination of two variables x and y is a constant, then the graph of all the points (x, y) is a line.

Recall the equation $2.5x + 2y = 30$ from the previous lesson. This equation represents allowable \$30 purchases of x hamburgers at \$2.50 each and y hot dogs at \$2.00 each from Harry's Hamburger Hovel. Because you do not buy fractions of sandwiches, both x and y are nonnegative integers. So a graph of the solution is a set of discrete points. However, if you allow x and y to be any real numbers, then the graph of $2.5x + 2y = 30$ is shown at the right.



The equation $2.5x + 2y = 30$ is of the form $Ax + By = C$, with $A = 2.5$, $B = 2$, and $C = 30$. When A and B are not both zero, the graph of $Ax + By = C$ is always a line.

Standard Form of an Equation of a Line Theorem

The graph of $Ax + By = C$, where A and B are not both zero, is a line.

Proof There are two cases to consider: (1) if $B = 0$ and (2) if $B \neq 0$.

(1) If $B = 0$, then $A \neq 0$, and the equation is simply $Ax = C$. Multiply both sides by $\frac{1}{A}$ to obtain the equivalent equation $x = \frac{C}{A}$.

The graph of this equation is a vertical line.

(2) If $B \neq 0$, then solve the given equation for y:

$Ax + By = C$	Given
$By = -Ax + C$	Add $-Ax$ to both sides.
$y = -\frac{A}{B}x + \frac{C}{B}$	Divide both sides by B.

Mental Math

Give a general variation equation based on the description.

a. The cost c of painting the interior of a house varies directly as the number n of rooms to be painted.

b. The amount p of paint needed for a wall varies jointly as the length ℓ and height h of the wall.

c. The time t it will take to paint varies inversely as the number n of painters hired.

d. The time d needed for the paint to dry varies directly as the thickness t of the paint applied and inversely as the square of the amount a of air circulation in the room.

STOP QY1

> **QY1**
> When $B \neq 0$, what are the slope and y-intercept of the line with equation $Ax + By = C$?

Graphing a Line Using Intercepts

Because the form $Ax + By = C$ can describe any line, it is called the *standard form of an equation for a line*. Although, if $B \neq 0$, you could rewrite such an equation in slope-intercept form in order to make a graph, it is often much quicker to graph such equations by hand using x- and y-intercepts. If A, B, and C are all nonzero, the line with equation $Ax + By = C$ has distinct x- and y-intercepts, so the intercepts can be used to graph the line.

Example 1

Graph the equation $-5x + 2y = 10$ using its x- and y-intercepts.

Solution To find the x-intercept, substitute 0 for y, and solve for x.

$$-5x + 2(0) = 10$$
$$x = -2$$

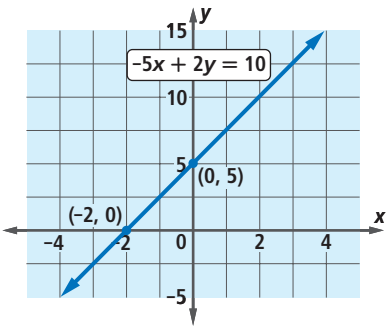
The x-intercept is -2.

To find the y-intercept, substitute 0 for x, and solve for y.

$$-5(0) + 2y = 10$$
$$y = 5$$

The y-intercept is 5.

Plot (–2, 0) and (0, 5) and draw the line containing them, as shown at the right.



Check Find a third ordered pair that satisfies $-5x + 2y = 10$. For example, when $x = 2$, $-10 + 2y = 10$, so $2y = 20$ and $y = 10$. Thus, (2, 10) should be on the graph. Is it? Yes. It checks.

This technique does not work when A, B, or C is zero. If $A = 0$, the slope is $-\frac{0}{B} = 0$. The line is horizontal, and so there is no x-intercept. If $B = 0$, the slope of the line is $-\frac{A}{0}$, which is undefined. The line is vertical, and so there is no y-intercept.

> **QY2**
> Why can you not use the x- and y-intercepts to graph $Ax + By = C$ when $C = 0$?

STOP QY2

Example 2

Graph $x + 0y = 3$.

a. Is this the graph of a line?

b. Is this the graph of a function?

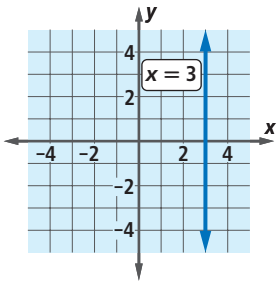
Solution

a. The equation simplifies to $x = 3$. The value of x is always 3, regardless of the value of y. The graph is a vertical line.

b. Create a table of values.

x	y
3	–5
3	0
3	2

It is not the graph of a function because more than one ordered pair has the same x-coordinate.



Equivalent Equations for Lines

One drawback of the standard form is that the same line can have many different, but equivalent, equations in standard form. Recall that multiplying both sides of an equation by a nonzero real number yields an equivalent equation. Since lines have a unique equation in slope-intercept form, you can test equations for equivalence by putting them in slope-intercept form.

GUIDED

Example 3

Find which equations below, if any, represent the same line.

(1) $4x + 1.5y = 12$ (2) $8x + 3y = 24$

(3) $8x + 3y = 12$ (4) $16x + 6y = 12$

Solution 1 Rewrite each line in slope-intercept form.

(1) y = ___?___ (2) y = ___?___

(3) y = ___?___ (4) y = ___?___

Equations ___?___ are equivalent. Equations ___?___ are not equivalent to any other given equations.

Solution 2 If I multiply both sides of Equation ___?___ by 2, Equation ___?___ results. So Equations ___?___ and ___?___ are equivalent. Since the right side of three of the given equations is 12, no other equations are equivalent.

Equations (2), (3), and (4) from Example 3 represent lines with the same slope but different y-intercepts. This suggests that the graphs of $Ax + By = C$ and $Ax + By = D$ are distinct parallel lines when $C \neq D$.

Questions

COVERING THE IDEAS

1. a. **Fill in the Blank** If A and B are not both 0, the graph of $Ax + By = C$ is a ___?___.
   b. If $A \neq 0$ but $B = 0$, what kind of line is the graph?
   c. If $B \neq 0$ but $A = 0$, what kind of line is the graph?
2. **Fill in the Blank** $Ax + By = C$ is in the ___?___ of an equation of a line.
3. What is true about the slope of a vertical line?
4. **True or False** Every line in standard form can be graphed by drawing the line containing its x- and y-intercepts.

In 5 and 6, an equation for a line is given.

a. Find its x-intercept.

b. Find its y-intercept.

c. Graph the line using your answers to Parts a and b.

5. $4x + 9y = 36$
6. $4x - 5y = 10$
7. Consider $Ax + By = C$.
   a. Find the x-intercept of the line. What happens when $A = 0$?
   b. Find the y-intercept of the line. What happens when $B = 0$?

APPLYING THE MATHEMATICS

8. Write an equation in standard form for the line graphed at the right.
9. Find the value of C such that the point $(4, -1)$ lies on the graph of $10x - 2y = C$.
10. Delaney's Deli makes ham and cheese sandwiches and turkey sandwiches. Each ham and cheese sandwich uses $\frac{1}{8}$ lb of cheese, while each turkey sandwich uses no cheese. Let x be the number of ham and cheese sandwiches the deli prepares. Let y be the number of turkey sandwiches the deli prepares.
    a. Write an equation stating that the total amount of cheese the deli uses is 5 lb.
    b. Graph your equation from Part a.

11. a. Graph the line with equation $0x + 4y = 14$.
    b. Find two ordered pairs satisfying the equation.
    c. Compute the slope of the line through the two points.

In 12 and 13, find an equation for a line in standard form with the given properties.

12. y-intercept -3 and slope 2
13. no y-intercept and passes through the point (17, 29.93)
14. Mallory combines n liters of a solution that is 3 mol/L chlorine with y liters of a solution that is 5 mol/L chlorine. She ends up with a mixture that contains 2 moles of chlorine.
    a. Write an equation relating n, y, and the total amount of chlorine in the mixture.
    b. Graph the equation you obtained in Part a by finding the n- and y-intercepts. Consider n to be the independent variable.
    c. Use your graph to find about how many liters of the 5 mol/L solution Mallory must add to 0.4 liter of the 3 mol/L solution to get the final mixture.
15. Consider the graphs of $Ax + By = C$ and $Ax + By = D$ if A and B are not both zero and $C \neq D$.
    a. Rewrite each equation in slope-intercept form.
    b. What is the relationship between the slopes of the lines? What does that tell you about the lines?
    c. What is the relationship between the y-intercepts of the lines? What does that tell you about the lines?
    d. Write the conclusions of Parts b and c as one if-then statement.
    e. Use the if-then statement from Part d to give the equations of several lines parallel to $16x - 13y = 11$.
16. Use a CAS `expand` command to show that multiplying the equation $Ax + By = C$ by a nonzero number k yields another equation in standard form. Why must this new equation describe the same line as the original?

REVIEW

17. At a library book sale, paperbacks are being sold for \$0.50 each and hardcover books are \$1. If you want to buy P paperbacks and H hardcover books, write a linear combination that expresses the amount you will have to pay. **(Lesson 3-2)**

18. Suppose your car uses 0.035 gallon of gas to travel 1 mile in the city and 0.027 gallon of gas to travel 1 mile on the highway. **(Lesson 3-2)**
    a. Write a linear combination to express the number of gallons of gas you would use to travel C miles in the city and H miles on the highway.
    b. If your car has 14 gallons of gas, how many city miles can you drive without refilling if you also make a 200-mile highway trip?

19. A car starts out 400 miles from St. Louis and drives directly toward St. Louis at 60 mph. **(Lesson 3-1)**
    a. Find an equation for the distance d from St. Louis as a function of time t in hours from the start of the trip.
    b. Does the equation in Part a describe a constant-increase or a constant-decrease situation?

Gateway Arch in St. Louis is the tallest national monument in the United States. The shape of the arch is known as a *catenary curve*.

20. The sum S of the measures of the interior angles of a convex polygon varies directly as $n - 2$, where n is the number of sides of the polygon. **(Lesson 2-1, Previous Course)**
    a. Find the constant of variation.
    b. Graph the function.

21. The independent variable of a function is given. State a reasonable domain for the function. **(Lesson 1-4)**
    a. h = number of hours worked in a day
    b. d = distance traveled away from home while on vacation
    c. t = temperature in Indianapolis, Indiana, in February

EXPLORATION

22. Consider the lines $Ax + By = C$ and $Bx + Ay = C$. Explore the connections between slopes and intercepts of these lines.

QY ANSWERS

1. The slope is $-\frac{A}{B}$, and the y-intercept is $\frac{C}{B}$.
2. When $C = 0$, the x- and y-intercepts are both zero. The line passes through the origin, and the x- and y-intercepts are not distinct.