Lesson 3-1 Constant Change and the Graph of $y = mx + b$

Vocabulary

y-intercept
linear function
slope-intercept form
x-intercept

BIG IDEA If y changes by a constant amount m as x increases by 1, then $y = mx + b$ for all (x, y) and the graph of the points (x, y) is a line with slope m and y-intercept b.

Constant-Increase and Constant-Decrease Situations

In many real-world situations, there is an initial condition and a constant increase or decrease applied to that condition. This type of situation can be modeled by a linear equation.

Example 1

Noah usually waits until he has only 1 gallon of gas left in his car before filling up. Suppose the gas pump pumps at a rate of 6 gallons per minute. Write an equation for a function representing the amount of gas in Noah's tank x minutes after he starts pumping gas. The tank holds 17.5 gallons when full.

Solution Let $A(x)$ be the amount of gas in Noah's tank x minutes after he starts pumping. The function will have the form

$A(x) =$ (amount in tank at start) $+$ (amount added).

The amount in the tank at the start is 1 gallon. Every minute, 6 gallons are added, so after x minutes, $6x$ gallons have been added. The function with equation

$$A(x) = 1 + 6x$$

or

$$A(x) = 6x + 1$$

gives the amount of gas in his tank after x minutes.

A graph of the function A in Example 1 is shown on the next page.

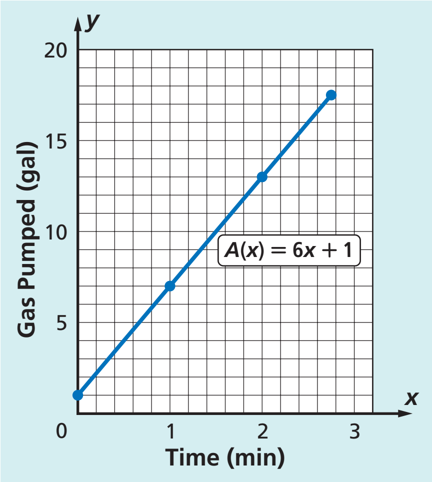
Mental Math

a. Rafael graduated high school in May 2007 at the age of 18. How old will he be in October 2020?

b. Rafael's birthday is August 19. How old was he in November 2005?

c. Rafael's sister Bianca is 2 years and 8 months younger than he is. How old was Bianca in November 2005?

Reighard's gas station, established in 1909, claims to be the oldest gas station in the United States.



STOP QY1

Outside the context of Noah's gas tank, $y = 6x + 1$ is an equation describing a function whose domain and range are each the set of real numbers. The graph of the equation is the line containing the segment graphed above.

The y-value of the point where a graph crosses the y-axis is called its ***y*-intercept**. The graph of $y = 6x + 1$ crosses the y-axis at the point $(0, 1)$, so its y-intercept is 1. The y-intercept is the initial condition, or starting point, of the situation modeled by the graph; in Noah's case it is 1 gallon.

The slope of this line is the rate of change, 6 gallons per minute. You can verify this by finding the slope between two points on the graph, for example $(0, 1)$ and $(2.75, 17.5)$.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(17.5 - 1)\text{gallons}}{(2.75 - 0)\text{minutes}} = \frac{16.5 \text{ gallons}}{2.75 \text{ minutes}} = 6\ \frac{\text{gallons}}{\text{minute}}$$

Notice that the unit of the slope matches the unit of the constant change.

Slope-Intercept Form

A function whose graph is a line or a part of a line is called a **linear function**. In general, a linear function is a function that can be represented by an equation in the form $y = mx + b$. The form $y = mx + b$, or $f(x) = mx + b$, is called the **slope-intercept form** of a linear equation. Although the letter b is commonly used for the y-intercept, and the letter m commonly represents the slope, any other letters could be used in their place. Example 2 illustrates how the slope and y-intercept can be used to graph a line.

QY1

In Example 1, how long will it take to fill up Noah's gas tank?

GUIDED

Example 2

A pond with about 26 acre-feet of water needs to be drained for repairs to its levy. Engineers determine that 8 acre-feet of water can be safely drained per day.

a. Write a linear equation to model this situation.

b. In how many days will the pond be empty?

c. What are the *y*-intercept and the slope of the line represented by your equation in Part a?

d. Graph the line and indicate the point where the reservoir is empty.

Solution

a. Let $M(t)$ be the amount of water in the pond after t days.

$M(t) =$ __?__ – __?__

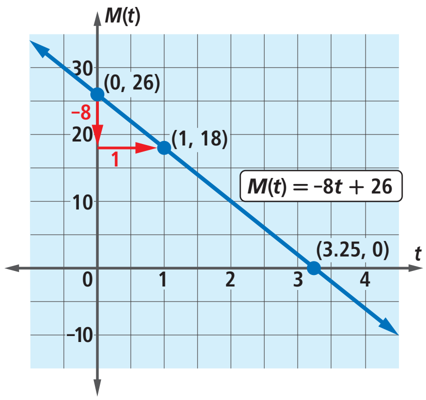
b. The pond will be empty when $M(t) = 0$.

Solve __?__ $= 0$ to find $t =$ __?__.

The reservoir will be empty after __?__ days.

c. The *y*-intercept is the starting point; the slope is the rate of change. So in this situation, the *y*-intercept is __?__ and the slope is __?__.

d. To graph this line, first locate the *y*-intercept 26, which corresponds to a full pond when $t = 0$. Use the slope to locate another point. A slope of $-8 = \frac{-8}{1}$ means that every horizontal change of 1 unit to the right corresponds to a vertical change of 8 units down. This gives the new point $(0 + 1, 26 - 8) = (1, 18)$. Plot (1, 18) and draw the line. Label it $M(t) = -8t + 26$. The pond is empty at the point where the line crosses the *x*-axis, at the point (3.25, 0).

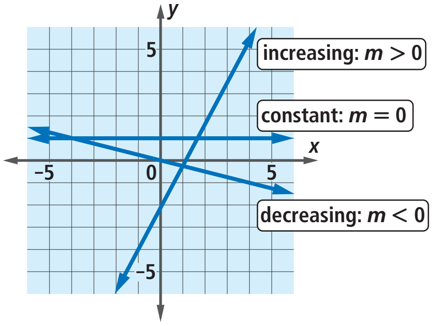


The *x*-value of the point where a line crosses the *x*-axis is called the ***x*-intercept** of the line. Because the line in Example 2 intersects the *x*-axis at (3.25, 0), 3.25 is the *x*-intercept of the line.

In constant-increase situations, such as Example 1, as the value of x increases, so does the value of y. The slope of the line is positive. The graph slants up to the right and the function is said to be *increasing*.

In constant-decrease situations, such as Example 2, as the value of x increases, the value of y decreases. The slope of the line is negative. The graph slants down to the right and the function is said to be *decreasing*.

When the slope $m = 0$, the graph does not rise or fall; the function is *constant*. A line with a slope of 0 is horizontal. Vertical lines do not represent functions and have an undefined slope.



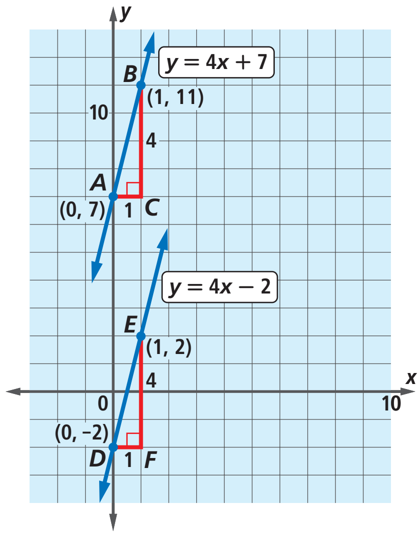
STOP **QY2**

▸ QY2

Why is a vertical line not the graph of a function?

Parallel Lines and Slope

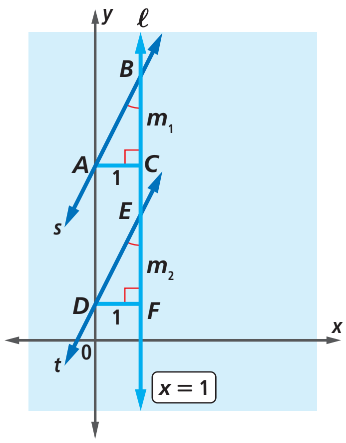
Consider the graphs of $y = 4x + 7$ and $y = 4x - 2$ at the right. Both lines have slope 4; on each line, as you move 1 unit to the right, the line rises 4 units. In the figure, right triangles $\triangle ABC$ and $\triangle DEF$ are congruent by the SAS Congruence Theorem. So $\angle CAB \cong \angle FDE$, and these lines form congruent corresponding angles with the *y*-axis at A and D. Consequently, $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ are parallel.



This argument can be repeated with any two lines that have the same slope. We also say that a line is parallel to itself. Thus, we can conclude the following:

If two lines have the same slope, then they are parallel.

The converse of this statement is: If two lines are parallel, then they have the same slope. This can be proved for all nonvertical lines as follows. Suppose lines s and t are parallel and not vertical, as shown at the right. Let m_1 be the slope of s, and m_2 be the slope of t. We want to show that $m_1 = m_2$. Draw line ℓ with equation $x = 1$. Note that ℓ is a transversal to the parallel lines. Draw horizontal segments from the *y*-intercepts of lines s and t to line ℓ as shown, forming right angles at C and F. Then $m_1 =$ slope of $s = \frac{AC}{BC} = \frac{AC}{1} = AC$, and $m_2 =$ slope of $t = \frac{DF}{EF} = \frac{DF}{1} = DF$. Recall from geometry that corresponding angles formed by parallel lines and a transversal are congruent. So $\angle ABC \cong \angle DEF$. Note also that $AC = DF$, so $\overline{AC} \cong \overline{DF}$. Since $\angle BCA \cong \angle EFD$, $\triangle ABC$ and $\triangle DEF$ are congruent by the AAS Congruence Theorem, and $BC = EF$. Because $m_1 = BC$ and $m_2 = EF$, the slopes are equal.



Consequently, we have shown that the following is true:

If two non-vertical lines are parallel, then they have the same slope.

Because the original statement and its converse above are both true, you can combine them into one biconditional (if and only if) statement that is important enough to be labeled as a theorem.

Parallel Lines and Slope Theorem

Two non-vertical lines are parallel if and only if they have the same slope.

Questions

COVERING THE IDEAS

1. **Fill in the Blanks** In the equation $y = mx + b$, the slope is __?__ and the y-intercept is __?__.

2. **Fill in the Blanks** A slope of 3 means a __?__ change of 3 units for every __?__ change of one unit.

3. Refer to Example 1. Suppose a car has 4 gallons in its tank and the tank holds 14.5 gallons. How long will it take to fill the tank?

4. Refer to Example 2.
   a. What does the y-intercept mean in the context of draining the pond?
   b. What does the slope mean in this context?
   c. How much water is in the pond at the end of the second day?

5. **Fill in the Blanks** In real-world constant-increase or constant-decrease situations, the initial condition can be represented by the __?__, and the constant change can be represented by the __?__.

In 6 and 7, find the slope of the line containing the points.

6. (7, 2) and (5, –4)

7. (32, 14) and (–8, –6)

8. Why are vertical lines excluded from the Parallel Lines and Slopes Theorem?

9. Write an equation for the line with y-intercept of –3 and slope of $\frac{2}{3}$.

10. Give the slope and y-intercept of $y = 5 - 2x$.

11. Which lines are parallel? (There may be more than one pair.)
    a. $y = 2x - 7$
    b. $y = -3x + 4$
    c. $2x - y = 7$
    d. $y = 12 - 3x$
    e. $-4x + 2y = 12$

APPLYING THE MATHEMATICS

12. A stack of Sunday newspapers sits on a skid 3" off the ground. Each newspaper is $1\frac{3}{4}$" thick.

a. How high is the stack if there are *n* newspapers in it?

b. If the top of the stack is 3' high, about how many newspapers are in the stack?

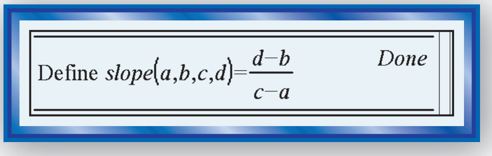
13. The Clear-Bell Cell phone company has a plan that costs customers $29.99 for a monthly service fee plus 3¢ per minute used.

a. Write a linear equation to model this situation.

b. Identify the slope and the *y*-intercept. Interpret the slope and *y*-intercept in the context of this problem.

c. If Polly talks for 136 minutes, how much would her cell phone bill be (excluding taxes)?

14. The function slope$(a,b,c,d) = \frac{(d-b)}{(c-a)}$ calculates the slope of the line connecting two points. To use this function, define slope on a CAS, with the coordinates of the points (a, b) and (c, d) as arguments. Use the function to find the slope of the line through



a. (2, -3) and (5, 3).

b. (4.2, 8.3) and (-3.4, 5.01).

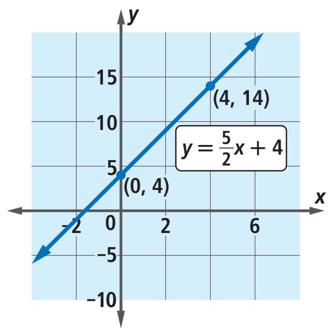
15. Find the slope of a line with *y*-intercept 4 that contains the point (2, 6).

In 16 and 17, use graph paper to draw an accurate graph of these lines.

16. $y = -\frac{2}{3}x + 4$

17. $y = 5$

18. The line with equation $y = \frac{5}{2}x + 4$ is graphed at the right.



a. What is its slope?

b. Find the average rate of change between (0, 4) and (4, 14) to verify your answer to Part a.

c. What is the *y*-intercept?

d. Is the point (-4, 3) on the line? Explain how you know.

19. Stephanie is a taxi driver. She rents her taxi for \$300/week. She feels she can take in \$120/day in fares. Let D be the number of days she drives her taxi in a week and T be the total earnings she expects after subtracting the rental charge.

a. Identify four ordered pairs (D, T).

b. Write an equation that gives T as a function of D.

c. What is the most she can earn in a year at this rate?

20. Suppose a varies directly as b, and that a is 10 when b is 3.

a. Find the constant of variation and write an equation describing the variation.

b. Make a table of values and graph the equation.

c. Write the variation equation in slope-intercept form and identify the slope and y-intercept.

d. Which type of variation does this represent: constant increase or constant decrease?

REVIEW

In 21–24, decide whether the variables in the equation exemplify direct, inverse, combined, or joint variation. (Lessons 2-1, 2-2, 2-9)

21. $ab = 5c$ **22.** $x = y$ **23.** $m = \frac{61g}{hz}$ **24.** $2 = pr$

25. Consider the table of data at the right, which relates the number p of snowplows out on city streets to the number c of car crashes during a snowy day. **(Lesson 2-7)**

a. Does c vary directly or inversely as a power of p?

b. Based on your answer to Part a , either $c = kp^n$ or $c = \frac{k}{p^n}$. Find n for the appropriate relation.

Plows p	Crashes c
1	50
2	25
3	16
4	12
5	10
6	7

26. Create the first five rows of a table for $y = \frac{3}{5}x$ with a start value of –10 and an increment of 5. **(Lesson 1-5)**

27. Solve $\frac{1}{3}y + 2x = 16$ for y. **(Lesson 1-7)**

EXPLORATION

28. Suppose you are going on vacation outside the United States. Find the roundtrip airfare from your local airport to this destination. Estimate your daily expenses for hotel and food. Use the information to write a function that gives the cost of a trip to stay for d days. Is this function a linear function?

QY ANSWERS

1. 2.75 min, or 2 min 45 sec

2. Two points on a vertical line have different second coordinates but the same first coordinate. So, these lines violate the definition of a function.