# Lesson 12-8 Factoring and Rational Expressions

**Vocabulary**

rational expression
lowest terms

**BIG IDEA** The factored form of a polynomial is useful in writing rational expressions in lowest terms and in performing operations on rational expressions.

In Lesson 12-7, you saw how factoring can help find the solutions to polynomial equations. Factoring can also help in working with fractions that have polynomials in their numerators and denominators.

**Mental Math**

**Estimate to the nearest dollar the interest earned in one year in a savings account with 2.02% annual interest rate containing**

**a.** \$501.

**b.** \$4,012.

**c.** \$9,998.

## Activity

**Step 1** Before reading Step 2, perform the addition of these fractions and write your answer in lowest terms. Do not use a calculator.

$$\frac{75}{100} + \frac{66}{99}$$

**Step 2** Describe how you did the addition. Did you find a common denominator? If so, then you probably worked with fractions with denominator 9,900. Or, did you put each fraction in lowest terms first? Then you could do the addition with a common denominator of 12. Compare the way you added these fractions with the ways that others in your class added them.

## Writing Rational Expressions in Lowest Terms

In the addition of fractions above, it is useful to write each fraction in lowest terms. As you know, this is done by dividing both the numerator and denominator of the fraction by one of their factors. Because 25 is a common factor of the numerator and denominator of the first fraction, $\frac{75}{100} = \frac{25 \cdot 3}{25 \cdot 4}$. Similarly, $\frac{66}{99} = \frac{33 \cdot 2}{33 \cdot 3} = \frac{2}{3}$.

The same idea can be used with *rational expressions*. A **rational expression** is the written quotient of two polynomials. Here are five examples of rational expressions.

$\frac{0.7819}{x + y}$ $\frac{3x^3y}{x^2y^4}$

$\frac{a + \pi}{b - \pi}$ $\frac{4n^2 + 4n + 1}{4n^2 - 1}$

$\frac{6k^3 - 12k^2 + 42k - 210}{3k^3 - 6k^2 + 21k - 105}$

A rational expression is in **lowest terms** when there is no polynomial that is a factor of its numerator and denominator. The second rational expression on the previous page is not in lowest terms because $x^2y$ is a common factor of the numerator and denominator.

$$\frac{3x^3y}{x^2y^4} = \frac{x^2y \cdot 3x}{x^2y \cdot y^3} = \frac{3x}{y^3}$$

Technically, $\frac{3x}{y^3}$ is not exactly equivalent to $\frac{3x^3y}{x^2y^4}$ because the $x$ in $\frac{3x}{y^3}$ can equal 0 while the $x$ in $\frac{3x^3y}{x^2y^4}$ cannot equal 0. Often this is taken for granted, but sometimes people will write $x \neq 0$ and $y \neq 0$ because fractions do not allow 0 as the denominator.

The rational expressions $\frac{4n^2 + 4n + 1}{4n^2 - 1}$ and $\frac{6k^3 - 12k^2 + 42k - 210}{3k^3 - 6k^2 + 21k - 105}$ look more complicated than the others on page 761. But each expression can be put in lowest terms by factoring out the common factors.

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## Example 1

**Write $\frac{4n^2 + 4n + 1}{4n^2 - 1}$ in lowest terms. Assume the denominator does not equal 0.**

**Solution**

**Step 1** Factor the numerator $4n^2 + 4n + 1$.

**Step 2** The denominator $4n^2 - 1$ is the difference of two squares. Use this information to factor it.

**Step 3** From Steps 1 and 2, fill in the blanks.

$$\frac{4n^2 + 4n + 1}{4n^2 - 1} = \frac{(\underline{\ ?\ }n + \underline{\ ?\ })(\underline{\ ?\ }n + \underline{\ ?\ })}{(\underline{\ ?\ }n + \underline{\ ?\ })(\underline{\ ?\ }n - \underline{\ ?\ })}$$

**Step 4** The numerator and denominator in Step 3 have a factor in common. Divide them by that factor.

$$\frac{4n^2 + 4n + 1}{4n^2 - 1} = \frac{(\underline{\ ?\ }n + \underline{\ ?\ })}{(\underline{\ ?\ }n - \underline{\ ?\ })}$$

Because the numerator and denominator on the right have no common factor, the fraction is in lowest terms.

**Check** Check your answer by substituting 3 for $n$ in the original rational expression and in the expression of Step 4.

STOP **QY**

A graphing calculator and CAS technology can be of great assistance when writing a fraction in lowest terms.

**QY**

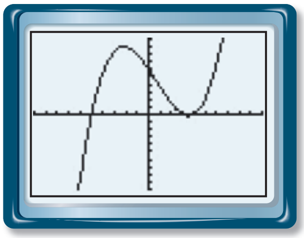
Explain why $\frac{6k^3 - 12k^2 + 42k - 210}{3k^3 - 6k^2 + 21k - 105}$ is not in lowest terms.

## Example 2

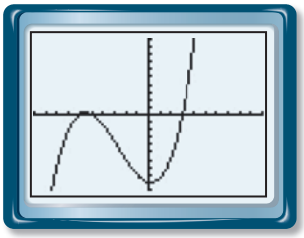
**Simplify the expression $\frac{x^3 - 2x^2 - 23x + 60}{x^3 + 8x^2 - 3x - 90}$.**

**Solution 1** To factor the numerator, graph $f(x) = x^3 - 2x^2 - 23x + 60$ with a graphing calculator. The graph has three $x$-intercepts: −5, 3, and 4, indicating that $(x - 4)$, $(x - 3)$, and $(x - -5)$ are factors of the numerator. To factor the denominator, graph $g(x) = x^3 + 8x^2 - 3x - 90$. This graph has $x$-intercepts 3, −5, and −6, indicating that $(x - 3)$, $(x - -5)$, and $(x - -6)$ are factors of the denominator. Thus,

$$\frac{x^3 - 2x^2 - 23x + 60}{x^3 + 8x^2 - 3x - 90} = \frac{(x + 5)(x - 4)(x - 3)}{(x - 3)(x + 5)(x + 6)} = \frac{x - 4}{x + 6}.$$

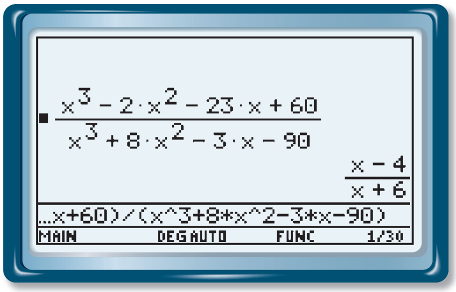


$-10 \le x \le 10, -100 \le y \le 100$

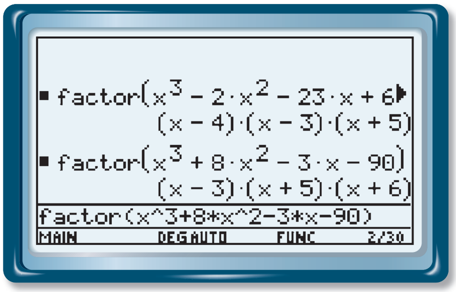


$-10 \le x \le 10, -100 \le y \le 100$

**Solution 2** Use a CAS. If you enter the given rational expression into a CAS, you may immediately see the expression in lowest terms.




**Solution 3** Use a CAS to show the steps of Solution 1. Factor the numerator and the denominator.




Thus $\frac{x^3 - 2x^2 - 23x + 60}{x^3 + 8x^2 - 3x - 90} = \frac{(x - 4)(x - 3)(x + 5)}{(x - 3)(x + 5)(x + 6)}$.

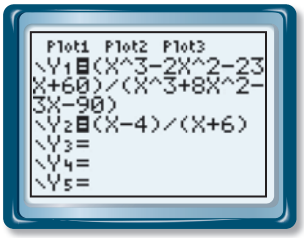
Divide numerator and denominator by the common factors.

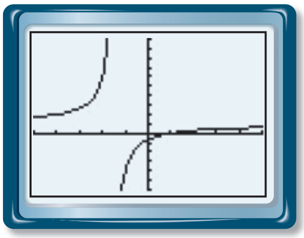
The solution is $\frac{x - 4}{x + 6}$.

*Caution:* Notice that you *cannot* simplify $\frac{x - 4}{x + 6}$ by dividing by $x$ because the $x$'s are terms, not factors.

***(continued on next page)***

**Check** Graph Y1 $= \frac{x^3 - 2x^2 - 23x + 60}{x^3 + 8x^2 - 3x - 90}$ and Y2 $= \frac{x-4}{x+6}$ to see whether they appear to be equivalent expressions. Use the WINDOW $-25 \le x \le 25$, $-6 \le y \le 6$.



The two appear to form the same graph, so the result checks.

## Adding and Subtracting Rational Expressions

To add $\frac{1}{9} + \frac{7}{15}$, you can either look for a calculator or look for a common denominator. If you do not have a calculator that does fractions, the least common denominator can be found by factoring each denominator and finding a product that will be a multiple of each denominator. Since $9 = 3 \cdot 3$ and $15 = 3 \cdot 5$, the least common denominator is $3 \cdot 3 \cdot 5$, or 45.

$$\begin{aligned}\frac{1}{9} + \frac{7}{15} &= \frac{1}{3 \cdot 3} + \frac{7}{3 \cdot 5} \\ &= \frac{1 \cdot 5}{3 \cdot 3 \cdot 5} + \frac{7 \cdot 3}{3 \cdot 5 \cdot 3} \\ &= \frac{5}{45} + \frac{21}{45} \\ &= \frac{26}{45}\end{aligned}$$

You may have done most of this process in your head. We show the steps because you can add or subtract rational expressions in the same way.

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### Example 3

**Write $\frac{a}{b} + \frac{c}{d}$ as a single rational expression.**

**Solution** Since $b$ and $d$ have no common factors, their least common multiple is $bd$. This is the common denominator.

$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot ?}{b \cdot d} + \frac{? \cdot c}{b \cdot d}$ Fraction Multiplication Property

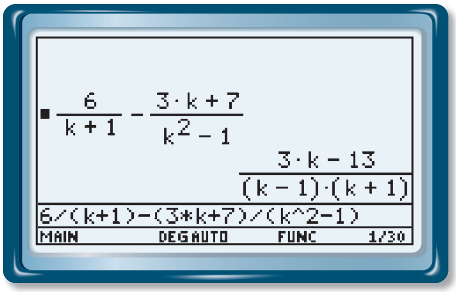
$= \frac{?}{b \cdot d}$ Distributive Property

**Check** You can check by substituting numbers for $a$, $b$, $c$, and $d$. Try $a = 1$, $b = 9$, $c = 7$, and $d = 15$, since you already know the sum is $\frac{26}{45}$. The rest is left as a question in the Questions section.

## Example 4

Write $\frac{6}{k+1} - \frac{3k+7}{k^2-1}$ as a single rational expression.

**Solution 1** Use a CAS. You must be careful to include parentheses to identify numerators and denominators of the fractions. In one calculator, we entered 6/(k+1)-(3*k+7)/(k^2-1). The calculator returned the expression $\frac{3k-13}{(k-1)(k+1)}$.




**Solution 2** Work by hand. Find the least common denominator. The denominator $k + 1$ is prime. The denominator $k^2 - 1$ equals $(k + 1)(k - 1)$. So the least common denominator is $(k + 1)(k - 1)$. Rewrite the first fraction with that denominator.

$$\frac{6}{k+1} - \frac{3k+7}{k^2-1} = \frac{6k-1}{(k+1)(k-1)} - \frac{3k+7}{(k+1)(k-1)}$$

Subtract the fractions as you would any fractions with the same denominator. But notice that the subtracted numerator must be treated as a quantity.

$$= \frac{6(k-1) - (3k+7)}{(k+1)(k-1)}$$

$$= \frac{6k - 6 - 3k - 7}{(k+1)(k-1)}$$

$$= \frac{3k-13}{(k-1)(k+1)}$$

**Check** You can check by substitution or by graphing. We leave these checks to you in the Questions section.

# Questions

## COVERING THE IDEAS

1. What is the definition of *rational expression*?

2. **Multiple Choice** Which of the following are *not* rational expressions?

   A $\frac{2x}{3}$ B $\frac{4y^0}{5y^{15}}$ C $\frac{z\sqrt{6}}{7}$ D $\frac{8\sqrt{w}}{9w}$

**In 3 and 4, simplify the rational expression and indicate all restrictions on values of the variables.**

3. $\frac{30a^4b^2c^3}{12a^4bc^5}$

4. $\frac{28x - 21x}{7xy}$

5. a. By factoring the numerator and denominator, write $\frac{2x^2 - 7x + 6}{x^2 - 4}$ in lowest terms.

   b. What values can $x$ not have in this expression?

   c. Check your answer by letting $x = 10$.

6. a. Write $\frac{7x^2 + 161x + 910}{14x^2 + 182x + 420}$ in lowest terms.

   b. Check your answer by graphing, as was done in Example 2.

7. Complete the check of Guided Example 3.

8. Check the answer to Example 4 by graphing.

9. Check the answer to Example 4 by substitution.

**In 10–13, write as a single rational expression. Check your answer.**

10. $\frac{a}{b} - \frac{c}{d}$

11. $\frac{3}{2n} + \frac{3}{8n}$

12. $\frac{z + 1}{4z - 3} + \frac{3z}{8z^2 - 18z + 9}$

13. $\frac{8x^2 + 16x + 8}{5x + 5} - \frac{x + 1}{3x^2 - 3}$

## APPLYING THE MATHEMATICS

**In 14 and 15, use a CAS or graphing calculator to write the expression in lowest terms.**

14. $\frac{-3x^3 - 15x^2 - 24x - 12}{x^2 + 3x + 2}$

15. $\frac{x^5 - 4x^4 - 37x^3 + 124x^2 + 276x - 720}{x^5 + 20x^4 + 155x^3 + 580x^2 + 1{,}044x + 720}$

**In 16 and 17, the sum $F$ of the integers from 1 to $n$ is given by the formula $F = \frac{n^2 + n}{2}$. The sum $S$ of the squares of the integers from 1 to $n$ is given by the formula $S = \frac{2n^3 + 3n^2 + n}{6}$. The sum $C$ of the cubes of the integers from 1 to $n$ is given by the formula $C = \frac{n^4 + 2n^3 + n^2}{4}$.**

16. a. Find the values of $C$, $F$, and $\frac{C}{F}$ when $n = 13$.

    b. Show that $\frac{C}{F} = F$ for all values of $n$.

17. a. Find the values of $S$, $F$, and $\frac{S}{F}$ when $n = 13$.

    b. Find a rational expression for $\frac{S}{F}$ in lowest terms.

    c. Explain why there are values of $n$ for which $\frac{S}{F}$ is not an integer.

18. Generalize the following pattern and use addition of rational expressions to show why your generalization is true.

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}, \quad \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \quad \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

## REVIEW

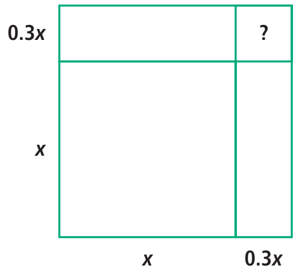
**19.** If an object is thrown upward from a height of 0 meters at a speed of $v\ \frac{\text{meters}}{\text{second}}$, then its height after $t$ seconds is $vt - 4.9t^2$ meters. **(Lesson 12-6)**

**a.** If an object is thrown upward with speed $v$, how long will it take it to hit the ground?

**b.** What must be true about $v$ if the time it takes the object to hit the ground in seconds is an integer? Give an example of a $v$ for which this happens and a $v$ for which it does not happen.

**20.** Is $3x^2 - 2x + 5$ a prime polynomial? How do you know? **(Lesson 12-6)**

**21.** Gerardo is cutting a shape out of paper. He begins with a square piece of paper, and cuts out the upper right corner, as in the figure at the right. What is the area of the piece he cut out? **(Lesson 12-2)**




**22.** Nikki wants to call her friend Adelaide. She remembers that the last three digits of her number are 3, 4, and 7, but she doesn't remember the correct order. If she guesses an order at random, what is the probability that she will get the right number? **(Lesson 11-7)**

**23.** Marcus took a test in which there were 20 questions. In this test, every correct answer gave 6 points, and every incorrect answer subtracted 3 points from the grade. If Marcus got a 93 on the test, how many questions did he get right? **(Lesson 10-5)**

**24.** Is there a convex polygon with exactly 43 diagonals? Explain how you know. **(Lesson 9-7)**

## EXPLORATION

**25.** A teacher, wanting to show students that their ideas could be used with very complicated rational expressions, used the following expression.

$$\frac{x^9 + 11x^8 - 84x^7 - 1{,}660x^6 - 4{,}874x^5 + 44{,}082x^4 + 400{,}140x^3\ 1{,}347{,}300x^2 + 2{,}156{,}625x + 1{,}366{,}875}{x^8 - 22x^7 + 90x^6 + 882x^5 - 5{,}508x^4 - 10{,}530x^3 + 74{,}358x^2 + 39{,}366x - 295{,}245}$$

However, the teacher forgot the operation sign between $x^3$ and 1,347,300.

**a.** If the numerator was meant to be factored, what is the operation?

**b.** Use the answer to Part a to write the expression in lowest terms.

**QY ANSWER**

The numerator and denominator have common factors.

$\frac{6k^3 - 12k^2 + 42k - 210}{3k^3 - 6k^2 + 21k - 105} = \frac{6(k^3 - 2k^2 + 7k - 35)}{3(k^3 - 2k^2 + 7k - 35)} = 2$