Lesson 12-7 Graphs of Polynomial Functions of Higher Degree

Vocabulary

cubic polynomial

BIG IDEA The factored form of a polynomial is useful in graphing and solving equations.

Some of the ideas that you have seen in earlier lessons of this chapter extend to the graphs of polynomial functions of degrees 3 and higher. In particular, factoring a polynomial can be a powerful tool to uncover interesting features of graphs of polynomial functions.

Mental Math

A baseball player has a batting average of .245. State whether his batting average goes up or down if he plays a game where he goes

a. 1 for 4.

b. 1 for 5.

c. 2 for 6.

How Are Factors and x-Intercepts Related?

Activity 1

In 1–6, a polynomial function is given in factored form.

a. Graph the function with a graphing calculator.

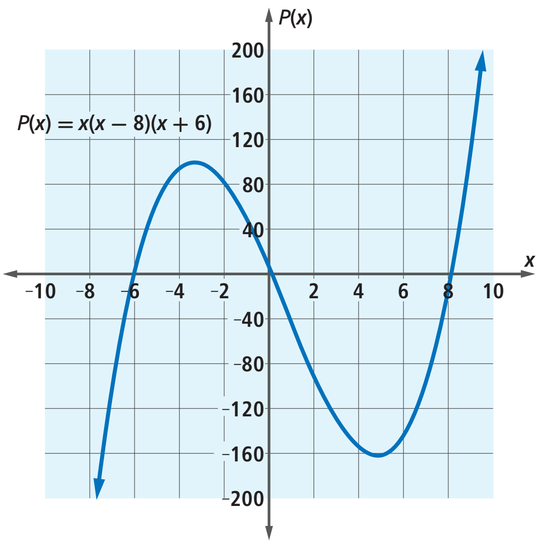
b. Identify the x-intercepts of the graph.

c. How are the x-intercepts of the graph related to the factors?

1. $f(x) = 5(x - 1)$
2. $g(x) = (x - 1)(x + 2)$
3. $h(x) = (x - 1)(x + 2)(x - 4)$
4. $k(x) = 3(x - 1)(x + 2)(x - 4)$
5. $m(x) = (x - 1)^2(x + 2)^2$
6. $q(x) = (x - 1)(x + 2)(x - 4)(2x + 15)$
7. Look back at your work. Make a conjecture about the factors of a polynomial and the x-intercepts of its graph. Test your conjecture with a different polynomial function than those above.

A graph of the function P with $P(x) = x(x - 8)(x + 6)$ is shown at the right, along with a table of values. Notice how the factors and the x-intercepts are related.

x	$P(x)$	x	$P(x)$	x	$P(x)$
−9	−459	−1	45	7	−13
−8	−256	0	0	8	0
−7	−105	1	−49	9	135
−6	0	2	−96	10	320
−5	65	3	−135	11	561
−4	96	4	−160	12	864
−3	99	5	−165		
−2	80	6	−144		



The graph of the function P has three x-intercepts: 0, 8, and –6. Would your conjecture from Activity 1 have predicted this result? Whether or not you predicted this, there is a simple but elegant relationship between factors and x-intercepts that holds for any polynomial function.

Factor Theorem

Let r be a real number and $P(x)$ be a polynomial in x.

1. If $x - r$ is a factor of $P(x)$, then $P(r) = 0$; that is, r is an x-intercept of the graph of P.
2. If $P(r) = 0$, then $x - r$ is a factor of $P(x)$.

The Factor Theorem is true because the x-intercepts of the graph of the function P are the values of x such that $P(x) = 0$. For the polynomial $P(x) = x(x - 8)(x + 6)$ graphed on the previous page, the equation $P(x) = 0$ means $x(x - 8)(x + 6) = 0$.

By the Zero Product Property, this is a true statement when

$x = 0$ or $x - 8 = 0$ or $x + 6 = 0$.

$x = 8$ or $x = -6$

So, because x, $x - 8$, and $x + 6$ are factors of $P(x)$, 0, 8, and –6 are the x-intercepts of the graph of P.

GUIDED

Example 1

A polynomial function P has x-intercepts 5, 7.8, –46, and –200. What is a possible equation for the function?

Solution The polynomial must have at least four factors.

Because 5 is an x-intercept, $x - 5$ is a factor of the polynomial.

Because 7.8 is an x-intercept, ___?___ is a factor of the polynomial.

Because –46 is an x-intercept, ___?___ is a factor of the polynomial.

Because –200 is an x-intercept, ___?___ is a factor of the polynomial.

Possibly, $P(x) = (x - 5)$(___?___)(___?___)(___?___).

Converting from Factored Form to Standard Form

The polynomial $P(x) = x(x - 8)(x + 6)$ is the product of three factors. To convert this polynomial to standard form, multiply any two of its factors. Then multiply the product of those factors by the third factor. Because multiplication is associative, it does not make any difference which two factors you multiply first.

$$\begin{aligned} P(x) &= x(x - 8)(x + 6) \\ &= (x^2 - 8x)(x + 6) \\ &= (x^2 - 8x) \cdot x + (x^2 - 8x) \cdot 6 \\ &= x^3 - 8x^2 + 6x^2 - 48x \\ &= x^3 - 2x^2 - 48x \end{aligned}$$

The standard form shows clearly that $P(x)$ is a polynomial of degree 3. It is a **cubic polynomial.** Just as a quadratic function has at most two x-intercepts, a cubic function has at most three x-intercepts.

The following example involves the function q from Activity 1. It is a polynomial function of 4th degree.

Example 2

Rewrite the polynomial $q(x) = (x - 1)(x + 2)(x - 4)(2x + 15)$ in standard form.

Solution There are four factors in the polynomial. Any two can be multiplied first. We multiply the first two and the last two.

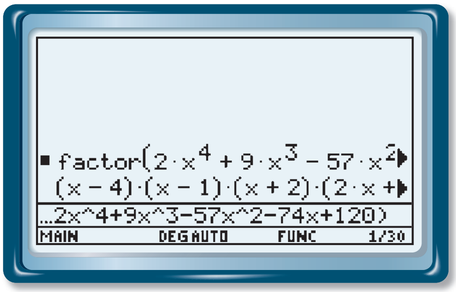
$$\begin{aligned} q(x) &= (x - 1)(x + 2)(x - 4)(2x + 15) \\ &= (x^2 + x - 2)(2x^2 + 7x - 60) \end{aligned}$$

Now use the Extended Distributive Property. Each term of the left factor must be multiplied by each term of the right factor. There are nine products.

$$\begin{aligned} &= x^2(2x^2 + 7x - 60) + x(2x^2 + 7x - 60) - 2(2x^2 + 7x - 60) \\ &= 2x^4 + 7x^3 - 60x^2 + 2x^3 + 7x^2 - 60x - 4x^2 - 14x + 120 \\ &= 2x^4 + 9x^3 - 57x^2 - 74x + 120 \end{aligned}$$

Check 1 In factored form it is easy to see that $q(1) = 0$. So substitute 1 for x in the standard form to see if the value of the polynomial is 0.
The value is $2 + 9 - 57 - 74 + 120 = 0$. It checks.

Check 2 Use a CAS to factor the answer. See if the original factored form appears. Our CAS gives $q(x) = (x - 1)(x + 2)(x - 4)(2x + 15)$.



In Activity 2, you are asked to explore what happens when the same factor appears twice in a polynomial.

Activity 2

Step 1 **a.** Graph the following polynomial functions on the window $-10 \le x \le 10$, $-500 \le y \le 500$. Make a sketch of each graph.

$f(x) = x(x - 3)(x + 7)$ $\quad g(x) = x(x - 3)(x + 7)^2$

$h(x) = x(x - 3)^2(x + 7)^2$ $\quad j(x) = x^2(x - 3)^2(x + 7)^2$

b. What are the *x*-intercepts of these graphs?

c. What is different about the graphs around the point (−7, 0) when $(x + 7)^2$ is a factor rather than just $(x + 7)$?

Step 2 **a.** Multiply each polynomial by −1 and graph the resulting functions.

b. Describe what happens to the *x*-intercepts.

Step 3 Multiplying $P(x)$ by −1 means graphing $-P(x)$. The graph wiggles in the middle. But in all these functions, as you go farther to the right, the graph heads either up or down. As you go farther to the left, the graph also heads either up or down.

a. Copy and complete the table.

Polynomial	Far Right: Up or Down?	Far Left: Up or Down?	Polynomial	Far Right: Up or Down?	Far Left: Up or Down?
$f(x) = x(x - 3)(x + 7)$	?	?	$-f(x)$	?	?
$g(x) = x(x - 3)(x + 7)^2$	?	?	$-g(x)$	?	?
$h(x) = x(x - 3)^2(x + 7)^2$	?	?	$-h(x)$	?	?
$j(x) = x^2(x - 3)^2(x + 7)^2$	?	?	$-i(x)$	?	?

b. Describe the general pattern.

Step 4 **a.** Experiment to find what happens to the graph of $P(x) = ax(x - 3)(x + 7)$ as values of *a* change from positive to negative.

b. Does the same situation hold for the graph of $j(x) = ax^2(x - 3)^2(x + 7)^2$?

The Importance of Polynomial Functions

In this course, you have studied polynomial functions of degrees 1 and 2 in detail. A polynomial function of degree 1 has an equation of the form $y = mx + b$. Its graph is a line. A polynomial function of degree 2 has an equation of the form $y = ax^2 + bx + c$. Its graph is a parabola. The graphs of polynomial functions of degrees 3 and 4 have more varied shapes and graphs of higher degrees have still more varied shapes. This makes polynomial functions very useful in approximating data of many kinds and very useful in approximating other functions. Polynomials also appear as formulas in a number of situations, a few of which are mentioned in the Questions for this lesson.

Questions

COVERING THE IDEAS

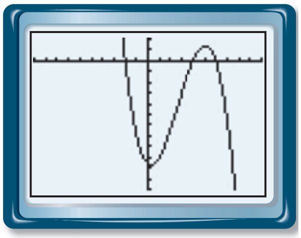
In 1–4, an equation for a function is given.

a. Identify the x-intercepts of the graph of the function.

b. Check your answer to Part a by graphing the function. Draw a rough sketch of the graph.

c. Write the equation in standard form.

1. $f(x) = (x + 5)(2x - 3)$
2. $y = (x + 5.8)(2x - 3.4)$
3. $y = -3(x - 1)(2x + 1)^2$
4. $g(x) = x(8x + 5)(10x + 2)(x - 1)$
5. Give an equation for a polynomial function whose graph intersects the x-axis at $(-9, 0)$, $(4, 0)$, and nowhere else.
6. Suppose the graph of a polynomial function has three x-intercepts: 1, –4, and 5.
   a. Give an equation in factored form for the polynomial function.
   b. Rewrite your equation from Part a in standard form.
7. a. Give an equation in factored form for a polynomial function with four x-intercepts: 2, –2, 5, and –5.
   b. Rewrite your equation from Part a in standard form.
8. Match the following equations with their possible graphs.
   a. $a(x) = (x + 2)(x - 4)^2(x - 6)$
   b. $b(x) = -1(x + 2)(x - 4)(x - 6)$
   c. $c(x) = -1(x + 2)^2(x - 4)(x - 6)$
   d. $d(x) = -(x + 2)(x - 4)^2(x - 6)$

i.


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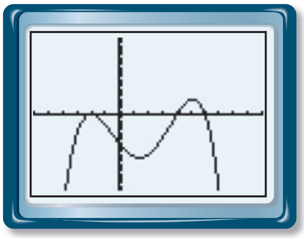

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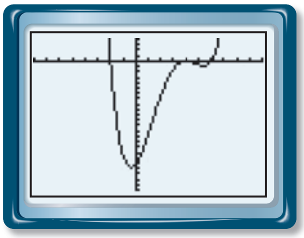

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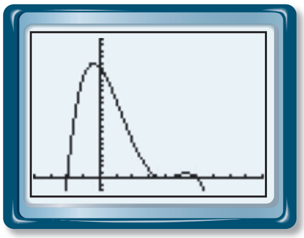
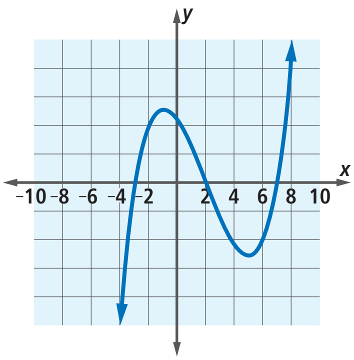


In 9–12, give a possible equation in factored form for each graph.

9.



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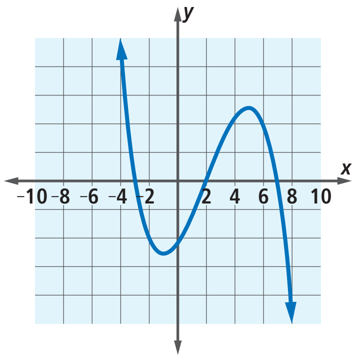

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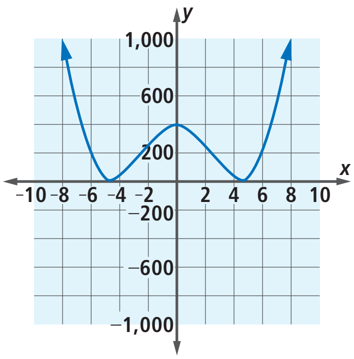

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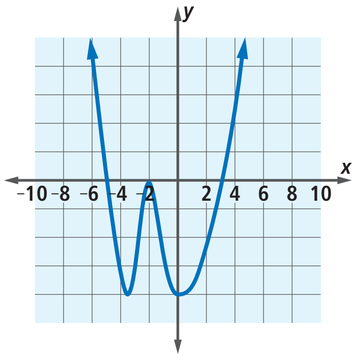


13. Let $S(n)$ = the sum of the squares of the integers from 1 to n. It is known that $S(n) = \frac{1}{6}n(n + 1)(2n + 1)$.

a. Find $S(5)$ using the formula and verify your answer by using the definition of $S(n)$.

b. What is the value of $S(n + 1) - S(n)$?

c. Write the formula for $S(n)$ in standard form.

14. Explain what you know about each of the following aspects of the graph of the function $h(x) = -8(x - 11)^2(x + 5)(x + 10)^2$.

a. x-intercepts

b. when the graph changes direction at an x-intercept

c. the direction of the far right side of the graph

APPLYING THE MATHEMATICS

15. a. Graph $y_1 = (x - 2)(x + 4)(x - 5)$ and $y_2 = (x - 2)^3(x + 4)(x - 5)$.

b. Use the results of Part a to predict a characteristic of the graph of $y_3 = (x - 2)(x + 4)(x - 5)^3$.

c. Use the results of Parts a and b to predict a characteristic of the graph of $y_4 = (x - 2)^3(x + 4)(x - 5)^3$.

d. Generalize the results of Parts a, b, and c.

REVIEW

In 16–18, factor the polynomial. (Lessons 12-5, 11-4)

16. $9x^2y^2 - 27x^3y + 19xy$

17. $4a^2 - 16b^2$

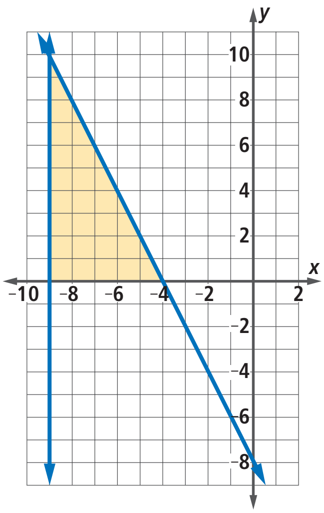
18. $15n^2 + 1 - 8n$

In 19–22, solve by using any method. (Lessons 12-5, 12-2, 9-5)

19. $a^2 + 6a = 55$

20. $-3w^2 - 7w + 11 = 0$

21. **Multiple Choice** Which system of inequalities is graphed at the right? (Lesson 10-9)



A $\begin{cases} y \geq -2x - 8 \\ y \geq 0 \\ x \leq -9 \end{cases}$ B $\begin{cases} y \leq -2x - 8 \\ y \geq 0 \\ x \geq -9 \end{cases}$ C $\begin{cases} y \geq -2x - 8 \\ y \leq 0 \\ x \leq -9 \end{cases}$ D $\begin{cases} y \leq -2x - 8 \\ y \geq -9 \\ x \geq 0 \end{cases}$

22. According to the U.S. Department of Labor, the following percentages of the adult population were employed in the particular years shown in the table at the right. Assume these trends continue. (Lesson 6-7)

Year	Male	Female
1974	74.9	42.6
1979	73.8	47.5
1984	70.7	49.5
1989	72.5	54.3
1995	70.8	55.6
2001	70.9	57.0
2002	69.7	56.3

a. Use linear regression to find a line that fits the pairs (year, male). Write the equation of a line of best fit.

b. Repeat Part a for the pairs (year, female). Write the equation of a line of best fit.

c. Using your equation from Part b, predict what percentage of adult women will be employed in the year 2010.

d. Using your equations from Parts a and b, predict when, if ever, the same percentage of adult men and adult women will be employed.

EXPLORATION

23. a. Explore the graphs of $y_1 = x^3$, $y_2 = x^4$, $y_3 = x^5$, and $y_4 = x^6$ and make a generalization about the graph of $f(x) = x^n$, when n is a positive integer.

b. If x is replaced by $x - 5$ in each of the graphs of Part a, what happens to the graphs?

c. Generalize the result in Part b.