Lesson 12-6

Which Quadratic Expressions Are Factorable?

▶ **BIG IDEA** A quadratic expression with integer coefficients is factorable over the integers if and only if its discriminant is a perfect square.

This lesson connects two topics you have seen in this chapter: factoring and solutions to quadratic equations. These topics seem quite different. Their relationship to each other is an example of how what you learn in one part of mathematics is often useful in another part.

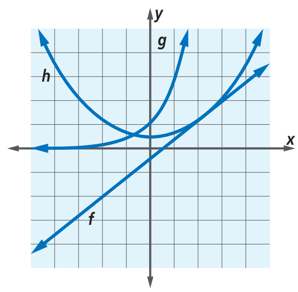
You have seen four ways to find the real-number values of x that satisfy $ax^2 + bx + c = 0$.

1. You can graph $y = ax^2 + bx + c$ and look for its x-intercepts.
2. You can use the Quadratic Formula.
3. You can factor $ax^2 + bx + c$ and use the Zero Product Property.
4. You can set $f(x) = ax^2 + bx + c$ to 0 and look for values of x such that $f(x) = 0$.

The first two ways can always be done. But you know that it is not always possible to factor $ax^2 + bx + c$ over the integers, so it useful to know when it is possible.

Mental Math

Match each function graphed below with its type.



a. exponential growth

b. linear

c. quadratic

A Quadratic Equation with Rational Solutions

Consider the equation $9x^2 + 14x - 8 = 0$. Use the Quadratic Formula.

Step 1 $x = \dfrac{-14 \pm \sqrt{14^2 - 4 \cdot 9 \cdot (-8)}}{2 \cdot 9}$

Step 2 $= \dfrac{-14 \pm \sqrt{484}}{18}$

Step 3 $= \dfrac{-14 \pm 22}{18}$

Step 4 So $x = \dfrac{-14 + 22}{18} = \dfrac{4}{9}$ or $x = \dfrac{-14 - 22}{18} = -2$

Notice that the solutions $\frac{4}{9}$ and -2 have no visible radical sign. This is because the number 484 under the square root sign is a perfect square (Step 2). So, after calculating the square root (Step 3), one integer is divided by another, and the solutions are rational numbers.

In general, for quadratic equations with integer coefficients, if the *discriminant* $b^2 - 4ac$ in the quadratic equation $ax^2 + bx + c = 0$ is a perfect square, then the solutions are rational. The square root of an integer that is not a perfect square is irrational. So the square root will remain in the solutions, and the solutions will be irrational. These results can be summarized in one sentence: *When a, b, and c are integers, the solutions to $ax^2 + bx + c = 0$ are rational numbers if and only if $b^2 - 4ac$ is a perfect square.*

Relating the Solving of $ax^2 + bx + c = 0$ to the Factoring of $ax^2 + bx + c$

Now we connect this with factoring. Consider the same equation as before: $9x^2 + 14x - 8 = 0$.

Factor the left side.

$(9x - 4)(x + 2) = 0$

Use the Zero Product Property.

$$9x - 4 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = \frac{4}{9} \quad \text{or} \quad x = -2$$

You can see from the equation above that when a quadratic equation in standard form is factorable, the solutions are rational numbers. Combining this observation with the facts on the previous page leads us to the following conclusion, which we call the *Discriminant Theorem.* A formal proof of this theorem is given in Chapter 13.

Discriminant Theorem

When *a, b,* and *c* are integers, with $a \neq 0$, either all three of the following conditions hold, or none of these hold.

1. $b^2 - 4ac$ is a perfect square.
2. $ax^2 + bx + c$ is factorable over the set of polynomials with integer coefficients.
3. The solutions to $ax^2 + bx + c = 0$ are rational numbers.

Example 1

Is $8x - 5x^2 + 21$ factorable into polynomials with integer coefficients?

Solution First rewrite this expression in the standard form of a polynomial.

$$-5x^2 + 8x + 21$$

Thus $a = -5$, $b = 8$, and $c = 21$.

(continued on next page)

Then $b^2 - 4ac = (8)^2 - 4 \cdot (-5) \cdot 21 = 64 + 420 = 484$.

Since $484 = 22^2$, 484 is a perfect square. So the expression is factorable.

STOP QY

QY

Verify Example 1 by finding the factorization of $8x - 5x^2 + 21$.

GUIDED

Example 2

Is the polynomial $2x^2 - 10 + 5x$ factorable?

Solution

Step 1 Write the polynomial in standard form. __?__

Step 2 Identify *a*, *b*, and *c*. $a =$ __?__, $b =$ __?__, and $c =$ __?__

Step 3 Calculate $b^2 - 4ac$. __?__

Step 4 Is $b^2 - 4ac$ a perfect square? __?__

Step 5 What is your conclusion?

The phrase "with integer coefficients" is necessary in Example 1 because every quadratic expression is then factorable if noninteger coefficients are allowed.

Example 3

What can be learned by applying the Discriminant Theorem to the quadratic equation $x^2 - 29 = 0$?

Solution In this case, $a = 1$, $b = 0$, and $c = -29$, so $b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot (-29) = 116$. Since 116 is not a perfect square, the solutions to $x^2 - 29 = 0$ are irrational and the polynomial $x^2 - 29$ cannot be factored into linear factors with integer coefficients.

Yet the polynomial $x^2 - 29$ in Example 3 can be factored as the difference of two squares.

$$x^2 - 29 = x^2 - \left(\sqrt{29}\right)^2 = \left(x - \sqrt{29}\right)\left(x + \sqrt{29}\right)$$

The factors do not have integer coefficients, so we say that $x^2 - 29$ is prime over the set of polynomials with integer coefficients, but not over the set of all polynomials. It is just like factoring 7 into $3 \cdot \frac{7}{3}$. The integer 7 is prime over the integers but can be factored into rational numbers.

Applying the Discriminant Theorem

Knowing whether an expression is factorable can help determine what methods are available to solve an equation.

Example 4

Solve $m^2 - 9m + 24 = 0$ by any method.

Solution Because the coefficient of m^2 is 1, it is reasonable to try to factor the left side. But first evaluate $b^2 - 4ac$ to see whether this is possible.

$a = 1$, $b = -9$, and $c = 24$. So, $b^2 - 4ac = (-9)^2 - 4 \cdot 1 \cdot (24) = -15$. This is not a perfect square, so the equation does not factor over the integers.

In fact, because $b^2 - 4ac$ is negative, there are no real solutions to this equation.

Check Graph $y = x^2 - 9x + 24$. You will see that the graph does not intersect the x-axis. There are no x-intercepts.

What percent of quadratic expressions are factorable? Try the following activity.

Activity

This activity can be done with a partner if a CAS is available, or as a whole-class activity otherwise.

There are infinitely many quadratic expressions of the form $ax^2 + bx + c$, but there are only 8,000 of these in which a, b, and c are nonzero integers from –10 to 10. What percent of these are factorable? There are too many to try to factor by hand, even with a CAS. It is possible to determine this number by programming a computer to factor all of them. But it is also possible to estimate the percent by sampling.

Step 1 Set a calculator to generate random integers from –10 to 10.

Step 2 Generate three such nonzero integers. Call them a, b, and c. Record them and the expression $ax^2 + bx + c$ in a table like the one shown below.

Trial	a	b	c	$ax^2 + bx + c$	Factorable?
1	2	–3	7	$2x^2 - 3x + 7$	Prime
2	?	?	?	?	?
3	?	?	?	?	?

(continued on next page)

Step 3 If you have a CAS, try to factor $ax^2 + bx + c$ over the integers. If you are working by hand, calculate $b^2 - 4ac$. If the expression is factorable, record the factors. If not, record the word *Prime*.

Step 4 Repeat Steps 2 and 3 at least 20 times. What percent of your quadratic expressions are factorable?

Step 5 Combine your results with those of others in the class. What is your class's estimate of the percent of these quadratic expressions that are factorable?

Questions

COVERING THE IDEAS

In 1–5, a quadratic expression is given. Calculate $b^2 - 4ac$ to determine if the quadratic is factorable or prime over the integers. If possible, factor the expression.

1. $x^2 - 9x - 22$
2. $36 - y^2$
3. $4n^2 - 12n + 9$
4. $-3 + m^2 + 2m$
5. $7x^2 - 13x - 6$
6. Consider the equation $ax^2 + bx + c = 0$ where a, b, and c are integers. If $b^2 - 4ac$ is a perfect square, explain why x is rational.
7. Suppose a, b, and c are integers. When will the x-intercepts of $y = ax^2 + bx + c$ be rational numbers?
8. Give an example of a quadratic expression that can be factored only if noninteger coefficients are allowed.

APPLYING THE MATHEMATICS

9. The sum of the integers from 1 to n is $\frac{1}{2}n(n + 1)$. Find n if the sum of the integers from 1 to n is 499,500.
10. What in this lesson tells you that the solutions to the quadratic equation $x^2 - 3 = 0$ are irrational? (This provides a way of showing that $\sqrt{3}$ is irrational.)
11. Find a value of k such that $4x^2 + kx - 5$ is factorable over the integers.
12. a. By multiplying, verify that $(x + 5 + \sqrt{2})(x + 5 - \sqrt{2}) = x^2 + 10x + 23$.
    b. Verify that the discriminant of the expression $x^2 + 10x + 23$ is not a perfect square.
    c. Parts a and b indicate that $x^2 + 10x + 23$ is factorable, yet its discriminant is not a perfect square. Why doesn't this situation violate the Discriminant Theorem?

REVIEW

In 13 and 14, factor the polynomial completely. (Lessons 12-5, 12-4)

13. $r^2 - 5r + 4$

14. $-2x^2 + 5x + 12$

15. Consider $w^2 + 9w + c$. (Lesson 12-2)
    a. Complete the square to find the value of c.
    b. Express the perfect square trinomial in factored form.

16. The surface area S of a cylinder with radius r and height h is given by the formula $S = 2\pi r^2 + 2\pi rh$. (Lesson 11-4)
    a. Factor the right hand side of this formula into prime factors.
    b. Calculate the exact surface area of a cylinder with a diameter of 12 cm and a height of 9 cm using either the given formula or its factored form. Which form do you think is easier for this purpose?

In 17–19, rewrite the expression with no negative exponents and each variable mentioned no more than once. (Lessons 8-5, 8-4, 8-3)

17. $\frac{8n^{-3}m^2}{6m^{-5}}$

18. $\left(\frac{a^2b}{4a^5}\right)^2$

19. $\left(\frac{3x^4y^{-1}}{15x^3y^0}\right)^2$

EXPLORATION

20. Look back at Question 11. Find *all* integer values of k such that $4x^2 + kx - 5$ is factorable in the set of polynomials over the integers. Explain how you know that you have found all values.

21. Use a CAS to factor the general expression $ax^2 + bx + c$. (You will likely have to indicate that x is the variable.)
    a. What factorization does the CAS give?
    b. Explain how you know that this factorization is correct.
    c. Why doesn't the existence of factors for any quadratic expression violate the Discriminant Theorem?

QY ANSWER

$-(5x + 7)(x - 3)$