Lesson 12-5 Factoring $ax^2 + bx + c$

BIG IDEA Some quadratic trinomials of the form $ax^2 + bx + c$ can be factored into two linear factors.

You have seen that some trinomials of the form $x^2 + bx + c$ can be factored into a product of two binomials.

$$x^2 + 0x - 100 = (x + 10)(x - 10)$$
$$x^2 + 12x + 36 = (x + 6)(x + 6)$$
$$x^2 - 9x + 14 = (x - 7)(x - 2)$$
$$x^2 + 7x - 8 = (x - 1)(x + 8)$$

In this lesson, we consider quadratic trinomials in which the coefficient of the square term is not 1. Again, we seek to factor the trinomial into binomials with integer coefficients.

In factoring such a trinomial, first check for a common factor of the three terms.

Mental Math

Find the greatest common factor of

a. $16t$ and 32.

b. $9a$, $6b$, and $10ab$.

c. x^2 and $4x^3$.

Example 1

Factor $50x^5 + 200x^4 + 200x^3$.

Solution $50x^3$ is a common factor of the three terms.

$$50x^5 + 200x^4 + 200x^3 = 50x^3(x^2 + 4x + 4)$$

To factor $x^2 + 4x + 4$, we need a binomial whose constant terms have a product of 4 and a sum of 4.

$$x^2 + 4x + 4 = (x + 2)(x + 2)$$

So, $50x^5 + 200x^4 + 200x^3 = 50x^3(x + 2)(x + 2)$.

The original polynomial is said to be factored completely.

Check

$$50x^3(x + 2)(x + 2) = (50x^4 + 100x^3)(x + 2)$$
$$= 50x^5 + 100x^4 + 100x^4 + 200x^3$$
$$= 50x^5 + 200x^4 + 200x^3$$

The factorization checks.

When the coefficient of the square term is not 1 and there is no common factor of the three terms, a different process is applied. Suppose $ax^2 + bx + c = (dx + e)(fx + g)$.

The product of d and f, from the first terms of the binomials, is a. The product of e and g, the constant terms of the binomials, is c. The task is to find d, e, f, and g so that the rest of the multiplication and addition gives b.

Example 2

Factor $5x^2 + 32x + 12$.

Solution

Step 1 Rewrite the expression as a product of two binomials.

$$5x^2 + 32x + 12 = (dx + e)(fx + g)$$

You need to find integers d, e, f, and g.

Step 2 The coefficient of $5x^2$ is 5, so $df = 5$. Assume either d or f is 5, and the other is 1. Now write the following.

$$5x^2 + 32x + 12 = (5x + e)(x + g)$$

The product of e and g is 12, so $eg = 12$. Because the middle term $32x$ is positive, e and g must be positive. Thus, e and g might equal 1 and 12, or 2 and 6, or 3 and 4, in either order. Try all six possibilities.

Can e and g be 1 and 12?

$$(5x + 1)(x + 12) = 5x^2 + 61x + 12$$
$$(5x + 12)(x + 1) = 5x^2 + 17x + 12$$

No, we want $b = 32$, not 61 or 17.

Can e and g be 3 and 4?

$$(5x + 3)(x + 4) = 5x^2 + 23x + 12$$
$$(5x + 4)(x + 3) = 5x^2 + 19x + 12$$

No. Again the middle term is not what we want.

Can e and g be 2 and 6?

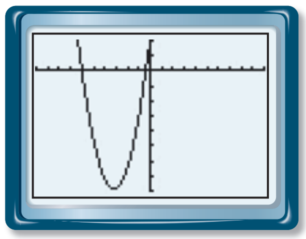
$$(5x + 2)(x + 6) = 5x^2 + 32x + 12$$

Yes; here $b = 32$. This is what we want.

$$5x^2 + 32x + 12 = (5x + 2)(x + 6)$$

Check 1 The multiplication $(5x + 2)(x + 6) = 5x^2 + 32x + 12$ is a check.

Check 2 Graph $y = 5x^2 + 32x + 12$ and $y = (5x + 2)(x + 6)$ on the same set of axes. The graphs should be identical. It checks.



$y = 5x^2 + 32x + 12$
and
$y = (5x + 2)(x + 6)$

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Check 3 Another check is to substitute a value for *x*, say 4.

Does $5x^2 + 32x + 12 = (5x + 2)(x + 6)$?

$5 \cdot 4^2 + 32 \cdot 4 + 12 = (5 \cdot 4 + 2)(4 + 6)$

$80 + 128 + 12 = 22 \cdot 10$

Yes. Each side equals 220.

In Example 2, there are not many possible factors because the coefficient of x^2 is 5 and all numbers are positive. Example 3 has more possibilities, but the idea is still the same. Try factors until you find the correct ones.

GUIDED

Example 3

Factor $15y^2 - 16y - 7$.

Solution First write down the form. $(ay + b)(cy + d)$. So $ac =$ _?_. Thus either *a* and *c* are 3 and 5 or they are _?_ and _?_. The product $bd = -7$. So *b* and *d* are either _?_ and _?_, or _?_ and _?_.

List all the possible factors with $a = 3$ and $c = 5$, and multiply.

$(3y + 7)(5y - 1) =$ _?_

$(3y - 7)(5y + 1) =$ _?_

$(3y - 1)(5y + 7) =$ _?_

$(3y + 1)(5y - 7) =$ _?_

List all the possible factors with $a = 1$ and $c = 15$.

(_?_$y + 7$)(_?_$y - 1$) = _?_

(_?_$y - 7$)(_?_$y + 1$) = _?_

(_?_$y - 1$)(_?_$y + 7$) = _?_

(_?_$y + 1$)(_?_$y - 7$) = _?_

At most, you need to do these eight multiplications. If one of them gives $15y^2 - 16y - 7$, then that is the correct factoring.

So $15y^2 - 16y - 7 =$ _?_.

In Example 3, notice that each choice of factors gives a product that differs only in the coefficient of *y* (the middle term). If the original problem were to factor $15y^2 - 40y - 7$, this process shows that no factors with integer coefficients will work. The quadratic $15y^2 - 40y - 7$ is a prime polynomial over the set of integers.

Once a quadratic trinomial $f(x)$ has been factored, then solving $f(x) = 0$ is easy using the Zero Product Property.

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Example 4

Solve $15y^2 - 16y - 7 = 0$.

Solution Use the factorization of $15y^2 - 16y - 7$ in Example 3.

$15^2 - 16y - 7 = 0$

$(\underline{\ \ ?\ \ })(\underline{\ \ ?\ \ }) = 0$

$\underline{\ \ ?\ \ } = 0$ or $\underline{\ \ ?\ \ } = 0$

$y = \underline{\ ?\ }$ or $y = \underline{\ ?\ }$

Questions

COVERING THE IDEAS

1. Perform the multiplications in Parts a–d.
   a. $(2x + 3)(4x + 5)$
   b. $(2x + 5)(4x + 3)$
   c. $(2x + 1)(4x + 15)$
   d. $(2x + 15)(4x + 1)$
   e. Explain how these multiplications are related to factoring $8x^2 + 26x + 15$.
2. Suppose $ax^2 + bx + c = (dx + e)(fx + g)$ for all values of x.
   a. The product of d and f is $\underline{\ ?\ }$.
   b. The product of $\underline{\ ?\ }$ and $\underline{\ ?\ }$ is c.
3. Factor the trinomials completely.
   a. $2x^2 + 14x + 2$
   b. $5n^2 + 35n - 50$

In 4–9, factor the trinomial, if possible.

4. $5A^2 + 7A + 2$
5. $-3x^2 + 11x - 6$
6. $y^2 - 10y + 16$
7. $14w^2 - 9w - 1$
8. $-4x^2 - 11x + 3$
9. $17k^2 - 36k + 19$
10. Check the solutions to Example 1 by substitution.
11. Solve $20x^2 + 11x - 3 = 0$ by factoring.

APPLYING THE MATHEMATICS

12. Jules solved the equation $2x^2 - 5x + 3 = 7$ in the following way.

Step 1 He factored $2x^2 - 5x + 3$ into $(2x - 3)(x - 1)$.

Step 2 He substituted the factored expression back into the equation $(2x - 3)(x - 1) = 7$.

Step 3 He considered all the possibilities: $2x - 3 = 7$ and $x - 1 = 1$, in this case $x = 5$ or $x = 2$; or $2x - 3 = 1$ and $x - 1 = 7$, in this case $x = 2$ or $x = 8$.

Step 4 He checked his work and found that none of these values of x check in the original equation.

What did Jules do wrong?

13. Find the vertex of the parabola with equation $y = 8x^2 - 6x + 1$ by first factoring to obtain the x-intercepts.

14. Consider the equation $6t^2 + 7t - 24 = 0$.
    a. Solve the equation by using the Quadratic Formula.
    b. Solve the equation by factoring.
    c. Which method do you prefer to solve this problem? Why?

15. a. Solve the equation $3n^2 = 2 - 5n$ using the Quadratic Formula.
    b. Check your solution to Part a by solving the same equation using factoring.

16. a. Factor $14x^3 - 21x^2 - 98x$ into the product of a monomial and a trinomial.
    b. Give the complete factorization of $14x^3 - 21x^2 - 98x$.

In 17 and 18, find the complete factorization.

17. $9p^2 + 30p^3 + 25p^4$

18. $-2x^2 + 23xy - 30y^2$

REVIEW

19. Rewrite $x^8 - 16$ as the product of
    a. two binomials.
    b. three binomials. (**Lessons 12-4, 11-6**)

20. Find two consecutive positive even integers whose product is 360. (**Lessons 12-4, 12-2**)

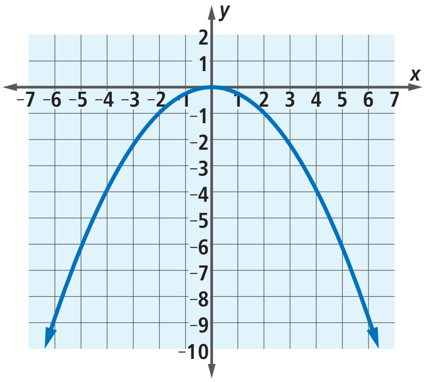
21. Find the x-intercepts of $y = x^2 - 8x + 3$ by completing the square. (**Lessons 12-3, 12-2**)

22. a. Here are three instances of a pattern. Describe the general pattern using two variables, x and y. **(Lessons 11-6, 1-2)**

$$(48 + 32)(24 - 16) = 2(24^2 - 16^2)$$
$$(10 + 20)(5 - 10) = 2(5^2 - 10^2)$$
$$(4 + 1)(2 - 0.5) = 2(2^2 - 0.5^2)$$

b. Does your general pattern hold for all real values of x and y? Justify your answer.

23. **Multiple Choice** Which equation is graphed below? **(Lesson 9-1)**



A $y = -\frac{1}{4}x^2$

B $y = 4x^2$

C $y = -4x^2$

D $y = \frac{1}{4}x^2$

24. When a fair, six-sided die is tossed, the probability of getting a 1 is $\frac{1}{6}$. If the die is tossed twice, the probability of getting a 1 both times is $\frac{1}{6} \cdot \frac{1}{6} = \left(\frac{1}{6}\right)^2$. **(Lessons 8-2, 8-1)**

a. Write an expression to represent the probability of rolling a die m times and getting a 1 each time.

b. Write your answer to Part a as a power of 6.

EXPLORATION

25. The polynomial $6x^3 + 47x^2 + 97x + 60$ can be factored over the integers into $(3x + a)(2x + b)(x + c)$. Find a, b, and c. (*Hint:* What is $a \cdot b \cdot c$?)