Lesson 12-4 Factoring $x^2 + bx + c$

Vocabulary

square term
linear term
constant term
prime polynomial over the integers

BIG IDEA Some quadratic trinomials of the form $x^2 + bx + c$ can be factored into two linear factors.

In Lesson 12-3 you saw the advantage of the factored form $y = a(x - r_1)(x - r_2)$ in finding the x-intercepts r_1 and r_2 of the graph of a quadratic function. In this lesson you will see how to convert quadratic expressions from the form $x^2 + bx + c$ into factored form.

Notice the pattern that results from the multiplication of the binomials of the forms $(x + p)$ and $(x + q)$. After combining like terms, the product is a trinomial.

	square term		linear term		constant term
$(x + 4)(x + 3) = x^2 + 3x + 4x + 12 =$	x^2	$+$	$7x$	$+$	12
$(x - 6)(x + 8) = x^2 + 8x - 6x - 48 =$	x^2	$+$	$2x$	$-$	48
$(x + p)(x + q) = x^2 + qx + px + pq =$	x^2	$+$	$(p + q)x$	$+$	pq

Examine the trinomials above. Their constant term pq is the product of the constant terms of the binomials. The coefficient $p + q$ of the linear term is the sum of the constant terms of the binomials. This pattern suggests a way to factor trinomials in which the coefficient of the square term is 1.

Mental Math

Use the discriminant to determine the number of real solutions to

a. $-5y^2 + 6y + 7 = 0$.

b. $3h^2 + 10 - h = 0$.

c. $-9x^2 - 12x - 4 = 0$.

Example 1

Factor $x^2 + 11x + 18$.

Solution To factor, you need to identify two binomials, $(x + p)$ and $(x + q)$, whose product equals $x^2 + 11x + 18$. You must find p and q, two numbers whose product is 18 and whose sum is 11. Because the product is positive and the sum is positive, both p and q are positive. List the positive pairs of numbers whose product is 18. Then calculate their sums.

Product is 18	Sum of Factors
1, 18	19
2, 9	11
3, 6	9

The sum of the numbers 2 and 9 is 11. So $p = 2$ and $q = 9$.

Thus, $x^2 + 11x + 18 = (x + 2)(x + 9)$.

Check Factoring can always be checked by multiplication.
$(x + 2)(x + 9) = x^2 + 9x + 2x + 18 = x^2 + 11x + 18$; it checks.

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Example 2

Factor $x^2 - x - 30$.

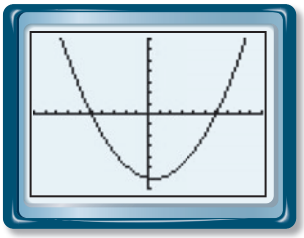
Solution Think of this trinomial as $x^2 + -1x + -30$. You need two numbers whose product is −30 and whose sum is −1. Since the product is negative, one of the factors is negative. List the possibilities.

Product is -30	Sum of Factors
-1, 30	29
-2, ?	?
-3, ?	?
-5, ?	?
-6, ?	?
-10, ?	?
?, ?	?
?, ?	?

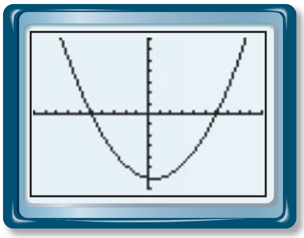
The only pair of factors of −30 whose sum is −1 is ? and ?.
So $x^2 - x - 30 = (x - ?)(x + ?)$.

Check 1 Multiply $(x - ?)(x + ?) = x^2 + ?x - ?x - 30 = x^2 - x - 30$. It checks.

Check 2 Graph $y = x^2 - x - 30$ and $y = (x - ?)(x + ?)$. The graphs should be identical. Below we show the output from a graphing calculator, with the window $-10 \le x \le 10$, $-35 \le y \le 35$.



$y = x^2 - x - 30$



$y = (x - 6)(x + 5)$

The graphs appear to be identical.

As you know, some trinomials are perfect squares. You can use the method from Examples 1 and 2 to solve perfect square trinomials.

Example 3

Factor $t^2 - 8t + 16$.

Solution Find factors of 16 whose sum is −8. Because the product is positive and the sum is negative, both numbers are negative. You need only to consider negative factors of 16.

Product is 16	Sum of Factors
-1, -16	-17
-2, -8	-10
-4, -4	-8

So $t^2 - 8t + 16 = (t - 4)(t - 4) = (t - 4)^2$.

Check Use a CAS to factor $t^2 - 8t + 16$.

Not all trinomials of the form $x^2 + bx + c$ can be factored into polynomials with integer coefficients. For example, to factor $t^2 - 12t + 16$ as two binomials $(t + p)(t + q)$, where p and q are integers, the product of p and q would have to be 16 and their sum would have to be –12. The table in Example 3 shows that there are no such pairs of numbers. We say that $t^2 - 12t + 16$ is *prime over the integers*. A **prime polynomial over the integers** is one that cannot be factored into factors of lower degree with integer coefficients.

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Example 4

Factor $m^2 + 5m - 24$.

Solution

1. Think of factors of _?_ whose sum is _?_.
2. Because the product is negative, how many of the factors are negative?
3. $m^2 + 5m - 24 = (\_?\_)(\_?\_)$

Check Check your solution by graphing $y = x^2 - 5x - 24$ and $y = (\_?\_)(\_?\_)$.

Activity

Use a CAS and the FACTOR command to complete this Activity. Work with a partner, a team, or your entire class.

The entries in the table on the next page are quadratic expressions of the form $x^2 + bx + c$. We want to factor them into polynomials with integer coefficients.

b	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$	$c = 6$	$c = 7$	$c = 8$	$c = 9$	$c = 10$
1	$x^2 + x + 1$	$x^2 + x + 2$	$x^2 + x + 3$	$x^2 + x + 4$	$x^2 + x + 5$	$x^2 + x + 6$	$x^2 + x + 7$	$x^2 + x + 8$	$x^2 + x + 9$	$x^2 + x + 10$
2	$x^2 + 2x + 1$	$x^2 + 2x + 2$	$x^2 + 2x + 3$	$x^2 + 2x + 4$	$x^2 + 2x + 5$	$x^2 + 2x + 6$	$x^2 + 2x + 7$	$x^2 + 2x + 8$	$x^2 + 2x + 9$	$x^2 + 2x + 10$
3	$x^2 + 3x + 1$	$x^2 + 3x + 2$	$x^2 + 3x + 3$	$x^2 + 3x + 4$	$x^2 + 3x + 5$	$x^2 + 3x + 6$	$x^2 + 3x + 7$	$x^2 + 3x + 8$	$x^2 + 3x + 9$	$x^2 + 3x + 10$
4	$x^2 + 4x + 1$	$x^2 + 4x + 2$	$x^2 + 4x + 3$	$x^2 + 4x + 4$	$x^2 + 4x + 5$	$x^2 + 4x + 6$	$x^2 + 4x + 7$	$x^2 + 4x + 8$	$x^2 + 4x + 9$	$x^2 + 4x + 10$
5	$x^2 + 5x + 1$	$x^2 + 5x + 2$	$x^2 + 5x + 3$	$x^2 + 5x + 4$	$x^2 + 5x + 5$	$x^2 + 5x + 6$	$x^2 + 5x + 7$	$x^2 + 5x + 8$	$x^2 + 5x + 9$	$x^2 + 5x + 10$
6	$x^2 + 6x + 1$	$x^2 + 6x + 2$	$x^2 + 6x + 3$	$x^2 + 6x + 4$	$x^2 + 6x + 5$	$x^2 + 6x + 6$	$x^2 + 6x + 7$	$x^2 + 6x + 8$	$x^2 + 6x + 9$	$x^2 + 6x + 10$
7	$x^2 + 7x + 1$	$x^2 + 7x + 2$	$x^2 + 7x + 3$	$x^2 + 7x + 4$	$x^2 + 7x + 5$	$x^2 + 7x + 6$	$x^2 + 7x + 7$	$x^2 + 7x + 8$	$x^2 + 7x + 9$	$x^2 + 7x + 10$
8	$x^2 + 8x + 1$	$x^2 + 8x + 2$	$x^2 + 8x + 3$	$x^2 + 8x + 4$	$x^2 + 8x + 5$	$x^2 + 8x + 6$	$x^2 + 8x + 7$	$x^2 + 8x + 8$	$x^2 + 8x + 9$	$x^2 + 8x + 10$
9	$x^2 + 9x + 1$	$x^2 + 9x + 2$	$x^2 + 9x + 3$	$x^2 + 9x + 4$	$x^2 + 9x + 5$	$x^2 + 9x + 6$	$x^2 + 9x + 7$	$x^2 + 9x + 8$	$x^2 + 9x + 9$	$x^2 + 9x + 10$
10	$x^2 + 10x + 1$	$x^2 + 10x + 2$	$x^2 + 10x + 3$	$x^2 + 10x + 4$	$x^2 + 10x + 5$	$x^2 + 10x + 6$	$x^2 + 10x + 7$	$x^2 + 10x + 8$	$x^2 + 10x + 9$	$x^2 + 10x + 10$

Step 1 Make a table like the one above with the same row and column headings, but keep the other cells blank.

Step 2 Factor each of the 100 entries in the table. Put the factored form in your table. If the quadratic cannot be factored over the integers, write "P," for prime, in the box. The expressions $x^2 + x + 1$ and $x^2 + 2x + 1$ have been done for you below.

b	$c = 1$
1	P
2	$(x + 1)(x + 1)$

Step 3 In your table, circle the factored expressions.

Step 4 For which values of c is there only one factorization of $x^2 + bx + c$ in that column? What type of numbers are these c values?

Step 5 When $c = 6$, there are two factorizations of $x^2 + bx + 6$. The factorizations occur when $b = 5$ and $b = 7$.

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

How are the 2 and 3 related to the 6?

How are the 2 and 3 related to the 5?

$$x^2 + 7x + 6 = (x + 1)(x + 6)$$

How are the 1 and 6 related to the 6?

How are the 1 and 6 related to the 7?

Step 6 If $x^2 + bx + c$ factors into $(x + p)(x + q)$ then $p + q =$ ___?___ and $pq =$ ___?___.

Step 7 For how many integer values of b is the expression $x^2 + bx + 20$ factorable? Explain. Give the values of b that allow $x^2 + bx + 20$ to be factored.

For how many integer values of b is the expression $x^2 + bx + 37$ factorable? Explain. Give the values of b that allow $x^2 + bx + 37$ to be factored.

Questions

COVERING THE IDEAS

1. a. In order to factor $x^2 + 10x + 24$, list the possible integer factors of the last term and their sums.
   b. Factor $x^2 + 10x + 24$.
   c. Check your work.
2. Suppose $(x + p)(x + q) = x^2 + bx + c$.
   a. What must pq equal?
   b. What must $p + q$ equal?
3. Sandra, Steve, and Simona each attempted to factor $n^2 + 2n - 48$. Which student's factorization is correct? Explain what mistake the other two students made.

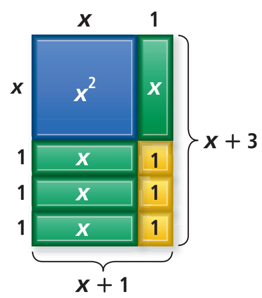
Sandra's	Steve's	Simona's
$(n + 6)(n - 8)$	$(n - 6)(n - 8)$	$(n - 6)(n + 8)$

In 4–9, write the trinomial as the product of two binomials.

4. $x^2 + 22x + 40$
5. $q^2 + 20q + 19$
6. $z^2 - z - 56$
7. $v^2 - 102v + 101$
8. $r^2 + 10r + 16$
9. $m^2 + 17m - 38$
10. Explain why the trinomial $x^2 + 6x + 4$ cannot be factored over the integers.
11. a. Factor $x^2 - 8x + 15$.
    b. What are the x-intercepts of the graph of $y = x^2 - 8x + 15$?

APPLYING THE MATHEMATICS

12. The diagram at the right uses tiles to the factorization of $x^2 + 4x + 3 = (x + 1)(x + 3)$.



    Make a drawing to show the factorization of $x^2 + 7x + 10$.
13. a. Find an equation of the form $y = (x + p)(x + q)$ whose graph is identical to the graph of $y = x^2 - 18x + 32$.
    b. Check your work by graphing both equations on the same set of axes.
14. a. Find the vertex of the parabola with equation $y = x^2 + 2x - 35$ by factoring to find the x-intercepts.
    b. Check Part a by completing the square to put the equation in vertex form.
15. Factor $-40 + 13x - x^2$.
16. If $m^2 + 13mn + 22n^2 = (m + pn)(m + qn)$, what are p and q?

REVIEW

17. Solve and check $(n - 10)\left(\frac{1}{2}n + 6\right) = 0$. **(Lesson 12-3)**

In 18 and 19, find the value of c that makes each trinomial a perfect square. (Lesson 12-2)

18. $x^2 - 14x + c$

19. $x^2 + 9x + c$

In 20 and 21, expand the expression. (Lesson 11-6)

20. $(4 - x)(4 + x)$

21. $(5a - 3)(5a + 3)$

22. Explain how $37^2 - 35^2$ can be calculated in your head. **(Lesson 11-6)**

In 23 and 24, simplify the expression. (Lessons 11-5, 11-4)

23. $\frac{9z^3 + 10z}{z}$

24. $(n^2 + m^2) - (n - m)^2$

25. Factor $28b^4 + 8b^2 + 40$ completely. **(Lesson 11-4)**

26. A certain type of glass allows 85% of the light hitting it to pass through 1 centimeter of glass. The fraction y of light passing through x centimeters of glass is then $y = (0.85)^x$. **(Lesson 7-3)**
    a. Draw a graph of this equation for $0 \le x \le 10$.
    b. Use the graph to estimate the thickness of this glass you would need to allow only a quarter of the light hitting it to pass through.

Light is shining through tinted glass.

In 27 and 28, give the slope and y-intercept for each line. (Lessons 6-8, 6-4)

27. $y = \frac{1}{4}x$

28. $12x - 3y = 30$

EXPLORATION

29. Using a CAS, make a table like that in the Activity but with integer values of c from –1 to –10. (Row 1 of the table is $x^2 + x - 1$, $x^2 + x - 2$, and so on.)
    a. Repeat Steps 1–3 from the Activity.
    b. How many of these 100 quadratic expressions can be factored over the integers?
    c. Describe a pattern in the table that could enable you to extend the table to more factors without doing any calculations.