Lesson

12-3 The Factored Form of a Quadratic Function

Vocabulary

factored form (of a quadratic function)

BIG IDEA The graph of the equation $y = a(x - r_1)(x - r_2)$ is a parabola that intersects the *x*-axis at $(r_1, 0)$ and $(r_2, 0)$.

Mental Math

Using one fair, 6-sided die, what is the probability of rolling

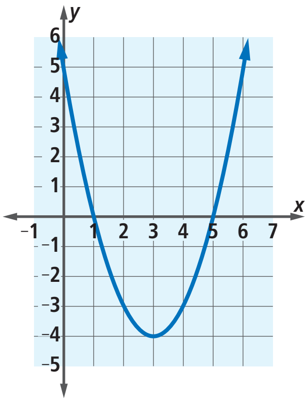
a. a 3?

b. an even number?

c. a number less than 3?

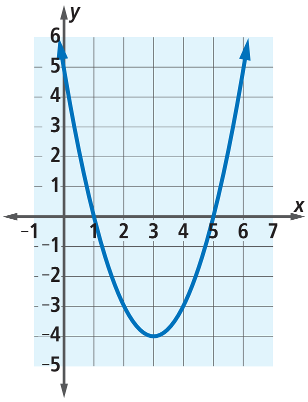
You have seen two forms of equations for a quadratic function: standard form and vertex form. In this lesson, you will see some advantages of a third form called *factored form*. Below are graphs of three equations: $y + 4 = (x - 3)^2$, $y = (x - 1)(x - 5)$, and $y = x^2 - 6x + 5$.

$y + 4 = (x - 3)^2$



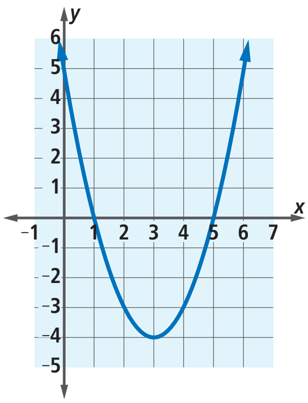
Vertex form

$y = (x - 1)(x - 5)$



Factored form

$y = x^2 - 6x + 5$



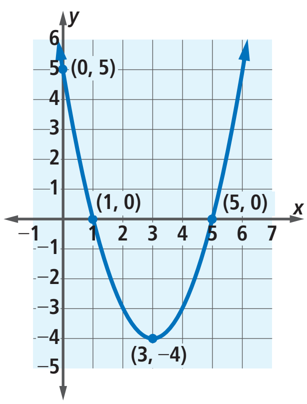
Standard form

They are in fact the same parabola described in three different ways. You can check this by converting the first two equations into standard form.

$$\begin{aligned} y + 4 &= (x - 3)^2 \\ y + 4 &= (x - 3)(x - 3) \\ y + 4 &= x^2 - 6x + 9 \\ y &= x^2 - 6x + 5 \end{aligned}$$

$$\begin{aligned} y &= (x - 1)(x - 5) \\ y &= x^2 - 1x - 5x + 5 \\ y &= x^2 - 6x + 5 \end{aligned}$$

Different key aspects of the graph are revealed by each form. From the vertex form, you can easily determine the vertex, (3, –4). From the factored form, you can easily determine the *x*-intercepts, 1 and 5. In standard form, the *y*-intercept is clearly 5.

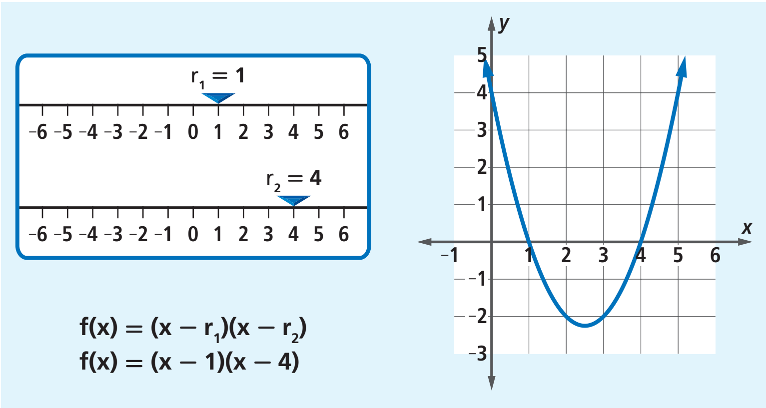


Activity 1

Use a dynamic graphing system.

Step 1 Create two sliders with values between −6 and 6. Label one r_1 and the other r_2.

Step 2 Slide bars so $r_1 = 1$ and $r_2 = 4$.



Step 3 Graph the function $f(x) = (x - r_1)(x - r_2)$.

Step 4 Give the points of intersection of the graph of f and the x-axis.

Step 5 Move the sliders to complete the table below.

r_1	r_2	$f(x) = (x - r_1)(x - r_2)$	Points of intersection of graph and x-axis
5	2	$f(x) = (x - 5)(x - 2)$	(?, 0) and (?, 0)
−4	−3	?	?
0	−1	?	?
3	3	?	?
?	?	$f(x) = (x + 2)(x - 4)$	?
?	?	$f(x) = (x + 5)(x + 5)$	?

Step 6 Explain how the factored form of a quadratic in the third column reveals the x-intercepts of the graph of that quadratic.

Step 7 Give the x-intercepts of the following functions using their graphs.

a. $f(x) = (x - 3)(x + 1)$ **b.** $g(x) = x(x + 6)$ **c.** $h(x) = (x - 2)(x - 2)$

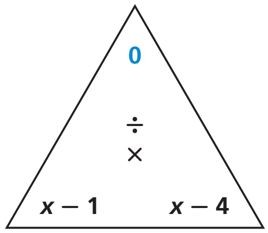
How the Factored Form Displays the x-Intercepts

The equation $y = ax^2 + bx + c$ is in **factored form** when it is written as $y = a(x - r_1)(x - r_2)$.

For the function with equation $y = (x - 1)(x - 4)$ graphed in Activity 1, $a = 1$, $r_1 = 1$, and $r_2 = 4$.

The x-intercepts of the function are the values of x for which $y = 0$. So they are the values of x that satisfy the equation $0 = (x - 1)(x - 4)$.

Recall the Zero Product Property from Lesson 2-8: When the product of two numbers is zero, at least one of the numbers must be 0. In symbols, if $ab = 0$, then $a = 0$ or $b = 0$. Consequently, $y = 0$ when either $x - 1 = 0$ or $x - 4 = 0$. So $y = 0$ when either $x = 1$ or $x = 4$.



In general, the *x*-intercepts of a parabola can be determined from factored form in the same way that the vertex can be determined from vertex form.

Factor Theorem for Quadratic Functions

The *x*-intercepts of the graph of $y = a(x - r_1)(x - r_2)$ are r_1 and r_2.

Example 1

Consider the equation $y = (x + 4)(x - 2)$.

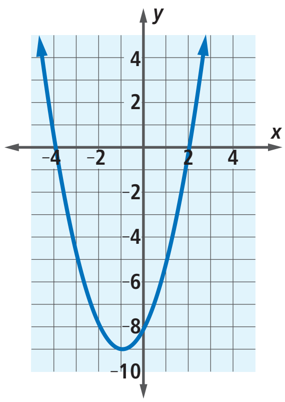
a. Find the *x*-intercepts of its graph.

b. Graph the equation.

Solutions

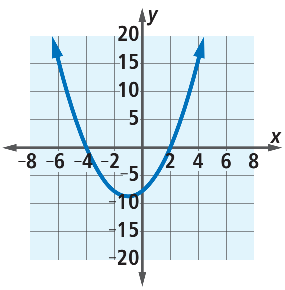
a. The *x*-intercepts occur when $y = 0$. So solve $(x + 4)(x - 2) = 0$. By the Zero Product Property, either $x + 4 = 0$ or $x - 2 = 0$, so either $x = -4$ or $x = 2$. So the *x*-intercepts are -4 and 2.

b. Recall that the *x*-coordinate of the vertex is the mean of the *x*-intercepts −4 and 2. So the vertex has *x*-coordinate −1. When $x = -1$, $y = (-1 + 4)(-1 - 2) = -9$. So the vertex is (−1, −9). With this information, you can sketch a graph.



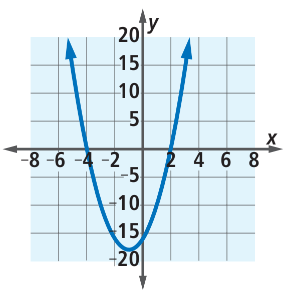
In the equation $y = (x + 4)(x - 2)$, the value of *a*, the coefficient of x^2, is 1. If the factors $x + 4$ and $x - 2$ remain the same but the value of *a* is changed, notice the similarities and changes in the graphs.

$y = (x + 4)(x - 2)$



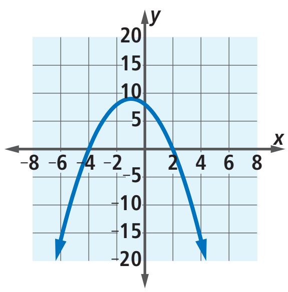
x-intercepts: –4 and 2
vertex: (–1, –9)

$y = 2(x + 4)(x - 2)$



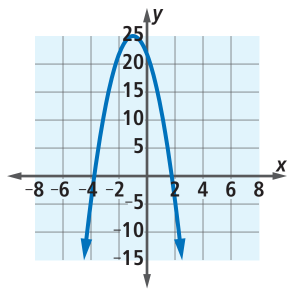
x-intercepts: –4 and 2
vertex: (–1, –18)

$y = -(x + 4)(x - 2)$



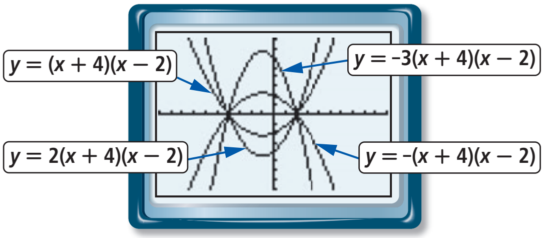
x-intercepts: –4 and 2
vertex: (–1, 9)

$y = -3(x + 4)(x - 2)$



x-intercepts: –4 and 2
vertex: (–1, 27)

To see them better, all four equations can be placed on the same set of axes.



All four graphs have the same pair of x-intercepts, –4 and 2, so each goes through the points (–4, 0) and (2, 0).

Activity 2

Use a dynamic graphing system. You can use the previous Activity's set-up for this Activity.

Step 1 Create two sliders with values between −6 and 6. Label one r_1 and the other r_2.

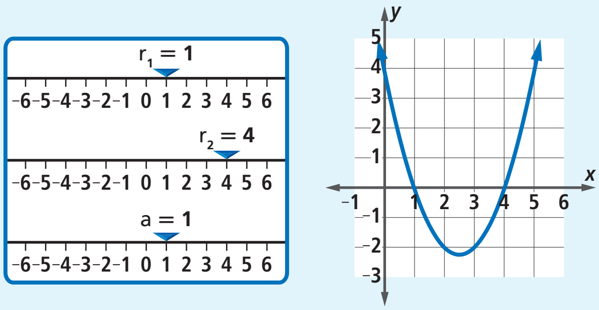
Step 2 Create a third slider with values between −6 and 6 and label it a.

Step 3 Slide bars so $r_1 = 1$, $r_2 = 4$, and $a = 1$.

Step 4 Plot the function $f(x) = a \cdot (x - r_1)(x - r_2)$.

Step 5 Slide a. Do the x-intercepts change?

Step 6 Make $r_1 = -3$ and $r_2 = -3$. Slide a. Do the x-intercepts change?



$f(x) = a \cdot (x - r_1)(x - r_2)$
$f(x) = 1 \cdot (x - 1)(x - 4)$

Step 7 Move r_1 and r_2 to other values and then slide a. Explain your observations about the relationship between the value of a and the x-intercepts.

Step 8 Slide a into the positive values. What is true about the shape of the parabola when a is positive?

Step 9 Slide a into the negative values. What is true about the shape of the parabola when a is negative?

Step 10 Slide a to zero. Describe what happens to the graph.

Step 11 Complete the table. Verify your results by graphing.

a	r_1	r_2	$f(x) = a(x - r_1)(x - r_2)$	Does the parabola open up or down?
2	2	–3	$f(x) = 2(x - 2)(x + 3)$	?
–1	–4.1	5	?	?
5	–6	–6	?	up
–3	–5	0	?	?

When a quadratic expression is in factored form and equal to 0, you can solve equations and find *x*-intercepts quite easily. You can also determine vertices and maximum and minimum values of the expression.

Example 2

a. Find the *x*-intercepts of the graph of $y = (3x - 5)(2x + 1)$.

b. Find the vertex of the parabola.

Solutions

a. Solve $0 = (3x - 5)(2x + 1)$. Use the Zero Product Property.

Either $3x - 5 = 0$ or $2x + 1 = 0$.

$3x = 5$ or $2x = -1$

$x = \frac{5}{3}$ or $x = -\frac{1}{2}$

Thus the x-intercepts are $\frac{5}{3}$ and $-\frac{1}{2}$.

b. The *x*-coordinate of the vertex is the mean of the *x*-intercepts.

$$\frac{\frac{5}{3} + \frac{-1}{2}}{2} = \frac{\frac{10}{6} - \frac{3}{6}}{2} = \frac{\frac{7}{6}}{2} = \frac{7}{12}$$

When $x = \frac{7}{12}$, $y = \left(3 \cdot \frac{7}{12} - 5\right)\left(2 \cdot \frac{7}{12} + 1\right) = \left(\frac{21}{12} - \frac{60}{12}\right)\left(\frac{7}{6} + \frac{6}{6}\right)$

$= -\frac{39}{12} \cdot \frac{13}{6} = -\frac{169}{24}$.

So the vertex of the parabola is $\left(\frac{7}{12}, -\frac{169}{24}\right) = \left(\frac{7}{12}, -7\frac{1}{24}\right)$.

Questions

COVERING THE IDEAS

1. Give the *x*-intercepts of the graph of $y = 3(x - 8)(x + 4)$.

2. If the product of two numbers is zero, what must be true of at least one of those numbers?

In 3–5, solve the equation.

3. $0 = -5(x - 32)(x + 89.326)$

4. $777(n + 198)(2n - 10) = 0$

5. $p(p + 19) = 0$

6. Consider the equations $y = 3(x - 20)(x + 80)$ and $y = -2(x - 20)(x + 80)$.
   a. What two points do the graphs of these equations have in common?
   b. What is the *x*-coordinate of the vertex of both graphs?
   c. What is the *y*-coordinate of the vertex for each graph?

In 7–10, an equation for a function is given.

a. Find the *x*-intercepts of the graph of the function.
b. Find the vertex of the graph of the function.
c. Sketch a graph of the function.
d. Check your work by writing the equation in standard form and graphing that equation.

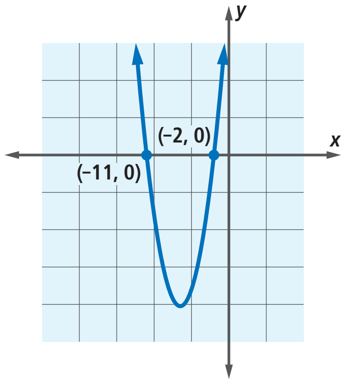
7. $y = (x + 15)(x + 7)$

8. $y = -3(x + 8)(2x - 9)$

9. $f(x) = -x(4x + 11)$

10. $g(x) = (x - 3)^2$

11. A quadratic function is graphed at the right.
    a. Give an equation for the axis of symmetry of the parabola.
    b. Give 3 possible equations in factored form for the graph.

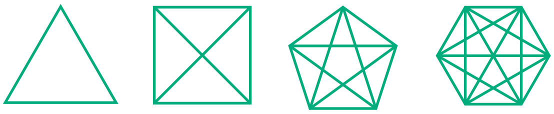


APPLYING THE MATHEMATICS

12. Down in a canyon there is a cannon that shoots cantaloupes straight up. Candice is standing on a cliff above the cannon. The cliff is at ground level or a height of 0 feet, so that the cantaloupes are fired from a negative starting height. The cantaloupe is shot up into the air higher than the cliff (on which Candice is standing) and then comes back down past the cliff back into the canyon. The cantaloupe passes by Candice 1 second after it is fired on the way up and 2 seconds after it was fired on the way down.
    a. What part of the situation represents the *x*-intercepts (or where the cantaloupe has a height of 0 feet)?
    b. In projectile problems where the units are in feet and seconds, $a = -16$. Write an equation for the situation in factored form.
    c. Give the axis of symmetry for this graph.
    d. Give the coordinates of the vertex of the graph.
    e. What does the vertex represent in the scenario about the cantaloupe?

13. The vertex of a parabola is (−2, −18) and one of the *x*-intercepts is 1.
    a. Give the other *x*-intercept.
    b. Write an equation for the parabola in factored, vertex, and standard forms.

14. A formula that describes how many diagonals d that can be drawn in a polygon with n sides is $d = \frac{1}{2}n(n - 3)$.



Number of sides	3	4	5	6	...
Number of diagonals	0	2	5	9	...

a. What are the n-intercepts of the formula's graph?

b. Why does the point $(2, -1)$ not make sense in this situation?

c. The graph of the formula is part of a parabola. Find its vertex.

REVIEW

15. Consider the equation $y = -x^2 + 10x - 20$. **(Lessons 12-2, 12-1)**

a. Rewrite the equation in vertex form.

b. Give the vertex of the parabola.

c. Graph the parabola.

In 16 and 17, multiply the expression. (Lessons 11-6, 11-5)

16. $(4a - 1)(3a + 6)$

17. $(5n + 8)(5n - 8)$

18. If the cost of 15 pads of paper is \$12.30, how many pads can be purchased with \$3.75? **(Lesson 5-5)**

19. Give the coordinates of the point of intersection of the two lines. **(Lesson 4-2)**

a. $x = 2, y = -4$ b. $x = a, y = 0$ c. $x = r, y = s$

20. A class of 34 students contains 2.5% of all the students in the school. How many students are in the school? **(Lesson 4-1)**

EXPLORATION

21. Consider the equation $y_1 = (x - 5)(x - 2)(x + 1)$.

a. Graph this equation using a graphing calculator.

b. Identify the x-intercepts of the graph.

c. Use the results of Parts a and b to graph $y_2 = -(x - 5)(x - 2)(x + 1)$ without a graphing calculator.

d. Use the results of Parts a and b to graph $y_3 = 3(x - 5)(x - 2)(x + 1)$ without a graphing calculator.

e. Write a few sentences generalizing Parts a through d.