Lesson 12-2 Completing the Square

Vocabulary

complete the square

BIG IDEA Completing the square is a process that converts an equation for a parabola from standard form into vertex form.

You have now seen two forms of equations whose graphs are parabolas.

$$\textbf{Standard form } y = ax^2 + bx + c$$

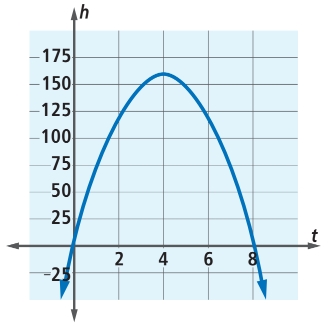
$$\textbf{Vertex form } y - k = a(x - h)^2$$

From the vertex form you can read the vertex of the parabola and also the maximum or minimum possible value of y. For example, from this form, you could tell the highest point that a baseball or a rocket reaches if you have an equation for its path.

But equations for paths are usually found in standard form $y = ax^2 + bx + c$. So the goal of this lesson is for you to learn how to convert an equation in standard form to one in vertex form.

Mental Math

A rocket's height *h* in feet *t* seconds after it is launched is shown below.



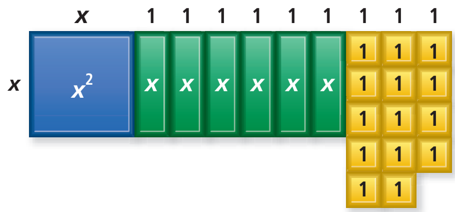
a. How long is the rocket in the air?

b. Estimate the greatest height it reaches.

c. When does it reach this greatest height?

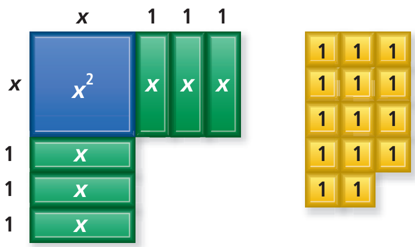
The Problem, Visually Stated

Consider the equation $y = x^2 + 6x + 14$. Visually, you can picture this quadratic expression as 1 square, 6 lengths, and 14 units as shown at the right.



$x^2 + 6x + 14$

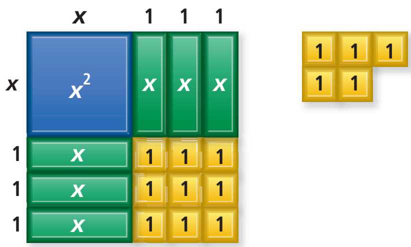
We want to convert it into vertex form. The idea is to move half of the lengths to try to create a bigger square as shown at the right.



$x^2 + 3x + 3x + 14$

It will take 9 of the units to fill in the bottom right corner to complete the square. The new bigger square, pictured below, has an area $x^2 + 3x + 3x + 9$. But its length and width are each $x + 3$. So it has area $(x + 3)^2$.

And because 5 units are left over, we have shown that $x^2 + 6x + 14 = (x + 3)^2 + 5$.



$(x + 3)^2 + 5$

Now, if $y = x^2 + 6x + 14$, then $y = (x + 3)^2 + 5$, which means that $y - 5 = (x + 3)^2$.

Activity

In 1–4, an equation for a parabola is given.

a. Using algebra tiles, build the given quadratic expression with a square, lengths, and units.

b. Rearrange the square, lengths, and units to convert the equation to vertex form.

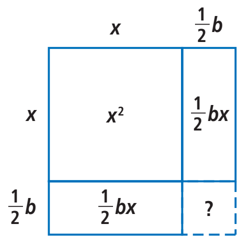
1. $y = x^2 + 4x + 18$
2. $y = x^2 + 10x + 30$
3. $y = x^2 + 10x + 25$
4. $y = x^2 + 14x + 52$

The General Process

In the expression $x^2 + 6x + 14$ on the previous page, we separated $6x$ into $3x + 3x$ and added 9 units to get the square. In general, the goal is to add a number to $x^2 + bx$ so that the right side of the equation contains a perfect square. We know, from the square of a binomial, that $(x + h)^2 = x^2 + 2hx + h^2$. Our goal is to find a number h^2 so that $x^2 + bx + h^2$ is a perfect square.

Comparing $x^2 + 2hx + h^2$ with $x^2 + bx$, we see that $b = 2h$. So $h = \frac{1}{2}b$. This means that $h^2 = \left(\frac{1}{2}b\right)^2$. And so $\left(x + \frac{1}{2}b\right)^2 = x^2 + bx + \left(\frac{1}{2}b\right)^2$.

Thus, to **complete the square** on $x^2 + bx$, add $\left(\frac{1}{2}b\right)^2$.



For example, in $x^2 + 6x$, $b = 6$ and $h = 3$. Then $h^2 = 9$.

Converting from Standard Form to Vertex Form

By completing the square, you can convert an equation in standard form to one in vertex form.

Example 1

a. Convert the equation $y = x^2 + 9x$ for a parabola into vertex form.

b. Find the vertex of this parabola.

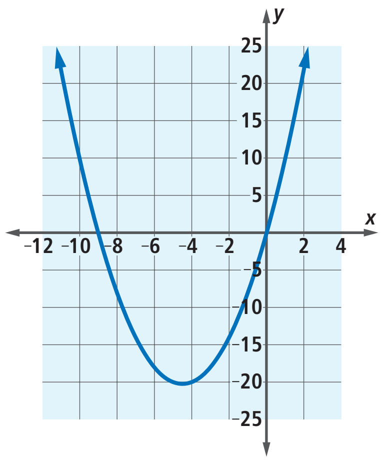
Solutions

a. Think of $x^2 + 9x$ as $x^2 + bx$. Then $b = 9$. So $\left(\frac{1}{2}b\right)^2 = (4.5)^2$. Thus, using the above argument, if you add 4.5^2 to $x^2 + 9x$, you will have the square of a binomial. But in an equation, you cannot add something to one side without adding it to the other.

$$y = x^2 + 9x$$
$$y + 4.5^2 = x^2 + 9x + 4.5^2 \quad \text{Add } (4.5)^2 \text{ to both sides.}$$
$$y + 4.5^2 = (x + 4.5)^2 \quad \text{Square of a binomial}$$

b. From Part a, we see that the vertex is $(-4.5, -4.5^2)$, that is, $(-4.5, -20.25)$.

Check Graph the two parabolas with equations $y = x^2 + 9x$ and $y + 4.5^2 = (x + 4.5)^2$ on the same grid. The graphs are identical to the one shown at the right, and the vertex is $(-4.5, -20.25)$.



Completing the Square on $y = x^2 - bx$

Recall from Lesson 11-6 that $(x - h)^2 = x^2 - 2hx + h^2$.

Consequently, to complete the square on $x^2 - 2hx$ you add h^2. That is, to complete the square on $x^2 - bx$ you add the same amount as you do to complete the square on $x^2 + bx$.

Example 2

Without graphing, find the minimum value for y when $y = x^2 - 6x - 13$.

Solution You can find the minimum value of y if you know the vertex of the parabola that is the graph of $y = x^2 - 6x - 13$. First add 13 to both sides to isolate $x^2 - 6x$ on the right side.

$$y = x^2 - 6x - 13$$
$$y + 13 = x^2 - 6x$$

Now complete the square on $x^2 - 6x$. Here $b = -6$, so add $\left(\frac{-6}{2}\right)^2$, or 9, to both sides.

$$y + 13 + 9 = x^2 - 6x + 9$$
$$y + 22 = (x - 3)^2$$

So the vertex is $(3, -22)$.

Consequently, the minimum value of y is -22.

Check 1 Try values of x near the vertex and see what values of y result.

When $x = 4$, $y = 4^2 - 6 \cdot 4 - 13 = -21$.

When $x = 2$, $y = 2^2 - 6 \cdot 2 - 13 = -21$ also.

The symmetry confirms that -22 is a minimum value for y when $x = 3$, because it is less than -21.

Check 2 Graph the equation $y = x^2 - 6x - 13$. We leave that to you.

Questions

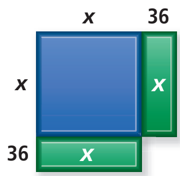
COVERING THE IDEAS

In 1 and 2, square the binomial.

1. $x + 7$
2. $n - 6.5$

3. **Fill in the Blanks** To complete the square for $x^2 + 20x$, add $\underline{\ ?\ }$. The result is the square of the binomial $\underline{\ ?\ }$.

4. **a.** Give the sum of the areas of the three rectangles below.
   b. What is the area of the undrawn rectangle needed to complete the large square?
   c. What algebraic expression will the completed large square below picture?



In 5–9, a quadratic expression is given.

a. What number must be added to the expression to complete the square?

b. After adding that number, the expression is the square of what binomial?

5. $x^2 + 2x$
6. $t^2 + 30t$
7. $r^2 - 7r$
8. $v^2 + bv$
9. $w^2 - bw$

10. **a.** Convert the equation $y = x^2 + 14x$ into vertex form.
    b. Find the vertex of this parabola.

11. **a.** Convert the equation $y = x^2 - 3x + 1$ into vertex form.
    b. Find the minimum value of y.

APPLYING THE MATHEMATICS

12. In this lesson, all the parabolas are graphs of equations of the form $y = x^2 + bx + c$. To deal with an equation of the form $y = -x^2 + bx + c$, first multiply both sides of the equation by -1, then complete the square, and finally multiply by -1 again so that y will be on the left side. Try this method to find the vertex of the parabola with equation $y = -x^2 + 5x + 2$.

13. In Lesson 9-4, the equation $h = -16t^2 + 32t + 6$ described the height h of a ball t seconds after being thrown from a height of 6 feet with an initial upward velocity of 32 feet per second. Put this equation into vertex form using the following steps.

Step 1 Substitute y for h and x for t.

Step 2 Divide both sides of the equation by –16 so that the coefficient of x^2 is 1.

Step 3 Complete the square on the right side of the equation and add the appropriate amount to the left side.

Step 4 Multiply both sides of the equation by –16 so that the coefficient of y on the left side of the equation is 1.

a. What is the vertex of the parabola?

b. Is this a minimum or a maximum?

Kevin Garnett shoots a free throw for the Minnesota Timberwolves of the National Basketball Association.

Source: Associated Press

14. The equation $h = -0.12x^2 + 2x + 6$ describes the path of a basketball free throw, where h is the height of the ball in feet when the ball is x feet forward of the free-throw line.

a. Use the steps in Question 13 to put this equation into vertex form.

b. What is the greatest height the ball reaches?

15. If $y = x^2 - x + 1$, can y ever be negative? Explain your answer.

16. The process of completing the square can be used to solve quadratic equations. Consider the equation $y^2 - 10y + 24 = 0$.

a. Add –24 to both sides.

b. Complete the square on $y^2 - 10y$ and add the constant term to both sides.

c. You now have an equation of the form $(y - 5)^2 = k$. What is k?

d. Solve the equation in Part c by taking the square roots of both sides.

17. Use the process described in Question 16 to solve $x^2 + 24x + 7 = 0$.

REVIEW

18. Consider the parabola with quadratic equation $y + 8 = 3(x + 2)^2$. **(Lesson 12-1)**

a. Find the vertex of the parabola.

b. Graph the parabola.

19. Two parents of blood type AB will produce children of three different blood types: A, B, and AB. One inheritance hypothesis argues that when parents of blood type AB produce children, 25% will have blood type A, 25% will have blood type B, and 50% will have blood type AB. Consider the table below that gives the blood types of 248 children born of 100 couples with both parents of blood type AB. Use a chi-square test to determine whether the data support the hypothesis. Justify your reasoning. **(Lesson 11-8)**

Blood Type	Number of Children
A	58
B	51
AB	139

20. **a.** How many solutions does the system $\begin{cases} y = |x| \\ y = 2 \end{cases}$ have?

b. Find the solutions. **(Lesson 10-1)**

In 21 and 22, solve. (Lessons 9-2, 5-2)

21. $\frac{4}{x} = \frac{8}{15}$

22. $\frac{m}{7} = \frac{20}{m}$

23. A watch company increases the price of its watches by 8%. If their watch now sells for \$130.50, what did it sell for before the increase? **(Lesson 4-1)**

24. Solve $6(3x^2 - 3x) - 9(2x^2 + 1) = 12$. **(Lessons 3-4, 2-1)**

EXPLORATION

25. Explain how the drawing below can be used to show $(x - b)^2 = x^2 - 2bx + b^2$.