Lesson 12-1 Graphing $y - k = a(x - h)^2$

Vocabulary

vertex form of an equation for a parabola

BIG IDEA The graph of the equation $y - k = a(x - h)^2$ is a parabola whose vertex can be easily found.

In Chapter 9, you graphed many parabolas with equations in the *standard form* $y = ax^2 + bx + c$. When an equation is in this form, the vertex of the parabola is not obvious. In Activity 1, you are asked to examine some parabolas with equations in the form $y - k = a(x - h)^2$. When an equation is in this form, its graph is a parabola whose vertex can be found rather easily.

Mental Math

Suppose $b(x) = 2|x| - 4$. Evaluate

a. $b(5)$.

b. $b(-5)$.

c. $b(5) - b(-5)$.

d. $\frac{b(5)}{b(-5)}$.

Activity 1

In 1–6, set a graphing calculator for the window $-15 \le x \le 15$, $-10 \le y \le 10$.

a. Graph the equation on your calculator. (You will have to solve the equation for y if it is not already solved for y.) Copy the graph by hand onto your paper.

b. Label the vertex with its coordinates.

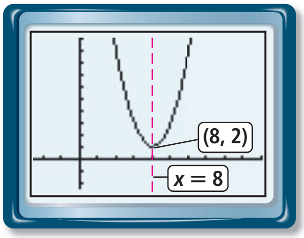
c. Draw the axis of symmetry as a dotted line. Label the axis of symmetry with its equation.

1. $y - 4 = (x - 3)^2$
2. $y + 3 = (x - 5)^2$
3. $y - 1 = (x + 4)^2$
4. $y + 8 = -(x + 6)^2$
5. $y - 12 = -(x - 4)^2$
6. $y = (x - 0.35)^2$
7. Look back at your graphs for Questions 1–6 and the equations that produced them. Explain how to look at an equation like $y - k = (x - h)^2$ to help determine the vertex of its graph.

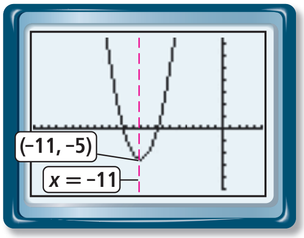
In 8 and 9, each graph is of an equation of the form $y - k = (x - h)^2$. The vertex and axis of symmetry of the parabola are given. Use what you learned in Questions 1–7.

a. Write an equation for the graph.

b. Check your equation by graphing it on your calculator. Do you get the graph you expected?

8. 

$-5 \le x \le 20, -5 \le y \le 20$

9. 

$-25 \le x \le 5, -10 \le y \le 14$

Example 1

Consider the graph of $y - 7 = 6(x - 15)^2$.

a. Explain why the graph is a parabola.

b. Find the vertex of this parabola without graphing.

Solution

a. If the equation can be written in the form $y = ax^2 + bx + c$, its graph is a parabola. Work with the given equation to get it into that form.

$y - 7 = 6(x - 15)^2$

$y - 7 = 6(x^2 - 30x + 225)$ Square of a Binomial

$y - 7 = 6x^2 - 180x + 1{,}350$ Distributive Property

$y = 6x^2 - 180x + 1{,}357$ Add 7 to both sides.

This equation is of the form $y = ax^2 + bx + c$, with $a = 6$, $b = -180$, and $c = 1{,}357$, so its graph is a parabola.

b. To find its vertex, examine the original equation $y - 7 = 6(x - 15)^2$. Add 7 to both sides.

$$y = 6(x - 15)^2 + 7$$

Since $(x - 15)^2$ is the square of a real number, $(x - 15)^2$ cannot be negative. The least value $(x - 15)^2$ can have is 0, and that occurs when $x = 15$. Thus the least value that $6(x - 15)^2$ can have is 0, and the least value that $6(x - 15)^2 + 7$ can have is 7. All of these least values occur when $x = 15$. As a consequence, the vertex of the parabola is at the point on the parabola with x-coordinate 15. When $x = 15$, substitution shows that $y = 7$. So the vertex is (15, 7). Since 7 is the least value of y, the parabola must open up.

Check You should graph the equation $y - 7 = 6(x - 15)^2$ with a graphing calculator, making sure that the window contains the point (15, 7). (You will likely have to solve the equation for y before graphing.)

The argument in Example 1 can be repeated in general. It demonstrates the theorem on the next page.

Parabola Vertex Theorem

The graph of all ordered pairs (x, y) satisfying an equation of the form $y - k = a(x - h)^2$ is a parabola with vertex (h, k).

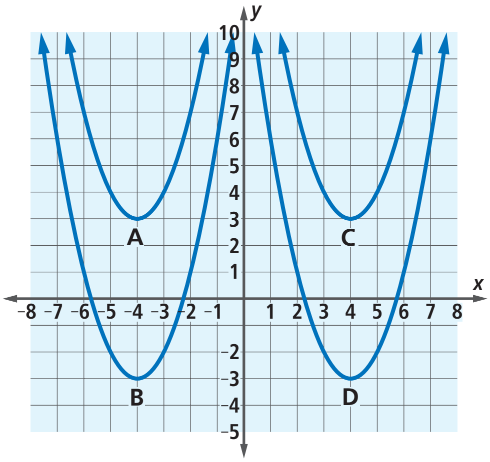
QY

The form $y - k = a(x - h)^2$ is called the **vertex form of an equation for a parabola** because you can easily find the vertex from the equation.

> **QY**
>
> Give the vertex of the parabola with equation $y - 8 = 3(x + 15)^2$.

Example 2

Multiple Choice Which of the four curves at the right is the graph of $y + 3 = (x - 4)^2$?



Solution Rewrite $y + 3 = (x - 4)^2$ so it corresponds to the general equation $y - k = a(x - h)^2$.

$$y - -3 = (x - 4)^2$$

The vertex is given by (h, k). So the vertex of the graph is (4, -3). The parabola with this vertex is choice D.

Activity 2

Use a dynamic graphing system to do the following.

Step 1 Create three sliders that include positive and negative numbers.

Name one slider *a*. (Or create a parameter *a* that is based on the slider.)

Name one slider *h*. (Or create a parameter *h* that is based on the slider.)

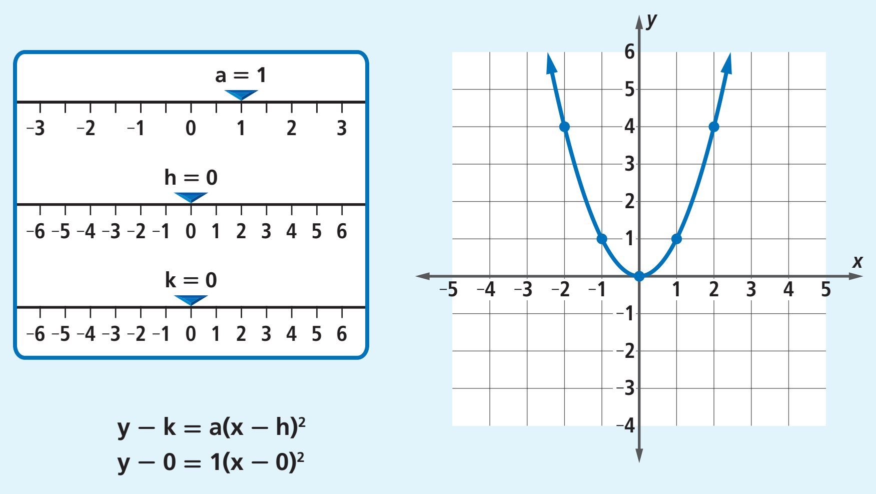
Name one slider *k*. (Or create a parameter *k* that is based on the slider.)

Step 2 Move the sliders so $a = 1$, $h = 0$, and $k = 0$.

Step 3 Create the equation $y - k = a(x - h)^2$, using the parameters for *a*, *h*, and *k*. Plot the equation. The graph should show $y - 0 = 1(x - 0)^2$ or $y = x^2$.

a. Make a table of values for $-3 \le x \le 3$.

b. Find the differences between the *y*-coordinates as the *x*-coordinates increase by 1. Does this pattern look familiar?



Step 4 Slowly move the *a* slider from 1 to 0.

a. What happens to the graph for $0 < a < 1$?

b. Move the *a* slider to 0.5. Make a table of values for $-3 < x < 3$. How do the *y*-coordinates compare to those in the table in Step 3?

c. Find the differences between the *y*-coordinates as the *x*-coordinates increase by 1. How do the differences compare to those in Step 3?

Step 5 Now move the *a* slider to the right of 1.

a. What happens to the graph when $a > 1$?

b. Move the *a* slider to 2. Make a table of values for $-3 < x < 3$. How do the *y*-coordinates compare to those in the table in Step 3?

c. Find the differences between the *y*-coordinates as the *x*-coordinates increase by 1. How do the differences compare to those in Step 3?

Step 6 Is the effect *a* has on the differences in the *y*-coordinates of $y = ax^2 + bx + c$ the same as or different from the effect *a* has on the differences in the *y*-coordinates of $y - k = a(x - h)^2$? Explain.

Step 7 Write a prediction of what you think happens to the graph when *a* is between –1 and 0.

Write a prediction of what you think happens to the graph when *a* is less than –1.

Step 8 Move the *a* slider to test your predictions. Were you correct? Explain how your predictions in Step 7 are similar to and different from what occurred in Steps 4 and 5.

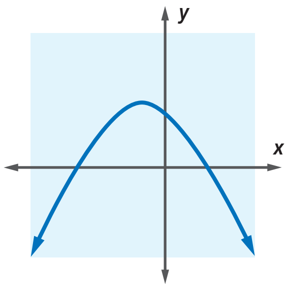
Step 9 Sketch what you think the graph of each function will look like.

$y - 5 = 2(x - 1)^2$ $y + 3 = 0.25(x - 2)^2$ $y = -3(x + 4)^2$

(continued on next page)

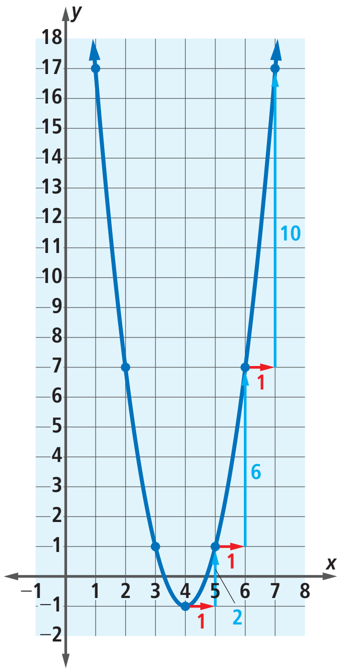
Step 10 Check your predictions from Step 9 by moving sliders *a*, *h*, and *k* to match the values in the function.

Step 11 Write a possible equation in vertex form for the graph of the parabola at the right.



Example 3

Write an equation in vertex form for the graph of the parabola at the right.



Solution 1 First locate the vertex and substitute it into the vertex form of an equation for a parabola, $y - k = a(x - h)^2$.

The vertex is at (4, -1), so $y - -1 = a(x - 4)^2$.

Next find the value of *a*. Compare the pattern in the change of the *y*-coordinates as the *x*-coordinates change by 1 in the graph to that of $y = x^2$.

Pattern change in *y*-coordinates for $y = x^2$: 1, 3, 5, ...

Pattern change in *y*-coordinates for graph: 2, 6, 10, ...

The pattern change for the graph is double the pattern change of $y = x^2$, so $a = 2$.

The equation for the parabola is $y + 1 = 2(x - 4)^2$.

Solution 2 As in Solution 1, first locate the vertex and substitute it into the vertex form of an equation for a parabola, $y - k = a(x - h)^2$.

The vertex is at (4, -1), so $y - -1 = a(x - 4)^2$.

Next find the value of *a*. Pick a point on the graph that is not the vertex and substitute it into the equation. Then solve for *a*. We pick point (6, 7).

$$7 + 1 = a(6 - 4)^2$$
$$8 = a(2)^2$$
$$8 = 4a$$
$$2 = a$$

Substitute *a* and the vertex into the equation.

The equation for the parabola is $y + 1 = 2(x - 4)^2$.

Check Pick a point on the graph and substitute it into the equation. We'll use the point (3, 1).

$$1 + 1 = 2(3 - 4)^2$$
$$2 = 2(-1)^2$$
$$2 = 2 \cdot 1$$
$$2 = 2 \text{ So it checks.}$$

Problems involving area can lead to parabolas. Consider the following problem.

GUIDED

Example 4

There are many possible rectangles with a perimeter of 24 units. Suppose the length of one side of such a rectangle is *L*.

a. Find the area *A* of the rectangle in terms of *L*.

b. Graph the equation from Part a.

c. Use the graph to determine the maximum area of a rectangle with perimeter of 24 units.

Solutions

a. You know that the perimeter *P* of a rectangle is given by the formula $P = 2L + 2W$. So in this case, $24 = 2L + 2W$.

Now solve this equation for *W*.

$\underline{\quad ? \quad} = 2W$

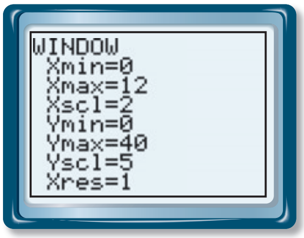
Divide both sides by 2.

$\underline{\quad ? \quad} = W$

Since the area $A = LW$, substituting $\underline{\quad ? \quad}$ for *W* gives the following formula for *A*.

$A = L\underline{\quad ? \quad}$

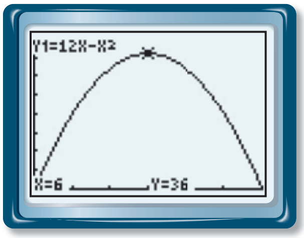
b. Letting $x = L$ and $y = A$, a graph of $y = x(12 - x)$ is shown here. The graph is a parabola because $y = 12x - x^2$ is of the form $y = ax^2 + bx + c$. The only part of the parabola that makes sense in this problem is for values of *x* between 0 and 12, so we use that window.



c. Each value of *y* in $y = x(12 - x)$ is the area of a particular rectangle. If $x = L = 2$, then $W = 12 - 2 = 10$ units. The area is $y = 2 \cdot 10 = 20$ units^2. That is, the point (2, 20) on the parabola means that when one side of the rectangle is 2 units, the area of the rectangle is 20 units^2.

If $x = L = 3$, then $W = \underline{\ ?\ }$. The area $y = \underline{\ ?\ } = \underline{\ ?\ }$.

If $x = L = 10$, then $W = \underline{\ ?\ }$. The area $y = \underline{\ ?\ } = \underline{\ ?\ }$.



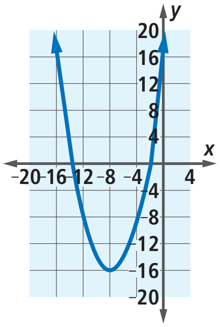
The maximum value of *y* is at the vertex of the parabola. From the graph the vertex is (6, 36). So the maximum area of the rectangle is $\underline{\quad ? \quad}$, occurring when $L = \underline{\quad ? \quad}$ and $W = \underline{\quad ? \quad}$, that is, when the rectangle is a square.

The equation graphed in Example 4 is not in vertex form. That makes it difficult to know the vertex. In Lesson 12-2, you will see how to convert an equation into vertex form.

Questions

COVERING THE IDEAS

1. The graph at the right shows a parabola.
   a. What are the coordinates of its vertex?
   b. Give an equation for its axis of symmetry.



2. Give the minimum value of each expression.
   a. x^2 b. $(x + 6)^2$ c. $3(x + 6)^2$ d. $3(x + 6)^2 - 8$
3. Explain why 7 is the maximum value of the expression $-4(x - 5)^2 + 7$.

In 4–7, an equation of a parabola is given.

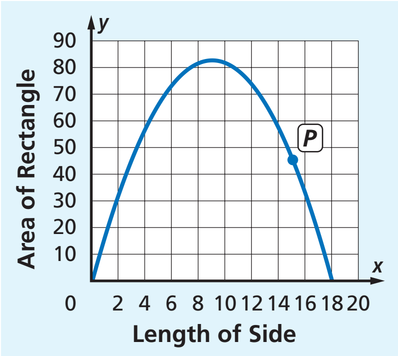
a. Find the coordinates of its vertex.

b. Write an equation for its axis of symmetry.

c. Tell whether the parabola opens up or down.

4. $y + 8 = -5(x - 9)^2$
5. $y - 21 = 0.2(x - 15)^2$
6. $y - 43 = 8x^2$
7. $y + \frac{1}{2} = -(x + 6)^2$

8. The equation $y = x(18 - x)$ gives the area of a rectangle with perimeter of 36 where x is the length of one of its sides. This equation is graphed at the right.
   a. Give the areas of the three rectangles for which $x = 4$, 5, and 11.
   b. Point P on the graph represents a rectangle. Give the rectangle's side lengths and area.
   c. What is the maximum area of a rectangle with perimeter 36?



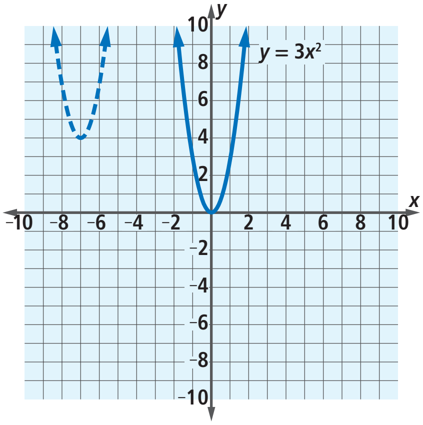
9. Explain the similarities and differences in the graphs of the functions.
   a. $y + 8 = (x - 4)^2$
   b. $y + 8 = 1.25(x - 4)^2$
   c. $y + 8 = 0.75(x - 4)^2$

APPLYING THE MATHEMATICS

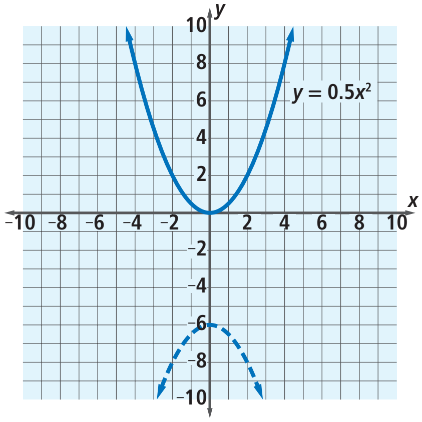
10. A rectangle has perimeter 60.
    a. Find a formula for its area A in terms of the length L of one side.
    b. Graph the formula you found in Part a.
    c. What is the maximum area of this rectangle?
11. A parabola has vertex (–12, 9) and contains the point (–10, 5). Give an equation for the parabola.

In 12 and 13, an equation of the form $y = ax^2$ and its graph are given. A translation image of the parabola is graphed with a dashed curve. Write an equation for the image.

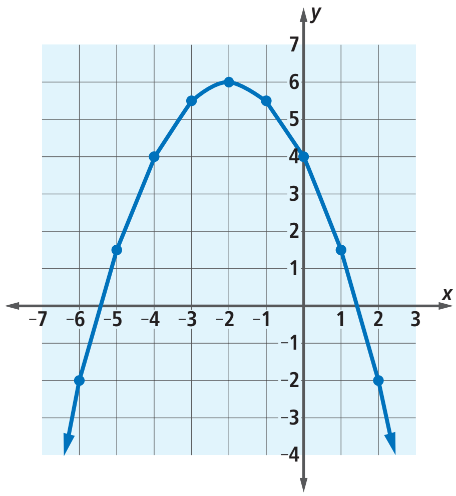
12.



13.



14. Write an equation for the graph of the parabola at the right.



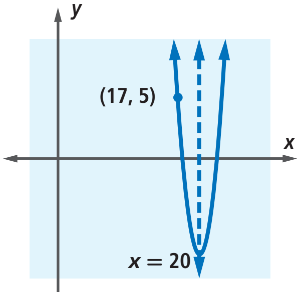
In 15 and 16, find equations for two different parabolas that fit the description.

15. The vertex is (5, –18) and the parabola opens down.

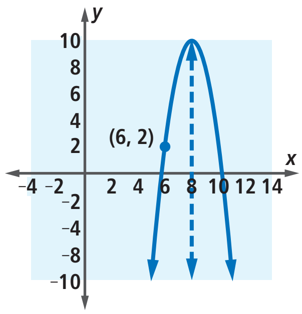
16. The axis of symmetry is $x = 2$ and the parabola opens up.

In 17 and 18, a graph of a parabola and a point on it are given. Find the coordinates of a second point on the parabola that has the same *y*-coordinate as the given point.

17.



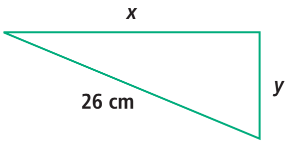
18.



REVIEW

19. Draw rectangles picturing $3a(a + 8) = 3a^2 + 24a$. **(Lesson 11-3)**

20. The sum of the legs of a right triangle is 34 cm. If the hypotenuse is 26 cm, calculate the length of each of the legs. **(Lessons 10-2, 8-6)**



21. Find all values of m that satisfy $(m^2)^2 - 15m^2 + 36 = 0$. (Lesson 9-5)

22. Suppose a basketball team wins 9 of its first 11 games during a season. At this rate, how many games would you expect the team to win in a 28-game season? (Lesson 5-9)

23. A climber is ascending Mount Kilimanjaro, the highest mountain in Africa. At 9 A.M. the climber is at an elevation of 15,416 feet and at 10:15 A.M. the climber is at an elevation of 16,004 feet. At this rate, when would the climber reach the 19,336-foot summit? (Lesson 5-5)

24. Simplify $5\pi \div \frac{4\pi}{3}$. (Lesson 5-2)

25. Solve the equation or inequality. (Lessons 4-5, 4-4)

a. $2x = 3x$

b. $2x > 3x$

Mount Kilimanjaro is not only the highest peak on the African continent; it is also the tallest freestanding mountain in the world at 19,336 feet.

Source: Mount Kilimanjaro National Park

EXPLORATION

26. Tiger Woods drives golf balls 300 yards before they hit the ground. Suppose one of his drives is 80 feet high at its peak, and that the path of the ball is a parabola.

a. With a suitable placement of coordinates, find an equation for this parabola.

b. How far from the tee (where the drive begins) is the ball 50 feet up in the air?

QY ANSWER

(–15, 8)