Kentucky Academic Standards for Mathematics: Calculus (+)

Calculus instructional time should focus on 3 critical areas:

1. Conceptual understanding and procedural fluency of limits, derivatives and integration.
2. Applications of derivatives and integrals.
3. Working with functions in a variety of ways: graphical, numerical, analytical and verbal.

Calculus Overview

Limits	Function Behavior	Continuity	Understanding the Derivative	Applications of the Derivative	Understanding Integration	Applications of Integration
• Understanding the concept of the limit of a function.	• Describe the asymptotic and unbounded behavior of functions.	• Develop an understanding of continuity as a property of functions.	• Demonstrate an understanding of the derivative.	• Apply differentiation techniques.	• Understand and apply the Fundamental Theorem of Calculus.	• Apply techniques of integration. • Use integration to solve problems.

Modeling Standards: Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice and specific modeling standards appear throughout the high school standards indicated by a star symbol (★). The star symbol sometimes appears on the heading for a group of standards; in that case, it should be understood to apply to all standards in that group.

(+): Calculus standards are not required standards for all Kentucky students; therefore, all Calculus standards would be considered (+) standards.

TWO plus signs (++) indicate a standard that is optional even for calculus.

Calculus-Limits

Standards for Mathematical Practice

MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.	MP.5. Use appropriate tools strategically. MP.6. Attend to precision. MP.7. Look for and make use of structure. MP.8. Look for and express regularity in repeated reasoning.

Cluster: Understand the concept of the limit of a function.

Standards	Clarifications
KY.HS.C.1 (+) Understand limits. a. Apply limits to a variety of functions, including piecewise functions. b. (++) Prove that the limit of a function exists, based upon the definition of a limit. **MP.2, MP.3**	
KY.HS.C.2 (+) Demonstrate an understanding of limits by estimating and finding the limit of a function at a point graphically, numerically and algebraically. **MP.5, MP.8**	Include analysis of limits in piecewise functions. Algebraic techniques include but are not limited to factoring, multiplying by the conjugate and finding the lowest common denominator.
KY.HS.C.3 (+) Apply properties and theorems of limits, including limits of indeterminate forms. **MP.2, MP.3**	Include sums, differences, products, quotients, composition of functions, special limits, Squeeze Theorem and L'Hospital's Rule.
KY.HS.C.4 (+) Communicate understanding of limits using precise mathematical symbols and language. **MP.3, MP.6**	Use of limits to predict the function value for an undefined value (hole in the graph). Apply the definition of a limit to margin of error. For example, if the weight of a golf ball needs to be within a certain range (ϵ), then the radius of the ball must be to a certain level of accuracy (δ).

Attending to the Standards for Mathematical Practice

Students can use technology to examine the graph of a function and determine whether or not the limit of the function exists at a point (**MP.5**). Students can use a table to find the value of a function for points that approach a given point, leading to conjectures about the limit of the function (**MP.8**).

The identified mathematical practices, coherence connections and clarifications are possible suggestions; however, they are not the only pathways.

Calculus-Function Behavior	
Standards for Mathematical Practice	
MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.	MP.5. Use appropriate tools strategically. MP.6. Attend to precision. MP.7. Look for and make use of structure. MP.8. Look for and express regularity in repeated reasoning.

Cluster: Describe the asymptotic and unbounded behavior of functions.	
Standards	**Clarifications**
KY.HS.C.5 (+) Describe asymptotic behavior (analytically and graphically) in terms of infinite limits and limits at infinity. **MP.2, MP.5**	
KY.HS.C.6 (+) Discuss the end behavior of functions; identify representative functions for each type of end behavior using precise mathematical symbols and language. **MP.2, MP.6**	$\lim_{x \to \infty} f(x) = 4$ implies a horizontal asymptote of $y = 4$ $\lim_{x \to \infty} f(x) = \infty$ implies right hand end behavior is positive infinity NOTE: odd functions result in end behavior similar to lines (opposite directions); even functions result in end behavior similar to parabolas (same direction)

Attending to the Standards for Mathematical Practice
Students use analytic methods to identify vertical and horizontal asymptotes (**MP.2**). Students use technology to examine the graph of a function, to determine the values for which it is defined and to convergence for increasingly large values in the domain (**MP.5**).

The identified mathematical practices, coherence connections and clarifications are possible suggestions; however, they are not the only pathways.

Calculus-Continuity	
Standards for Mathematical Practice	
MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.	MP.5. Use appropriate tools strategically. MP.6. Attend to precision. MP.7. Look for and make use of structure. MP.8. Look for and express regularity in repeated reasoning.

Cluster: Develop an understanding of continuity as a property of functions.

Standards	Clarifications
KY.HS.C.7 (+) Understand and use the limit definition of continuity to determine whether a given function is continuous at a specific point. **MP.2, MP.3**	If a function is continuous at $x = c$, then $\lim_{x \to c} f(x) = f(c)$.
KY.HS.C.8 (+) Define and identify different types of discontinuity – removable (hole) or non-removable (jump, asymptote) – in terms of limits. **MP.3, MP.6**	Non-removable discontinuity is identified by vertical asymptotes (infinite discontinuity) and jumps (non-agreement of left- and right-hand limits). Removable discontinuity is represented by a hole in the graph (agreement of left- and right-hand limits). Include analysis of special limits, such as $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$
KY.HS.C.9 (+) Understand and apply continuous function theorems. a. Apply the Intermediate Value Theorem to continuous functions. b. Apply the Extreme Value Theorem to continuous functions. **MP.2, MP.3**	a. Intermediate Value Theorem illustration: Sarah's mom measures her height every year on her birthday. On her 10th birthday, Sarah was 48 inches tall and on her 11th birthday she measured 52 inches. Her cousin said, "You were the same height as me sometime during this year." How tall is Sarah's cousin? Justify your answer. b. The Extreme Value Theorem is contingent on the concept of continuity, but will not be addressed in sequence until the concept of derivatives and critical numbers is established.
KY.HS.C.10 (+) Communicate an understanding of continuity using precise mathematical symbols and language. **MP.2, MP.6**	Continuity on a closed interval $[a, b]$ requires continuity on the open interval (a, b), $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$

Attending to the Standards for Mathematical Practice

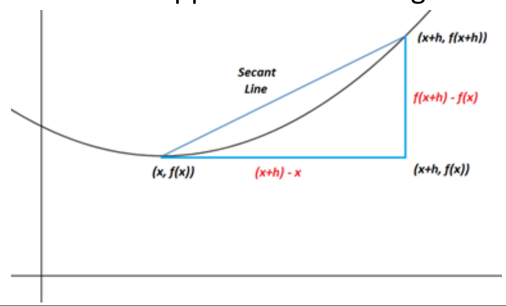
Students explain why a function is continuous or continuous at a point or over an interval (**MP.3**). Students use technology to examine the graph of a function and determine whether it is continuous in a given interval (**MP.5**).

Calculus-Understanding the Derivative

Standards for Mathematical Practice

MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.	MP.5. Use appropriate tools strategically. MP.6. Attend to precision. MP.7. Look for and make use of structure. MP.8. Look for and express regularity in repeated reasoning.

Cluster: Demonstrate an understanding of the derivative.

Standards	Clarifications
KY.HS.C.11 (+) Define derivatives. a. Define the derivative of a function as the limit of the difference quotient. b. Understand this limit of the difference quotient can be interpreted as an instantaneous rate of change or the slope of a tangent line. **MP.5, MP.8**	The difference quotient $\frac{f(x+h)-f(x)}{h}$ represents the slope of the secant line between $(x, f(x))$ and $(x+h, f(x+h))$ as shown below. The secant line approaches the tangent line as *h* approaches 0. 
KY.HS.C.12 (+) Use average rate of change to estimate the derivative from a table of values or a graph. **MP.2, MP.8**	
KY.HS.C.13 (+) Understand the derivative as a function. **MP.2, MP.5**	Analysis of the derivative as a function implies characteristics of the original function. This analysis can be done analytically or graphically. Students move fluently between graphs of functions and derivatives using each to predict what the other would look like. (For example, positive/negative values of derivative imply increasing/decreasing of original function.) Differentiability of original function implies the derivative is a continuous function.
KY.HS.C.14 (+) Apply the definition of derivative to find derivative values and derivative functions.	Include the formal definition of a derivative: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

Standards	Clarifications
MP.2, MP.3	The alternate form of this formal definition is to calculate the derivative at one particular value.
KY.HS.C.15 (+) Explain why differentiability implies continuity yet continuity does not imply differentiability. **MP.3, MP.6**	
KY.HS.C.16 (+) Understand and apply the Mean Value Theorem, including numerical, graphical and algebraic representations. **MP.2, MP.5**	
KY.HS.C.17 (+) Understand the relationship between the concavity of a function and the sign of the second derivative. **MP.2, MP.3**	
KY.HS.C.18 (++) Understand Rolle's Theorem as a special case of the Mean Value Theorem. **MP.2, MP.3**	

Attending to the Standards for Mathematical Practice
Students use analytic strategies to determine characteristics of functions as they relate to derivatives (**MP.2**) and technology to confirm the analytic results (**MP.5**). Students use tables of values to examine the average rate of change of a function over smaller and smaller intervals, leading to the derivative as the instantaneous rate of change (**MP.8**).

Calculus-Applications of Derivatives	
Standards for Mathematical Practice	
MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.	MP.5. Use appropriate tools strategically. MP.6. Attend to precision. MP.7. Look for and make use of structure. MP.8. Look for and express regularity in repeated reasoning.
Cluster: Apply differentiation techniques.	
Standards	**Clarifications**
KY.HS.C.19 (+) Efficiently find derivatives of functions with and without technology. **MP.2, MP.5**	Functions include linear, quadratic, polynomial, exponential, logarithmic (including bases other than e), trigonometric (including inverses), square root and other root functions. Efficiently finding a derivative involves selecting the most appropriate formula. For example, the derivative of $f(x) = x/4$ can be found using the quotient rule, but it is more efficient to use the power rule to find the derivative of $f(x) = \frac{1}{4}x$.
KY.HS.C.20 (+) Understand and use derivative rules for sums, differences, products and quotients of two functions and calculate the derivative of a composite function using the chain rule. **MP2, MP.3**	
KY.HS.C.21 (+) Use implicit differentiation to find a derivative in an equation of two variables. **MP.1, MP.2**	Include a variety of functions (such as polynomial, root, logarithmic, exponential and trigonometric. Implicit differentiation can be used to explore rules such as exponential and logarithmic for bases other than e.
KY.HS.C.22 (+) Use implicit differentiation to find the derivative of the inverse of a function. **MP.2, MP.3**	
KY.HS.C.23 (+) Understand the relationship between the increasing and decreasing behavior of a function and the sign of the first derivative of the function. **MP.1, MP.2**	

Standards	Clarifications
KY.HS.C.24 (+) Use the first derivative to analyze curves and identify relative extrema. **MP.2, MP.3**	The Extreme Value Theorem is useful in optimization problems involving closed intervals since absolute extrema may occur at endpoints. (For example, consider using wire to create a circle and square of maximum value. The maximum area is obtained by using all the wire on the circle).
KY.HS.C.25 (+) Understand the relationship of concavity to the second derivative. **MP.2, MP.5**	
KY.HS.C.26 (+) Use the second derivative to find points of inflection. **MP.2, MP.3**	Points of inflection must be defined values for the function.
KY.HS.C.27 (+) Use the second derivative to analytically locate intervals on which a function is concave up, concave down or neither. **MP.2, MP.3**	
KY.HS.C.28 (+) Describe how graphical characteristics of a given function, the first derivative of that function and the second derivative of that function interrelate. **MP.2, MP.5**	
KY.HS.C.29 (+) Use derivatives to express rate of change in a variety of contexts. **MP.2, MP.4**	Examples include but are not limited to exponential growth (population) and decay (half-life), logistic growth, continuous interest and Newton's Law of Cooling.
KY.HS.C.30 (+) Use derivatives to solve a variety of problems including related rates, optimization, tangent line approximations and growth and decay models. **MP.1, MP.4**	Related rate examples include but are not limited to relating variables using the Pythagorean Theorem, relating variables using trigonometric relationships and relating variables using geometric formulas. Tangent line approximations (linearization): ● Tangent lines make good approximations of function values close to the point of tangency. ● Tangent line approximations will be an overestimate if the function is concave down. ● Tangent line approximations will be an underestimate if the function is concave up. Growth and Decay:

Standards	Clarifications
	● Use the derivative to calculate the rate of change of growth or decay at a specific time.
KY.HS.C.31 (+) Use differentiation to solve problems involving velocity, speed and acceleration. **MP.1, MP.2**	
KY.HS.C.32 (+) Understand and apply differential equations. a. Verify solutions to differential equations and use them to model real-world problems with and without technology. b. Solve separable differential equations and use them in modeling real-world problems with and without technology. **MP.1, MP.4**	Solving separable equations requires integration, however, students establish patterns for recognizing what makes a solution work. ● Students create differential equations by starting with the answers ● Students understand what makes the differential equation separable.

Attending to the Standards for Mathematical Practice
Students use derivatives to identify and describe the characteristics of a function (**MP.2, MP.6).** Contextual questions about optimal or extreme values can be identified by representing problem situations in a variety of ways (**MP.4**) and applying appropriate tools and techniques to solve the questions that are posed (**MP.1**).

The identified mathematical practices, coherence connections and clarifications are possible suggestions; however, they are not the only pathways.

Calculus-Understanding Integration	
Standards for Mathematical Practice	
MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.	MP.5. Use appropriate tools strategically. MP.6. Attend to precision. MP.7. Look for and make use of structure. MP.8. Look for and express regularity in repeated reasoning.
Cluster: Demonstrate understanding of a definite integral.	
Standards	**Clarifications**
KY.HS.C.33 (+) Understand the definite integral of a function over an interval. Interpret a definite integral as a limit of Riemann Sums and as net accumulation of change. **MP.2, MP.5**	
KY.HS.C.34 (+) Write a Riemann sum that represents the definition of a definite integral. **MP.2, MP.3**	
KY.HS.C.35 (+) Calculate the values of Riemann Sums over equal subdivisions to approximate definite integrals of functions represented graphically and numerically (using tables). Use left-hand sums, right-hand sums, midpoint sums and trapezoidal sums. **MP.2, MP.3**	For strictly increasing functions, a right-hand sum overestimates and left-hand sum underestimates. For strictly decreasing, the opposite is true.
KY.HS.C.36 (+) Recognize differentiation and integration as inverse operations. **MP.2, MP.8**	Integration rules can be established by reversing derivative rules. Many integration rules can be developed using implicit derivatives and/ or substitution.
KY.HS.C.37 (+) Understand how the Fundamental Theorem of Calculus connects differentiation and integration and use it to evaluate definite and indefinite integrals and to represent particular antiderivatives. **MP.2, MP.3**	Include understanding and applying the Second Fundamental Theorem of Calculus.
KY.HS.C.38 (+) Perform analytical and graphical analysis of functions using the Fundamental Theorem of Calculus. **MP.2, MP.5**	Use integration capabilities of graphing utilities to verify solutions obtained by applying the Fundamental Theorem of Calculus.

Standards	Clarifications
KY.HS.C.39 (+) Understand and use the definite integral of a function over an interval and understand its use as a mathematical tool. **MP.1, MP.2**	

Attending to the Standards for Mathematical Practice
Students understand how graphical displays of functions (**MP.5**) and the application of limits (**MP.2**) lead to the concept of integration.

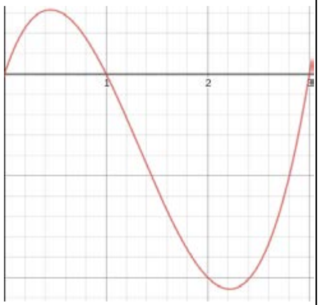
Calculus-Applications of Integration	
Standards for Mathematical Practice	
MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.	MP.5. Use appropriate tools strategically. MP.6. Attend to precision. MP.7. Look for and make use of structure. MP.8. Look for and express regularity in repeated reasoning.
Cluster: Apply techniques of integration.	
Standards	**Clarifications**
KY.HS.C.40 (+) Find antiderivatives of a variety of basic functions including power, exponential, logarithmic and trigonometric and apply basic properties of definite integrals. **MP.2, MP.7**	
KY.HS.C.41 (+) Use substitution techniques and change of limits of integration to find antiderivatives. **MP.2, MP.3**	Combining substitution techniques with basic rules allows for a broad spectrum of additional functions to be integrated. Substitution is the derivative equivalent of the chain rule and may be used to develop basic integration rules.
KY.HS.C.42 (+) Find particular antiderivatives given initial conditions. **MP.1, MP.2**	
Attending to the Standards for Mathematical Practice	
When applying techniques of integration represent problem situations (**MP.1**)**,** students identify whether an available techniques (**MP.7**) is applicable to a given integral expression (**MP.2**)**.**	

Calculus-Applications of Integration

Standards for Mathematical Practice

MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.	MP.5. Use appropriate tools strategically. MP.6. Attend to precision. MP.7. Look for and make use of structure. MP.8. Look for and express regularity in repeated reasoning.

Cluster: Define trigonometric ratios and solve problems involving right triangles.

Standards	Clarifications/Illustrations
KY.HS.C.43 (+) Model, solve and interpret applications of antiderivatives including finding area, velocity, acceleration and volume of a solid. **MP.1, MP.4**	Include area under a curve and area between two curves. Students calculate intersection points and note when functions "switch" requiring two integrals. Students calculate horizontal area. Students calculate volume using the dish, washer and shell methods.
KY.HS.C.44 (+) Apply integration to solve problems including particle motion and exponential growth and decay. **MP.1, MP.4**	Include particle motion problems, such as the velocity function below. • Where does particle change direction? • When is it moving to the left? • When is it moving to the right? • How far does it move to the left? • How far does it move to the right? • What is the displacement of the particle? • What is the total distance traveled? • If the particle started at x = 5, where is it at the end of the first 3 seconds? 
KY.HS.C.45 (+) Describe the application of integration to a variety of problems using precise mathematical language and notation. **MP.4, MP.6**	Use definite integrals to represent displacement, total distance traveled and average value of a function. Integrals are solutions to differential equations, such as $\frac{dy}{dx} = ky$ is the solution to $y = Ce^{kt}$.

Attending to the Standards for Mathematical Practice

Students recognize that a variety of applied problems can be represented using integral expressions (**MP.4**) and identify appropriate integration strategies (**MP.2**) to solve these problems (**MP.1**).

The identified mathematical practices, coherence connections and clarifications are possible suggestions; however, they are not the only pathways.