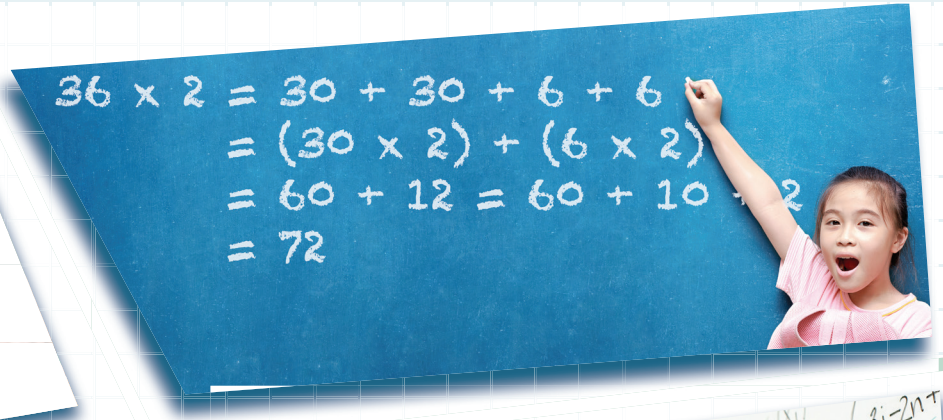
Mathematics Framework

for California Public Schools

Kindergarten Through Grade Twelve



Adopted by the California State Board of Education, November 2013

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Developed by the Instructional Quality Commission

Adopted by the California State Board of Education
November 2013

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The guidance in the *Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* (2015 edition) is not binding on local educational agencies or other entities. Except for the statutes, regulations, and court decisions that are referenced herein, the document is exemplary, and compliance with it is not mandatory. (See *Education Code* Section 33308.5.)

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Foreword

With the *Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* (framework), we take the next steps on a path toward our goal of college and career readiness for all of California's students. This journey began with the adoption of the California Common Core State Standards for Mathematics (CA CCSSM) in August 2010, which sparked exciting and important shifts in mathematics instruction and learning. The journey will continue as schools and districts bring the CA CCSSM to life in classrooms throughout the state. This framework is part of that effort.

Focus, coherence, and rigor—the three major instructional shifts under the CA CCSSM—are the principles underlying a new direction for student learning. The promise of the CA CCSSM is instruction and learning that promotes problem solving, communication skills, and critical thinking. To meet this promise, students must experience a balanced approach to instruction and learning that supports conceptual understanding, procedural skill and fluency, and application of mathematics to real-world problems.

Mathematics is essential to living in and understanding the world. We apply mathematics when we check our pockets to be sure we have enough money for a purchase, measure ingredients for making dinner, or evaluate the evidence in a debate on local government spending. Mathematicians and scientists apply mathematics to determine how much thrust is needed to launch a spacecraft, to plot its course into space, and ensure that it lands safely. Understanding and being able to apply mathematics opens the doors to college and to careers in fields as varied as automobile mechanics, architecture, construction, medicine, engineering, economics, and the arts. In addition, exploring mathematical concepts, working collaboratively on engaging tasks, and presenting and critiquing arguments—all of which are required by the CA CCSSM—help prepare students for life after high school.

It will take a concerted effort for all students to meet the goal of college and career readiness, and the path will not always be easy. Teachers, administrators, other educators, parents and family members, community members, education stakeholders, policymakers, institutes of higher education, early learning programs, and students all have vital roles to play. Working together, embracing the challenge and the promise of providing all students with high-quality, standards-based mathematics instruction, we can help our children reach their personal college and career goals.

We invite you to join us on the journey toward college and career readiness for California's students and offer this mathematics framework as a guide.

TOM TORLAKSON
State Superintendent of Public Instruction

MICHAEL W. KIRST
President, California State Board of Education

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Acknowledgments

This edition of the *Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* was adopted by the California State Board of Education (SBE) on November 6, 2013. When this edition was approved, the following persons were serving on the SBE:

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Ilene Straus, Vice President
Sue Burr
Carl A. Cohn
Bruce Holaday
Aida Molina
Patricia Ann Rucker
Nicolasa Sandoval
Trish Boyd Williams
Jesse Zhang, Student Member

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*Affiliations listed were current at the time of each member's appointment.

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Introduction

The highest form of pure thought is mathematics.
—Plato (427–347 BCE)

Focus, coherence, and rigor, the underlying principles of the California Common Core State Standards for Mathematics (CA CCSSM), hold the promise of preparing all California students for college, careers, and civic life—and developing mathematically competent individuals who can use mathematics as a tool for making wise decisions in their personal lives, a foundation for rewarding work, and a means for comprehending and influencing the world in which they will live. This framework supports these ambitious goals by emphasizing mathematical instruction and learning that focus on key topics, build mathematical understanding and fluency in a coherent manner, and develop students' ability to apply mathematics creatively to analyze and solve complex problems.

Why Is Mathematics Important?

Mathematics impacts everyday life, future careers, and good citizenship. A solid foundation in mathematics prepares students for future occupations in fields such as business, medicine, science, engineering, and technology. Students' understanding of probability and the ability to quantify and analyze information enable them to interpret economic data, participate in political discussions, and make wise personal financial decisions. Mathematical modeling is a tool for solving everyday problems, making informed decisions, improving life skills (e.g., logical thinking, reasoning, and problem solving), planning, designing, predicting, and developing financial literacy.

Success in mathematics education provides students with college and career options and increases prospects for future income. Knowledge and understanding of high school mathematics correlates to access to college, graduation from college, and earnings in the top quartile of income from employment. The value of such preparation promises to be even greater in the future. The National Science Board indicates that the growth of jobs in the mathematics-intensive science and engineering workforce is outpacing overall job growth by a 3-to-1 ratio (National Mathematics Advisory Panel 2008).

Mathematics Achievement

With regard to achievement in mathematics, students in the United States have not kept pace with their international peers. Achievement gaps still exist throughout the country, college remediation rates are too high, and some students are unprepared to perform and thrive in the workforce. California's student achievement data reflect similar challenges for some students. The 2011 National Assessment of Educational Progress (NAEP) results indicate that California's fourth- and eighth-grade students continue to make incremental gains in their mathematics scores; however, too many students also continue to place at the "Basic" achievement level, which denotes partial mastery of fundamental skills (California Department of Education [CDE] 2011).

Standards Implementation

The CA CCSSM resemble the standards of the highest-achieving nations and reflect the importance of focus, coherence, and rigor. California's implementation of the CA CCSSM demonstrates a commitment to providing a world-class education for all students, narrowing the achievement gap, supporting lifelong learning, and helping students develop the skills and knowledge necessary to fully participate in the global economy of the twenty-first century. The CA CCSSM build on California's standards-based educational system in which standards, curriculum, instruction, assessment, and accountability are aligned to support student attainment of the standards. Teachers and local school officials, in collaboration with families and community partners, use standards to help students achieve academic success (CDE 2012c).

California Common Core State Standards for Mathematics

For more than a decade, research conducted on mathematics education in high-performing countries has pointed to the conclusion that the mathematics curriculum in the United States must become substantially more focused and coherent to improve mathematics achievement in this country. The national Common Core State Standards for Mathematics, as well as the CA CCSSM, were established to address the problem of having a curriculum that is "a mile wide and an inch deep" (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010c).

These standards were informed by international benchmarking and began with research-based learning progressions detailing what is known about how students' mathematical knowledge, skills, and understanding develop over time. The progression from kindergarten standards to standards for higher mathematics exemplifies the three principles of focus, coherence, and rigor that underlie the CA CCSSM. The standards stress conceptual understanding, procedural skill and fluency, and application to ensure that students will learn and absorb the critical information necessary to succeed at higher levels of mathematics and can apply their learning in increasingly complex situations.

The CA CCSSM include two types of standards: Standards for Mathematical Practice, which are the same at each grade level; and Standards for Mathematical Content, which are different at each grade level. These two types of standards address both "habits of mind" that students should develop to foster mathematical understanding and expertise, and skills and knowledge—what students need to know and be able to do. The standards also call for mathematical practices and mathematical content to be connected as students engage in mathematics. The Standards for Mathematical Practice are defined in the Overview of the Standards Chapters. In addition, the Standards for Mathematical Content and the Standards for Mathematical Practice are listed at the end of each grade level (K–8) and higher mathematics course.

Guiding Principles for Mathematics Programs in California

Five guiding principles[1] underlie the Standards for Mathematical Practice, Standards for Mathematical Content, and other resources in this framework; see table IN-1. These philosophical statements should guide the construction and evaluation of mathematics programs in schools and the broader community. The Standards for Mathematical Practice are interwoven throughout the guiding principles.

Table IN-1. Guiding Principles for Mathematics Programs in California

Guiding Principle 1: Learning Mathematical ideas should be explored in ways that stimulate curiosity, create enjoyment of mathematics, and develop depth of understanding.
Guiding Principle 2: Teaching An effective mathematics program is based on a carefully designed set of content standards that are clear and specific, focused, and articulated over time as a coherent sequence.
Guiding Principle 3: Technology Technology is an essential tool that should be used strategically in mathematics education.
Guiding Principle 4: Equity All students should have a high-quality mathematics program that prepares them for college and careers.
Guiding Principle 5: Assessment Assessment of student learning in mathematics should take many forms to inform instruction and learning.

Guiding Principle 1: Learning

Mathematical ideas should be explored in ways that stimulate curiosity, create enjoyment of mathematics, and develop depth of understanding.

Students need to understand mathematics deeply and use it effectively. The Standards for Mathematical Practice describe ways in which students increasingly engage with the subject matter as they grow in mathematical maturity and expertise through the elementary, middle, and high school years.

For students to achieve mathematical understanding, instruction and learning must balance mathematical procedures and conceptual understanding. Students should be actively engaged in doing meaningful mathematics, discussing mathematical ideas, and applying mathematics in interesting, thought-provoking situations. Student understanding is further developed through ongoing reflection about cognitively demanding and worthwhile tasks.

1. The guiding principles were adapted from the Massachusetts Curriculum Frameworks and are included by permission of the Massachusetts Department of Elementary and Secondary Education. The complete and current version of each Massachusetts curriculum framework is available at http://www.doe.mass.edu/frameworks/current.html (accessed May 9, 2014).

Tasks should be designed to challenge students in multiple ways. Short- and long-term investigations that connect procedures and skills with conceptual understanding are integral components of an effective mathematics program. Activities should build upon students' curiosity and prior knowledge and enable them to solve progressively deeper, broader, and more sophisticated problems; see **MP.1** (Make sense of problems and persevere in solving them) in table OV-2 of the Overview of the Standards Chapters. Mathematical tasks reflecting sound and significant mathematics should generate active classroom discourse, promote the development of conjectures, and lead to an understanding of the necessity for mathematical reasoning; see **MP.2** (Reason abstractly and quantitatively) in table OV-2 of the Overview of the Standards Chapters.

Guiding Principle 2: Teaching

An effective mathematics program is based on a carefully designed set of content standards that are clear and specific, focused, and articulated over time as a coherent sequence.

The sequence of topics and instruction should be based on what is known about how students' mathematical knowledge, skill, and understanding develop over time. What and how students are taught should reflect not only the topics within mathematics but also the key ideas that determine how knowledge is organized and generated within mathematics; see **MP.7** (Look for and make use of structure) in table OV-2 of the Overview of the Standards Chapters. Students should be asked to apply their learning and to show their mathematical thinking and understanding. This high-quality instruction requires teachers to have a deep knowledge of mathematics.

Mathematical problem solving is the hallmark of an effective mathematics program. Skill in mathematical problem solving requires practice with a variety of mathematical problems as well as a firm grasp of mathematical techniques and their underlying principles. Armed with this deeper knowledge, students can use mathematics in flexible ways to attack various problems and devise different ways to solve any particular problem; see **MP.8** (Look for and express regularity in repeated reasoning) in table OV-2 of the Overview of the Standards Chapters. Mathematical problem solving calls for reflective thinking, persistence, learning from the ideas of others, and reviewing one's own work with a critical eye. Students should be able to construct viable arguments and critique the reasoning of others. They should analyze situations and justify their conclusions, communicate their conclusions to others, and respond to the arguments of others; see **MP.3** (Construct viable arguments and critique the reasoning of others) in table OV-2 of the Overview of the Standards Chapters. Students at all grades should be able to listen to or read the arguments of others, decide whether they make sense, and ask questions to clarify or improve the arguments.

Mathematical problem solving provides students with experiences to develop other mathematical practices. Success in solving mathematical problems helps to create an abiding interest in mathematics. Students learn to model with mathematics and to apply the mathematics that they know to solve problems arising in everyday life, society, and the workplace; see **MP.4** (Model with mathematics) in table OV-2 of the Overview of the Standards Chapters.

For a mathematics program to be effective, it must be taught by knowledgeable teachers. According to Liping Ma, "The real mathematical thinking going on in a classroom, in fact, depends heavily on the teacher's understanding of mathematics" (Ma 2010). Research on the relationship between teachers' mathematical knowledge and student achievement confirms the importance of teachers' content knowledge (National Mathematics Advisory Panel 2008). The message from the research is clear: having knowledgeable teachers really does matter, and teacher expertise in a subject drives student achievement. As Liping Ma states, "Improving teachers' content subject matter knowledge and improving students' mathematics education are thus interwoven and interdependent processes that must occur simultaneously" (Ma 2010). See the Instructional Strategies chapter and the Supporting High-Quality Common Core Mathematics Instruction chapter for more information.

Guiding Principle 3: Technology

Technology is an essential tool that should be used strategically in mathematics education.

Technology enhances the mathematics curriculum in many ways. Tools such as measuring instruments, manipulatives (such as base-ten blocks and fraction pieces), scientific and graphing calculators, and computers with appropriate software, if properly used, contribute to a rich learning environment for investigating, exploring, developing, and applying mathematical concepts. Appropriate use of calculators is essential; calculators should not be used as a replacement for basic understanding and skills. Elementary students should learn how to perform the basic arithmetic operations independent of the use of a calculator (National Center for Education Statistics 1995). The use of a graphing calculator can help middle school and secondary students visualize properties of functions and their graphs. Graphing calculators should be used to enhance—not replace—student understanding and skills.

When presenting or solving mathematical problems, teachers and students should consider the tools available to them. Students should be familiar with tools appropriate for their grade level so that they can make sound decisions about which tools will be helpful; see **MP.5** (Use appropriate tools strategically) in table OV-2 of the Overview of the Standards Chapters.

Technology enables students to communicate ideas in the classroom or to search information sources such as the Internet, which is an important addition to a school's internal library resources. Technology can also be especially helpful in assisting students with special needs in the classroom, at home, and in the community.

Technology changes the mathematics to be learned, as well as when and how it is learned. For example, currently available technology provides a dynamic and exploration-driven approach to mathematical concepts such as functions, rates of change, geometry, and averages that was not possible in the past. Some mathematics becomes more important because technology requires it, some becomes less important because technology replaces it, and some becomes possible because technology allows it. See the Technology in the Teaching of Mathematics chapter for additional information.

Guiding Principle 4: Equity

All students should have a high-quality mathematics program that prepares them for college and careers.

All California students should have a high-quality mathematics program that meets the goals and expectations of the CA CCSSM and addresses students' individual interests and talents. The standards provide clear signposts along the way to the goal of college and career readiness for all students; they also accommodate a broad range of students, from those requiring a significant amount of extra support in mathematics to others needing minimal support or enrichment opportunities. To promote achievement of these standards, teachers should plan for, instruct, model, and support classroom discourse, reflection, use of multiple problem-solving strategies, and a positive disposition toward mathematics. They should have high expectations for all students. At every level of the education system, teachers should act on the belief that every child can and should learn challenging mathematics. Teachers and guidance personnel should advise students and parents about why it is important to take advanced courses in mathematics and how this will prepare students for success in college and the workplace.

All students should have the benefit of quality instructional materials, good libraries, and adequate technology—and all students must have the opportunity to learn and meet the same high standards. In order to meet the needs of the greatest range of students, mathematics programs should provide the necessary intervention and support for those students who are below or above grade-level expectations. Practice and enrichment should extend beyond the classroom. Tutorial sessions, mathematics clubs, competitions, robotics, and apprenticeships are examples of mathematics activities that promote learning.

Because mathematics is the cornerstone of many disciplines, a comprehensive curriculum should include applications to everyday life and modeling activities that demonstrate the connections among disciplines. Schools should also provide opportunities for communicating with experts in applied fields to enhance students' knowledge of these connections; see **MP.4** (Model with mathematics) in table OV-2 of the Overview of the Standards Chapters.

An important part of preparing students for college and careers is to ensure that they have the mathematics and problem-solving skills necessary to make sound financial decisions in everyday life—for example, to set up a bank account, learn about saving money and earning interest, understand student loans, read credit and debit statements, select the best bargains when shopping, and choose the most cost-effective cell phone plan based on monthly usage. See the Universal Access chapter and appendixes A and B for additional information.

Guiding Principle 5: Assessment

Assessment of student learning in mathematics should take many forms to inform instruction and learning.

A comprehensive assessment program is an integral component of an instructional program. It provides students with frequent feedback on their performance, teachers with diagnostic tools for gauging

students' depth of understanding of mathematical concepts and skills, parents with information about their children's performance in the context of program goals, and administrators with a means for measuring student achievement.

Assessments take a variety of forms, require different amounts of time, and address various aspects of student learning. Gaps in knowledge and errors in reasoning can be identified when students "think aloud" or talk through their reasoning. By observing and questioning students as they work, teachers can gain insight into students' abilities to apply appropriate mathematical concepts and skills, make conjectures, and draw conclusions. Homework, mathematics journals, portfolios, oral presentations, and group projects offer additional means for capturing students' thinking, knowledge of mathematics, facility with the language of mathematics, and ability to communicate what they know to others. Tests and quizzes assess knowledge of mathematical concepts, operations, and skills and their efficient application to problem solving; they can also pinpoint areas that require more practice or teaching. Taken together, the results of these different forms of assessment provide rich profiles of students' achievements in mathematics and serve as the basis for identifying curricula and instructional approaches to best develop students' talents.

Assessment should also be a major component of the learning process. As students help identify goals for lessons or investigations, they gain greater awareness of what they need to learn and how they will demonstrate that learning. Engaging students in this kind of goal setting can help them reflect on their work, understand the standards to which they are held accountable, and take ownership of their learning. See the Assessment chapter for additional information.

Supporting Twenty-First-Century Learning

California is part of a growing national movement to teach students the problem-solving skills and critical thinking they need for college, careers, and civic life. The Partnership for 21st Century Skills (P21) developed a framework for twenty-first-century learning comprising student outcomes and support systems. The student outcomes consist of the following elements:

1. Core subjects and twenty-first-century interdisciplinary themes, which include global awareness; financial, economic, business, and entrepreneurial literacy; civic literacy; health literacy; and environmental literacy
2. Life and career skills, which include flexibility and adaptability, initiative and self-direction, social and cross-cultural skills, productivity and accountability, and leadership and responsibility
3. Learning and innovation skills, often referred to as the "4 Cs": creativity and innovation, critical thinking and problem solving, communication, and collaboration
4. Information, media, and technology skills, which include information literacy, media literacy, and ICT (information, communications, and technology) literacy

Support systems provided by P21 include standards and assessments, curriculum and instruction, professional development, and learning environments.

California educators need to intentionally include the 4 Cs in mathematics instruction. A fundamental goal is to promote higher-order mathematical thinking skills and interdisciplinary approaches that integrate the use of supportive technologies, inquiry, and problem-based learning to provide contexts for pupils to apply learning in relevant, real-world scenarios and that prepare all pupils for college, careers, and citizenship in the twenty-first century. Mathematics instruction and learning are instrumental to mastering P21 interdisciplinary themes, particularly financial, economic, business, and entrepreneurial literacy. Resources connecting the Partnership for 21st Century Skills with the Common Core State Standards are available at http://www.p21.org/ (accessed May 15, 2014).

Purpose of the Framework

The *Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* is meant to guide teachers in curriculum development and instruction as they work to ensure that all students meet or exceed the CA CCSSM. The framework also provides educators and developers of instructional materials with a context for implementing the standards. Building on the standards, the framework addresses how all students in California public schools can best meet those standards. California's mathematics framework is available online and, as such, will remain a "living" document that will be updated regularly.

Implementation of the CA CCSSM will take time and effort, but it also provides a new opportunity to ensure that California's students are held to the same high expectations in mathematics as their national and global peers. Educators and administrators, as well as parents, guardians, and community members, are challenged to become familiar with the standards and to support raising the bar for student achievement through rigorous curriculum and instruction that develops students' conceptual understanding, procedural skill and fluency, and application of mathematics to solve problems.

Overview of the Standards Chapters

These Standards are not intended to be new names for old ways of doing business.
—National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA/CCSSO) 2010f

In 2009, the Council of Chief State School Officers (CCSSO) and the National Governors Association Center for Best Practices (NGA) committed to developing a set of standards that would help prepare students for success in careers and college. The Common Core State Standards Initiative was a voluntary, state-led effort coordinated by the CCSSO and NGA to establish clear and consistent education standards. Development of the standards began with research-based learning progressions detailing what is known about how students' mathematical knowledge, skills, and understanding develop over time.

In June 2010, the State of California replaced its existing mathematics standards by adopting the California Common Core State Standards for Mathematics (CA CCSSM). The state's previous mathematics standards had been in place since 1997. In January 2013, in accordance with Senate Bill 1200, the California State Board of Education (SBE) adopted modifications to the CA CCSSM, which included organizing the standards into model courses for higher mathematics aligned with Appendix A of the Common Core State Standards Initiative. Standards that are unique to California (California additions) are identified by boldface type and followed by the abbreviation **CA**.

California's new standards define what students should understand and be able to do in the study of mathematics. The state's implementation of the CA CCSSM demonstrates a continued commitment to providing a world-class education for all students that supports lifelong learning and the skills and knowledge necessary to participate in the global economy of the twenty-first century.

Understanding the California Common Core State Standards for Mathematics

The CA CCSSM were designed to help students gain proficiency with and understanding of mathematics across grade levels. The standards call for learning mathematical content in the context of real-world situations, using mathematics to solve problems, and developing "habits of mind" that foster mastery of mathematics content as well as mathematical understanding.

The standards for kindergarten through grade eight (K–8) prepare students for higher mathematics, beginning with Mathematics I or Algebra I, and serve as the foundation on which more advanced mathematical knowledge can be built. The standards for higher mathematics (high school–level standards) prepare students for college, careers, and productive citizenship. In short, the standards are a progression of mathematical learning.

The standards are based on three major principles: **focus, coherence, and rigor**. These principles are meant to fuel greater achievement in a rigorous curriculum, in which students acquire conceptual understanding, procedural skill and fluency, and the ability to apply mathematics to solve problems.

Major Principles of the California Common Core State Standards for Mathematics

- **Focus**—Place strong emphasis where the standards focus.
- **Coherence**—Think across grades, and link to major topics in each grade.
- **Rigor**—In major topics, pursue with equal intensity:
  - conceptual understanding;
  - procedural skill and fluency;
  - application.

Focus is necessary so that students have sufficient time to think about, practice, and integrate new ideas into their growing knowledge structure. Focus is also a way to allow time for the kinds of rich classroom discussion and interaction that support the Standards for Mathematical Practice (MP) and develop students' broader mathematical understanding. Instruction should focus deeply on only those concepts that are emphasized in the standards so that students can build a strong foundation in conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems inside and outside the mathematics classroom.

Coherence arises from mathematical connections. Some of the connections in the standards knit topics together at a single grade level. Most connections are vertical, as the standards support a progression of increasing knowledge, skill, and sophistication across the grades.

- Thinking across grades: The standards are designed to help administrators and teachers connect learning within and across grades. For example, the standards develop fractions and multiplication across grade levels, so that students can build new understanding on foundations that were established in previous years. Thus each standard is an extension of previous learning, not a completely new concept.
- Linking to major topics: Connections between the standards at a single grade level can be used to improve the instructional focus by linking additional or supporting topics to the major work of the grade. For example, in grade three, bar graphs are not "just another topic to cover." Students use information presented in bar graphs to solve word problems using the four operations of arithmetic. (For lists of Major and Additional/Supporting topics, see the Cluster-Level Emphases charts in each grade-level chapter.)

Grades	Priorities in Support of Rich Instruction: Expectations of Fluency and Conceptual Understanding in the CA CCSSM
K–2	Addition and subtraction—concepts, skills, problem solving, and place value
3–5	Multiplication and division of whole numbers and fractions—concepts, skills, and problem solving
6	Ratios and proportional reasoning; early expressions and equations
7	Ratios and proportional reasoning; arithmetic of rational numbers
8	Linear algebra

Adapted from Achieve the Core 2012.

Rigor requires that conceptual understanding, procedural skill and fluency, and application be approached with equal intensity.

- Conceptual understanding: The word *understand* is used in the standards to set explicit expectations for conceptual understanding. Teachers focus on much more than "how to get the answer"; they support students' ability to access concepts from a number of different perspectives. Students might demonstrate deep conceptual understanding of core mathematics concepts by solving short conceptual problems, applying mathematics in new situations, and speaking and writing about their understanding. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, such students may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, help other students understand a given method or find and correct an error, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

Examples of Understanding in the CA CCSSM	
Grade/Level	**Standards**
K	**Understand** that each successive number name refers to a quantity that is one larger (**K.CC.4c**).
2	**Understand** that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds (**2.NBT.7**).
4	**Understand** addition and subtraction of fractions as joining and separating parts referring to the same whole (**4.NF.3a**).
6	**Understand** the concept of a ratio and use ratio language to describe a ratio relationship between two quantities (**6.RP.1**).
8	**Understand** that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output (**8.F.1**).
Higher Mathematics	**Understand** that a function from one set (called the *domain*) to another set (called the *range*) assigns to each element of the domain exactly one element of the range (**F-IF.1**). (*Note:* This is only a portion of the complete standard.)
Higher Mathematics	Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles (**G-SRT.6**).

- Procedural skill and fluency: The standards are explicit where fluency is expected. In kindergarten through grade six (K–6), students should make steady progress toward procedural skill and computational fluency (being accurate and reasonably fast), including knowing single-digit products and sums from memory (for example, see **2.OA.2** and **3.OA.7**). As used in the standards, the word *fluently* refers to fluency with a written or mental method, not a method using manipulatives or concrete representations. Progress toward fluency should be woven into instruction in grade-appropriate ways, along with developing conceptual understanding of the four operations.[1]

1 . For more information about how students develop fluency in tandem with understanding, see the University of Arizona's Progressions documents on Operations and Algebraic Thinking and on Number and Operations in Base Ten (the University of Arizona Progressions Documents for the Common Core Math Standards [UA Progressions] 2011–13).

Manipulatives and concrete representations such as diagrams that enhance conceptual understanding can help students make connections to written and symbolic methods (e.g., see **1.NBT.1**). Methods and algorithms should be general and based on principles of mathematics (e.g., place value and properties of operations).

Developing fluency with single-digit computations can involve a mixture of just knowing some answers, knowing some answers from understanding patterns, and knowing some answers from understanding and using strategies. In grades four, five, and six, moving to fluency with multi-digit computations and operations with decimals and fractions requires developing a base of understanding in previous years about how to use place value in carrying out and interpreting operations with single digits within a multi-digit number and understanding how to use unit fractions and equivalence for meaningful fraction operations. Students examine various methods and relate them to visual models, but from the beginning students develop, discuss, and use efficient, accurate, and generalizable methods that are or will lead to a variation of the standard algorithm. Students drop the visual models when they can, although they may continue to use models if needed. *Fluency* means working without visual models. Sufficient practice and extra support should be provided at each grade to allow all students to meet the standards that explicitly call for fluency.

Grade	Examples of Expectations of Fluency in the K–6 CA CCSSM
K	Add/subtract within 5
1	Add/subtract within 10
2	Add/subtract within 20 (using mental strategies) Add/subtract within 100 (using strategies[2])
3	Multiply/divide within 100 Add/subtract within 1,000 (using algorithms[3])
4	Add/subtract whole numbers within 1,000,000 (using the standard algorithm[4])
5	Multiply multi-digit numbers (using the standard algorithm) Add/subtract fractions
6	Divide multi-digit numbers (using the standard algorithm) Perform multi-digit decimal operations (add, subtract, multiply, and divide using the standard algorithm for each operation)

Adapted from Achieve the Core 2012.

2. These strategies would be based on place value, properties of operations, and/or the relationship between addition and subtraction.

3. A range of algorithms may be used.

4. Minor variations of writing the standard algorithm are acceptable.

- Application: Students are expected to use mathematics to solve "real-world problems." In the standards, the phrase *real-world problems* and the star symbol (★) are used to set expectations and flag opportunities for applications and modeling (which is a Standard for Mathematical Practice as well as a Conceptual Category in higher mathematics). Real-world problems and standards that support modeling are also opportunities to provide activities related to careers and everyday life. Teachers in content areas outside of mathematics—particularly science—ensure that students use mathematics at all grade levels to make meaning of and access content (adapted from Achieve the Core 2012).

Progression to Higher Mathematics

The progression from kindergarten standards to standards for higher mathematics, beginning with Mathematics I or Algebra I, exemplifies the three principles of focus, coherence, and rigor that are central to the CA CCSSM.

In kindergarten through grade five (K–5), the focus is on addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals, with a balance of concepts, skills, and problem solving. Arithmetic is viewed as an important set of skills and also as a thinking subject that prepares students for higher mathematics. Measurement and geometry develop alongside number and operations and are tied specifically to arithmetic along the way.

In middle school, multiplication and division develop into the powerful forms of ratio and proportional reasoning. The properties of operations take on prominence as arithmetic matures into algebra. The theme of quantitative relationships also becomes explicit in grades six through eight, developing into the formal concept of a function by grade eight. Meanwhile, the foundations of deductive geometry are laid in the middle grades. Finally, the gradual development of data representations in kindergarten through grade five leads to statistics in middle school: the study of shape, center, and spread of data distributions; possible associations between two variables; and the use of sampling in making statistical decisions.

In higher mathematics, algebra, functions, geometry, and statistics develop with an emphasis on modeling. Students continue to take a thinking approach to algebra, learning to see and make use of structure in algebraic expressions of growing complexity (Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

Mathematics is a logically progressing discipline that has intricate connections among the various domains and clusters in the standards. Sustained practice is required to master grade-level and course-level content. The major work (or emphases) in the standards for kindergarten through grade eight is noted in the Cluster-Level Emphases charts presented in each of the grade-level chapters that follow. Further, table OV-1 (adapted from Achieve the Core 2012) summarizes an important subset of the major work in kindergarten through grade eight, as the progression of learning in the standards leads toward Mathematics I or Algebra I.

Table OV-1. Progression to Algebra I and Mathematics I in Kindergarten Through Grade Eight

Kindergarten	Grade One	Grade Two	Grade Three	Grade Four	Grade Five	Grade Six	Grade Seven	Grade Eight
Know number names and the count sequence Count to tell the number of objects Compare numbers Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from Work with numbers 11–19 to gain foundations for place value	Represent and solve problems involving addition and subtraction Understand and apply properties of operations and the relationship between addition and subtraction Add and subtract within 20 Work with addition and subtraction equations Extend the counting sequence Understand place value Use place-value understanding and properties of operations to add and subtract Measure lengths indirectly and by iterating length units	Represent and solve problems involving addition and subtraction Add and subtract within 20 Understand place value Use place-value understanding and properties of operations to add and subtract Measure and estimate lengths in standard units Relate addition and subtraction to length	Represent and solve problems involving multiplication and division Understand properties of multiplication and the relationship between multiplication and division Multiply and divide within 100 Solve problems involving the four operations, and identify and explain patterns in arithmetic Develop understanding of fractions as numbers Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects Geometric measurement: understand concepts of area, and relate area to multiplication and to addition	Use the four operations with whole numbers to solve problems Generalize place-value understanding for multi-digit whole numbers Use place-value understanding and properties of operations to perform multi-digit arithmetic Extend understanding of fraction equivalence and ordering Build fractions from unit fractions by applying and extending previous understandings of operations Understand decimal notation for fractions, and compare decimal fractions	Understand the place-value system Perform operations with multi-digit whole numbers and decimals to hundredths Use equivalent fractions as a strategy to add and subtract fractions Apply and extend previous understandings of multiplication and division to multiply and divide fractions Geometric measurement: understand concepts of volume, and relate volume to multiplication and to addition Graph points in the coordinate plane to solve real-world and mathematical problems*	Apply and extend previous understandings of multiplication and division to divide fractions by fractions Apply and extend previous understandings of numbers to the system of rational numbers Understand ratio concepts and use ratio reasoning to solve problems Apply and extend previous understandings of arithmetic to algebraic expressions Reason about and solve one-variable equations and inequalities Represent and analyze quantitative relationships between dependent and independent variables	Apply and extend previous understanding of operations with fractions to add, subtract, multiply, and divide rational numbers Analyze proportional relationships and use them to solve real-world and mathematical problems Use properties of operations to generate equivalent expressions Solve real-life and mathematical problems using numerical and algebraic expressions and equations	Work with radicals and integer exponents Understand the connections between proportional relationships, lines, and linear equations Analyze and solve linear equations and pairs of simultaneous linear equations Define, evaluate, and compare functions Use functions to model relationships between quantities

Adapted from Achieve the Core 2012.

*Indicates a cluster that is well thought of as part of a student's progress to algebra, but that is currently not designated as Major by one or both of the assessment consortia (PARCC and Smarter Balanced) in their draft materials. Apart from the one exception marked by an asterisk, the clusters listed here are a subset of those designated as Major in both of the assessment consortia's draft documents.

Two Types of Standards

The CA CCSSM include two types of standards: Standards for Mathematical Practice and Standards for Mathematical Content. These standards address "habits of mind" that students should develop to foster mathematical understanding and expertise, as well as concepts, skills, and knowledge—what students need to understand, know, and be able to do. The standards also require that mathematical practices and mathematical content be connected. These connections are essential to support the development of students' broader mathematical understanding, as students who lack understanding of a topic may rely too heavily on procedures. The Standards for Mathematical Practice must be taught as carefully and practiced as intentionally as the Standards for Mathematical Content are. Neither type should be isolated from the other; mathematics instruction is most effective when these two aspects of the CA CCSSM come together as a powerful whole.

The eight **Standards for Mathematical Practice** (**MP**) describe the attributes of mathematically proficient students and expertise that mathematics educators at all levels should seek to develop in their students; see table OV-2. Mathematical practices provide a vehicle through which students engage with and learn mathematics. As students move from elementary school through high school, mathematical practices are integrated in the tasks as students engage in doing mathematics and master new and more advanced mathematical ideas and understandings.

Standards for Mathematical Practice (**MP**)

These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the National Council of Teachers of Mathematics' process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition (NGA/CCSSO 2010q).

Table OV-2. Standards for Mathematical Practice (MP)

MP.1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP.2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically, and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize,* to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meanings of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

MP.3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. **Students build proofs by induction and proofs by contradiction. CA.3.1** (for higher mathematics only).

Table OV-2 *(continued)*

MP.4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

MP.5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a Web site, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

MP.6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, expressing numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school, they have learned to examine claims and make explicit use of definitions.

MP.7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square, and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Table OV-2 *(continued)*

MP.8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Table OV-3 summarizes the eight MP standards and provides examples of questions that teachers might use to support mathematical thinking and student engagement (as called for in the MP standards).

Table OV-3

Summary of the Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
MP.1 Make sense of problems and persevere in solving them. • Mathematically proficient students interpret and make meaning of the problem to find a starting point. • Analyze what is given in order to explain to themselves the meaning of the problem. • Plan a solution pathway instead of jumping to a solution. • Monitor their own progress and change the approach if necessary. • See relationships between various representations. • Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another. • Continually ask themselves, "Does this make sense?" • Can understand various approaches to solutions.	• How would you describe the problems in your own words? • How would you describe what you are trying to find? • What do you notice about _____? • What information is given in the problem? • Describe the relationship between the quantities. • Describe what you have already tried. What might you change? • Talk me through the steps you have used to this point. • What steps in the process are you most confident about? • What are some other strategies you might try? • What are some other problems that are similar to this one? • How might you use one of your previous problems to help you begin? • How else might you [organize, represent, show, etc.] _____?

Table OV-3 *(continued)*	
Summary of the Standards for Mathematical Practice	**Questions to Develop Mathematical Thinking**
MP.2 Reason abstractly and quantitatively. • Mathematically proficient students make sense of quantities, and the relationships between quantities, in problem situations. • Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships. • Understand the meaning of quantities and flexibly use operations and their properties. • Create a logical representation of the problem. • Attend to the meaning of quantities, not just how to compute them.	• What do the numbers used in the problem represent? • What is the relationship of the quantities? • How is _____ related to _____? • What is the relationship between _____ and _____? • What does _____ mean to you? (e.g. symbol, quantity, diagram) • What properties might we use to find a solution? • How did you decide that you needed to use _____ in this task? • Could we have used another operation or property to solve this task? Why or why not?
MP.3 Construct viable arguments and critique the reasoning of others. • Mathematically proficient students analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments. • Justify conclusions with mathematical ideas. • Listen to the arguments of others, and ask useful questions to determine if an argument makes sense. • Ask clarifying questions or suggest ideas to improve or revise the argument. • Compare two arguments and determine if the logic is correct or flawed.	• What mathematical evidence would support your solution? • How can we be sure that _____? How could you prove that _____? • Will it still work if _____? • What were you considering when _____? • How did you decide to try that strategy? • How did you test whether your approach worked? • How did you decide what the problem was asking you to find? (What was unknown?) • Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not? • What is the same and what is different about _____? • How could you demonstrate a counter-example? • I think it might be clearer if you said _____. Is that what you meant? • Is your method like Shawna's method? If not, how is your method different?

Table OV-3 *(continued)*	
Summary of the Standards for Mathematical Practice	**Questions to Develop Mathematical Thinking**
MP.4 Model with mathematics. • Mathematically proficient students understand this is a way to reason quantitatively and abstractly (able to decontextualize and contextualize). • Apply the mathematics they know to solve everyday problems. • Simplify a complex problem and identify important quantities to look at relationships. • Represent mathematics to describe a situation either with an equation or a diagram, and interpret the results of a mathematical situation. • Reflect on whether the results make sense, possibly improving or revising the model. • Ask themselves, "How can I represent this mathematically?"	• What math drawing or diagram could you make and label to represent the problem? • What are some ways to represent the quantities? • What is an equation or expression that matches the [diagram, number line, chart, table, etc.]? • Where did you see one of the quantities in the task in your equation or expression? • How would it help to create a [diagram, graph, table, etc.]? • What are some ways to visually represent _____? • What formula might apply in this situation?
MP.5 Use appropriate tools strategically. • Mathematically proficient students use available tools including visual models, recognizing the strengths and limitations of each. • Use estimation and other mathematical knowledge to detect possible errors. • Identify relevant external mathematical resources to pose and solve problems. • Use technological tools to deepen their understanding of mathematics.	• What mathematical tools could we use to visualize and represent the situation? • What information do you have? • What do you know that is not stated in the problem? • What approach would you consider trying first? • What estimate did you make for the solution? • In this situation, would it be helpful to use a [graph, number line, ruler, diagram, calculator, manipulatives, etc.]? • Why was it helpful to use _____? • What can using a _____ show us that _____ may not? • In what situations might it be more informative or helpful to use _____?

Table OV-3 *(continued)*	
Summary of the Standards for Mathematical Practice	**Questions to Develop Mathematical Thinking**
MP.6 Attend to precision. • Mathematically proficient students communicate precisely with others and try to use clear mathematical language when discussing their reasoning. • Understand the meanings of symbols used in mathematics and can label quantities appropriately. • Express numerical answers with a degree of precision appropriate for the problem context. • Calculate efficiently and accurately.	• What mathematical terms apply in this situation? • How did you know your solution was reasonable? • Explain how you might show that your solution answers the problem. • What would be a more efficient strategy? • How are you showing the meaning of the quantities? • What symbols or mathematical notations are important in this problem? • What mathematical language, definitions, properties (and so forth) can you use to explain _____? • Can you say it in a different way? • Can you say it in your own words? And now say it in mathematical words. • How could you test your solution to see if it answers the problem?
MP.7 Look for and make use of structure. • Mathematically proficient students look for the overall structures and patterns in mathematics and think about how to describe these in words, mathematical symbols, or visual models. • See complicated things as single objects or as being composed of several objects. Compose and decompose conceptually. • Apply general mathematical patterns, rules, or procedures to specific situations.	• What observations can you make about _____? • What do you notice when _____? • What parts of the problem might you [eliminate, simplify, etc.]? • What patterns do you find in _____? • How do you know if something is a pattern? • What ideas that we have learned before were useful in solving this problem? • What are some other problems that are similar to this one? • How does this relate to _____? • In what ways does this problem connect to other mathematical concepts?

Table OV-3 *(continued)*	
Summary of the Standards for Mathematical Practice	**Questions to Develop Mathematical Thinking**
MP.8 Look for and express regularity in repeated reasoning. • Mathematically proficient students see repeated calculations and look for generalizations and shortcuts. • See the overall process of the problem and still attend to the details in the problem-solving steps. • Understand the broader application of patterns and see the structure in similar situations. • Continually evaluate the reasonableness of their intermediate results.	• Explain how this strategy works in other situations. • Is this always true, sometimes true, or never true? • How would we prove that _____? • What do you notice about _____? • What is happening in this situation? • What would happen if _____? • Is there a mathematical rule for _____? • What predictions or generalizations can this pattern support? • What mathematical consistencies do you notice? • How is this situation like and different from other situations using this operation?

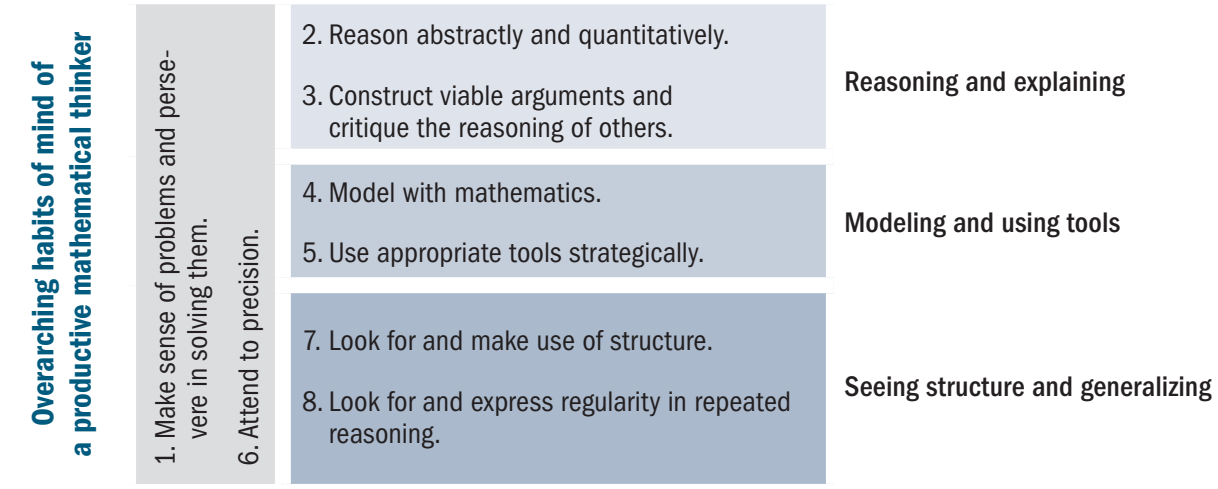
Adapted from Kansas Association of Teachers of Mathematics 2012, 3rd Grade Flipbook.

Ideally, several MP standards will be evident in each lesson as they interact and overlap with each other. The MP standards are not a checklist; they are the basis for mathematics instruction and learning. To help students persevere in solving problems (**MP.1**), teachers need to allow their students to struggle productively, and they must be attentive to the type of feedback they provide to students. Dr. Carol Dweck's research (Dweck 2006) revealed that feedback offering praise of effort and perseverance seems to engender and reinforce a "growth mindset."[5] In Dweck's estimation, "[g]rowth-minded teachers tell students the truth [about being able to close the learning gap between them and their peers] and then give them the tools to close the gap" (Dweck 2006).

Structuring the MP standards can help educators recognize opportunities for students to engage with mathematics in grade-appropriate ways. In figure OV-1, the eight MP standards are grouped into four categories. These four pairs of standards can also be given names, beginning with the rectangle on the far left and then moving from the bottom to the top with the other three rectangles. These names can become a sentence teachers might ask at the end of every day—for example, "Did I *Make Sense of Math* and *Math Structure* by using *Math Drawings* to support *Math Reasoning*?" This approach can help teachers to continually incorporate the core of the MP standards into classroom practices.

5. According to Dweck, a person with a growth mindset believes that intelligence is something that can be nurtured and gained. When people with this type of mindset do not meet the expected level of performance on a test or an assignment or have difficulty understanding a concept, they work hard at it, believing that if they just try hard enough, they will achieve the desired outcome.

Figure OV-1. Structuring the Standards for Mathematical Practice (MP)



Source: McCallum 2011.

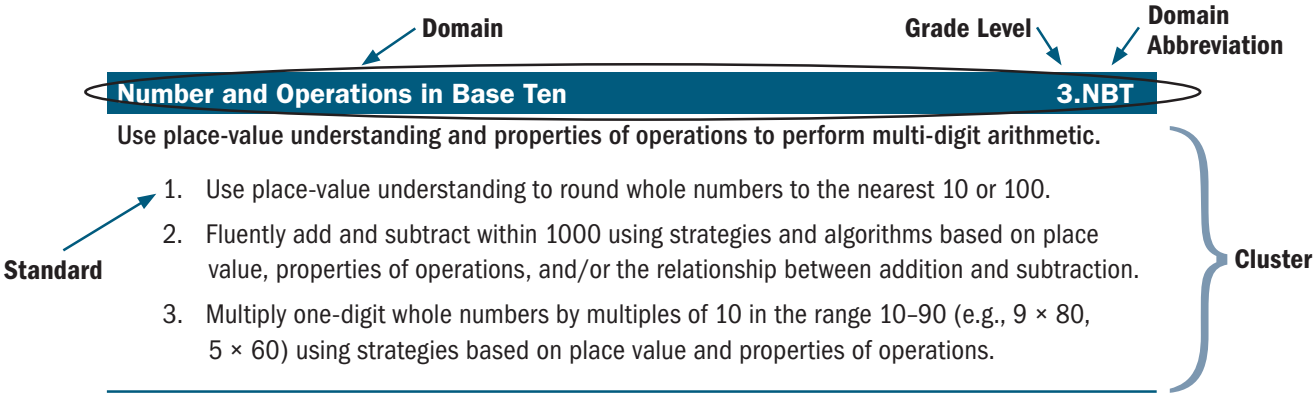
The **Standards for Mathematical Content** were built on progressions of topics across a number of grade levels, informed both by research on children's cognitive development and by the logical structure of mathematics.

Kindergarten Through Grade Eight

In kindergarten through grade eight, the standards are organized by grade level and then by **domains** (clusters of standards that address "big ideas" and support connections of topics across the grades), **clusters** (groups of related standards inside domains), and finally by the **standards** (what students should understand and be able to do). The standards do not dictate curriculum or pedagogy. For example, just because Topic A appears before Topic B in the standards for a given grade, it does not mean that Topic A must be taught before Topic B (NGA/CCSSO 2010c).

Throughout this framework, specific standards or groups of standards are identified in the narrative. For example, as shown in figure OV-2, a narrative reference to **3.NBT.1–3** signifies a standard at the third-grade level, the domain Number and Operations in Base Ten (**NBT**), and standards 1, 2, and 3 in the first cluster.

Figure OV-2. How to Read the Standards for Kindergarten Through Grade Eight



Higher Mathematics

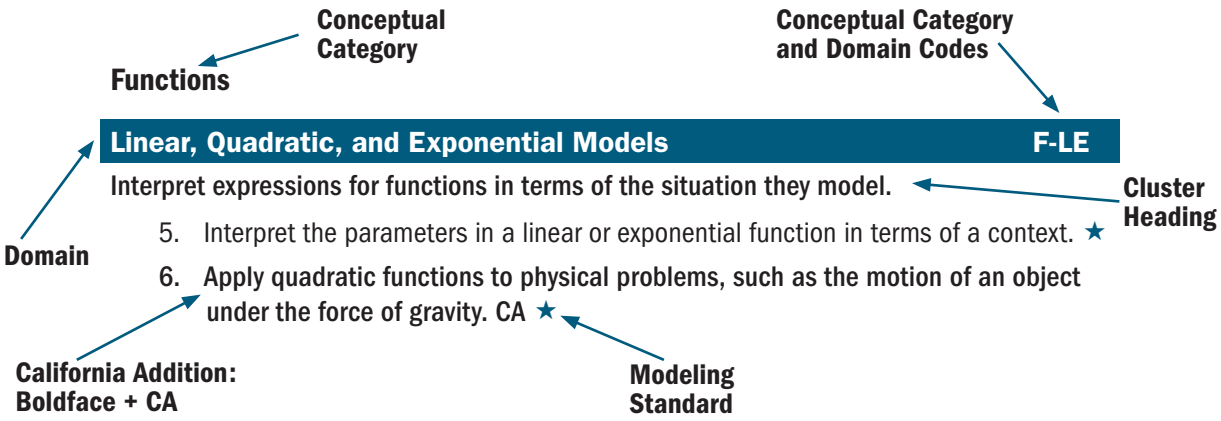
The standards for higher mathematics are organized differently than the K–8 standards. When developed by the NGA/CCSSO, the higher mathematics standards were not organized into courses; instead, they were listed according to the following conceptual categories:

- Number and Quantity (N)
- Algebra (A)
- Functions (F)
- Modeling (★)
- Geometry (G)
- Statistics and Probability (S)

Conceptual categories present a coherent view of higher mathematics; a student's work with functions, for example, crosses a number of traditional course boundaries, potentially up through and including calculus. With the exception of Modeling (see explanation following figure OV-3), each conceptual category is further subdivided into several domains, and each domain is subdivided into clusters. This structure is similar to that of the grade-level content standards.

Each higher mathematics standard begins with the identifier for the conceptual category (N, A, F, G, S), followed by the domain code, and then the standard number.

Figure OV-3. How to Read the Standards for Higher Mathematics



The two standards in figure OV-3 would be referred to as **F-LE.5** and **F-LE.6**, respectively. The star symbol (★) indicates that both standards are also Modeling standards. Modeling is best interpreted not as a collection of isolated topics, but rather in relation to other standards. Readers are encouraged to refer to appendix B for an extensive explanation of the Modeling conceptual category.

Table OV-4 illustrates how the domains and conceptual categories are distributed across the K–12 mathematical content standards. The corresponding abbreviations for each are also identified—for example, Geometry (G).

Table OV-4. Mathematical Content Domains (K–8) and Conceptual Categories (Higher Mathematics)

<table>
<tr><th>Grade</th><th>K</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th colspan="2">Higher Mathematics Conceptual Categories</th></tr>
<tr><td rowspan="6">K–8 Domains</td><td colspan="6">Counting and Cardinality (CC)</td><td colspan="3">Ratios and Proportional Relationships(RP)</td><td>Functions (F)</td><td>Functions (F)</td><td rowspan="6">Modeling (★)</td></tr>
<tr><td colspan="9">Operations and Algebraic Thinking (OA)</td><td colspan="2">Expression and Equations (EE)</td><td>Algebra (A)</td></tr>
<tr><td colspan="9">Number and Operations in Base Ten (NBT)</td><td colspan="2" rowspan="2">The Number System (NS)</td><td rowspan="2">Number and Quantity (N)</td></tr>
<tr><td colspan="6"></td><td colspan="3">Number and Operations—Fractions (NF)</td></tr>
<tr><td colspan="9">Measurement and Data (MD)</td><td colspan="2">Statistics and Probability (SP)</td><td>Statistics and Probability (S)</td></tr>
<tr><td colspan="9">Geometry (G)</td><td colspan="2">Geometry (G)</td><td>Geometry (G)</td></tr>
</table>

Overview: K–8 Chapters

The chapters covering kindergarten through grade eight provide guidance on instruction and learning aligned with the CA CCSSM. Each chapter presents a brief summary of prior learning and an overview of what students learn at that grade level. A section on the Standards for Mathematical Content highlights the instructional focus of the standards at the grade and includes a Cluster-Level Emphases chart that designates clusters of standards as "Major" or "Additional/Supporting" work at the grade level. The Connecting Mathematical Practices and Content section provides grade-level explanations and examples of how the MP standards may be integrated into grade-level-appropriate tasks.

The largest section of each chapter is a description of Standards-Based Learning organized by domains and clusters, with exemplars to explain the content standards, highlight connections to the various mathematical practice standards, and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. Also noted are opportunities to link concepts in the Additional/Supporting clusters to Major work at the grade (based on the grade-specific Cluster-Level Emphases charts) and examples of focus, coherence, and rigor. Finally, each chapter presents "Essential Learning for the Next Grade" to highlight important knowledge, skills, and understanding that students will need to succeed in future grades. The grade-level content standards are embedded throughout the narrative and at the end of each chapter. Standards that are unique to California (California additions) are identified by boldface type and followed by the abbreviation **CA**.

Overview: Higher Mathematics Chapters

When first adopted in August 2010, the CA CCSSM for higher mathematics were organized differently than the K–8 standards—by conceptual categories rather than in courses. In January 2013, the SBE adopted modifications to the CA CCSSM, including organizing content standards into model courses for higher mathematics, in accordance with Senate Bill 1200 (*Education Code* Section 60605.11, Chapter 654, Statues of 2012).

The model courses are organized into two pathways: Traditional and Integrated. The framework includes a description of these courses. The content of these courses is the same, regardless of the grade level at which they are taught.

Standards for Mathematical Practice

The MP standards are interwoven throughout the higher mathematics curriculum. Instruction should focus equally on developing students' ability to engage in the practice standards and on developing conceptual understanding of and procedural fluency in the content standards. The MP standards are the same at each grade level, with the exception of an additional practice standard included only in the CA CCSSM for higher mathematics:

MP.3.1 CA: Students build proofs by induction and proofs by contradiction.

This standard can be seen as an extension of Mathematical Practice 3, in which students construct viable arguments and critique the reasoning of others.

In the higher mathematics courses, the levels of sophistication of each MP standard increase as students integrate grade-appropriate mathematical practices with the content standards. Examples of the MP standards appear in each higher mathematics course narrative.

Standards for Mathematical Content

The entire catalog of higher mathematics standards is presented in the *California Common Core State Standards: Mathematics* (CDE 2013a), organized by both model courses and conceptual category. In this framework, the standards are organized into model courses that were adopted by the SBE in January 2013. The higher mathematics content standards specify the mathematics that all students should study in order to be college- and career-ready. Additional mathematical content that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics is indicated by a (+) symbol, as in this example:

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

All standards without a (+) symbol should be included in the common mathematics curriculum for all college- and career-ready students. Standards with a (+) symbol may also appear in courses intended for all students.

Higher Mathematics Chapters

The higher mathematics chapters are organized into courses according to two pathways:

- **Traditional Pathway** — consists of the higher mathematics standards organized along more traditional lines into Algebra I, Geometry, and Algebra II courses. In this sequence, almost the entire Geometry conceptual category is separated into a single course and treated as a separate subject. Although these courses have the same names as their traditional counterparts, it is important to note that the nature of the CA CCSSM yields very different courses. In the past, the label "Geometry" referred to a specific course, but now it may also refer to the conceptual category. Care will be taken throughout the higher mathematics chapters to make the distinction clear.
- **Integrated Pathway** — consists of the courses Mathematics I, II, and III. The integrated pathway presents higher mathematics as a connected subject, in that each course contains standards from all six of the conceptual categories. For example, in Mathematics I, students will focus on linear functions. Students contrast linear functions with exponential functions, solve linear equations, and model with functions. They also investigate the geometric properties of graphs of linear functions (lines) and model statistical data with lines of best fit. This is the way in which most other high-performing countries present higher mathematics, and it maintains the theme developed in kindergarten through grade eight of mathematics being a connected, multifaceted subject.

As noted earlier, regardless of the grade level at which a course is taught, the content of these courses is the same; for example, an Algebra I course or Mathematics I course is aligned with the Algebra I or Mathematics I course presented in the higher mathematics chapters of the framework. This is also true for advanced courses mentioned below.

In addition, the framework contains suggested courses in Precalculus and Statistics and Probability composed of CA CCSSM and an appendix on Mathematical Modeling (see appendix B). The Precalculus course mainly consists of standards with a (+) symbol, about two-thirds of which have not yet been taught in either the Integrated or Traditional Pathway; the course is designed to provide appropriate preparation for Calculus. The 1997 Calculus and Advanced Placement Probability and Statistics courses are also included.

Local educational agencies are not limited to offering the higher mathematics courses described in this framework. Beyond providing the courses necessary for students to fulfill the state requirements for high school graduation, local districts make decisions about which courses to offer their students. For example, career technical education (CTE) courses that integrate the higher mathematics CA CCSSM with technical and work-related knowledge and skills can make mathematics more relevant to students and can be an alternate yet rigorous pathway which prepares students for technical education programs after high school. CTE courses provide opportunities for students to engage in hands-on activities, problem solving, and decision making while learning in an occupational setting. The *California Career Technical Education Model Curriculum Standards* are a vital resource for designing CTE courses that

incorporate the CA CCSSM.[6] There are also CTE courses developed by groups of educators at the University of California Curriculum Integration (UCCI) Institutes that balance academic rigor with career technical content and meet the mathematics component of the A–G requirements for college admission.[7] In addition, appendix B provides guidance to assist local educational agencies in designing a higher mathematics course in modeling.

The *Statement on Competencies in Mathematics Expected of Entering College Students,* issued by the Intersegmental Committee of the Academic Senates of the California Community Colleges, the California State University, and the University of California (ICAS 2013), is another document that local educational agencies may want to consult as they determine which courses to offer and what content to incorporate into the courses. This document describes the characteristics, skills, and knowledge students need in order to succeed in college.

Each CA CCSSM course is described in its own chapter, starting with an overview of the course followed by a detailed description of the mathematics content standards that are included in the course. Throughout, there are examples that illustrate the mathematical ideas and connect the MP standards to the content standards. In particular, standards that are expected to be new to existing secondary teachers are explained more fully than standards that have appeared in the curriculum prior to the adoption of the CA CCSSM.

It is important to note that some CA CCSSM standards are broad in scope and, as a result, are included in more than one course. When this occurs, a parenthetical comment is included with the standard to clarify the intent of the standard for that course. For example, the following standard appears in both Algebra I and Algebra II and has a different parenthetical comment for each course:

Algebra I

Arithmetic with Polynomials and Rational Expressions **A-APR**

Perform arithmetic operations on polynomials. [Linear and quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Algebra II

Arithmetic with Polynomials and Rational Expressions **A-APR**

Perform arithmetic operations on polynomials. [Beyond quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

6. California's CTE model curriculum standards are viewable at http://www.cde.ca.gov/ci/ct/sf/ctemcstandards.asp (accessed April 9, 2014).

7. For additional information, go to http://www.ucop.edu/agguide/career-technical-education/index.html (accessed April 9, 2014).

In Algebra I, the notation specifies that the standard applies to linear and quadratic expressions. In Algebra II, the notation specifies that the standard applies to all expressions beyond quadratic.

California's new mathematics framework is a vital document that teachers will reference often; it is not a publication offering "business as usual." This framework embodies the belief that all students can learn mathematics and contains essential information for teachers and other stakeholders about universal access to the curriculum, teaching strategies, assessment, technology, modeling, and instructional materials. The framework also provides school and district administrators with information about how to support high-quality instruction. It is important for teachers of a single grade level to read not only their respective grade-level or course chapter in the framework, but also the grade level or chapter immediately preceding *and* following their particular area of focus. This will help teachers to plan a coherent, focused, and rigorous course of study.

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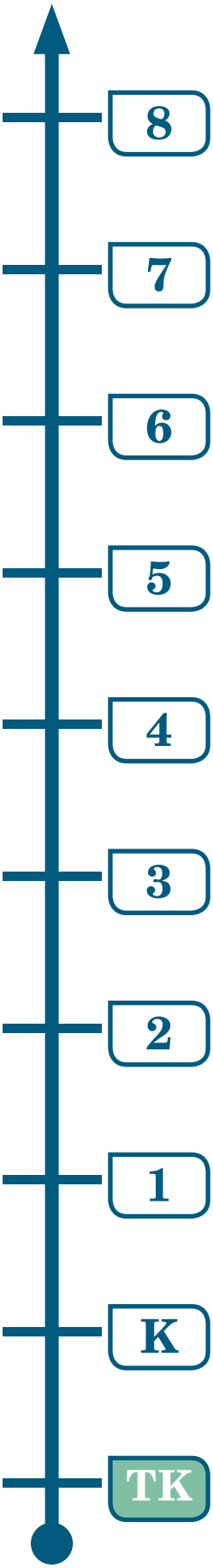


Transitional Kindergarten

The Kindergarten Readiness Act of 2010 (Senate Bill 1381, Chapter 705, Statutes of 2010) changed the entry-age requirements for kindergarten in California's public schools.[1] It also required local educational agencies to offer transitional kindergarten (TK) classes in addition to traditional kindergarten classes starting in the 2012–13 school year. Transitional kindergarten is defined in California *Education Code* section 48000(d) as "the first year of a two-year kindergarten program that uses a modified kindergarten curriculum that is age and developmentally appropriate." Traditional kindergarten, by contrast, is a one-year program with grade-specific curriculum.

The Kindergarten Readiness Act requires districts to provide students in TK programs with instruction in a modified kindergarten curriculum that is age and developmentally appropriate, but it does not specify what that curriculum should be. Districts must determine what "age and developmentally appropriate" means in terms of curriculum. The law also defines transitional kindergarten as the first year of a two-year kindergarten program, so preschool programs and TK programs within a district should be distinct. To determine the

1. California *Education Code* section 48000(a) specifies that a child shall be admitted to kindergarten at the beginning of the school year if the child will have his or her fifth birthday on or before the following dates: November 1 for the 2012–13 school year, October 1 for the 2013–14 school year, and September 1 for the 2014–15 school year.

type of modified curriculum to implement, each district needs to consider how transitional kindergarten fits into its early education system and how TK education differs from instruction in preschool and traditional kindergarten classes. Instructional leadership at both the district and school levels is necessary to ensure that TK programs meet the instructional and developmental needs of young learners. Ideally, teachers and other professionals who know mathematics content and are well versed in child development theories will develop the TK curriculum. If transitional kindergarten will truly serve as a bridge between preschool and traditional kindergarten, coordination and articulation between preschool programs (in the district and the community) and traditional kindergarten classes must occur.

Student Learning in Transitional Kindergarten

Unlike preschool or kindergarten, transitional kindergarten does not have grade-specific content standards. Therefore, the guidelines in this chapter reflect the range of abilities that students may possess in the period between preschool and kindergarten, but they are not specific to a grade-level standard. Each domain section in the chapter includes particular California Preschool Learning Foundations (for children at age 60 months) and corresponding kindergarten standards from the California Common Core State Standards for Mathematics (CA CCSSM). Sample activities that illustrate connections between the foundations and the standards are provided.

To build the foundation for success in traditional kindergarten and beyond, instructional time for mathematics in transitional kindergarten should focus on two critical areas: (1) representing, relating, and operating on whole numbers; and (2) geometry, with a focus on identifying and describing shapes and space, as well as analyzing, comparing, and composing shapes (California County Superintendents Educational Services Association [CCSESA] 2011b). To help students gain a deeper understanding of mathematics, the Standards for Mathematical Practice should be connected to content instruction.

> **"Instructional time should focus on two critical mathematical areas. One area is representing, relating, and operating on whole numbers . . . The second important area is geometry with a focus on identifying and describing shapes and space; and analyzing, comparing, creating, and comparing shapes. These two areas are intricate and complex and build the foundation for future learning in mathematics. While both prepare the young learner for more formal mathematics instruction, learning time should be devoted to number sense more than any other topic in mathematics."**
>
> **—CCSESA 2011b, 26**

It is important to remember that all students need instruction that is appropriate for their developmental level and provides opportunities for growth. In the classroom, this means that if a student is struggling with some of the California Preschool Learning Foundations, he or she should be provided with opportunities to develop abilities in those areas. Similarly, if a student meets some or all of the kindergarten standards, that student should be provided with learning opportunities that extend beyond the standards.

In transitional kindergarten, developmentally appropriate instruction involves hands-on activities for students and learning experiences in small- to medium-size groups. Particularly important are opportunities to support mathematical vocabulary acquisition in teacher–student and student–student interactions. Questions of all kinds support mathematical thinking and problem solving, especially open-ended and more challenging questions. Although the CA CCSSM do not emphasize calendar-time activities, these activities may be valuable for students' social and academic development if they support important goals such as developing social skills (group participation, taking turns, and cooperation), language skills, understanding of sequence, and number concepts.

High-quality support of mathematics education includes these important factors:

- A mathematically rich environment
- Frequent opportunities for mathematical discourse
- Engaging and meaningful mathematics activities
- Explicit instruction
- Modeling of mathematical thinking
- Nurturing of students' mathematical explorations

A *mathematically rich environment* includes a mathematics center that is refreshed on a regular basis; posters (e.g., showing shapes and numbers with sets) or wall sections devoted to interesting mathematics problems (*Are there more ducks or geese? Fewer brown birds or gray birds?*); a variety of manipulatives (teddy bears, snap-together cubes, dinosaurs, vehicles, and the like); unit blocks; shopping paraphernalia (money, cash register, labels for prices, grocery store items); two- and three-dimensional shapes; attribute blocks; and so forth.

Frequent opportunities for mathematical discourse and "math talk" build mathematical understanding and vocabulary. *Mathematical discourse* requires thinking on one's feet, knowing mathematics vocabulary, and having definitions of mathematical terms and concepts that make sense to students. The following examples of mathematical discourse are rich with mathematical vocabulary (e.g., *triangle, sides, corners, count, more, bigger, longer,* and number words for 1–7).

Example 1

Student Andrew: "Is this a triangle?" (*Holds up a square.*)

Teacher: "What do you think, children?" (*Asks other children in the small group to contribute.*)

Students, in unison: "No!"

Teacher: "Why not?"

Student Zahra: "Because a triangle doesn't have four sides."

Teacher: "That's right. How many sides does a triangle have?"

Student Alexander: "Three!"

Teacher: "How many corners does a triangle have?"

Student Alexander: "Three, just like the sides!"

Example 2

Student Nora: "Sami isn't being fair. He has more trains than I do."

Teacher: "How do you know?"

Student Nora: "His pile looks bigger!"

Student Sami: "I don't have more!"

Teacher: "How can we figure out if one of you has more?"

Student Nora: "We could count them."

Teacher: "Okay, let's have both of you count your trains."

Student Sami: "One, two, three, four, five, six, seven."

Student Nora: "One, two, three, four, five, six, seven." (*Fails to tag and count one of her eight trains.*)

Student Sami: "She skipped one! That's not fair!"

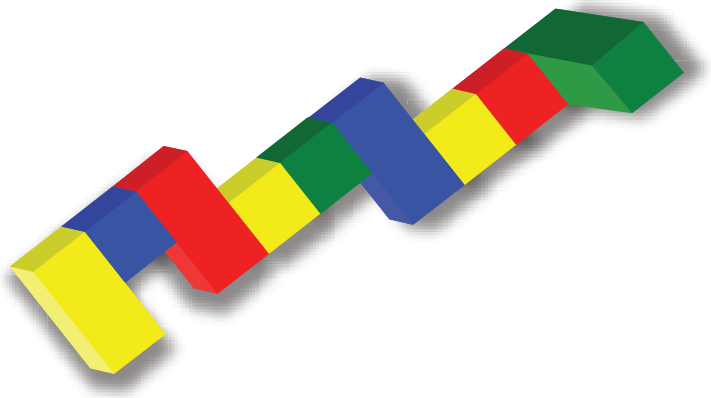
Teacher: "You are right; she did skip one. We could count again and be very careful to make sure not to skip—but can you think of another way that we can figure out if one of you has more?"

Student Sami: "We could line them up against each other and see who has a longer train."

Teacher: "Okay, show me how you do that. Sami, you line up your trains, and Nora, you line up your trains."

Engaging and meaningful mathematics activities are those that encourage students to think mathematically about the world around them. These frequently require careful planning. For a student who is interested in dinosaurs, helping him or her make a t-chart of herbivores and carnivores (using pictures or toy versions of the dinosaurs or writing the names of the dinosaurs from a book) and then having the student count the number of dinosaurs in each category may be a highly engaging activity. Some students enjoy the challenge of recreating structures with building blocks that connect or snap together or with magnetic builders (see figure TK-1). Create a structure with one of these sets, and ask a student to recreate it, including exact shapes, colors, and positions. Then ask a student to create a set that you or other students duplicate—tell the student to make it as difficult as possible. Then have them analyze whether you and the other students recreated it correctly.

Figure TK-1. A Structure Made of Building Blocks



Explicit instruction is vital in transitional kindergarten. It allows teachers to support students' acquisition of concepts that may not come up in play or other classroom activities. Explicit instruction does not mean *didactic;* rather, it means purposefully providing activities that support the understanding of a mathematical concept. The previous example of the dinosaur sorting activity is explicit (and yet highly engaging to a dinosaur fan!). Other examples of explicit instruction include dividing a set of toy trucks into three equal (and fair) shares and measuring how many children, lying end to end on the floor, it would take to equal the length of a whale shark. Then the teacher could ask the students to figure out how many mice it would take to equal the length of the shark: *Does it take more mice or more children?* All of these activities are purposeful, explicit, and contain important mathematical concepts.

Modeling of mathematical thinking provides students with strategies, techniques, and a path to deeper and more flexible understanding of mathematical concepts. There are many ways to sort students into groups—by color of clothing, laced shoes versus non-laced shoes, counting off, or by using the first or last letter of students' names. Visually and verbally modeling these sorting techniques helps students understand that there is more than one way to solve a problem. Teachers encounter mathematics problems throughout the day. Pencils are needed at each table (*How many at each table? What is the total number of pencils needed?*). More milk cartons are needed from the cafeteria (*How many more?*). Other questions arise: *How many minutes before lunch time? How many cotton balls are needed for this activity?* Solving these and other problems out loud with students allows students to see the usefulness of mathematics in real-world situations. In addition, visual aids such as a list of numbers with dots in 5-patterns above them may support analysis and learning of number words and quantities.

Finally, *nurturing of students' mathematical explorations* may create a classroom atmosphere where students believe they can solve problems and learn fun new concepts. Discovering repeating numbers in a hundreds chart is eye-opening for a young student. It can also be magical for a student to realize that 1 plus any whole number equals the next number in the counting sequence. Activities like these nurture students' interest and encourage future mathematical investigations.

Creating a learning environment that supports foundational mathematics is critical for the acquisition of later, more complex mathematical knowledge and skills. Research shows that early mathematics skills at entry to kindergarten are predictive of later academic success in both reading and mathematics (Duncan et al. 2007). Transitional kindergarten provides an excellent opportunity to continue building on students' mathematical understandings. Because children arrive at school with varied mathematical experiences, differentiated instruction is an essential part of classroom teaching; see the Universal

Access chapter for more information. Understanding each student's development and fine-tuning instruction to meet each student's needs are critical to providing quality education for all students. This is also true for English learners and students with disabilities. Although whole-group activities may be useful for introducing a concept or playing a game, smaller groups or one-on-one interactions are necessary for student acquisition of in-depth knowledge of concepts and teachers' understanding of each student's mathematical thinking, knowledge, and skills.

References to the Standards for Mathematical Practice (MP) are woven throughout the activity examples in this chapter. The MP standards describe how mathematically proficient students engage in mathematics and suggest behaviors to nurture in students. The MP standards are appropriate for transitional kindergarten students and should be integrated throughout instruction. Examples of these practices that are specific to transitional kindergarten are provided in table TK-3 at the end of the chapter.

One approach to developing a modified curriculum that is age and developmentally appropriate is to consider the intersections between the California Preschool Learning Foundations and the CA CCSSM for kindergarten (CCSESA 2011b). Transitional kindergarten may be thought of as an opportunity to introduce students to some of the kindergarten standards rather than expecting students to strive for mastery of those standards. Especially at the beginning of a TK program, a modified curriculum could provide more hands-on activities, more learning through play and exploration, and more time to develop students' mathematical skills and conceptual understandings in core lessons about smaller numbers. It should focus on developing skills and habits of mind that lead to success in traditional kindergarten, including problem solving, persistence, and reasoning.

In 2012, the California Department of Education published a document—*The Alignment of the California Preschool Learning Foundations with Key Early Education Resources* (http://www.cde.ca.gov/sp/cd/re/documents/psalignment.pdf) [CDE 2012a]—that connects the California Infant/Toddler Learning and Development Foundations, Head Start Child Development and Early Learning Framework, California Preschool Learning Foundations, and CA CCSSM. Table TK-1 shows the connection between the California Preschool Learning Foundations and the Standards for Mathematical Practice. Similarly, table TK-2 shows the alignment between the California Preschool Learning Foundations and the kindergarten CA CCSSM.

Table TK-1. Alignment Between the California Preschool Learning Foundations and the Standards for Mathematical Practice

California Preschool Learning Foundations	Standards for Mathematical Practice for Grades K–12 (MP)
Mathematical Reasoning Children use mathematical thinking to solve problems in their everyday environment.	**MP.1** Make sense of problems and persevere in solving them. **MP.2** Reason abstractly and quantitatively. **MP.3** Construct viable arguments and critique the reasoning of others. **MP.4** Model with mathematics. **MP.5** Use appropriate tools strategically. **MP.6** Attend to precision. **MP.7** Look for and make use of structure. **MP.8** Look for and express regularity in repeated reasoning.

Table TK-2. Alignment Between the California Preschool Learning Foundations and the California Common Core State Standards for Mathematics (Kindergarten)

California Preschool Learning Foundations	California Common Core State Standards—Kindergarten
Mathematics	**Mathematics**
Number Sense	**Counting and Cardinality**
Children understand numbers and quantities in their everyday environment.	Know number names and the count sequence Count to tell the number of objects Compare numbers
Children understand number relationships and operations in their everyday environment.	**Operations and Algebraic Thinking** Understand addition as putting together and adding to, and subtraction as taking apart and taking from **Number and Operations in Base Ten** Work with numbers 11–19 to gain foundations for place value
Algebra and Functions (Classification and Patterning)	**Measurement and Data**
Children sort and classify objects in their everyday environment. Children recognize/expand understanding of simple repeating patterns.	Classify objects and count the number of objects in categories
Measurement	**Measurement and Data**
Children compare, order, and measure ojects.	Describe and compare measurable attributes
Geometry	**Geometry**
Children identify and use shapes.	Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres). Analyze, compare, create, and compose shapes.
Children understand positions in space.	Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

Integration of Domains

The following tables integrate the California Preschool Learning Foundations for children at around 60 months of age and the corresponding kindergarten domains from the CA CCSSM. The tables are provided to facilitate district-level discussions on the development of a modified curriculum for mathematics instruction in transitional kindergarten that is age and developmentally appropriate. Each table includes these elements:

- California Preschool Learning Foundations and corresponding CA CCSSM kindergarten standards
- Vocabulary—a list of vocabulary words that students should acquire as they expand their understanding of the concepts
- What it looks like—examples of what understanding the concepts might look or sound like in the classroom (in ascending order of complexity)
- Big ideas—some of the main ideas involved in grasping the concepts related to the standard(s)
- Instructional issues—misconceptions or common conceptual difficulties that students might have
- Activities—classroom exercises that support the acquisition of the abilities embodied in the California Preschool Learning Foundations and CA CCSSM (in ascending order of complexity)

California Preschool Learning Foundations (at around 60 months of age) *Number Sense*	CA CCSSM – Kindergarten *Counting and Cardinality (CC)*
Children expand their understanding of numbers and quantities in their everyday environment. **PLF.NS–1.1** Recite numbers in order to 20 with increasing accuracy.[2]	**Know number names and the count sequence.** **K.CC.1** Count to 100 by ones and by tens. **K.CC.2** Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
Vocabulary: Number words (e.g., *one, two, three,* and so on, from 1 to 100), *count, count by, count from, number, next number, How did you figure that out?*	
What it looks like: • While playing hide-and-seek, Ezra counts to 20 before looking for the other children. • When asked to count as high as she can, Melia counts to 50. • When asked how old he is, Kenji answers, "I'm five, and then I'll be six, seven, eight, nine, ten! (**MP.7, MP.8**)	

2. In the *California Preschool Learning Foundations, Volume 1* (CDE 2008), this foundation is listed only as "1.1" in the Number Sense strand. A naming pattern was created for the mathematics framework to make clearer comparisons between the preschool foundations and California's Common Core State Standards. For **PLF.NS–1.1,** *PLF* stands for Preschool Learning Foundations, *NS* stands for Number Sense, and *1.1* is the specific foundation referenced. Also note that numerals, not the spelled-out number names that appear in the published PLF document, are used throughout the mathematics framework in places where preschool foundations are listed.

Big ideas: Students learn to recite numbers before they can apply one-to-one concepts to counting objects or understand cardinality (i.e., the last number counted represents the numerosity of the set). Encourage students to slow down as they count. After students have had experiences counting from 1, have them start counting in the middle of the counting sequence to encourage conceptual understanding of the order of numbers.

Instructional issues: An important goal in early mathematics instruction is for students to achieve fluency with the counting sequence. Students may learn a short sequence of numbers (*"four-five-six"*) and not understand that they are separate numbers (similar to the *"l-m-n-o-p"* issue when learning the alphabet). Numbers 11 through 15 may be difficult for students to learn because these numbers do not follow the pattern of 16 through 19 (the number followed by *teen*). Use discussions about how these numbers are kind of funny—calling attention to the irregularity of these number names may make it easier for students to remember that the names do not follow the regular naming pattern.

Note: Saying the counting numbers is sometimes referred to as verbal or rote counting and does not indicate an understanding of object counting with one-to-one correspondence.

Activities: Transition times are useful for providing opportunities to learn the counting numbers. Students may count how long it takes to clean up the blocks, sit in a circle, and so on. These are not precise measures of time; rather, they provide students with chances to exercise their newfound rote counting abilities. (**MP.4**)

Using a puppet named George, tell the students a story about how George has difficulty remembering how to count. Tell them that you want them to help George figure out when his counting is incorrect. Ask the students to raise their hands when they hear George make a mistake and to remember George's counting mistake. In George's voice, count to 10, skipping or repeating one number in the sequence. Call on the students who raise their hands to describe George's mistake. Ask questions to make sure students thoroughly describe the mistake and how George can fix it. (**MP.2, MP.3, MP.4, MP.6**)

During whole-group time, ask the students to sit in a circle and tell them that they are going to play a counting game. Tell them that this is a fancy game of counting called "Everybody Gets a Number." Choose a child to start the counting sequence. That child says "One" aloud; the child sitting to his or her left (going clockwise around the circle) says the next number, and the counting continues with each child in the circle. When a child says an incorrect number or does not know the next number, ask for the child to his or her right to help out. If that child does not know the number, ask the child to his or her right (keep asking the next student to the right until you find a child who can help). As students advance in knowledge, increase the difficulty of the game by asking the students to count faster, make the number goal higher, start with a number other than 1, or count by tens. (**MP.1, MP.4, MP.7**)

California Preschool Learning Foundations (at around 60 months of age) *Number Sense*	CA CCSSM – Kindergarten *Counting and Cardinality (CC)*
Children expand their understanding of numbers and quantities in their everyday environment. **PLF.NS–1.2** Recognize and know the names of some written numerals.	**Know number names and the count sequence.** **K.CC.3** Write numbers from 0 to 20. Represent a number of objects with a written numeral 0–20 (with 0 representing a count of no objects).
Vocabulary: *Zero*	
What it looks like: • Thomas sees the numeral 4 on the wall and says, "I'm that number!" • Zeke paints the numeral 5 several times at the easel. • Using a puzzle that involves matching numbers with objects, Susan correctly matches the numerals 6 through 10 with pictures of sets of animals that number 6 through 10. (**MP.2, MP.4, MP.6**) • After drawing a pumpkin with four teeth, Maria draws a pumpkin with no teeth, laughs, and then says, "Look, *zero* teeth!" (**MP.2, MP.4**)	
Big ideas: Numerals (written or printed numbers) can describe the numerosity of a set of objects. Zero represents an empty set (in other words, no objects to count).	
Instructional issues: Students learn to count with smaller sets before they learn to count larger sets. Students may draw numbers backwards or confuse numbers that look similar to each other (e.g., 6 and 9). The concept of *zero* is difficult for students to understand and may require many examples and experiences.	
Activities: Ask students to go on a Number Hunt around the classroom. The game may be played in a variety of ways. Students could look for any numeral and then name it when called upon. Alternatively, the teacher might ask students to look for particular numerals. Number cards (one for each student in the classroom; some numbers may appear on more than one card) may be hidden around the room, and then the teacher can ask each student to find a number card and name the number on his or her card when called upon. Students create their own number cards (with the numerals 0 through 10), decorating them as they wish (using construction paper, index cards, card stock, plain white paper, and the like). The teacher then asks the students to put the cards in order and varies the activity by having students trade sets and put their new cards in order. (**MP.2, MP.4, MP.6**) Give students a number card (or let them choose) and ask them to find the same number of objects in their environment. Students bring their card and objects back to a central location (perhaps a rug or table) and share their findings with each other. With multiple students and different number cards, they can order their number cards and objects. To help students understand the concept of *zero,* hold up a number card with the numeral 0 and encourage the students to discuss how many objects they could match with the card. If appropriate, discuss with students how they know their numeral card matches the set they have displayed. (**MP.2, MP.3, MP.4, MP.6**)	

California Preschool Learning Foundations (at around 60 months of age) *Number Sense*	CA CCSSM – Kindergarten *Counting and Cardinality (CC)*
Children expand their understanding of numbers and quantities in their everyday environment. **PLF.NS–1.3** Identify, without counting, the number of objects in a collection of up to four objects (i.e., subitize).* **PLF.NS–1.4** Count up to 10 objects, using one-to-one correspondence (one object for each number word) with increasing accuracy. **PLF.NS–1.5** Understand, when counting, that the number name of the last object counted represents the total number of objects in the group (i.e., cardinality). **The Alignment of the California Preschool Learning Foundations with Key Early Education Resources* (CDE 2012a) places this foundation in a separate category that is not aligned with the CA CCSSM. It is retained here to show the connection between naming the numerosity of a set attained through subitizing and learning to count a set.	**Count to tell the number of objects.** **K.CC.4** Understand the relationship between numbers and quantities; connect counting to cardinality. a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object. b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted. c. Understand that each successive number name refers to a quantity that is one larger. **K.CC.5** Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1 to 20, count out that many objects.
Vocabulary: *How many, one more, all together, in all, total*	
What it looks like: • Nathan glances at the number cube on the table and says, "Look! I got three!" • DeSean lines up his eight toy cars and, touching each one, counts accurately 1 through 8. (**MP.6**) • The teacher asks Talia to count how many students are in the group. Talia counts six students and then announces, "There are six." (**MP.8**) • When one more student joins the group, the teacher asks, "Now how many are in the group?" Talia answers, "That's easy—one more, that's seven!" (**MP.2, MP.8**) • Diamond is passing out pencils at each table; she accurately places six pencils on each of the four tables. (**MP.2, MP.4, MP.6, MP.8**)	
Big ideas: Students at this age can subitize (immediately, and without counting, perceive a quantity) up to about four objects. This may increase to six when the objects are in a stereotypical arrangement (e.g., six pips on a domino). Cardinality refers to the ability to determine the numerosity of a set. Students initially count each item in a set, but when asked "How many?", they will count the set again. When students gain an understanding of cardinality, they will answer with the last number that they counted instead of counting the set again and know that the last number tells how many there are.	

Instructional issues: It may take a while for students to construct strategies to keep track of what has been counted in a set. Two of these strategies are touching each object in a row until the end has been reached and moving aside the objects already counted. Counting an existing set is easier for students than creating a smaller set from a larger set (e.g., taking exactly six teddy bears from a large container of many bears) because they have to remember the number to which they are counting while counting. When beginning activities that require a particular number of objects (e.g., five cards), teachers can encourage conceptual understanding of cardinality by having the students count out their cards from the larger set of cards instead of doing it for them.

Activities: Create a Number Wall with a "Number of the Week" where students display sets of pictures (magazine, drawings, stickers, and so on) that are equal in number (e.g., six magazine pictures of trees). Arrange pictures so that they are in groups that can be subitized. Each week, change the "Number of the Week." (**MP.2, MP.4, MP.6**)

An activity that encourages both numeral recognition and object counting can be created using a numeral strip (piece of paper containing a row of boxes with numerals printed in them, beginning with the numeral 1), number cubes or a spinner, and counters. Students take turns rolling the number cubes (or spinning) and then count out that many counters (teddy bears, cars, and the like) to show they are correct; then they cross out that numeral on their numeral strip. The game ends when all students have all numerals crossed out on their numeral strips. Note that numeral strips should have all possible number cubes or spinner numerals in order. (**MP.1, MP.2, MP.4, MP.6, MP.7**)

Play board games that require students to count spaces; such games usually come with number cubes or spinners. Note that the use of a number line is not formally introduced in the CA CCSSM until grade two. All number lists or number paths should have the numbers within a shape (usually squares) that may or may not be connected to adjacent shapes. (**MP.2, MP.4, MP.6, MP.7**)

California Preschool Learning Foundations (at around 60 months of age) *Number Sense*	CA CCSSM – Kindergarten *Counting and Cardinality (CC)*
Children expand their understanding of number relationships and operations in their everyday environment. **PLF.NS–2.1** Compare, by counting or matching, two groups of up to five objects and communicate "more," "same as," or "fewer" (or "less").	**Compare numbers.** **K.CC.6** Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies. **K.CC.7** Compare two numbers between 1 and 10 represented as written numerals.
Vocabulary: *More, fewer, less, same as, greater than, less than, more than*	

What it looks like:

- Jasmin and Lucas are playing in the block area, trying to divide the long blocks equally between themselves. In lining them up in one-to-one correspondence, Lucas says, "You have five and I only have four. You have more than I do!" (**MP.2, MP.3, MP.5, MP.6**)
- John and Jamal are playing with trains. After counting the trains several times, Jamal says, "I have eight trains, and you only have six! That's not fair. If I give you one of my trains, then we'll both have seven." (**MP.2, MP.3, MP.6**)
- Angel and Lisa are looking at graphs showing numbers of students who like various fruits. Angel remarks, "Look, more like apples than oranges. See? Apples has an 8 and oranges only has a 2." (**MP.2, MP.3, MP.4, MP.6, MP.8**)

Big ideas: Students may initially compare sets perceptually ("That one has more!") or by lining them up in one-to-one correspondence, or they may count both sets to compare quantitatively ("One, two, three. One, two, three—we both have three!"). Moving from comparing two sets of objects to comparing numerical symbols may be difficult for students. Encourage conceptual understanding by offering many opportunities to use numerals with matching sets of objects.

Instructional issues: Fair sharing in the classroom can play a big part in providing opportunities for students to compare quantities (everyone wants a fair share!). Some students may be confused by the length or size of a set of objects when comparing it to another set. Lining up objects in one-to-one correspondence may help students ascertain whether one set is larger than another. Students may struggle to understand that the numerosity of a set does not change if nothing is added or taken away. Using the question "Is this group really more, or does it just look like more?" may be helpful. Teachers should use the words *fewer* and *less* more frequently than the word *more,* because children typically have fewer opportunities to learn the words *fewer* and *less*.

Activities: Card game of Compare (comparing numerals or sets of icons on cards). Each student receives a set of cards with numerals or sets of objects on them (within 5). Working with a partner, each student flips over one card (like the card game "War"). The students decide which card represents *more* or *fewer,* or if the cards are *the same as.* (**MP.2**)

Play a game in which you create a set of counters (1–9). Count the counters with a small group of students. Then either add one more or take away one counter from the set. Then ask the students to figure out how many there are in the set. Involve the students in a discussion of how they can figure out the answer (there are several ways) and how they know the answer; involve the entire group in the discussion, making sure that all students participate. To support students' understanding, display a number list or path in the classroom—at the students' eye level—showing the numerals in order from 1 (these are the counting numbers) with dots in groups of 5 above them. (**MP.1, MP.2, MP.3, MP.4, MP.8**)

In a small group, play a game with counters (teddy bears, cars, and the like) with the students. Create a set and ask the students to create their own sets of the same number. Work with 1 to 10 counters. To increase the complexity of the game, ask the students to create a set that is one more or one less than your set, or have them create a set that has more (or fewer) objects than yours. Have a discussion with the students, asking each child to describe her or his set—whether it is larger or smaller than yours and how they know. If students do not use the phrase "the extra" during the discussion, pointing to the objects that are "the extra" may be helpful. The group with more has extra objects. (**MP.1, MP.2, MP.3, MP.4, MP.7, MP.8**)

California Preschool Learning Foundations (at around 60 months of age) *Number Sense*	CA CCSSM – Kindergarten *Operations and Algebraic Thinking (OA)*
Children expand their understanding of number relationships and operations in their everyday environment. **PLF.NS–2.2** Understand that adding one or taking away one changes the number in a small group of objects by exactly one. **PLF.NS–2.3** Understand that putting two groups of objects together will make a bigger group and that a group of objects can be taken apart into smaller groups. **PLF.NS–2.4** Solve simple addition and subtraction problems with a small number of objects (sums up to 10), usually by counting.	**Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.** **K.OA.1** Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations. **K.OA.2** Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem. **K.OA.3** Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1). **K.OA.4** For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation. **K.OA.5** Fluently add and subtract within 5.
Vocabulary: *Bigger, smaller, add, subtract, take away, addition, subtraction, adding, subtracting, make 10, all together, equals, the same as, in all, total, amount left*	
What it looks like: • Tony announces, "Look, if we put your blocks with my blocks, we have a bigger pile! We have more." (This is an example of an Add To/Result Unknown situation. See table GL-4 in the glossary.) (**MP.8**) • Miriam says, "I have three cows and two pigs. That makes one, two, three, four, five. Five animals!" (This is an example of a Put Together/Total Unknown addition situation. See table GL-4 in the glossary.) (**MP.2, MP.4, MP.6**) • While playing in the block area, José says to Antonio, "If we put your cylinders with my cylinders, we'll have, one, two three, four, five, six cylinders—enough for the factory smokestacks!" (This is another example of a Put Together/Total Unknown addition situation.) (**MP.2, MP.4**) • Oscar says, "There are five cars, but two are broken, so we can only use three of them." (This is an example of a Take From/Results Unknown subtraction situation. See table GL-4 in the glossary.) (**MP.2, MP.4**)	

Big ideas: Most young students use a counting-all strategy to solve addition problems with objects. That is, they count all of the objects in both sets. However, some students will go on to learn the more advanced grade-one strategy of counting on from the larger set (e.g., when adding four and two objects, they begin with "Four" and continue, "five, six"). Provide students with opportunities to take apart groups of objects and examine how many they started with, how many were taken away, and how many are left. Table GL-4 in the glossary illustrates the variety of addition and subtraction situations and difficulty level. Students in transitional kindergarten may work with problems involving *Add To/Take From with Result Unknown, Put Together/Take Apart with Total Unknown, and Both Addends Unknown.*

Instructional issues: Students can directly model addition and subtraction situations given by the teacher or taken from their own lives. Provide frequent opportunities to engage in addition and subtraction activities involving story situations; students should learn to tell such stories and not just solve them. Initially working within addends less than 5 encourages in-depth understanding of addition and subtraction concepts. Encourage problem solving through the use of fingers, drawings, and manipulatives. When introducing the equal sign (=), emphasize and illustrate that the symbol means equal (not "the answer is"). Stress that the *quantities* represented on the left and right sides of this symbol must be the same (they can be objects, numerals, or expressions). Use the equation form $5 = 3 + 2$ when taking apart a number to show both addends. Using the word *partners* for addends helps students to conceptualize these numbers as hiding inside a number.

Activities: While reading books, ask questions about numbers. For instance, in a book about dogs, on the page showing a picture of two dogs, ask how many dogs there are, and then ask questions such as these: *How many legs does one dog have? How many legs do two dogs have? If one dog left the page, how many legs would be left?* (**MP.1, MP.2, MP.4, MP.6, MP.7**)

During small-group or whole-group time, have students represent with their fingers the addends in a story problem. Call on individual students to explain how they decided how many fingers to choose for each hand. *Example:* "One day, two baby dinosaurs hatched out of their eggs. The mama triceratops was so excited that she called to her auntie to come and see. Then four more baby dinosaurs hatched! How many dinosaurs hatched all together? Mirasol, can you show me how many fingers you used?" Note that children from different cultures learn to show numbers on their fingers in different ways. Children may start with the thumb, the little finger, or the pointing finger. Support all of these ways of showing numbers with fingers. (**MP.2, MP.4, MP.5**)

Present students with story problems and encourage the students to solve the problems with manipulatives or drawings. Initially, talking about how one can represent the problem on paper or with manipulatives might be useful. *Example:* "Four cars are waiting to be repaired at the repair shop. These blocks will be the cars [puts four blocks in front of the students]. Paul said that his garage has only two car lifts. If we put these two cars up on the lifts [moves the blocks away from the group of four], how many cars are waiting for their turn on the lifts?" (**MP.2, MP.4, MP.5**)

California Preschool Learning Foundations (at around 60 months of age) *Measurement*	CA CCSSM – Kindergarten *Measurement and Data (MD)*
Children expand their understanding of comparing, ordering, and measuring objects. **PLF.M–1.1** Compare two objects by length, weight, or capacity directly (e.g., putting objects side by side) or indirectly (e.g., using a third object). **PLF.M–1.2** Order four or more objects by size. **PLF.M–1.3** Measure length using multiple duplicates of the same-size concrete units laid end to end.	**Describe and compare measurable attributes.** **K.MD.1** Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object. **K.MD.2** Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*
Vocabulary: *Longest, shortest, largest, smallest, heaviest, lightest, highest, lowest, most, least, more than, less than, same as, shorter than, longer than, larger than, smaller than, heavier than, lighter than*	
What it looks like: • Jake lines up the four twigs he found on the playground in order of length. (**MP.4, MP.7**) • Tyrone pulls his train up beside Malik's with the engines lined up and says, "My train is longer than your train." (**MP.4, MP.6, MP.7**) • Dylan and Kiara are comparing the pumpkins they have drawn and cut out. Kiara puts Dylan's pumpkin on top of hers and says, "My pumpkin is bigger than yours." Kiara then measures her pumpkin's mouth with her index finger and compares it to the mouth on Dylan's pumpkin. She says, "Your pumpkin has a longer mouth than mine!" (**MP.4, MP.5**)	
Big ideas: For students in later grades, measuring usually means assigning a numerical quantity to an object (e.g., 4 pounds or 6 inches). Generally, this is referred to as *formal measurement.* However, for younger students, directly comparing these attributes *(informal measurement)* forms an important foundation for later understanding. Using duplicates of concrete objects that are the same size prepares students for thinking about repeating units (such as inches on a ruler or measuring tape). Emphasizing that these units are all the same size is an important concept that provides scaffolding for later formal measurement.	
Instructional issues: Students may not understand that in order to compare the length or height of objects, all of the objects must have the same starting point (e.g., in measuring the height of four objects, they are all placed upright on a table). Help students develop this ability. For vertical measurements, use the table or floor as the starting point. For horizontal measurement, mark the starting point with tape (by drawing a line) or with a straight stick. When students are measuring with non-standard units of measure, encourage them to use the same unit to measure each item.	

Activities: Make balance scales available in your classroom. Encourage students to use them to compare the weight of objects. Before students weigh two different objects, ask them to estimate which object is heavier and then check their guesses after the items are weighed. (**MP.5, MP.6**)

Divide the class into four to six groups of students. Choose items from your classroom with measurable attributes—for example, pencils, dolls, or trucks for length; pumpkins, balls, or beanbags for weight. Provide each group of students with one of these items. After distributing, ask each group to find objects that are shorter/longer or heavier/lighter than the item you have given them (one item per student). Give the groups about five to 10 minutes to complete this task. Then ask each group to make two piles of objects they have collected—one that consists of longer/heavier objects than the initial object you have provided and one that is shorter/lighter. The goal is for all students in the group to agree on which pile each item belongs in. Then have all students in the classroom listen as each group reports to the whole group on their decisions. In order to keep all students engaged, tell them to listen carefully to the decisions made by other groups, to be prepared to say whether they agree or not, and to explain how they might check the accuracy of the decisions. (**MP.3, MP.4, MP.8**)

Young students are fascinated by large creatures. Find a children's book about dinosaurs or elephants that talks about the size of these animals. Create an activity in which the dinosaur or elephant is drawn to scale on the playground. Ask the students how they could measure the animal with their bodies ("How many children tall is a dinosaur? How many hands tall?"). Have a discussion about whether all children are the same height or not, and ask if it would it make a difference if you measured a dinosaur with different-sized children ("Would it take fewer third-grade students laid end to end than it would if transitional kindergarten students were laid end to end?"). Be sure to include discussion of the ability or inability to measure exactly the number of children (for instance, if it takes four children and a part of a child). This is not to teach fractions, but to highlight the unit of measurement and underscore the importance of using a standardized unit (e.g., "four Sams high" is different from "four Susies high"). (**MP.2, MP.3, MP.4, MP.8**)

California Preschool Learning Foundations (at around 60 months of age) *Algebra and Functions*	CA CCSSM – Kindergarten *Measurement and Data (MD)*
Children expand their understanding of sorting and classifying objects in their everyday environment. **PLF.AF–1.1** Sort and classify objects by *one or more* attributes, into two or more groups, with increasing accuracy (e.g., may sort first by one attribute and then by another attribute).	**Classify objects and count the number of objects in each category.** **K.MD.3** Classify objects into given categories; count the numbers of objects in each category and sort the categories by count.
Vocabulary: *Sort, group, same, different*	
What it looks like: • Jane is playing with the teddy-bear counters. The counters come in three sizes and four colors. First she divides them by color into four groups. Then she says, "Now I'm gonna put all the Daddies together and the Mamas together and the babies together." She sorts all of the bears into these three sizes. (**MP.1, MP.3, MP.4, MP.6, MP.8**) • Garrett is sorting buttons. First he sorts by color, then by size. (**MP.4, MP.6, MP.8**) • Demetrius is sorting the trains into engines, coal carriers, and flat train cars. He announces, "I have more flat cars than coal carriers, and I only have two engines." (**MP.2, MP.4, MP.6**)	
Big ideas: Objects may be sorted by more than one attribute.	
Instructional issues: Being able to sort a group of objects by more than one attribute is an important ability. Help students develop this ability by encouraging this activity in a variety of settings, not just with manipulatives. If you go on a walk, ask the students to think about how many ways trees could be grouped (e.g., by leaf shape, trunk color, type of fruit, and so forth). While eating lunch, ask students to give different ways for grouping vegetables (by color, softness or hardness, and so on).	
Activities: Attribute blocks provide a good way to encourage students to think of different ways to categorize objects. They may be grouped by color, shape, size, and thickness. (**MP.1, MP.4, MP.8**) Have students sit in four to six small groups. Have each group gather 10 to 12 objects from the classroom, and ask the students to figure out different ways to sort the items. Encourage discussion within each group as part of the decision-making process. Then have each group present its categories to the class and explain why the objects belong in the category. (**MP.1, MP.3, MP.4, MP.8**) Place 10 objects in two groups; items in each group should be related to one another in some way. Ask students, "How are these objects grouped together? Why were these objects placed in the same group? Why are there two groups of objects?" Encourage students to discuss the attributes they notice for each group and to explain their reasoning. This activity could be simple or complex, depending on the readiness of the students. (**MP.1, MP.3, MP.6, MP.7**)	

California Preschool Learning Foundations (at around 60 months of age) *Geometry*	CA CCSSM – Kindergarten *Geometry (G)*
Children identify and use a variety of shapes in their everyday environment. **PLF.G–1.1** Identify, describe, and construct a variety of different shapes, including variations of a circle, triangle, rectangle, square, and other shapes. **PLF.G–1.2** Combine different shapes to create a picture or design. **PLF.G–2.1** Identify positions of objects and people in space, including in/on/under, up/down, inside/outside, beside/between, and in front/behind.	**Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).** **K.G.1** Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above, below, beside, in front of, behind,* and *next to.* **K.G.2** Correctly name shapes regardless of their orientations or overall size. **K.G.3** Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid"). **Analyze, compare, create, and compose shapes.** **K.G.4** Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length). **K.G.5** Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes. **K.G.6** Compose simple shapes to form larger shapes. *For example, "Can you join these two triangles with full sides touching to make a rectangle?"*
Vocabulary: *In, on, under, up, down, inside, outside, beside, between, in front, behind, below, next to, flat, solid, square, circle, triangle, rectangle, hexagon, cube, cone, cylinder, sphere, side, corner, vertex, vertices*	
What it looks like: • Xavier says, "Look, the window is a rectangle, and it has rectangles in it!" (**MP.7**) • In a class discussion about shapes, Veronica says, "A sphere is just like a ball— round all around!" (**MP.2**) • In a discussion about the prepositions *above* and *below,* Cho says, "That's funny, things can be both! Everything is above the floor and below the ceiling!" (**MP.2, MP.3**)	
Big ideas: Shapes have fixed attributes, such as the number of sides and corners. Knowledge of three-dimensional shapes is important; do not limit exposure to two-dimensional shapes. Two or more shapes can be put together to make new shapes.	

Instructional issues: Shapes should be provided in all orientations and all permutations (long rectangles, triangles with a vertex pointing down, isosceles triangles, scalene triangles), and these should be discussed to help students focus on the central attributes. Help students to understand the difference between actual representations of shapes and common objects with similar characteristics (e.g., an apple has round characteristics, but it is not a sphere). Help students begin to understand that some shapes are special cases of a larger shape category (e.g., a square is a special rectangle that has all sides of equal lengths). Students should compose and decompose shapes with right angles and not just pattern blocks made from equilateral triangles.

Activities: In a whole group or small group, talk about words that describe where something is. Examples are *in, on, under, up, down, inside, outside, beside, between, in front, behind, below,* and *next to.* Ask each student to find an example of these positions/prepositions in the classroom. One example is *over:* Ask students to find things that are over something else. Give students about five minutes to find examples. Go around the room and ask each student what the object is and what it is over (e.g., *"The exit sign is over the door"*). (**MP.1, MP.4**)

Provide opportunities for sorting by shapes. For students who are just learning about shapes, a shape sorter —a container with different-shaped openings through which corresponding three-dimensional pieces (typically made of plastic or wood) can be pushed—may be useful. Pattern blocks and attribute blocks are also useful for sorting. (**MP.4**)

Provide an activity center where students create and work with shapes. This center might include shape magnets, clay balls and toothpicks, chopsticks, paper, pencils, and scissors. Encourage students to talk about what they are creating. Provide tangram sets with pictures to compose and parquet blocks for creating designs. (**MP.4, MP.5, MP.7**)

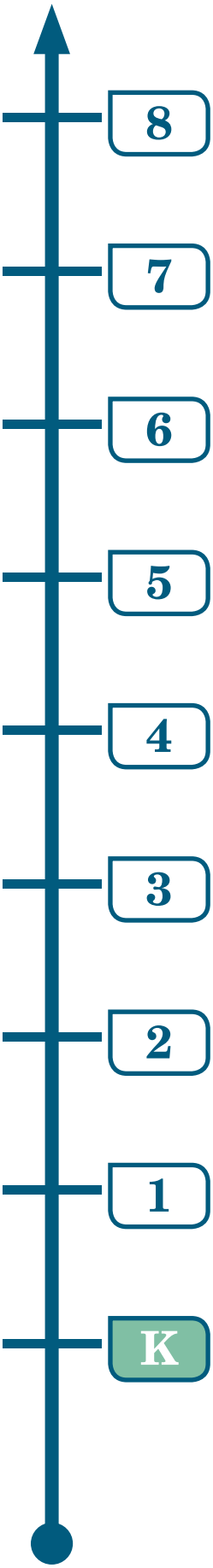
Gather a collection of two- and three-dimensional shapes. In a whole or small group, ask students to describe the shapes one by one. For instance, hold up a triangle and ask the students to describe it. At first, students might need help learning the vocabulary words listed above. To prompt students' descriptions, ask them how many sides, corners, vertices, or faces they see when looking at a particular shape. After students are comfortable providing these sorts of descriptions, change the activity by describing a hidden shape and asking the students to guess which shape it is. (**MP.3, MP.4, MP.6, MP.7, MP.8**)

Table TK-3. Standards for Mathematical Practice—Explanation and Examples for Transitional Kindergarten

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Transitional kindergarten provides an opportunity for teachers to instill the joy of mathematical problem solving. Mathematical activities should be both meaningful and challenging. Some of these activities are games (e.g., board games, card-number games, dominoes) that are useful because mathematics is being employed to solve problems. Consider using games in which no one "wins" until every student has finished, as well as games that require collaboration. Encourage students to persevere in solving problems; students often find that problems requiring a bit of time to solve can be the most rewarding. Possible prompts: *How do you know? What do you know about _____ ? What would happen if _____ ?*
MP.2 Reason abstractly and quantitatively.	Counting things for a reason—or just to get better at it—is important. Young students love to count things and to practice the counting sequence. Competence is the motivation. Many experiences in the manipulative-centered activities of transitional kindergartners are natural environments that require quantitative reasoning. Fair sharing, in particular, promotes this sort of thinking in the classroom. As students become more familiar with quantitative reasoning with objects, they become more able to reason abstractly—for example, "You have five trucks and I have four trucks, and since five is more than four, you have more trucks than I do. That's not fair!" Possible prompts: *What do you know about the number _____ ? Let's make a story about these numbers.*
MP.3 Construct viable arguments and critique the reasoning of others.	Young students are very capable of stating a point of view and defending it. Help students transfer these abilities to the domain of mathematics. Ask students how they arrived at their answer, and have them discuss with others not only the correct answer, but also the strategies used to find the answer. There are many problems with more than one correct answer (e.g., "What number is greater than five?") and more than one strategy for finding a correct answer. Model how to explain answers and discuss other solutions with classmates. Possible prompts: *How did you figure that out? What do you think about _____ ?*
MP.4 Model with mathematics.	*Modeling with mathematics* means that teachers provide models (solving a problem aloud and with manipulatives) and that students use objects to demonstrate their thinking. Possible prompt: *What could we use to _____ ?* Solve mathematical problems aloud. For example, divide a box of pencils so that each table receives one pencil for each student seated: "Let's see, there are four of you here, so we will need four pencils. One, two, three, four." Encourage students to use manipulatives to show their thinking ("Mica, can you show me how you know you shared these eight trucks fairly with Charlie?").
MP.5 Use appropriate tools strategically.	The transitional kindergarten classroom is filled with tools. These include instruments such as balance scales and measuring tapes, as well as the manipulatives and objects that students and teachers use to model mathematics. Students should have frequent opportunities to ponder which of these is appropriate to the task at hand. Possible prompts: *What could you use to help you with _____ ? How could you use a _____ to help you with _____ ?*

Table TK-3 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
MP.6 Attend to precision.	*Precision* entails more than arriving at a correct answer. It also involves being able to describe strategies, arguments, and decisions with increasing skill. Descriptions become more and more precise. Triangle descriptions change from "Because it looks like a triangle" to "It has three sides and three corners." Students learn that if they do not provide accurate representations during problem solving (e.g., when drawing 3 + 5 they draw only two and five objects), then they will have problems determining accurate answers. There is a beauty in precision, and many students are entranced by this beauty (e.g., 2 + 3 is always 5!). Possible prompts: *What do you know about _____ ? What else do you notice?*
MP.7 Look for and make use of structure.	Students in transitional kindergarten will begin to see patterns as they gain experience in mathematics. For instance, 1 plus any number will always equal the next [whole] number in the sequence. Possible prompts: *What do you notice about _____ ? How is this the same as _____ ? What are two different ways we can look at these objects? Tell me about your pattern.*
MP.8 Look for and express regularity in repeated reasoning.	Young students delight in finding patterns. For example, to solve addition problems, one can always count all the objects in both sets. One can also count on from the larger set. In number decomposition, students may find (especially if they record the addends) that if the first addend is decreased by 1, then the second is increased by 1 (3 + 7 = 10; 2 + 8 = 10; 5 = 3 + 2; 5 = 2 + 3). Asking questions of students that help them examine the strategies with which they solve problems will help them see regularity in the way they solve these problems. Possible prompt: *What do you notice?*



Kindergarten

Students in preschool and transitional kindergarten programs who have been exposed to important mathematical concepts—such as representing, relating, and operating on whole numbers and identifying and describing shapes—will be better prepared for kindergarten mathematics and for later learning.

Critical Areas of Instruction

In kindergarten, instructional time should focus on two critical areas: (1) representing and comparing whole numbers, initially with sets of objects; and (2) describing shapes and space. More learning time in kindergarten should be devoted to numbers rather than to other topics (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010p). Kindergarten students also work toward fluency with addition and subtraction of whole numbers within 5.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade-level examples of focus, coherence, and rigor are indicated throughout the chapter.

The standards do not give equal emphasis to all content for a particular grade level. Cluster headings can be viewed as the most effective way to communicate the **focus** and **coherence** of the standards. Some clusters of standards require a greater instructional emphasis than others based on the depth of the ideas, the time needed to master those clusters, and their importance to future mathematics or the later demands of preparing for college and careers.

Table K-1 highlights the content emphases at the cluster level for the kindergarten standards. Most of the instructional time should be spent on "Major" clusters and the standards within them, which are indicated throughout the text by a triangle symbol (▲). However, standards in the "Additional/Supporting" clusters should not be neglected; to do so would result in gaps in students' learning, including skills and understandings they may need in later grades. Instruction should reinforce topics in major clusters by using topics in the additional/supporting clusters and including problems and activities that support natural connections between clusters.

Teachers and administrators alike should note that the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences presented in a coherent manner (adapted from Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

Table K-1. Kindergarten Cluster-Level Emphases

Counting and Cardinality — K.CC

Major Clusters

- Know number names and the count sequence. (**K.CC.1–3** ▲)
- Count to tell the number of objects. (**K.CC.4–5** ▲)
- Compare numbers. (**K.CC.6–7** ▲)

Operations and Algebraic Thinking — K.OA

Major Clusters

- Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from. (**K.OA.1–5** ▲)

Number and Operations in Base Ten — K.NBT

Major Clusters

- Work with numbers 11–19 to gain foundations for place value. (**K.NBT.1** ▲)

Measurement and Data — K.MD

Additional/Supporting Clusters

- Describe and compare measurable attributes. (**K.MD.1–2**)
- Classify objects and count the number of objects in categories. (**K.MD.3**)

Geometry — K.G

Additional/Supporting Clusters

- Identify and describe shapes. (**K.G.1–3**)
- Analyze, compare, create, and compose shapes. (**K.G.4–6**)

Explanations of Major and Additional/Supporting Cluster-Level Emphases

Major Clusters (▲) — Areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than others based on the depth of the ideas, the time needed to master them, and their importance to future mathematics or the demands of college and career readiness.

Additional Clusters — Expose students to other subjects; may not connect tightly or explicitly to the major work of the grade.

Supporting Clusters — Designed to support and strengthen areas of major emphasis.

Note of caution: Neglecting material, whether it is found in the major or additional/supporting clusters, will leave gaps in students' skills and understanding and will leave students unprepared for the challenges they face in later grades.

Adapted from Achieve the Core 2012.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (MP) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The MP standards represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students.

Although the description of the MP standards remains the same at all grade levels, the way these standards look as students engage with and master new and more advanced mathematical ideas does change. Table K-2 presents examples of how the MP standards may be integrated into tasks appropriate for students in kindergarten. (Refer to the Overview of the Standards Chapters for a description of the MP standards.)

Table K-2. Standards for Mathematical Practice—Explanation and Examples for Kindergarten

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	In kindergarten, students begin to build the understanding that doing mathematics involves solving problems and discussing how they solved them. Students explain to themselves the meaning of a problem and look for ways to solve it. Real-life experiences should be used to support students' ability to connect mathematics to the world. To help students connect the language of mathematics to everyday life, ask students questions such as "How many students are absent?" or have them gather enough blocks for the students at their table. Younger students may use concrete objects or pictures to help them conceptualize and solve problems. They may check their thinking by asking themselves, "Does this make sense?", or they may try another strategy.
MP.2 Reason abstractly and quantitatively.	Younger students begin to recognize that a number represents a specific quantity and connect the quantity to written symbols. Quantitative reasoning entails creating a representation of a problem while attending to the meanings of the quantities. For example, a student may write the numeral 11 to represent an amount of objects counted, select the correct number card 17 to follow 16 on a calendar, or build two piles of counters to compare the numbers 5 and 8. In addition, kindergarten students begin to draw pictures, manipulate objects, or use diagrams or charts to express quantitative ideas. Students need to be encouraged to answer questions such as "How do you know?"—which reinforces their reasoning and understanding and helps student develop mathematical language.
MP.3 Construct viable arguments and critique the reasoning of others.	Younger students construct arguments using actions and concrete materials, such as objects, pictures, and drawings. They begin to develop their mathematical communication skills as they participate in mathematical discussions involving questions such as "How did you get that?" and "Why is that true?" They explain their thinking to others and respond to others' thinking. They begin to develop the ability to reason and analyze situations as they consider questions such as "Are you sure that _____?", "Do you think that would happen all the time?", and "I wonder why _____?"

Table K-2 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
MP.4 Model with mathematics.	In early grades, students begin to represent problem situations in multiple ways—by using numbers, objects, words, or mathematical language, acting out the situation, making a chart or list, drawing pictures, creating equations, and so forth. For example, a student may use cubes or tiles to show the different number pairs for 5, or place three objects on a 10-frame and then determine how many more are needed to "make a ten." Students rely on manipulatives (or other visual and concrete representations) while solving tasks and record an answer with a drawing or equation.
MP.5 Use appropriate tools strategically.	Younger students begin to consider tools available to them when solving a mathematical problem and decide when certain tools might be helpful. For instance, kindergartners may decide to use linking cubes to represent two quantities and then compare the two representations side by side, or later, make math drawings of the quantities. Students decide which tools may be helpful to use depending on the problem or task and explain why they use particular mathematical tools.
MP.6 Attend to precision.	Kindergarten students begin to develop precise communication skills, calculations, and measurements. Students describe their own actions, strategies, and reasoning using grade-level-appropriate vocabulary. Opportunities to work with pictorial representations and concrete objects can help students develop understanding and descriptive vocabulary. For example, students analyze and compare two- and three-dimensional shapes and sort objects based on appearance. While measuring objects iteratively (repetitively), students check to make sure that there are no gaps or overlaps. During tasks involving number sense, students check their work to ensure the accuracy and reasonableness of solutions. Students should be encouraged to answer questions such as, "How do you know your answer is reasonable?"
MP.7 Look for and make use of structure.	Younger students begin to discern a pattern or structure in the number system. For instance, students recognize that 3 + 2 = 5 and 2 + 3 = 5. Students use counting strategies, such as counting on, counting all, or taking away, to build fluency with facts to 5. Students notice the written pattern in the "teen" numbers—that the numbers start with 1 (representing 1 ten) and end with the number of additional ones. Teachers might ask, "What do you notice when _____?"
MP.8 Look for and express regularity in repeated reasoning.	In the early grades, students notice repetitive actions in counting, computations, and mathematical tasks. For example, the next number in a counting sequence is 1 more when counting by ones and 10 more when counting by tens (or 1 more group of 10). Students should be encouraged to answer questions such as, "What would happen if _____?" and "There are 8 crayons in the box. Some are red and some are blue. How many of each could there be?" Kindergarten students realize 8 crayons could include 4 of each color (8 = 4 + 4), 5 of one color and 3 of another (8 = 5 + 3), and so on. For each solution, students repeatedly engage in the process of finding two numbers to join together to equal 8.

Adapted from Arizona Department of Education (ADE) 2010 and North Carolina Department of Public Instruction (NCDPI) 2013b.

Standards-Based Learning at Kindergarten

The following narrative is organized by the domains in the Standards for Mathematical Content. It highlights some necessary foundational skills and provides exemplars to explain the content standards, highlight connections to the various Standards for Mathematical Practice (**MP**), and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. A triangle symbol (▲) indicates standards in the major clusters (see table K-1).

Domain: Counting and Cardinality

A critical area of instruction in kindergarten is representing, relating, and operating on whole numbers, initially with sets of objects.

Counting and Cardinality	K.CC

Know number names and the count sequence.

1. Count to 100 by ones and by tens.
2. Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
3. Write numbers from 0 to 20. Represent a number of objects with a written numeral 0–20 (with 0 representing a count of no objects).

Several learning progressions originate in knowing number names and the count sequence. One of the first major concepts in a student's mathematical development is *cardinality*. Cardinality can be explained as knowing that the number word spoken tells the quantity and that the number on which a person ends when counting represents the entire amount counted. The idea is that numbers mean *amount,* and no matter how you arrange and rearrange the items, the amount is the same. Students can generally say the counting words up to a given number before they can use these numbers to count objects or to tell the number of objects (adapted from the University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2011a and Georgia Department of Education [GaDOE] 2011).

Kindergarten students are introduced to the counting sequence (**K.CC.1–2▲**). When counting orally by ones, students begin to understand that the next number in the sequence is one more. Similarly, when counting by tens, the next number in the sequence is "10 more."

Examples: Counting Sequences for Forward Counting to 100 by Ones **K.CC.1**

- The "ones" (1–10)
- The "teens" (10, 11, 12, 13, 14, 15, 16, 17, 18, 19)
- "Crossing the decade" (15, 16, 17, 18, 19, 20, 21, 22, 23, 24, or, similarly, 26–34, 35–44, and so forth)

Students often have trouble with counting forward sequences that cross the decade. Focusing on short counting sequences may be helpful.

Adapted from Kansas Association of Teachers of Mathematics (KATM) 2012, Kindergarten Flipbook.

Initially, students might think of counting as a string of words, but gradually they transition to using counting as a tool to describe amounts in their world. Counting can be reinforced throughout the school day.

Examples **K.CC.1**

- Count the number of chairs of students who are absent.
- Count the number of stairs, shoes, and so on.
- Count groups of 10, such as "fingers in the classroom" (10 fingers per student). (**MP.6, MP.7, MP.8**)

Kindergarten students also count forward—beginning from a given number—instead of starting at 1. Counting forward (or "counting on") may be confusing for young students, because it conflicts with the initial strategy they learned about counting from the beginning. Activities or games that require students to add on to a previous count to reach a targeted number may encourage development of this concept (adapted from KATM 2012, Kindergarten Flipbook).

Kindergarten students learn to write numbers from 0 to 20 (**K.CC.3▲**) and represent a number of objects with a written numeral in the 0–20 range (using numerals as symbols for quantities). They understand that 0 represents a count of no objects. Students need multiple opportunities to count objects and recognize that a number represents a specific quantity. As this understanding develops, students begin to read and write numerals. The emphasis should first be on quantity and then on connecting quantities to the written symbols.

Example: A Learning Sequence for Understanding Numbers

A specific learning sequence might consist of these steps:

1. Count up to 20 objects in many settings and situations over several weeks.
2. Start to recognize, identify, and read the written numerals, and match the numerals to given sets of objects.
3. Write the numerals to represent counted objects.

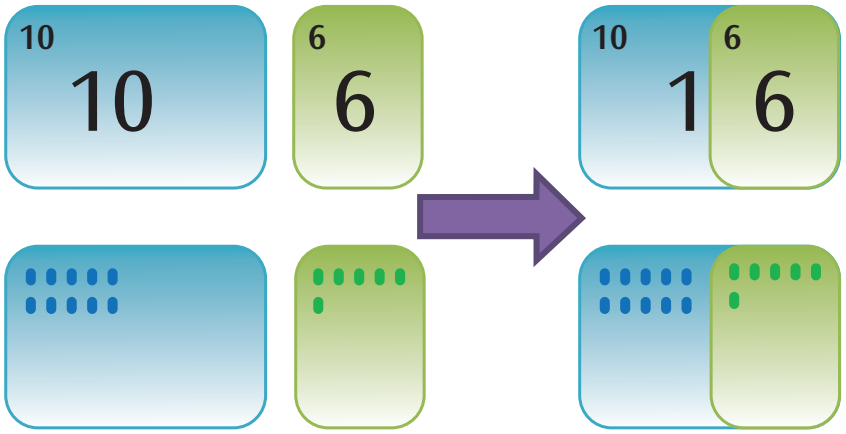
Adapted from ADE 2010.

As students connect quantities and written numerals, they also develop mathematical practices such as reasoning abstractly and quantitatively (**MP.2**). They use precise vocabulary to express how they know that their count is accurate (**MP.6**). They also use the structures and patterns of the number system and apply this understanding to counting (**MP.7, MP.8**) [adapted from ADE 2010].

Common Misconceptions

- Some students might not see zero (0) as a number. Ask students to write 0 and say "zero" to represent the number of items left when all items have been taken away. Avoid using the word *none* to represent this situation.
- Teen numbers can also be confusing for young students. To help avoid confusion, these numbers should be taught as a bundle of 10 ones and some extra ones. This approach supports a foundation for understanding both the place-value concept and symbols that represent each teen number. Layered place-value cards may help students understand the difficult teen numbers; see figure K-1.

Figure K-1. Layering Place-Value Cards to Illustrate Teen Numbers



Adapted from KATM 2012, Kindergarten Flipbook.

Counting and Cardinality K.CC

Count to tell the number of objects.

4. Understand the relationship between numbers and quantities; connect counting to cardinality.
   a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.
   b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.
   c. Understand that each successive number name refers to a quantity that is one larger.
5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

In kindergarten, students develop an understanding of the relationship between numbers and quantities and connect counting to cardinality (**K.CC.4**). Learning to count is a complex mental and physical activity that requires staying connected to the objects that are being counted. Children must understand that the count sequence has meaning when counting objects: that the last count word indicates the amount or the cardinality of the set (Van de Walle 2007). Kindergarten students use their understanding of the relationship between numbers and quantities to count a set of objects and see sets and numerals in relationship to one another, rather than as isolated concepts.

There are numerous opportunities for students to manipulate concrete objects or visual representations (e.g., dot cards, 10-frames) and connect number names with their quantities, which can help students master the concept of counting (adapted from NCDPI 2013b).

As students learn to count a group of objects, they pair each word said with one object (**K.CC.4a▲**). This is usually facilitated by an indicating act (such as touching, pointing to, or moving objects) that keeps each word said paired to only one object (the one-to-one-correspondence principle). Students learn that the last number named tells the number of objects counted (the cardinality principle) and that the number of objects is the same regardless of their arrangement or the order in which they were counted (the order-irrelevance principle). They also understand that each successive number name refers to a quantity that is 1 larger (**K.CC.4.b–c▲**) [adapted from UA Progressions Documents 2011a].

To develop their understanding of the relationship between numbers and quantities, students might count objects, placing one more object in the group at a time.

Example	**K.CC.4**
Using cubes, students count an existing group and then place another cube in the set to continue counting. Students continue placing one more cube in the set at a time and then identify the new total number of cubes. Students see that the counting sequence results in a quantity that increases by one each time another cube is placed in the group. Students may need to recount from one, but the goal is for students to count on from the existing number of cubes—a conceptual start for the grade-one skill of counting to 120, starting at any number less than 120.	

To count accurately, students rely on:

- knowing patterns and arbitrary parts of the number–word sequence;
- assigning one number word to one object (one-to-one correspondence);
- keeping track of objects that have already been counted (adapted from ADE 2010 and GaDOE 2011).

Five Major Principles: Development of Students' Understanding of How to Count and What to Count

1. **One-to-One-Correspondence Principle.** Students assign one, and only one, distinct counting word to each of the items to be counted. To follow this principle, students partition and re-partition the collection of objects to be counted into two categories: those that have been allocated a number name and those that have not. Students model numbers with objects, and each object is assigned a unique number name based on one-to-one correspondence between each object and the number name. If an item is not assigned a number name or is assigned more than one number name, the resulting count will be incorrect; refer to standard **K.CC.4a.**
2. **Standard-Order (of Number Names) Principle.** Students recite a number-name list in a fixed order (e.g., students count "One, two, three" for a collection of three objects). In other words, students can rote-count; refer to standard **K.CC.4a.**
3. **Cardinal Principle.** Students understand that the last number name used for the final object in a collection represents the number of items in that collection. This rule connects counting with "how many"; refer to standard **K.CC.4b.**
4. **Order-Irrelevance Principle.** Students understand that the order in which objects are counted has no effect on the total number of objects and that the quantity of a group of objects remains constant even when the objects are rearranged; refer to standard **K.CC.4b.**
5. **Abstraction Principle.** Students realize that the above four principles of counting apply to any collection of objects, whether tangible (e.g., marbles or blocks) or not (e.g., sounds or actions). They also realize that objects may have similar attributes (e.g., "All of these marbles are yellow") or different attributes (e.g., "These toys are different types and sizes"); refer to standard **K.CC.4.**

Adapted from Thompson 2010.

Students answer questions such as "How many are there?" by counting objects in a set and understanding that the last number stated represents the total amount of objects (cardinality, **K.CC.5▲**). Over time, students realize that the same set counted several different times will be the same amount each time. Counting objects arranged in a line is easiest; with more practice, students learn to count objects in more difficult arrangements, such as rectangular arrays, circles, and scattered configurations.

Scattered arrangements are the most challenging for students, and therefore kindergarten students count only up to 10 objects if arranged this way. Given a number from 1 to 20, kindergarten students also count out that many objects. This is also more difficult for students than simply counting the total number of objects, because as students count, they need to remember the number of objects to be counted out (adapted from UA Progressions Documents 2011a and NCDPI 2013b).

Examples of Counting Strategies **K.CC.4.a–b**

There are numerous counting strategies that students may use, depending on how objects are arranged. Here are a few examples:

- Move objects as each object is counted.
- Line up objects to count.
- Touch objects in a scattered arrangement as each object is counted.
- Count objects in a scattered arrangement by visually scanning each object without touching.

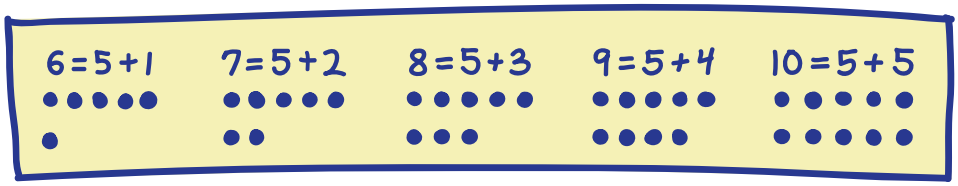
Adapted from KATM 2012, Kindergarten Flipbook.

Focus, Coherence, and Rigor

As students use various counting strategies when they participate in counting activities, they reinforce their understanding of the relationship between numbers and quantities and support mathematical practices such as modeling with mathematics (**MP.4**), the use of precise language (**MP.6**), and repeated reasoning to find a solution (**MP.8**).

Students come to quickly perceive the number of items in small groups—such as recognizing dot arrangements in different patterns without counting the objects. This is known as *perceptual subitizing,* a fundamental skill in the development of students' understanding of numbers. Perceptual subitizing develops into *conceptual subitizing*—recognizing a collection of objects as a composite of subparts and as a whole (e.g., seeing a five-dot domino and thinking 1 and 4 or seeing a set with two subsets of 2 and saying 4) [adapted from UA Progressions Documents 2011a]. Particularly important is the $5 + n$ pattern, in which one row of 5 circles has 1, 2, 3, 4, or 5 dots below to show 6, 7, 8, 9, and 10; see figure K-2. These rows are separated more than the individual dots to ensure students see the group of 5 and the extra dots.

Figure K-2. The $5+n$ pattern



Source: UA Progressions Documents 2011a.

Subitizing supports the development of addition and subtraction strategies, such as counting on and composing and decomposing numbers. Students need practice to develop competency in perceptual subitizing.

Example	K.CC.5
The teacher might place different amounts of beans on a mat (beginning with amounts of 4 or fewer) and then ask students to say how many beans they see. As students become proficient, dot cards can also be utilized to develop fluency. For example, the teacher can show a large dot card to students, and students then take the number counters they think they need to cover the dots on the card. Then one child places his or her counters on the dots while the rest of the class counts and checks. Eventually, the teacher briefly shows one large dot card and puts it down quickly. Then students try to recognize the number of dots without counting.	

Counting and Cardinality — K.CC

Compare numbers.

6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.[1]
7. Compare two numbers between 1 and 10 presented as written numerals.

In kindergarten, students compare the number of objects in one group (with up to 10 objects) to the number of objects in another group (**K.CC.6▲**). Students need a strong sense of the relationship between quantities and numerals to accurately compare groups and answer related questions. They may use matching strategies or counting strategies to determine whether one group is greater than, less than, or equal to the number of objects in another group.

1. Includes groups with up to 10 objects.

Example: More Triangles or More Squares?		K.CC.6▲ (MP.1, MP.2)
Student 1	**Student 2**	**Student 3**
I lined up 1 square with 1 triangle. Since there is 1 extra triangle, there are more triangles than squares.	I counted the squares and got 8. Then I counted the triangles and got 9. Since 9 is bigger than 8, there are more triangles than squares.	I put them in a pile. I then took away objects. Every time I took a square, I also took a triangle. When I had taken almost all of the shapes away, there was still a triangle left. That means that there are more triangles than squares.

Adapted from KATM 2012, Kindergarten Flipbook.

Matching and Counting Strategies for Comparing Groups of Objects

- **Matching.** Students use one-to-one correspondence, repeatedly matching one object from one set with one object from the other set to determine which set has more objects.
- **Counting.** Students count the objects in each set and then identify which set has more, less, or an equal number of objects.
- **Observation.** Students may use observation to compare two quantities. For example, by looking at two sets of objects, they may be able to tell which set has more or less without counting.
- **Benchmark Numbers.** Introduce the use of 0, 5, and 10 as benchmark numbers to help students further develop their sense of quantity as well as their ability to compare numbers. Benchmarks of 5 and 10 are especially useful with the $5 + n$ patterns.

Adapted from KATM 2012, Kindergarten Flipbook.

An important level of understanding is reached when students can compare two numbers from 1 to 10 represented as written numerals, without counting (**K.CC.7▲**). Students demonstrate their understanding of numbers when they can justify their answers (**MP.3**).

Example	K.CC.7▲
When a student gives an answer, the teacher may ask a probing question such as "How do you know?" to elicit student thinking and reasoning (**MP.3, MP.8**). Students might justify their answer (e.g., 7 is greater than 5) by demonstrating a one-to-one match, counting again, or using similar approaches that help to explain or verify the answer (adapted from KATM 2012, Kindergarten Flipbook).	

Focus, Coherence, and Rigor

Comparing numbers and groups in kindergarten will progress to comparing addition and subtraction situations in grade one. For example, "Which is more?" or "Which is less?" will progress to "How many more?" or "How many less?"

Domain: Operations and Algebraic Thinking

Kindergarten students are introduced to addition and subtraction with small numbers, and they work toward fluency with these operations for numbers within 5.

Operations and Algebraic Thinking — K.OA

Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

1. Represent addition and subtraction with objects, fingers, mental images, drawings,[2] sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.
2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.
3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).
4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.
5. Fluently add and subtract within 5.

Kindergarten students develop their understanding of addition and subtraction by making sense of word problems (**MP.1, MP.2**). Students experience a variety of addition situations that involve *putting together* and *adding to* and a variety of subtraction situations that involve *taking apart* and *taking from* (**K.OA.1–2▲**). Students use objects (such as two-color counters, clothespins on hangers, connecting cubes, 5-frames, and stickers), fingers, mental images, sounds, drawings, verbal explanations and acting out the situation to represent these operations (**MP. 1, MP.2, MP.4, MP.5**) [adapted from KATM 2012, Kindergarten Flipbook].

Students use both mathematical and non-mathematical language to explain their interpretation of a problem and the solution. Initially, students work with numbers within 5, which helps them move from perceptual subitizing to conceptual subitizing, in which they say the addends and the total (e.g., 2 and 1 make 3). Students will generally use fingers to keep track of addends and parts of addends and should develop rapid visual and kinesthetic recognition of numbers up to 5 on their fingers. Eventually, students will expand their work in addition and subtraction from within 5 to within 10.

2. Drawings need not show details, but should show the mathematics in the problem. (This applies wherever drawings are mentioned in the Standards.)

Students are introduced to expressions and equations using appropriate symbols, including +, –, and =. Teachers may write expressions (e.g., 3 – 1) or equations (e.g., 3 – 1 = ☐, or 3 = 1 + 2) that represent operations and problems with real-world contexts to reinforce students' understanding of these concepts. Teachers should emphasize that an equal sign (=) means "is the same as." Students should see these equations and be encouraged to write them; however, they are not required to write equations. In kindergarten, the use of formal vocabulary for both addition and subtraction (such as *minuend, subtrahend,* and *addend*) is not necessary. For English learners, phonologically identical words (e.g., *sum* and *some, whole* and *hole*) may be challenging; thus it is better to use the word *total* instead of *sum* for all students in kindergarten and grade one. Using the word *partners* instead of *addends* is also a helpful conceptual support for children in these grades. To support English learners, these words should be explicitly taught as they are introduced (adapted from UA Progressions Documents 2011a). For more information, refer to the Universal Access chapter.

Focus, Coherence, and Rigor

When students represent addition and subtraction, this also supports mathematical practices as they use objects or pictures to represent quantities (**K.OA.1▲**), reason quantitatively to make sense of quantities and develop a clear representation of the problem (**MP.2**), mathematize a real-world situation (**MP.4**), and use tools appropriately to model the problem (**MP.5**). Math drawings also facilitate student reflection and discussion and help young students justify answers (**MP.3**).

Word problems with real-life applications provide students with a context to develop their understanding of addition and subtraction (**K.OA.2▲**). Kindergarten students learn that addition is *putting together* and *adding to* and subtraction is *taking apart* and *taking from*. Kindergartners use objects or math drawings (with simple shapes such as circles) to model word problems (adapted from ADE 2010).

The most common types of addition and subtraction problems for kindergarten students are displayed with dark shading in table K-3. Students add and subtract within 10 to solve these types of problems.

Table K-3. Types of Addition and Subtraction Problems (Kindergarten)

Type of Problem	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ☐	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were 5 bunnies. How many bunnies hopped over to the first two? 2 + ☐ = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were 5 bunnies. How many bunnies were on the grass before? ☐ + 3 = 5
Take from	Five apples were on the table. I ate 2 apples. How many apples are on the table now? 5 – 2 = ☐	Five apples were on the table. I ate some apples. Then there were 3 apples. How many apples did I eat? 5 – ☐ = 3	Some apples were on the table. I ate 2 apples. Then there were 3 apples. How many apples were on the table before? ☐ – 2 = 3

	Total Unknown	Addend Unknown	Both Addends Unknown
Put together/ Take apart	Three red apples and 2 green apples are on the table. How many apples are on the table? 3 + 2 = ☐	Five apples are on the table. Three are red, and the rest are green. How many apples are green? 3 + ☐ = 5, 5 – 3 = ☐	Grandma has 5 flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	("How many more?" version): Lucy has 2 apples. Julie has 5 apples. How many more apples does Julie have than Lucy?	(Version with *more*): Julie has 3 more apples than Lucy. Lucy has 2 apples. How many apples does Julie have?	(Version with *more*): Julie has 3 more apples than Lucy. Julie has 5 apples. How many apples does Lucy have?
	("How many fewer?" version): Lucy has 2 apples. Julie has 5 apples. How many fewer apples does Lucy have than Julie? 2 + ☐ = 5, 5 – 2 = ☐	(Version with *fewer*): Lucy has 3 fewer apples than Julie. Lucy has 2 apples. How many apples does Julie have? 2 + 3 = ☐, 3 + 2 = ☐	(Version with *fewer*): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ☐, ☐ + 3 = 5

Note: Kindergarten students solve problem types with the darkest shading; students in grades one and two solve problems of all subtypes. Unshaded problems are the most difficult; first-grade students work with these problems but do not master them until grade two (adapted from NGA/CCSSO 2010d and UA Progressions Documents 2011a).

To solve word problems, students learn to apply various computational methods. Kindergarten students generally use Level 1 methods, moving on to Level 2 and Level 3 methods in later grades. The three levels are summarized in table K-4 and explained more thoroughly in appendix C.

Table K-4. Methods Used for Solving Single-Digit Addition and Subtraction Problems

Level 1: Direct Modeling by Counting All or Taking Away
Represent the situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Level 2: Counting On
Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total, but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. The count is tracked and monitored in some way (e.g., with fingers, objects, mental images of objects, body motions, or other count words).

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as the unknown addend).

Level 3: Converting to an Easier Equivalent Problem
Decompose an addend and compose a part with another addend.

Adapted from UA Progressions Documents 2011a.

Students learn that a set of objects may be broken into two sets in multiple ways. For example, a set of 5 objects may be separated into two sets—3 and 2 or 4 and 1 (**K.OA.3▲**). Thus, when breaking apart a set (decomposing), students develop the understanding that a smaller set of objects exists within that larger set. Students should have numerous experiences with decomposing sets of objects and recording with pictures and numbers, and the teacher should make connections between the drawings and symbols ($5 = 4 + 1$, $5 = 3 + 2$, $5 = 2 + 3$, $5 = 1 + 4$, and $5 = 5 + 0$), showing the total on the left and the two addends on the right. Students can find patterns in all of the decompositions of a given number and eventually summarize these patterns for several numbers. Experience with decomposing also emphasizes that the equal sign (=) means "is the same as."

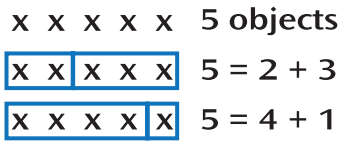
Students may use objects such as cubes, two-color counters, or square tiles to show different number pairs for a given number. For example, for the number 5, students may split a set of 5 objects into 1 and 4, 2 and 3, and 5 and 0. Students may also use drawings to show different number pairs for a given number (**MP.1, MP.2, MP.4**).

Example: Decomposing 5 K.OA.3▲

Students may draw 5 objects, showing how to decompose in several ways.

They may write equations involving 5 and its decompositions, such as:

5 = 4 + 1
3 + 2 = 5
2 + 3 = 4 + 1



Students can systematically list all the possible number partners for a given number. For example, they may list all number partners for 5 (0 + 5, 1 + 4, 2 + 3, 3 + 2, 4 + 1, and 5 + 0) and describe the pattern in the addends—that is, each number is one less or one more than the previous addend.

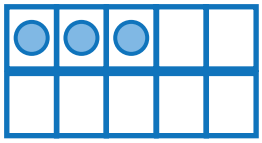
Adapted from KATM 2012, Kindergarten Flipbook.

Working with equations with one number on the left and an operation on the right (e.g., 5 = 2 + 3) to record groups of 5 things decomposed as groups of 2 and 3 things (**K.OA.3▲**) helps students to understand that equations indicate quantities on both sides of the equal sign have the same value (**MP.7**). Understanding the meaning of mathematical symbols allows students to develop precision in their communication about mathematics (**MP.6**). The equation can also be reversed so that an operation is on the left and the number is on the right (e.g., 2 + 3 = 5). Such equations model "add to" situations (adapted from UA Progressions Documents 2011a).

Number pairs that total 10 are foundational for students' ability to work fluently within base-ten numbers and operations. In kindergarten, students find the number that makes 10 when added to the given number for any number from 1 to 9. Students use objects or drawings and record their answers with a drawing or equation (**K.OA.4▲**). Students use different models, such as 10-frames, cubes, and two-color counters to help them visualize these number pairs for 10 (**MP.1, MP.2, MP.4**).

Examples: Tools and Strategies for Making a Ten K.OA.4▲

A student places 3 objects on a 10-frame and then determines how many more are needed to "make a ten." Students may use electronic versions of 10-frames to develop this skill (**MP.5**).



A student snaps 10 cubes together to make a pretend train.

- The student breaks the train into two parts. He or she identifies how many cubes are in each part and records the associated equation (10 = _____ + _____).
- The student breaks the train into two parts. He or she counts how many cubes are in one part and determines how many are in the other part without directly counting that part. Then the student records the associated equation (if the counted part has 4 cubes, the equation would be 10 = 4 + _____).
- The student covers up part of the train, without counting the covered part. He or she counts the cubes that are showing and determines how many are covered up. Then the student records the associated equation (if the counted part has 7 cubes, the equation would be 10 = 7 + _____) [**MP.8**].
- The student tosses 10 two-color counters on the table and records how many of each color are facing up (**MP.8**).

Adapted from KATM 2012, Kindergarten Flipbook.

Later in the year, students solve addition and subtraction equations for numbers within 5 (for example, 2 + 1 = □ or 3 − 1 = □) while still connecting these equations to situations verbally or with drawings. Experience with decompositions of numbers and with "add to" and "take from" situations enables students to begin to fluently add and subtract within 5 (**K.OA.5▲**).

FLUENCY

In the standards for kindergarten through grade six, there are individual content standards that set expectations for fluency in computation (e.g., "Fluently add and subtract within 5" [**K.OA.5▲**]). Such standards are culminations of progressions of learning that often span several grades and involve conceptual understanding, thoughtful practice, and extra support where necessary.

The word *fluent* is used in the standards to mean "reasonably fast and accurate" and the ability to use certain facts and procedures with enough facility that using them does not slow down or derail the problem solver as he or she works on more complex problems. Procedural fluency requires skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Developing fluency in each grade may involve a mixture of simply knowing some answers, knowing some answers from patterns, and knowing some answers from the use of strategies.

Adapted from UA Progressions Documents 2011a.

Below are several strategies that kindergarten students may use to attain fluency with addition and subtraction within 5:

- Visualizing the small numbers involved
- Counting on (e.g., for 3 + 2, students will say "3," then count on two more, "4, 5," and finish by saying the solution is "5")
- Counting back (e.g., for 4 – 1, students will say "4," then count back one, "3," and state that the solution is "3")
- Counting up to subtract (e.g., for 5 – 3 , students will say "3," then count up until they get to 5, keeping track of how many they counted up, stating that the solution is "2")
- Using doubles (e.g., for 2 + 3, students may say, "I know that 2 + 2 is 4, and 1 more is 5")
- Using the commutative property (e.g., students may say, "I know that 2 + 1 = 3, so 1 + 2 = 3")
- Using fact families (e.g., students may say, "I know that 2 + 3 = 5, so 5 – 3 = 2") [adapted from KATM 2012, Kindergarten Flipbook]

Example: Demonstrating Conceptual Understanding, Application, and Connection to the Mathematical Practices **K.OA.5▲**

Shake and Spill

Students use 5 two-color counters (e.g., red on one side and yellow on the other) and a cup (optional). The students put the counters in the cup, shake it, and spill them onto a table. The students determine how many of each color is showing and record the sum by using drawings or equations. The students should "shake and spill" several times to show different pairs of numbers that sum to 5.

Source: Illustrative Mathematics 2013a.

Domain: Number and Operations in Base Ten

Number and Operations in Base Ten	K.NBT

Work with numbers 11–19 to gain foundations for place value.

1. Compose and decompose numbers from 11 to 19 into 10 ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of 10 ones and one, two, three, four, five, six, seven, eight, or nine ones.

Kindergarten teachers help their students lay the foundation for understanding the base-ten system by drawing special attention to the number 10. Students compose and decompose numbers from 11 to 19 into 10 ones and some further ones. Students use objects or drawings and record each composition or decomposition with a drawing or equation (e.g., 16 = 9 + 7) [**K.NBT.1▲**].

Students describe, explore, and explain how the counting numbers from 11 through 19 are composed of 10 ones and some more ones. For example, when focusing on the number 14, students count out 14 objects using one-to-one correspondence and then use those objects to compose one group of 10 ones and 4 additional ones. Students connect the representation to the symbol "14" and recognize the written pattern in these numbers—that the numbers start with 1 (represents 1 ten) and end with the number of additional ones (**MP.1, MP.2, MP.4, MP.5, MP.6, MP.7, MP.8**) [adapted from UA Progressions Documents 2012b].

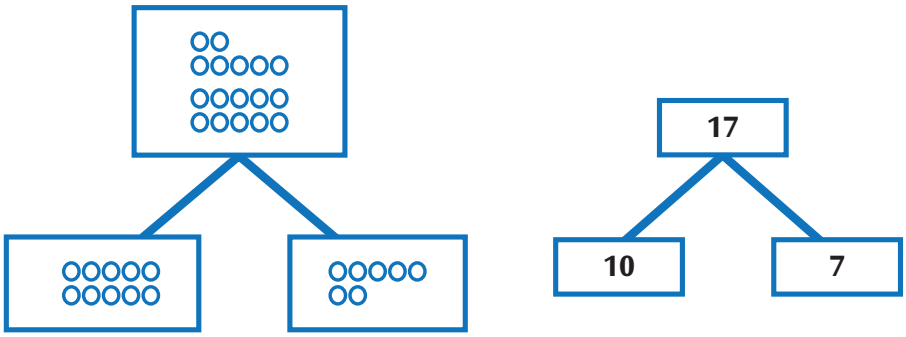
Students may have difficulty understanding that as a singular word, *ten* means "10 things." For many students, understanding that a group of 10 things can be replaced by a single word and that they both represent 10 is confusing. Students learn that this set of numbers (11–19) does not follow a consistent pattern in the verbal counting sequence. For example:

- *Eleven* and *twelve* are special number words.
- *Teen* means 1 ten plus ones.
- The verbal counting sequence for teen numbers is backwards—we say the ones digit before the tens digit. For example, 27 reads tens to ones (twenty-seven), but 17 reads ones to tens (seven-teen). To develop student understanding of written teen numbers, students read numbers as well as describe quantities. For example, for the number 17, students read "seventeen," decompose the number as "1 group of 10 ones and 7 additional ones," and record their understanding as 17 = 10 + 7 or use math drawings. This clarifies the pattern for them. Kindergarten students should see addition and subtraction equations. Student writing of equations in kindergarten is encouraged, but it is not required (adapted from ADE 2010).

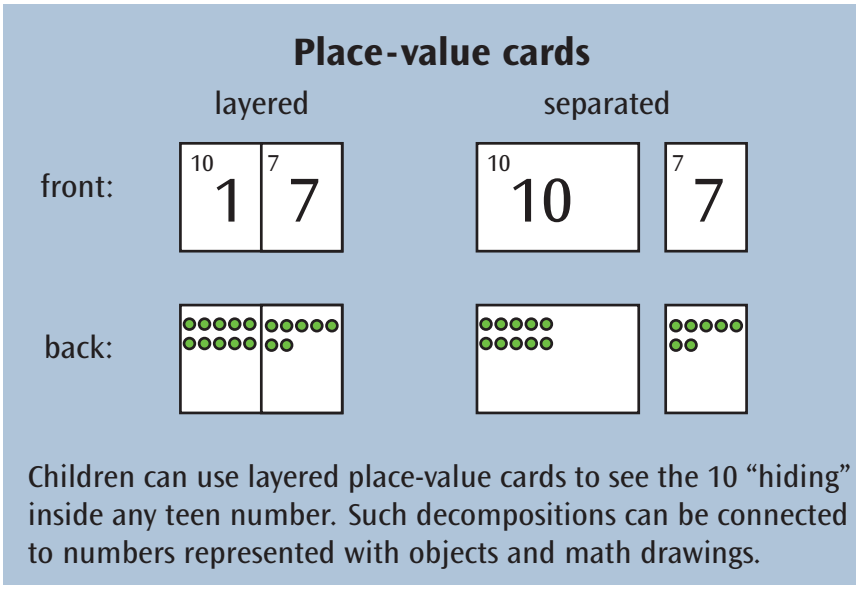
Examples: Understanding Teen Numbers **K.NBT.1▲**

Math drawings and other activities can help students develop place-value understanding of teen numbers.

Using 10-frames and number-bond diagrams



Using layered place-value cards



Children can use layered place-value cards to see the 10 "hiding" inside any teen number. Such decompositions can be connected to numbers represented with objects and math drawings.

Source: UA Progressions Documents 2012b.

Domain: Measurement and Data

Measurement and Data K.MD

Describe and compare measurable attributes.

1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

Students recognize and distinguish measurable attributes (e.g., length, area, volume) from non-measurable attributes (e.g., big or bigger) [**K.MD.1**]. Initially, many students will not be able to differentiate between these two types of attributes. Students will say one object is "bigger" than another without clarifying that it is longer, greater in area or volume, and so forth.

For students to accurately describe attributes such as length and weight, they need multiple opportunities to informally explore these attributes. Teachers encourage students' conversations to extend from describing objects as *big, small, long, tall,* or *high* to naming, discussing, and demonstrating with gestures the appropriate attribute (e.g., length, area, volume, or weight).

For example, a student might describe the measurable attributes of an empty can or milk carton by talking about how tall, wide, and heavy the can is, or how much liquid will fit inside the container. All of these are measurable attributes. By contrast, non-measurable attributes include designs, words, colors, or pictures on the can. As students discuss these situations and compare objects using different attributes, they learn to distinguish, label, and describe several measurable attributes of a single object (**MP.4, MP.5, MP.6, MP.**7).

Students directly compare two objects with a measurable attribute in common, to see which object has "more" or "less of" the attribute and describe the difference (**K.MD.2**). For example, students directly compare the heights of two children and describe one child as taller or shorter. Language plays an important role in this standard, as students describe the similarities and differences of measurable attributes of objects with terms such as *shorter than, taller than, lighter than, the same as,* and so forth (**MP.2, MP.4, MP.6, MP.**7).

When making direct comparisons for length, students must attend to the "starting point" of each object (e.g., the ends need to be lined up at the same point) or students need to compensate when the starting points are not lined up. Students develop an understanding of conservation of length (if an object is moved, its length does not change), an important concept when comparing the lengths of two objects (adapted from ADE 2010 and UA Progressions Documents 2012a).

With practice, students become increasingly competent at direct comparison—comparing the amount of an attribute in two objects without measurement. For example, when comparing the volume of two different boxes, ask students to discuss and justify their answers to these questions: *Which box will hold more? Which box will hold the least? Will the two boxes hold the same amount? How could you find out?* Students can decide to fill one box with dried beans and then pour the beans into the other box to determine the answers to these questions (adapted from KATM 2012, Kindergarten Flipbook).

Table K-5 presents a sample classroom activity that connects the Standards for Mathematical Content and Standards for Mathematical Practice.

Table K-5. Connecting to the Standards for Mathematical Practice—Kindergarten

Standards Addressed	Explanation and Examples
Connections to Standards for Mathematical Practice **MP.2.** Students reason abstractly when they imagine the attributes of given objects and attempt to compare them, even in the absence of physical objects at hand. Students mentally attribute quantities to features of objects when they compare these objects. **MP.3.** Students may be asked to justify why they think a comparison is correct, and if students disagree, they can try to explain their reasoning. **Standards for Mathematical Content** **K.MD.2.** Directly compare two objects with a measurable attribute in common, to see which object has "more of/less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*	**Task: The Comparison Game.** For this game, students have packs of 4 pairs of comparison cards, each pair corresponding to the following comparisons: heavier/lighter, taller/shorter, holds more/holds less, longer/shorter. In addition, each card pair has sample pictures on them that indicate the comparison, and furthermore, the words may be color-coded to aid students who cannot yet read the words on the cards (examples are shown at right). At the front of the room, the teacher shows the students two objects in sequence; the students must raise the appropriate card to compare the *second object to the first.* Several rounds are played with several different objects. Heavier Lighter Taller Shorter **Classroom Connections.** In alignment with standard **K.MD.2,** the purpose of this task is to give students several opportunities to compare measurable attributes of objects. Teachers can use a variety of objects and a variety of attributes, comparing two different objects and even the same object (e.g., evaluating the width of a cereal box versus its length). These informal comparisons of attributes lead to the development of estimation strategies for measurement and the use of standard units (e.g., "How many smaller unit squares fit into a given rectangle?"). The vocabulary on the cards may be too difficult for students. The teacher may introduce the lesson by holding up real objects, such as a book or a pencil, and state, "This book is heavier than the pencil," and then pass the two objects to the students to hold. After students become familiar with the concepts (e.g., heavier or lighter) through hands-on experiences, the teacher can reinforce the concept by using the cards.

Measurement and Data K.MD

Classify objects and count the number of objects in each category.

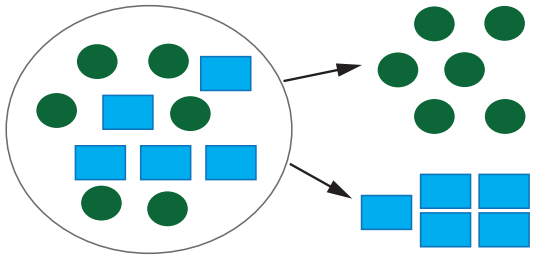
3. Classify objects into given categories; count the number of objects in each category and sort the categories by count.[3]

Kindergarten students connect counting and ordering skills and understandings to help them classify objects or people into given categories, count the number of objects in each category, and sort the categories by count (**K.MD.3**).

Students identify similarities and differences between objects (e.g., size, color, shape) and use these attributes to sort a collection of objects (**MP.2, MP.6, MP.7**).

When the objects are sorted, students count the objects in each set and then order each of the sets by the amount in each set.

Example:



For example, when given a collection of buttons, students separate buttons into different piles based on color. Next, they count the number of buttons in each pile (e.g., blue [5], green [4], orange [3], and purple [4]). Finally, they organize the groups by the quantity in each group—for example, from the smallest group (orange) to the largest group (blue), and groups with the same number (green and purple) are placed together.

Students should be able to explain their thinking. Teachers may use prompts such as these to ask students to explain their thought processes:

- Explain how you sorted the objects.
- Explain how you labeled each set with a category.
- Answer a variety of counting questions (such as "How many _____?").
- Compare the sorted groups using words such as *most, least, same,* and *different* (adapted from KATM 2012, Kindergarten Flipbook).

Focus, Coherence, and Rigor

As kindergartners classify objects, they build a foundation for collecting data and creating and analyzing graphical representations in later grades. Also, as students count the number of objects in each category and order the categories by count, they reinforce important skills and understanding in comparing numbers, which are part of the major work at this grade in the Counting and Cardinality domain (**K.CC.4–7▲**). Students can also reinforce mathematical practices as they make sense of problems by counting and recounting (**MP.1**) and explaining their process and reasoning (**MP.3**).

3. Limit category counts to be less than or equal to 10.

Domain: Geometry

A critical area of instruction in kindergarten is for students to describe shapes and space. Students develop geometric concepts and spatial reasoning from experience with the shapes of objects and the relative positions of objects.

Geometry	K.G

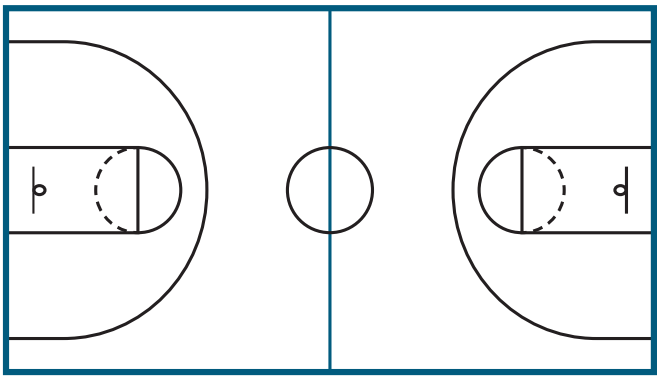
Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above, below, beside, in front of, behind,* and *next to.*
2. Correctly name shapes regardless of their orientations or overall size.
3. Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Students use positional words to describe objects in the environment (**K.G.1**). Examples of positional words include *in* and *out, inside* and *outside, down* and *up, above* and *below, over* and *under, before* and *after, top* and *bottom, front* and *back, right* and *left, on* and *off, begin* and *end,* and *near* and *far.*

Students develop spatial sense by connecting geometric shapes to their everyday lives. Students need opportunities to identify and name two- and three-dimensional shapes in and outside of the classroom and describe relative positions by answering questions such as these:

- Which way is the cafeteria? (*The cafeteria is to the right.*)
- Which shape is near the rectangle? (*The circle is near the rectangle.*)
- Where is the green ball? (*The green ball is on top of the cupboard.*)
- What types of shapes do you see on the floor of the basketball court? (*I see a rectangle and a circle on the basketball court.*)



Students begin to name and describe three-dimensional shapes with mathematical vocabulary, using words such as *sphere, cube, cylinder,* and *cone,* and answer related questions (**MP.6, MP.7**). Examples for standard **K.G.1** include the following:

- Ask students to find rectangles in the classroom and describe the relative positions of the rectangles they see. (*Possible answer:* The rectangle [a poster] is over the sphere [globe]).
- The teacher holds up objects—such as an ice-cream cone, a number cube, or a ball—and asks students to identify each shape.

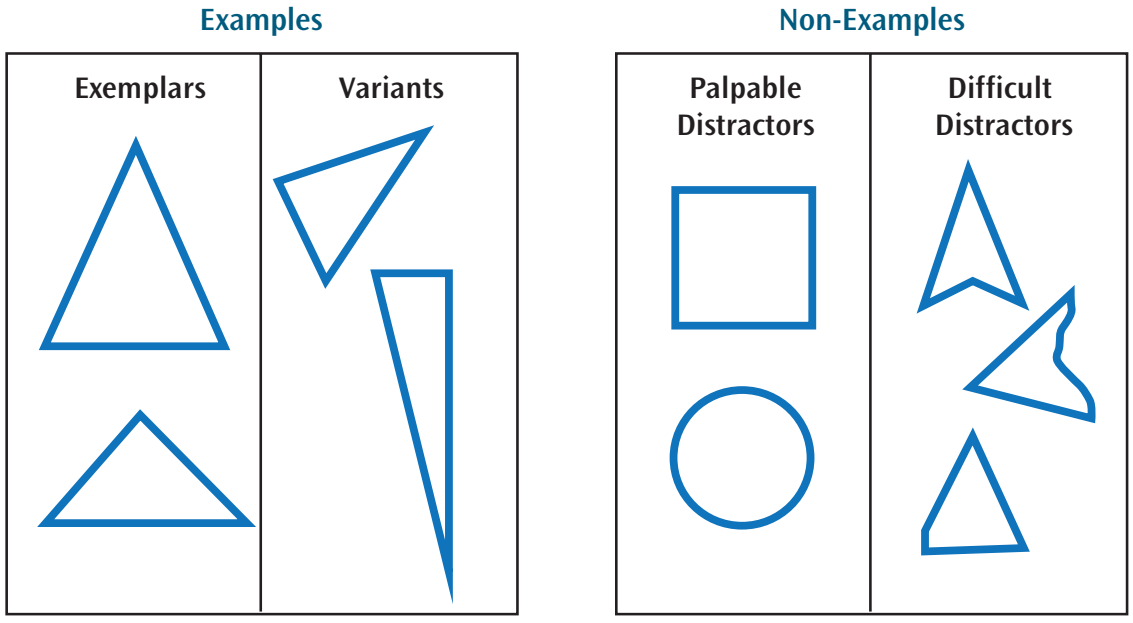
- The teacher places an object next to, behind, above, below, beside, or in front of another object and asks positional questions such as "Where is the object?" (adapted from ADE 2010; KATM 2012, Kindergarten Flipbook; and UA Progressions Documents 2012c).

Kindergarten students work with a variety of shapes that have different sizes. They learn to match two-dimensional shapes even when the shapes have different orientations (**K.G.2**). Students name shapes that occur in everyday situations, such as circles, triangles, and squares, and distinguish them from non-examples of these categories.

Students develop an intuitive image of each shape category. Figure K-3 includes examples and non-examples of triangles, as described below:

- Examples
  - Exemplars—typical visual prototypes of the shape category
  - Variants—other examples of the shape category
- Non-Examples
  - Palpable distractors—non-examples with little or no overall resemblance to the exemplars
  - Difficult distractors—visually similar to examples, but lack at least one defining attribute

Figure K-3. Examples and Non-Examples of Triangles



Adapted from UA Progressions Documents 2012c.

Common Misconceptions

- Most kindergarten students are unable to recognize an "upside-down triangle" as a triangle, because of its orientation. However, students should be exposed to many types of triangles, in many different orientations, to eliminate the misconception that a triangle is always vertex-up and equilateral.
- A square with a vertex pointing downward is often referred to as a "diamond." This needless introduction of a new shape name should be avoided, as it only serves to confuse the fact that such a shape is still a square, though its orientation is atypical.

Below are several strategies to help kindergarten students learn about shapes (**MP.6, MP.7**):

- Students form visual templates or refer to models for shape categories (e.g., children recognize a shape as a rectangle because it looks like a door).
- Students see examples of rectangles that are long and skinny and contrast rectangles with non-rectangles that appear to be similar, but lack an important defining attribute.
- Students see examples of triangles that have sides with three different lengths and then contrast triangles with non-triangles.
- The teacher hands out pairs of paper shapes in different sizes. Each student is given one shape. Then students need to find the partner who has the same shape.
- The teacher brings in a variety of spheres (a tennis ball, basketball, globe, table-tennis ball, and so on) to demonstrate that size does not change the name of a shape (adapted from ADE 2010; KATM 2012, Kindergarten Flipbook; and UA Progressions Documents 2012c).

Students identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid") [**K.G.3**] and differentiate between two-dimensional and three-dimensional shapes (**MP.6, MP.7**). For example:

- Students name a picture of a shape as two-dimensional because it is flat and can be measured in only **two** ways (by its length and width).
- Students name an object as three-dimensional because it is not flat (it is a solid object or shape) and can be measured by length, width, and height (or depth) [adapted from ADE 2010].

Geometry	**K.G**
Analyze, compare, create, and compose shapes.	
4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).	
5. Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.	
6. Compose simple shapes to form larger shapes. *For example, "Can you join these two triangles with full sides touching to make a rectangle?"*	

Kindergarten students connect their work with identifying and classifying simple shapes (refer to standards **K.G.1–3**) to help them compare shapes and manipulate two or more shapes to create a new shape. This understanding also builds foundations for students to "reason with shapes and their attributes" in grade one (refer to standards **1.G.1–3**).

Students describe similarities and differences between and among shapes using informal language (**K.G.4**). These experiences help young students begin to understand how three-dimensional shapes are composed of two-dimensional shapes—for example, the base and the top of a cylinder is a circle, the face of a cube is a square, a circle is formed in the shadow of a sphere. In early explorations of geometric properties, students discover how categories of shapes are subsumed within other categories.

Example: Sorting Shapes	**K.G.4**
Students sort a pile of squares and rectangles into two groups. They discuss how the rectangles and squares are alike and how they are different. After students demonstrate an understanding of the differences between squares and rectangles, the teacher hands out either a square or a rectangle cutout to each student. The teacher directs students with the square cutouts to one side of the room and the students with the rectangle cutouts to the opposite side of the room. The rectangle and square cutouts differ in size and color so that students focus on the shape attributes. To avoid the misconception that a square is not a rectangle, students learn informal language such as "A square is a special rectangle that has four sides of equal length."	

Students work with various triangles, rectangles, and hexagons with sides that are not all congruent. Initially, students describe shapes using everyday language and then expand their vocabulary to include geometric terms such as *sides* and *vertices* (or *corners*). Opportunities to work with pictorial representations and concrete objects, as well as technology, will help students develop their understanding and descriptive vocabulary for both two- and three-dimensional shapes (**MP.4, MP.6, MP.7**).

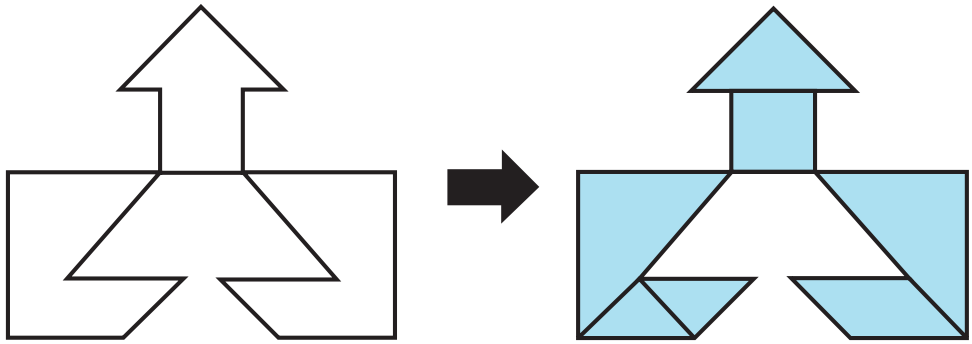
In kindergarten, students model shapes they observe in everyday life by building shapes from various components (e.g., clay, glue, tape, sticks, paper, straws) and by drawing shapes (**K.G.5**). Two-dimensional shapes are flat, and three-dimensional shapes are not flat (and can be "solid"), so students should *draw* or *create* two-dimensional shapes and *build* three-dimensional shapes (**MP.1, MP.4, MP.7**).

Students compose simple shapes to form larger shapes and answer questions such as, "Can you join these two triangles with full sides touching to make a rectangle?" (**K.G.6**). Composing shapes is an important concept in kindergarten. Students move from identifying and classifying simple shapes to manipulating two or more shapes to create a new shape. Students rotate, flip, and arrange puzzle pieces, and they move shapes to make a design or picture. Finally, students manipulate simple shapes to make a new shape (**MP.1, MP.3, MP.4, MP.7**) [adapted from KATM 2012, Kindergarten Flipbook].

Puzzles provide opportunities for students to apply spatial relationships and develop problem-solving skills in an entertaining and meaningful way. Pattern blocks and tangrams are often utilized when students work with two-dimensional shapes.

Example: Exploring Shapes with Tangrams **K.G.6**

Students make a schoolhouse using tangrams. The teacher models how to place the pieces and discusses how it is necessary to turn over, rotate, or slide pieces to complete a puzzle.



Each student or pair of students is then provided with a set of tangrams and a simple puzzle, such as the outlined version of the schoolhouse above. Students use their pieces to complete the puzzle.

Adapted from National Council of Teachers of Mathematics Illuminations 2013d.

Composing and decomposing shapes with right angles (squares, rectangles, and right triangles that also make isosceles triangles) provides important foundations for central geometric concepts (such as transformations) in later grades.

Examples of interactive tangram puzzles are available at the National Council of Teachers of Mathematics Web site (http://www.nctm.org/standards/content.aspx?id=25012 [accessed July 31, 2014]).

Essential Learning for the Next Grade

In kindergarten through grade five, the focus is on the addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals, with a balance of concepts, skills, and problem solving. Arithmetic is viewed as an important set of skills and also as a thinking subject that, done thoughtfully, prepares students for algebra. Measurement and geometry develop alongside number and operations and are tied specifically to arithmetic along the way.

In kindergarten through grade two, students focus on addition, subtraction, and measurement using whole numbers. To be prepared for grade-one mathematics, students should be able to demonstrate that they have acquired specific mathematical concepts and procedural skills by the end of kindergarten and have met the fluency expectations. For kindergartners, the expected fluencies are to add and subtract within 5 (**K.OA.5▲**). Addition and subtraction are introduced in kindergarten, and these fluencies and the conceptual understandings that support them are foundational for work in later grades.

It is particularly important for kindergarten students to attain the concepts, skills, and understandings necessary to know the number names and the count sequence (**K.CC.1–3▲**); count to tell the number of objects (**K.CC.4–5▲**); compare numbers (**K.CC.6–7▲**); understand addition as putting together and

adding to; and understand subtraction as taking apart and taking from (**K.OA.1–5▲**). Also, working with numbers to gain foundations for place value (**K.NBT.1▲**) is essential to understanding the base-ten number system.

Counting and Cardinality

In kindergarten, students learn to count. Students should connect counting to *cardinality*—knowing that the number word tells the quantity and that the number on which a person ends when counting represents the entire amount counted. Until this concept is developed, counting is merely a routine procedure done when a number is needed, and students will not understand how to apply numbers to solve problems.

By the end of kindergarten, important number concepts and skills for students include counting by ones and tens to 100 (rote counting); continuing a counting sequence when beginning from a number greater than 1 (counting on); counting objects to 20; writing numbers to 20; understanding one-to-one correspondence; identifying a quantity using both numerals and words; representing numbers with numerals (and pictures and words); understanding numbers and the relationships between numbers and quantities; and understanding the concepts of *more* and *less*. Counting to 100 and representing numbers with numerals (0 to 20) will prepare students to read and write numbers to 120 in grade one.

Addition and Subtraction

By the end of kindergarten, students are expected to add and subtract within 10 and solve addition and subtraction word problems. Students are also expected to be fluent with addition and subtraction within 5. Fluency with addition and subtraction will prepare students to add within 100 in grade one. Addition and subtraction constitute a major instructional focus for kindergarten through grade two.

Kindergarten Overview

Counting and Cardinality

- Know number names and the count sequence.
- Count to tell the number of objects.
- Compare numbers.

Operations and Algebraic Thinking

- Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

Number and Operations in Base Ten

- Work with numbers 11–19 to gain foundations for place value.

Measurement and Data

- Describe and compare measurable attributes.
- Classify objects and count the number of objects in categories.

Geometry

- Identify and describe shapes.
- Analyze, compare, create, and compose shapes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Counting and Cardinality K.CC

Know number names and the count sequence.

1. Count to 100 by ones and by tens.
2. Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
3. Write numbers from 0 to 20. Represent a number of objects with a written numeral 0–20 (with 0 representing a count of no objects).

Count to tell the number of objects.

4. Understand the relationship between numbers and quantities; connect counting to cardinality.
   a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.
   d. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.
   e. Understand that each successive number name refers to a quantity that is one larger.
5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

Compare numbers.

6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.[4]
7. Compare two numbers between 1 and 10 presented as written numerals.

Operations and Algebraic Thinking K.OA

Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

1. Represent addition and subtraction with objects, fingers, mental images, drawings,[5] sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.
2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.
3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).
4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.
5. Fluently add and subtract within 5.

4. Includes groups with up to 10 objects.
5. Drawings need not show details, but should show the mathematics in the problem. (This applies wherever drawings are mentioned in the Standards.)

Number and Operations in Base Ten — K.NBT

Work with numbers 11–19 to gain foundations for place value.

1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Measurement and Data — K.MD

Describe and compare measurable attributes.

1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

Classify objects and count the number of objects in each category.

3. Classify objects into given categories; count the numbers of objects in each category and sort the categories by count.[6]

Geometry — K.G

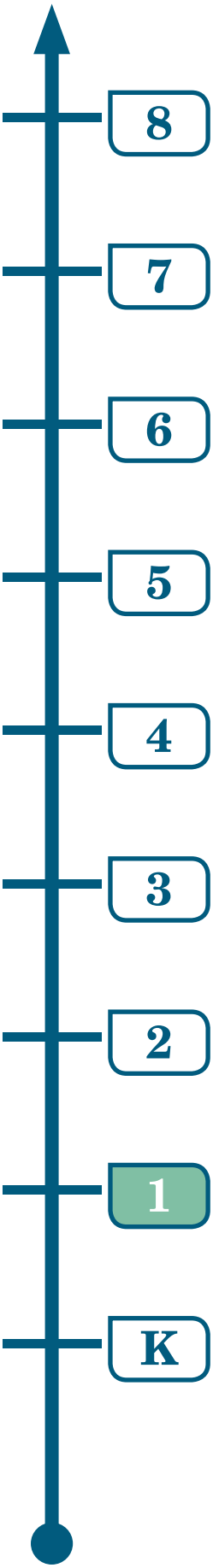
Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above, below, beside, in front of, behind,* and *next to.*
2. Correctly name shapes regardless of their orientations or overall size.
3. Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Analyze, compare, create, and compose shapes.

4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).
5. Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.
6. Compose simple shapes to form larger shapes. *For example, "Can you join these two triangles with full sides touching to make a rectangle?"*

6. Limit category counts to be less than or equal to 10.



Grade One

Grade-one students begin to develop the concept of place value by viewing 10 ones as a unit called a *ten*. This basic but essential idea is the underpinning of the base-ten number system. In kindergarten, students learned to count in order, count to find out "how many," and to add and subtract with small sets of numbers in different kinds of situations. They also developed fluency with addition and subtraction within 5. They saw teen numbers as composed of 10 ones and more ones. Additionally, kindergarten students identified and described geometric shapes and created and composed shapes (adapted from Charles A. Dana Center 2012).

Critical Areas of Instruction

In grade one, instructional time should focus on four critical areas: (1) developing understanding of addition, subtraction, and strategies for addition and subtraction within 20; (2) developing understanding of whole-number relationships and place value, including grouping in tens and ones; (3) developing understanding of linear measurement and measuring lengths as iterating length units; and (4) reasoning about attributes of and composing and decomposing geometric shapes (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010h). Students also work toward fluency in addition and subtraction with whole numbers within 10.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade-level examples of focus, coherence, and rigor are indicated throughout the chapter.

The standards do not give equal emphasis to all content for a particular grade level. Cluster headings can be viewed as the most effective way to communicate the focus and coherence of the standards. Some clusters of standards require a greater instructional emphasis than others based on the depth of the ideas, the time needed to master those clusters, and their importance to future mathematics or the later demands of preparing for college and careers.

Table 1-1 highlights the content emphases at the cluster level for the grade-one standards. The bulk of instructional time should be given to "Major" clusters and the standards within them, which are indicated throughout the text by a triangle symbol (▲). However, standards in the "Additional/Supporting" clusters should not be neglected; to do so would result in gaps in students' learning, including skills and understandings they may need in later grades. Instruction should reinforce topics in major clusters by using topics in the additional/supporting clusters and including problems and activities that support natural connections between clusters.

Teachers and administrators alike should note that the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences presented in a coherent manner (adapted from Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

Table 1-1. Grade One Cluster-Level Emphases

Operations and Algebraic Thinking — 1.OA

Major Clusters

- Represent and solve problems involving addition and subtraction. (**1.OA.1–2▲**)
- Understand and apply properties of operations and the relationship between addition and subtraction. (**1.OA.3–4▲**)
- Add and subtract within 20. (**1.OA.5–6▲**)
- Work with addition and subtraction equations. (**1.OA.7–8▲**)

Number and Operations in Base Ten — 1.NBT

Major Clusters

- Extend the counting sequence. (**1.NBT.1▲**)
- Understand place value. (**1.NBT.2–3▲**)
- Use place-value understanding and properties of operations to add and subtract. (**1.NBT.4–6▲**)

Measurement and Data — 1.MD

Major Clusters

- Measure lengths indirectly and by iterating length units. (**1.MD.1–2▲**)

Additional/Supporting Clusters

- Tell and write time. (**1.MD.3**)
- Represent and interpret data. (**1.MD.4**)

Geometry — 1.G

Additional/Supporting Clusters

- Reason with shapes and their attributes. (**1.G.1–3**)

Explanations of Major and Additional/Supporting Cluster-Level Emphases

Major Clusters (▲) — Areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than others based on the depth of the ideas, the time needed to master them, and their importance to future mathematics or the demands of college and career readiness.

Additional Clusters — Expose students to other subjects; may not connect tightly or explicitly to the major work of the grade.

Supporting Clusters — Designed to support and strengthen areas of major emphasis.

Note of caution: Neglecting material, whether it is found in the major or additional/supporting clusters, will leave gaps in students' skills and understanding and will leave students unprepared for the challenges they face in later grades.

Adapted from Achieve the Core 2012.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (MP) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The MP standards represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students.

Although the description of the MP standards remains the same at all grade levels, the way these standards look as students engage with and master new and more advanced mathematical ideas does change. Table 1-2 presents examples of how the MP standards may be integrated into tasks appropriate for students in grade one. (Refer to the Overview of the Standards Chapters for a description of the MP standards.)

Table 1-2. Standards for Mathematical Practice—Explanation and Examples for Grade One

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	In first grade, students realize that doing mathematics involves solving problems and discussing how they solved them. Students explain to themselves the meaning of a problem and look for ways to solve it. Younger students may use concrete objects or math drawings to help them conceptualize and solve problems. They may check their thinking by asking themselves, "Does this make sense?" They are willing to try other approaches.
MP.2 Reason abstractly and quantitatively.	Younger students recognize that a number represents a specific quantity. They connect the quantity to written symbols. Quantitative reasoning entails creating a representation of a problem while attending to the meanings of the quantities. First-grade students make sense of quantities and relationships while solving tasks. They represent situations by decontextualizing tasks into numbers and symbols. For example, "There are 14 children on the playground, and some children go line up. If there are 8 children still playing, how many children lined up?" Students translate the problem into the situation equation 14 – ___ = 8, then into the related equation 8 + ___ = 14, and then solve the task. Students also contextualize situations during the problem-solving process. For example, students refer to the context of the task to determine they need to subtract 8 from 14, because the number of children in line is the total number less the 8 who are still playing. To reinforce students' reasoning and understanding, teachers might ask, "How do you know" or "What is the relationship of the quantities?" Students might also reason about ways to partition two-dimensional geometric figures into halves and fourths.
MP.3 Construct viable arguments and critique the reasoning of others.	First-graders construct arguments using concrete referents, such as objects, pictures, drawings, and actions. They practice mathematical communication skills as they participate in mathematical discussions involving questions such as "How did you get that?" or "Explain your thinking" and "Why is that true?" They explain their own thinking and listen to the explanations of others. For example, "There are 9 books on the shelf. If you put some more books on the shelf and there are now 15 books on the shelf, how many books did you put on the shelf?" Students might use a variety of strategies to solve the task and then share and discuss their problem-solving strategies with their classmates.

Table 1-2 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
MP.4 Model with mathematics	In the early grades, students experiment with representing problem situations in multiple ways, including writing numbers, using words (mathematical language), drawing pictures, using objects, acting out, making a chart or list, or creating equations. Students need opportunities to connect the different representations and explain the connections. They should be able to use any of these representations as needed. First-grade students model real-life mathematical situations with an equation and check to make sure equations accurately match the problem context. Students use concrete models and pictorial representations while solving tasks and also write an equation to model problem situations. For example, to solve the problem, "There are 11 bananas on the counter. If you eat 4 bananas, how many are left?", students could write the equation 11 – 4 = 7. Students should be encouraged to answer questions such as "What math drawing or diagram could you make and label to represent the problem?" or "What are some ways to represent the quantities?"
MP.5 Use appropriate tools strategically.	Students begin to consider the available tools (including estimation) when solving a mathematical problem and decide when particular tools might be helpful. For instance, first-graders decide it might be best to use colored chips to model an addition problem. Students use tools such as counters, place-value (base-ten) blocks, hundreds number boards, concrete geometric shapes (e.g., pattern blocks or three-dimensional solids), and virtual representations to support conceptual understanding and mathematical thinking. Students determine which tools are appropriate to use. For example, when solving 12 + 8 = ____, students might explain why place-value blocks are appropriate to use to solve the problem. Students should be encouraged to answer questions such as "Why was it helpful to use _____?"
MP.6 Attend to precision.	As young children begin to develop their mathematical communication skills, they try to use clear and precise language in their discussions with others and when they explain their own reasoning. In grade one, students use precise communication, calculation, and measurement skills. Students are able to describe their solution strategies for mathematical tasks using grade-level-appropriate vocabulary, precise explanations, and mathematical reasoning. When students measure objects iteratively (repetitively), they check to make sure there are no gaps or overlaps. Students regularly check their work to ensure the accuracy and reasonableness of solutions.
MP.7 Look for and make use of structure.	First-grade students look for patterns and structures in the number system and other areas of mathematics. While solving addition problems, students begin to recognize the commutative property—for example, 7 + 4 = 11, and 4 + 7 = 11. While decomposing two-digit numbers, students realize that any two-digit number can be broken up into tens and ones (e.g., 35 = 30 + 5, 76 = 70 + 6). Grade-one students make use of structure when they work with subtraction as an unknown addend problem. For example, 13 – 7 = ____ can be written as 7 + ____ = 13 and can be thought of as "How much more do I need to add to 7 to get to 13?"

Table 1-2 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
MP.8 Look for and express regularity in repeated reasoning.	In the early grades, students notice repetitive actions in counting and computation. When children have multiple opportunities to add and subtract 10 and multiples of 10, they notice the pattern and gain a better understanding of place value. Students continually check their work by asking themselves, "Does this make sense?" Grade-one students begin to look for regularity in problem structures when solving mathematical tasks. For example, students add three one-digit numbers by using strategies such as "make a ten" or doubles. Students recognize when and how to use strategies to solve similar problems. For example, when evaluating 8 + 7 + 2, a student may say, "I know that 8 and 2 equals 10, then I add 7 to get to 17. It helps if I can make a ten out of two numbers when I start." Students use repeated reasoning while solving a task with multiple correct answers—for example, the problem "There are 12 crayons in the box. Some are red and some are blue. How many of each color could there be?" For this particular problem, students use repeated reasoning to find pairs of numbers that add up to 12 (e.g., the 12 crayons could include 6 of each color [6 + 6 = 12], 7 of one color and 5 of another [7 + 5 = 12], and so on). Students should be encouraged to answer questions such as "What is happening in this situation?" or "What predictions or generalizations can this pattern support?"

Adapted from Arizona Department of Education (ADE) 2010 and North Carolina Department of Public Instruction (NCDPI) 2013b.

Standards-Based Learning at Grade One

The following narrative is organized by the domains in the Standards for Mathematical Content and highlights some necessary foundational skills from previous grade levels. It also provides exemplars to explain the content standards, highlight connections to Standards for Mathematical Practice (**MP**), and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. A triangle symbol (▲) indicates standards in the major clusters (see table 1-1).

Domain: Operations and Algebraic Thinking

In kindergarten, students added and subtracted small numbers and developed fluency with these operations with whole numbers within 5. A critical area of instruction for students in grade one is to develop an understanding of and strategies for addition and subtraction within 20. First-grade students also become fluent with these operations within 10.

Students in first grade represent word problems (e.g., using objects, drawings, and equations) and relate strategies to a written method to solve addition and subtraction word problems within 20 (**1.OA.1–2▲**). Grade-one students extend their prior work in three major and interrelated ways:

- They use Level 2 and Level 3 problem-solving methods to extend addition and subtraction problem solving from within 10, to problems within 20 (see table 1-3).
- They represent and solve for all unknowns in all three problem types: add to, take from, and put together/take apart (see table 1-4).
- They represent and solve a new problem type: "compare" (see table 1-5).

To solve word problems, students learn to apply various computational methods. Kindergarten students generally use Level 1 methods, and students in first and second grade use Level 2 and Level 3 methods. The three levels are summarized in table 1-3 and explained more thoroughly in appendix C.

Table 1-3. Methods Used for Solving Single-Digit Addition and Subtraction Problems

Level 1: Direct Modeling by Counting All or Taking Away

Represent the situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Level 2: Counting On

Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total, but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. The count is tracked and monitored in some way (e.g., with fingers, objects, mental images of objects, body motions, or other count words).

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as the unknown addend).

Level 3: Converting to an Easier Equivalent Problem

Decompose an addend and compose a part with another addend.

Adapted from the University of Arizona (UA) Progressions Documents for the Common Core Math Standards 2011a.

Operations and Algebraic Thinking 1.OA

Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.[1]
2. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

In kindergarten, students worked with the following types of addition and subtraction situations: add to (with result unknown); take from (with result unknown); and put together/take apart (with total unknown and both addends unknown). First-graders extend this work to include problems with larger numbers and unknowns in all positions (see table 1-4). In first grade, students are also introduced to a new type of addition and subtraction problem—"compare" problems (see table 1-5).

Students in first grade add and subtract within 20 (**1.OA.1–2▲**) to solve the types of problems shown in tables 1-4 and 1-5 (**MP.1, MP.2, MP.3, MP.4, MP.5, MP.6**). A major goal for grade-one students is the use of "Level 2: Counting On" methods for addition (find the total) and subtraction (find the unknown addend). Level 2 methods represent a new challenge for students, because when students "count

1. See glossary, table GL-4.

on," an addend is already embedded in the total to be found; it is the named starting number of the "counting on" sequence. The new problem subtypes with which grade-one students work support the development of this "counting on" strategy. In particular, "compare" problems that are solved with tape diagrams can serve as a visual support for this strategy, and they are helpful as students move away from representing all objects in a problem to representing objects solely with numbers (adapted from UA Progressions Documents 2011a).

Initially, addition and subtraction problems include numbers that are small enough for students to make math drawings to solve problems that include all the objects. Students also use the addition symbol (+) to represent "add to" and "put together" situations, the subtraction symbol (−) to represent "take from" and "take apart" situations, and the equal sign (=) to represent a relationship regarding equality between one side of the equation and the other.

Table 1-4. Grade-One Addition and Subtraction Problem Types (Excluding "Compare" Problems)

Type of Problem	Result Unknown	Change Unknown	Start Unknown
Add to	Chris has 11 toy cars. José gave him 5 more. How many does Chris have now? This problem could be represented by 11 + 5 = ☐. General Case: $A + B = ☐$.	Bill had 5 toy robots. His mom gave him some more. Now he has 9 robots. How many toy robots did his mom give him? In this problem, the starting quantity is provided (5 robots), a second quantity is added to that amount (some robots), and the result quantity is given (9 robots). This question type is more algebraic and challenging than the "result unknown" problems and can be modeled by a situational equation (5 + ☐ = 9), which can be solved by counting on from 5 to 9. [Refer to standard **1.OA.6** for examples of addition and subtraction strategies that students use to solve problems.] General Case: $A + ☐ = C$.	Some children were playing on the playground, and 5 more children joined them. Then there were 12 children. How many children were playing before? This problem can be represented by ☐ + 5 = 12. The "start unknown" problems are difficult for students to solve because the initial quantity is unknown and therefore cannot be represented. Children need to see both addends as making the total, and then some children can solve this by 5 + ☐ = 12. General Case: $☐ + B = C$.

Table 1-4 *(continued)*

Type of Problem	Result Unknown	Change Unknown	Start Unknown
Take from	There were 20 oranges in the bowl. The family ate 5 oranges from the bowl. How many oranges are left in the bowl? This problem can be represented by $20 - 5 = \square$. General Case: $C - B = \square$.	Andrea had 8 stickers. She gave some stickers away. Now she has 2 stickers. How many stickers did she give away? This question can be modeled by a situational equation ($8 - \square = 2$) or a solution equation ($8 - 2 = \square$). Both the "take from" and "add to" questions involve actions. General Case: $C - \square = A$.	Some children were lining up for lunch. Four (4) children left, and then there were 6 children still waiting in line. How many children were there before? This problem can be modeled by $\square - 4 = 6$. Similar to the previous "add to (start unknown)" problem, the "take from" problems with the start unknown require a high level of conceptual understanding. Children need to see both addends as making the total, and then some children can solve this by $4 + 6 = \square$. General Case: $\square - B = A$.

	Total Unknown	Addend Unknown	Both Addends Unknown†
Put together/ Take apart§	There are 6 blue blocks and 7 red blocks in the box. How many blocks are there? This problem can be represented by $7 + 6 = \square$. General Case: $A + B = \square$.	Roger puts 10 apples in a fruit basket. Four (4) are red and the rest are green. How many are green? There is no direct or implied action. The problem involves a set and its subsets. It can be modeled by $10 - 4 = \square$ or $4 + \square = 10$. This type of problem provides students with opportunities to understand addends that are hiding inside a total and also to relate subtraction and an unknown-addend problem. General Case: $A + \square = C$. General Case: $C - A = \square$.	Grandma has 9 flowers. How many can she put in her green vase and how many in her purple vase? Students will name all the combinations of pairs that add to nine: $9 = 0 + 9$ $9 = 9 + 0$ $9 = 1 + 8$ $9 = 8 + 1$ $9 = 2 + 7$ $9 = 7 + 2$ $9 = 3 + 6$ $9 = 6 + 3$ $9 = 4 + 5$ $9 = 5 + 4$ Being systematic while naming the pairs is efficient. Students should notice that the pattern repeats after $5 + 4$ and know that they have named all possible combinations. General Case: $C = \square + \square$.

Note: In this table, the darkest shading indicates the problem subtypes introduced in kindergarten. Grade-one and grade-two students work with all problem subtypes. The unshaded problems are the most difficult subtypes that students work with in grade one, but students need not master these problems until grade two.

†These take-apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign (=), help children understand that the = sign does not always mean *makes* or *results in,* but does always mean *is the same number as.*

§Either addend can be unknown, so there are three variations of these problem situations. "Both Addends Unknown" is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

"Compare" problems are introduced in first grade (see table 1-5 for examples). In a compare situation, two quantities are compared to find "How many more" or "How many less." One reason "compare" problems are more advanced than the other two major problem types is that in "compare" problems, one of the quantities (the difference) is not present in the situation physically; it must be conceptualized and constructed in a representation by showing the "extra" that, when added to the smaller unknown, makes the total equal to the bigger unknown, or by finding this quantity embedded in the bigger unknown.

Table 1-5. Grade-One Addition and Subtraction Problem Types ("Compare" Problems)

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	Pat has 9 peaches. Lynda has 4 peaches. How many more peaches does Lynda have than Pat? "Compare" problems involve relationships between quantities. Although most adults might use subtraction to solve this type of Compare problem ($9 - 4 = \square$), students will often solve this problem as an unknown-addend problem ($4 + \square = 9$) or by using a "counting up" or matching strategy. In all mathematical problem solving, what matters is the explanation a student gives to relate a representation to a context—not the representation separated from its context. General Case: $A + \square = C$. General Case: $C - A = \square$.	**"More" version** Theo has 7 action figures. Rosa has 2 more action figures than Theo. How many action figures does Rosa have? This problem can be modeled by $7 + 2 = \square$.	**"Fewer" version** Bill has 8 stamps. Lisa has 2 fewer stamps than Bill. How many stamps does Lisa have? This problem can be modeled as $8 - 2 = \square$.
		"Fewer" version—with misleading language Lucy has 8 apples. She has 2 fewer apples than Marcus. How many apples does Marcus have? This problem can be modeled as $8 + 2 = \square$. The misleading word *fewer* may lead students to choose the wrong operation. General Case: $A + B = \square$.	**"More" version—with misleading language** David has 7 more bunnies than Keisha. David has 8 bunnies. How many bunnies does Keisha have? This problem can be modeled as $8 - 7 = \square$. The misleading word *more* may lead students to choose the wrong operation. General Case: $C - B = \square$. General Case: $\square + B = C$.

Note: This table shows that grade-one and grade-two students work with all "compare" problem types. The unshaded problems are the most difficult problem types that students work with in grade one, but students need not master these problems until grade two.

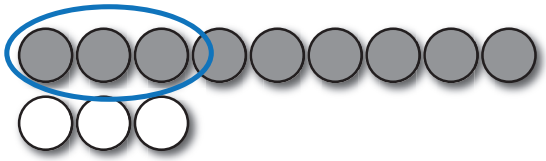
Adapted from NGA/CCSSO 2010d and UA Progressions Documents 2011a.

The language of these problems may also be difficult for students. For example, "Julie has 3 more apples than Lucy" states that both (a) Julie has more apples and (b) the difference is 3. Many students "hear" the part of the sentence about *who* has more, but do not initially hear the part about *how many more.* Students need experience hearing and saying a separate sentence for each of the two parts to help them comprehend and say the one-sentence form.

Example: Strategies for Solving "Compare" Problems **1.OA.1▲**

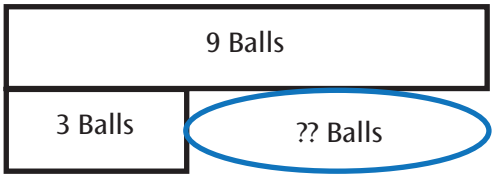
Abel has 9 balls. Susan has 3 balls. How many more balls does Abel have than Susan?

Students use objects to represent the two sets of balls and compare them.

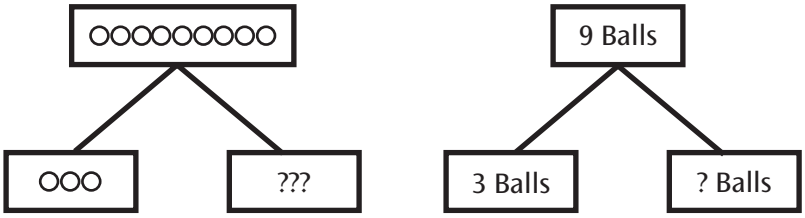


Teachers may also ask the related question, "How many fewer balls does Susan have than Abel?"

Students also use comparison bars. Rather than representing the actual objects with manipulatives or drawings, they use the numbers in the problem to represent the quantities.



Finally, students also work with number-bond diagrams, such as those shown below. They might use drawings that represent quantities or drawings that show only the numbers presented in a problem.



Although most adults know to solve "compare" problems with subtraction, students often represent these problems as missing-addend problems (e.g., representing the previous example involving Abel and Susan as 3 + □ = 9). Student methods such as these should be explored, and the connection between addition and subtraction made explicit (adapted from UA Progressions Documents 2011a).

As mentioned previously, the language and conceptual demands of "compare" problems are challenging for students in grade one. Some students may also have difficulty with the conceptual demands of "start unknown" problems. Grade-one students should have the opportunity to solve and discuss such problems, but proficiency with these most difficult subtypes should not be expected until grade two.

Literature can be incorporated into problem solving with young students. Many literature books include mathematical ideas and concepts. Books that contain problem situations involving addition and subtraction with the numbers 0 through 20 would be appropriate for grade-one students (Kansas Association of Teachers of Mathematics [KATM] 2012, 1st Grade Flipbook).

Focus, Coherence, and Rigor

Problems that provide opportunities for students to explain their thinking and use objects and drawings to represent word problems (**1.OA.▲**) also reinforce the Standards for Mathematical Practice, such as making sense of problems (**MP.1**), reasoning quantitatively to make sense of quantities and their relationships in problems (**MP.2**), and justifying conclusions (**MP.3**).

Common Misconceptions

- Some students misunderstand the meaning of the equal sign. The equal sign means *is the same as,* but many primary students think the equal sign means *the answer is coming up* to the right of the equal sign. When students are introduced only to examples of number sentences with the operation to the left of the equal sign and the answer to the right, they overgeneralize the meaning of the equal sign, which creates this misconception. First-graders should see equations written in multiple ways—for example, 5 + 7 = 12 and 12 = 5 + 7. The put together/take apart (with both addends unknown) problems are particularly helpful for eliciting equations such as 12 = 5 + 7 (with the sum to the left of the equal sign). Consider this problem: "Robbie puts 12 balls in a basket. Some of the balls are orange and the rest are black. How many are orange and how many are black?" These equations can be introduced in kindergarten with small numbers (e.g., 5 = 4 + 1), and they should be used throughout grade one.
- Many students assume key words or phrases in a problem suggest the same operation every time. For example, students might assume the word *left* always means they need to subtract to find a solution. To help students avoid this misconception, include problems in which key words represent different operations. For example, "Joe took 8 stickers he no longer wanted and gave them to Anna. Now Joe has 11 stickers left. How many stickers did Joe have to begin with?" Facilitate students' understanding of scenarios represented in word problems. Students should analyze word problems (**MP.1, MP.2**) and not rely on key words.

Adapted from KATM 2012, 1st Grade Flipbook.

Grade-one students solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20 (**1.OA.2▲**). Students can collaborate in small groups to develop problem-solving strategies. Grade-one students use a variety of strategies and models—such as drawings, words, and equations with symbols for the unknown numbers—to find solutions. Students explain, write, and reflect on their problem-solving strategies (**MP.1, MP.2, MP.3, MP.4, MP.6**). For example, each student could write or draw a problem in which three groups of items (whose sum is within 20) are to be combined. Students might exchange their problems with other students, solve them individually, and then discuss their models and solution strategies. The students work together to solve each problem using a different strategy. The level of difficulty for these problems also may be differentiated by using smaller numbers (up to 10) or larger numbers (up to 20).

Operations and Algebraic Thinking 1.OA

Understand and apply properties of operations and the relationship between addition and subtraction.

3. Apply properties of operations as strategies to add and subtract.[2] *Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)*
4. Understand subtraction as an unknown-addend problem. *For example, subtract 10 – 8 by finding the number that makes 10 when added to 8.*

First-grade students build their understanding of the relationship between addition and subtraction. Instruction should include opportunities for students to investigate, identify, and then apply a pattern or structure in mathematics. For example, pose a string of addition and subtraction problems involving the same three numbers chosen from the numbers 0 to 20 (e.g., 4 + 6 = 10 and 6 + 4 = 10; or 10 – 6 = 4 and 10 – 4 = 6). These are *related facts*—a set of three numbers that can be expressed with an addition or subtraction equation. Related facts help develop an understanding of the relationship between addition and subtraction and the commutative and associative properties.

Students apply properties of operations as strategies to add and subtract (**1.OA.3▲**). Although it is not necessary for grade-one students to learn the names of the properties, students need to understand the important ideas of the following properties:

- **Identity property of addition** (e.g., 6 = 6 + 0) — adding 0 to a number results in the same number.
- **Identity property of subtraction** (e.g., 9 – 0 = 9) — subtracting 0 from a number results in the same number.
- **Commutative property of addition** (e.g., 4 + 5 = 5 + 4) — the order in which you add numbers does not matter.
- **Associative property of addition** (e.g., 3 + (9 + 1) = (3 + 9) + 1 = 12 + 1 = 13) — when adding more than two numbers, it does not matter which numbers are added together first.

Example	**1.OA.3▲**
To show that order does not change the result in the operation of addition, students build a tower of 8 green cubes and 3 yellow cubes, and another tower of 3 yellow cubes and 8 green cubes. Students can also use cubes of 3 different colors to demonstrate that (2 + 6) + 4 is equivalent to 2 + (6 + 4) and then to prove 2 + (6 + 4) = 2 + 10.	

Adapted from KATM 2012, 1st Grade Flipbook.

2. Students need not use formal terms for these properties.

Focus, Coherence, and Rigor

Students apply the commutative and associative properties as strategies to solve addition problems (**1.OA.3▲**); these properties do not apply to subtraction. They use mathematical tools, such as cubes and counters, and visual models (e.g., drawings and a 100 chart) to model and explain their thinking. Students can share, discuss, and compare their strategies as a class (**MP.2, MP.7, MP.8**).

Students understand subtraction as an unknown-addend problem (**1.OA.4▲**). Word problems such as put together/take apart (with addend unknown) afford students a context to see subtraction as the opposite of addition by finding an unknown addend. Understanding subtraction as an unknown-addend addition problem is one of the essential understandings students will need in middle school to extend arithmetic to negative rational numbers (adapted from ADE 2010 and UA Progressions Documents 2011a).

Common Misconceptions

Students may assume that the commutative property applies to subtraction. After students have discovered and applied the commutative property of addition, ask them to investigate whether this property works for subtraction. Have students share and discuss their reasoning with each other; guide them to conclude that the commutative property does *not* apply to subtraction (adapted from KATM 2012, 1st Grade Flipbook). This may be challenging. Students might think they can switch the addends in subtraction equations because of their work with related-fact equations using the commutative property for addition. Although $10 - 2 = 8$ and $10 - 8 = 2$ are related equations, they do not constitute an example of the commutative property because the differences are not the same. Students also need to understand that they cannot switch the total and an addend (for example: $10 - 2$ and $2 - 10$) and get the same difference.

Operations and Algebraic Thinking 1.OA

Add and subtract within 20.

5. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).
6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

Primary students come to understand addition and subtraction as they connect counting and number sequence to these operations (**1.OA.5▲**). First-grade students connect *counting on* and *counting back* to addition and subtraction. For example, students count on (3) from 4 to solve the addition problem $4 + 3 = 7$. Similarly, students count back (3) from 7 to solve the subtraction problem $7 - 3 = 4$. The "counting all" strategy requires students to count an entire set. The "counting on" and "counting back" strategies occur when students are able to hold the start number in their head and count on from that number. Students generally have difficulty knowing where to begin their count when counting backward,

so it is much better to restate the subtraction as an unknown addend and solve by counting on: "7 – 3 means 3 + □ = 7, so 4, 5, 6, 7 . . . I counted on 4 more to get to 7, so 4 is the answer." Solving subtraction problems by counting on helps to reinforce the concept that subtraction problems are missing-addend problems, which is important for students' later understanding of operations with rational numbers.

Students will use different strategies to solve problems if given the time and space to do so. Teachers should explore the various methods that arise as students work to understand general properties of operations.

Example: Students use different strategies to solve a problem.	**1.OA.6▲**
There are crayons in a box. There are 4 green crayons, 5 blue crayons, and 6 red crayons. How many crayons are in the box? Explain to others how you found your answer. **Student 1** ***(Adding with a 10-frame and counters)*** I put 4 counters on a 10-frame for the green crayons. Then I put 5 different-colored counters on the 10-frame for the blue crayons. And then I put another 6 color counters out for the red crayons. Only one of the crayons fit, so I had 5 left over. One 10-frame and 5 left over make 15 crayons (**MP.2, MP.3, MP.5**) (**1.OA.2▲**). **Student 2** ***(Making tens)*** I know that 4 and 6 equal 10, so the green and red equal 10 crayons. Then I added the 5 blue crayons to get 15 total crayons (**MP.2. MP.6**) (**1.OA.3▲**). **Student 3** ***(Counting on)*** I counted on from 6, first counting on 5 to get 11 and then counting on 4 to get 15. I used my fingers to keep track of the 5 and the 4. But now I see that because 5 and 4 make 9, I could have counted on 6 from 9. So there were 15 total crayons (**MP.1, MP.2**) (**1.OA.6▲**).	

First-grade students use various strategies to add and subtract within 20 (**1.OA.6▲**). Students need ample opportunities to model operations using various strategies and explain their thinking (**MP.2, MP.7, MP.8**).

Example: 8 + 7 = _____	**1.OA.6▲**
Student 1 ***(Making 10 and decomposing a number)*** I know that 8 plus 2 is 10, so I decomposed (broke up) the 7 into a 2 and a 5. First I added 8 and 2 to get 10, and then I added the 5 to get 15. 8 + 7 = (8 + 2) + 5 = 10 + 5 = 15	**Student 2** ***(Creating an easier problem with known sums)*** I know 8 is 7 + 1. I also know that 7 and 7 equal 14. Then I added 1 more to get 15. 8 + 7 = (7 + 7) + 1 = 15
Example: 14 – 6 = _____	**1.OA.6▲**
Student 1 ***(Decomposing the number you subtract)*** I know that 14 minus 4 is 10, so I broke up the 6 into a 4 and a 2. 14 minus 4 is 10. Then I take away 2 more to get 8. 14 – 6 = (14 – 4) – 2 = 10 – 2 = 8	**Student 2** ***(Relationship between addition and subtraction)*** I know that 6 plus 8 is 14, so that means that 14 minus 6 is 8. 6 + 8 = 14, so 14 – 6 = 8. If I didn't know 6 + 8 = 14, I could start by making a ten: 6 + 4 is 10, and 4 more is 14, and 4 plus 4 is 8.

Adapted from ADE 2010 and Georgia Department of Education (GaDOE) 2011.

Students begin to develop algebraic understanding when they create equivalent expressions to solve a problem (such as when they write a situation equation and then write a solution equation from that) or use addition or subtraction combinations they know to solve more difficult problems.

FLUENCY

California's Common Core State Standards for Mathematics (K–6) set expectations for fluency in computation (e.g., *fluently* add and subtract within 10) [**1.OA.6▲**]. Such standards are culminations of progressions of learning, often spanning several grades, involving conceptual understanding, thoughtful practice, and extra support where necessary. The word *fluent* is used in the standards to mean "reasonably fast and accurate" and possessing the ability to use certain facts and procedures with enough facility that using such knowledge does not slow down or derail the problem solver as he or she works on more complex problems. Procedural fluency requires skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Developing fluency in each grade may involve a mixture of knowing some answers, knowing some answers from patterns, and knowing some answers through the use of strategies.

Adapted from UA Progressions Documents 2011a.

Some strategies to help students develop understanding and fluency with addition and subtraction include the use of 10-frames or math drawings, comparison bars, and number-bond diagrams. The use of visuals (e.g., hundreds charts and base-ten representations) can also support fluency and number sense.

Students continue to develop meanings for addition and subtraction as they encounter problem situations in kindergarten through grade two. They expand their ability to represent problems, and they use increasingly sophisticated computation methods to find answers. In each grade, the situations, representations, and methods should foster growth from one grade to the next.

Operations and Algebraic Thinking 1.OA

Work with addition and subtraction equations.

7. Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. *For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 – 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.*
8. Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = □ – 3, 6 + 6 = □.*

Students need to understand the meaning of the equal sign (**1.OA.7▲**) and know that the quantity on one side of the equal sign must be the same quantity as on the other side of the equal sign. Interchanging the language of *equal to* and *is the same as,* as well as *not equal to* and *is not the same as,* will help students grasp the meaning of the equal sign.

To avoid common pitfalls such as the equal sign meaning "to do something" or the equal sign meaning "the answer is," students should be able to:

- express their understanding of the meaning of the equal sign;
- realize that sentences other than $a+b=c$ are true (e.g., $a=a$, $c=a+b$, $a=a+0$, $a+b=b+a$);
- know the equal sign represents a relationship between two equal quantities;
- compare expressions without calculating. For example, a student evaluates 3 + 4 = 3 + 3 + 2. She says, "I know this statement is false because there is a 3 on both sides of the equal sign, but the right side has 3 + 2, and that makes 5, which is more than 4. So the two sides can't be equal."

True/False Statements for Developing Understanding of the Equal Sign 1.OA.7▲	
7 = 8 – 1	9 + 3 = 10
8 = 8	5 + 3 = 10 – 2
1 + 1 + 3 =7	3 + 4 + 5 = 3 + 5 + 4
4 + 3 = 3 + 4	3 + 4 + 5 = 7 + 5
6 – 1 = 1 – 6	13 = 10 + 4
12 + 2 – 2 = 12	10 + 9 + 1 = 19

Initially, students develop an understanding of the meaning of equality using models. Students can justify their answers, make conjectures (e.g., if you start with zero and add a number and then subtract that same number, you always get zero), and use estimation to support their understanding of equality (adapted from ADE 2010 and KATM 2012, 1st Grade Flipbook).

Domain: Number and Operations in Base Ten

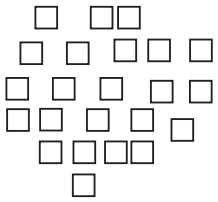
In kindergarten, students developed an important foundation for understanding the base-ten system: they viewed "teen" numbers as composed of 10 ones and some more ones. A critical area of instruction in grade one is to extend students' place-value understanding to view 10 ones as a unit called a *ten* and two-digit numbers as amounts of tens and ones (UA Progressions Documents 2012b).

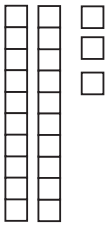
Number and Operations in Base Ten 1.NBT

Extend the counting sequence.

1. Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.

First-grade students extend reading and writing numerals beyond 20—to 120 (**1.NBT.1▲**). Students use objects, words, and symbols to express their understanding of numbers. For a given numeral, students count out the given number of objects, identify the quantity that each digit represents, and write and read the numeral (**MP.2, MP.7, MP.8**). For example:





Tens	Ones
2	3

23

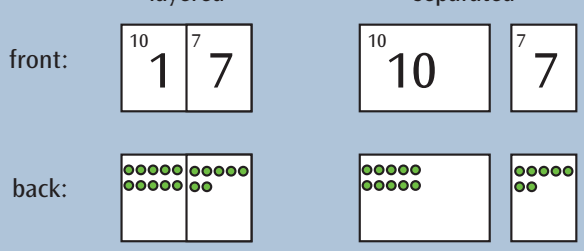
Twenty-three

Group of ones	Group of 2 tens and 3 ones	Place-value table	Write the number	Read and say the number

Source: Ohio Department of Education (ODE) 2011.

Seeing different representations can help students develop an understanding of numbers. Posting the number words in the classroom helps students to read and write the words. Extending hundreds charts to 120 and displaying them in the classroom can help students connect place value to the numerals and the words for the numbers 1 to 120. Students may need extra support with decade and century numbers when they orally count to 120. These transitions will be signaled by a 9 and require new rules to generate the next set of numbers. Students need experience counting from different starting points (e.g., start at 83 and count to 120).

Place-value cards



Children can use layered place-value cards to see the 10 "hiding" inside any teen number. Such decompositions can be connected to numbers represented with objects and math drawings.

Notice the power of the vertical hundreds chart: You can see all 9 of the tens in the numbers 91 to 99.

Part of a number list

91	101	111
92	102	112
93	103	113
94	104	114
95	105	115
96	106	116
97	107	117
98	108	118
99	109	119
100	110	120

In the classroom, a list of the numerals from 1 to 120 can be shown in columns of 10 to help highlight the base-ten structure. The numbers 101, . . . , 120 may be especially difficult for children to write.

Source: UA Progressions Documents 2012b.

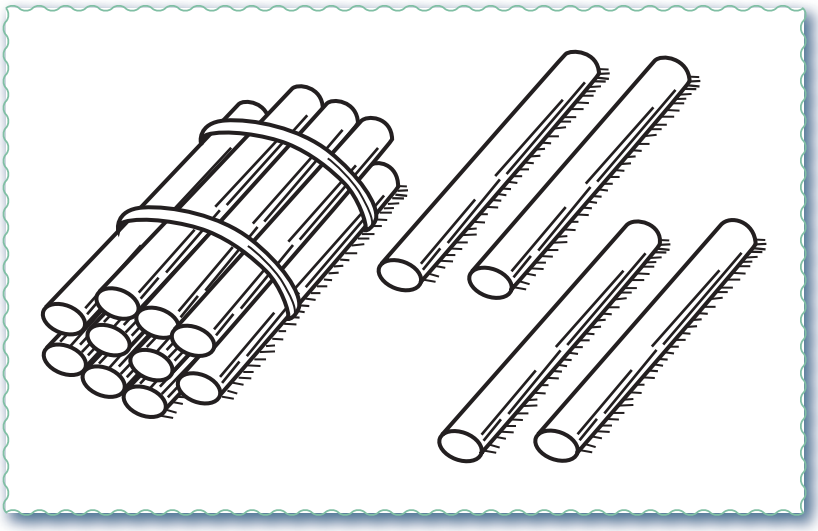
Number and Operations in Base Ten — 1.NBT

Understand place value.

2. Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:
   a. 10 can be thought of as a bundle of ten ones—called a "ten."
   b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.
   c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).
3. Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.

Grade-one students learn that the two digits of a two-digit number represent amounts of tens and ones (e.g., 67 represents 6 tens and 7 ones) (**1.NBT.2▲**).

Understanding the concept of a *ten* is fundamental to young students' mathematical development. This is the foundation of the place-value system. In kindergarten, students thought of a group of 10 cubes as 10 individual cubes. First-grade students understand 10 cubes as a bundle of 10 ones, or a *ten* (**1.NBT.2a▲**). Students can demonstrate this concept by counting 10 objects and "bundling" them into one group of 10 (**MP.2, MP.6, MP.7, MP.8**).



Students count between 10 and 20 objects and can make a bundle of 10 with or without some left over, which can help students write teen numbers (**1.NBT.2b▲**). They can continue counting any number of objects up to 99, making bundles of tens with or without leftovers (**1.NBT.2c▲**). For example, a student represents the number 14 as one bundle (one group of 10) with four left over.

Students can also use models to express larger numbers as bundles of tens and 0 ones or some leftover ones. Students explain their thinking in different ways. For example:

> **Teacher:** For the number 42, do you have enough to make 4 tens? Would you have any left? If so, how many would you have left?
>
> **Student 1:** I filled 4 10-frames to make 4 tens and had 2 counters left over. I had enough to make 4 tens with some left over. The number 42 has 4 tens and 2 ones.
>
> **Student 2:** I counted out 42 place-value cubes. I traded each group of 10 cubes for a 10-rod (stick). I now have 4 10-rods and 2 cubes left over. So the number 42 has 4 tens and 2 ones (adapted from ADE 2010).

Students learn to read 53 as *fifty-three* as well as 5 tens and 3 ones. However, some number words require extra attention at first grade because of their irregularities. Students learn that the decade words (e.g., *twenty, thirty, forty,* and so on) indicate 2 tens, 3 tens, 4 tens, and so on. They also realize many decade number words sound much like teen number words. For example, *fourteen* and *forty* sound very similar, as do *fifteen* and *fifty,* and so on to *nineteen* and *ninety.* Students learn that the number words from 13 to 19 give the number of ones before the number of tens. Students also frequently make counting errors such as "twenty-nine, twenty-*ten,* twenty-*eleven,* twenty-*twelve*" (UA Progressions Documents 2012b). Because of these complexities, it can be helpful for students to use regular tens words as well as English words—for example, "The number 53 is 5 tens, 3 ones, and also fifty-three."

Grade-one students use base-ten understanding to recognize that the digit in the tens place is more important than the digit in the ones place for determining the size of a two-digit number (**1.NBT.3▲**). Students use models that represent two sets of numbers to compare numbers. Students attend to the number of tens and then, if necessary, to the number of ones. Students may also use math drawings of tens and ones and spoken or written words to compare two numbers. Comparative language includes but is not limited to *more than, less than, greater than, most, greatest, least, same as, equal to,* and *not equal to* (**MP.2, MP.6, MP.7, MP.8**) [adapted from ADE 2010].

Table 1-6 presents a sample classroom activity that connects the Standards for Mathematical Content and Standards for Mathematical Practice.

Table 1-6. Connecting to the Standards for Mathematical Practice—Grade One

Standards Addressed	Explanation and Examples
Connections to Standards for Mathematical Practice **MP.2.** Students reason abstractly and quantitatively as they move between the written representation of numbers and the base-ten block representation of numbers. **MP.5.** Students develop an understanding of the use of base-ten blocks that will lay a foundation for using these blocks to develop and understand algorithms for operations. **MP.7.** Students begin to see that the numbers 0–9 can be represented with units only and that while the same is true for larger numbers, they can use bundles of ten units to represent them in a more organized way. This leads to the recording of numbers in the way that we do (e.g., 12 = 10 + 2, 1 stick and 2 units).	**Task.** The teacher has a spinner with the digits 0–9 on it. Each student has a collection of base-ten block units and rods (or "sticks"). The object of the task is for students to use their base-ten blocks to represent numbers spun by the teacher, add the resulting numbers, and then represent the sum using the base-ten blocks, exchanging 10 units for a rod when appropriate. For example, the teacher's first spin is a 6. She asks the students to represent 6 on the left side of their desk (or a provided mat). Then the teacher spins an 8, and students represent an 8 on the other side of their desk or mat. The teacher then instructs students to add the number of units together. Students will most likely combine the two piles and count the resulting number of units: 14. The teacher should then encourage students to exchange 10 units for a rod to emphasize that the number 14 represents 1 ten and 4 ones (that is, "1 rod and 4 units"). This can be repeated for several turns so that students represent larger numbers, adding and bundling more as the numbers increase.
Standards for Mathematical Content **1.OA.6▲.** Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten; decomposing a number leading to a ten; using the relationship between addition and subtraction; and creating equivalent but easier or known sums. **1.NBT.2▲.** Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases: a. 10 can be thought of as a bundle of ten ones—called a "ten." b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones. c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90, refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones). *Extension* **1.NBT.3▲.** Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.	**Possible Extensions** • Teachers could use spinners with different numbers on them (e.g., 0–19), and students can represent the numbers and compare them. • Teachers can ask students to subtract the smaller number from the larger number. • Teachers can use a spinner with 0–9, and students can count the indicated number of rods and name the number—for example, the teacher spins a 6, then the students take out 6 rods and record and name the resulting number (60). • The first spin could represent the number of units, and the second spin could represent the number of sticks. **Classroom Connections.** A firm foundation in understanding the base-ten structure of the number system is essential for student success with operations, decimals, proportional reasoning, and later algebra. Experiences such as these give students ample practice in representing and explaining why numbers are written the way they are. Students can begin to associate mental images of why numbers have the value that they do (e.g., why the number 20 is different from and larger than the number 2).

Number and Operations in Base Ten — 1.NBT

Use place-value understanding and properties of operations to add and subtract.

4. Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.
5. Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.
6. Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Students develop understandings and strategies to add within 100 using visual models to support understanding (**1.NBT.4▲**). In grade one, students focus on developing, discussing, and using efficient, accurate, and generalizable methods to add within 100, and they subtract multiples of 10. Students might also use strategies they invent that are not generalizable.

Focus, Coherence, and Rigor

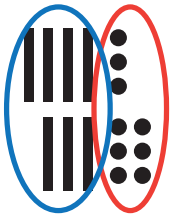
Grade-one students develop understanding of addition and subtraction within 20 using various strategies (**1.OA.6▲**), and they generalize their methods to add within 100 using concrete models and drawings (**1.NBT.4▲**). Reasoning about strategies and selecting appropriate strategies are critical to developing conceptual understanding of addition and subtraction in all situations (**MP.1, MP.2, MP.3**) [adapted from Charles A. Dana Center 2012].

Students should be exposed to problems that are in and out of context and presented in horizontal and vertical forms. Students solve problems using language associated with proper place value, and they explain and justify their mathematical thinking (**MP.2, MP.6, MP.7, MP.8**).

Students use various strategies and models for addition. Students relate the strategy to a written method and explain the reasoning used (**MP.2, MP.7, MP.8**).

Examples: Models, Written Methods, and Other Addition Strategies **1.NBT.4▲**

1. Solve 43 + 36. Students may total the tens and then the ones. Place-value blocks or other counters support understanding of how to record the written method:

43	36	43 + 36 = (40 + 30) + (3 + 6) = 70 + 9 = 79
		

Students circle like units in the drawings and represent the results numerically.

2. Find the sum.

$$\begin{array}{r} 28 \\ +\ 34 \\ \hline \end{array}$$

$$\begin{array}{r} 28 \\ +\ 34 \\ \hline 1 \\ 52 \\ \hline 62 \end{array}$$

Student thinks: *"Counting the ones, I get 10 plus 2 more. I mark the ten with a little one. Adding the tens I had gives me 2 tens plus 3 tens, which is 5 tens. Finally, 5 tens plus 1 more ten is 6 tens, or 60, and 2 more makes 62."*

3. Add 45 + 18.

Student thinks: *"Four (4) tens and 1 ten is 5 tens, which is 50. To add the ones, I can make a ten by thinking of 5 as 3 + 2, then the 2 combines with the 8 to make 1 ten. So now I have 6 tens altogether, or 60, and 3 ones left—so the total is 63."*

4. Add 29 + 14.

Student thinks: *"Since 29 is 1 away from 30, I'll just think of it as 30. Since 30 + 14 = 44, I know that the answer is 1 too many, so the answer is 43."*

Adapted from ADE 2010.

Grade-one students engage in mental calculations, such as mentally finding 10 more or 10 less than a given two-digit number without counting by ones (**1.NBT.5▲**). Drawings and place-value cards can illustrate connections between place value and written numbers. Prior use of models (such as connecting cubes, base-ten blocks, and hundreds charts) helps facilitate this understanding. It also helps students see the pattern involved when adding or subtracting 10. For example:

- 10 more than 43 is 53 because 53 is 1 more ten than 43.
- 10 less than 43 is 33 because 33 is 1 ten less than 43.

Students may use interactive or electronic versions of models (base-ten blocks, hundreds charts, and so forth) to develop conceptual understanding (adapted from ADE 2010).

Grade-one students need opportunities to represent numbers that are multiples of 10 (e.g., 90) with models or drawings and to subtract multiples of 10 (e.g., 20) using these representations or strategies based on place value (**1.NBT.5▲**). These opportunities help develop fluency with addition and subtraction facts and reinforce counting on and counting back by tens. As with single-digit numbers, counting back is difficult—so initially, forward methods of counting on by tens should be emphasized rather than counting back.

Domain: Measurement and Data

A critical area of instruction for grade-one students is to develop an understanding of linear measurement and that lengths are measured by iterating length units.

Measurement and Data	**1.MD**
Measure lengths indirectly and by iterating length units.	
1. Order three objects by length; compare the lengths of two objects indirectly by using a third object.	
2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*	

In grade one, students order three objects by length and compare the lengths of two objects indirectly by using a third object (**1MD.1▲**). Students indirectly compare the lengths of two objects by comparing each to a benchmark object of intermediate length. This concept is referred to as *transitivity.*

To compare objects, students learn that length is measured from one endpoint to another endpoint. They measure objects to determine which of two objects is longer, by physically aligning the objects. Based on length, students might describe objects as *taller, shorter, longer,* or *higher.* If students use less precise words such as *bigger* or *smaller* to describe a comparison, they should be encouraged to further explain what they mean (**MP.6, MP.7**). If objects have more than one measurable length, students also need to identify the length(s) they are measuring. For example, both the length and the width of an object are measurements of lengths.

Examples: Comparing Lengths **1MD.1▲**

Direct Comparisons. Students can place three items in order, according to length:

- Three students are ordered by height.
- Pencils, crayons, or markers are ordered by length.
- Towers built with cubes are ordered from shortest to tallest.
- Three students draw line segments and then order the segments from shortest to longest.

Indirect Comparisons. Students make clay "snakes." Given a tower of cubes, each student compares his or her snake to the tower. Then students make statements such as, "My snake is longer than the cube tower, and your snake is shorter than the cube tower. So my snake is longer than your snake."

Adapted from ADE 2010.

Students gain their first experience with measuring length as the iteration of a smaller, uniform length called a *length unit* (**1.MD.2▲**). Students learn that measuring the length of an object in this way requires placing length units (manipulatives of the same size) end to end without gaps or overlaps, and then counting the number of units to determine the length. The University of Arizona's Geometric Measurement Progression recommends beginning with actual standard units (e.g., 1-inch cubes or centimeter cubes, referred to as *length units*) to measure length (UA Progressions Documents 2012c). In order to fully understand the subtlety of using non-standard units, students need to understand relationships between units of measure, a concept that will appear in the curriculum in later grades.

Standard **1.MD.2▲** limits measurement to whole numbers of length, though not all objects will measure to an exact whole unit. Students will need to adjust their answers because of this. For example, if a pencil actually measures between 6 and 7 centimeter cubes long, the students could state the pencil is "about [6 or 7] centimeter cubes long"; they would choose the closer of the two numbers. As students measure objects (**1.MD.1–2▲**), they also reinforce counting skills and understandings that are part of the major work at grade one in the Number and Operations in Base Ten domain.

Measurement and Data	**1.MD**
Tell and write time.	
3. Tell and write time in hours and half hours using analog and digital clocks.	

Grade-one students understand several concepts related to telling time (**1.MD.3**), such as:

- Within a day, the hour hand goes around a clock twice (the hand moves only in one direction). A day starts with both hands of the clock pointing up.
- When the hour hand of a clock points exactly to a number, the time is exactly on the hour.
- Time on the hour is written in the same manner as it appears on a digital clock.
- The hour hand on a clock moves as time passes, so when it is halfway between two numbers, it is at the half hour.
- There are 60 minutes in one hour, so when the hour hand is halfway between two hours, 30 minutes have passed.
- A half hour is indicated in written form by using "30" after the colon.

Students need experiences exploring how to tell time in half hours and hours. For example, the clock at left in the following illustration shows that the time is 8:30. The hour hand is between the 8 and 9, but the hour is 8 since it is not yet on the 9.

Examples: Telling Time			**1.MD.3**
"The hour hand is halfway between 8 o'clock and 9 o'clock. It is 8:30."		"It is 4 o'clock because the hour hand points to 4."	

The idea that 30 minutes is "halfway" is a difficult concept for students because they have to choose the hour that has passed. Understanding that two 30s make 60 is easy if students make drawings of tens or think about 3 tens and 6 tens. Students can also explore the concept of *half* on a clock when they work on standard **1.G.3,** finding half of a circle (adapted from ADE 2010; KATM 2012, 1st Grade Flipbook; and NCDPI 2013b).

Measurement and Data 1.MD

Represent and interpret data.

4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

Students can use graphs and charts to organize and represent data (**1.MD.4**) about things in their lives (e.g., favorite colors, pets, shoe types, and so on).

Representing Data **1.MD.4 (MP.2, MP.4, MP.5)**

Tally Chart

Shoes We Wear		
Shoes	Tally	Total
	𝍸	5
	𝍸	3
	𝍸	4

Picture Chart

Charts may be constructed by groups of students as well as by individual students. These activities will help prepare students for work in grade two when they draw picture graphs and bar graphs (adapted from ADE 2010; GaDOE 2011; and KATM 2012, 1st Grade Flipbook).

When students collect, represent, and interpret data, they reinforce number sense and counting skills. When students ask and answer questions about information in charts or graphs, they sort and compare data. Students use addition and subtraction and comparative language and symbols to interpret graphs and charts (**MP.2, MP.3, MP.4, MP.5, MP.6**).

Focus, Coherence, and Rigor

When working in the cluster "Represent and interpret data," students organize, represent, and interpret data with up to three categories (**1.MD.4**). This work can also connect to student work with geometric shapes (**1.G.1**) as students collect and sort different shapes and pose and answer related questions—such as, *How many triangles are in the collection? How many rectangles are there? How many triangles and rectangles are there? Which category has the most items? How many more? Which category has the least? How many less?* Students' work with data also supports major work in the cluster "Represent and solve problems involving addition and subtraction" as students solve problems involving addition and subtraction with three whole numbers (**1.OA.1–2▲**).

Domain: Geometry

In grade one, a critical area of instruction is for students to reason about attributes of geometric shapes and about composing and decomposing these shapes.

Geometry 1.G

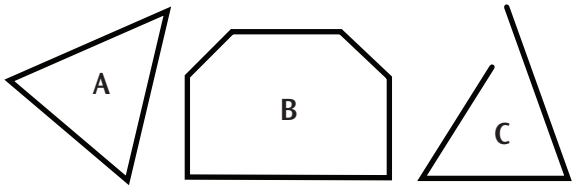
Reason with shapes and their attributes.

1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.
2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[3]
3. Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves, fourths,* and *quarters,* and use the phrases *half of, fourth of,* and *quarter of.* Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

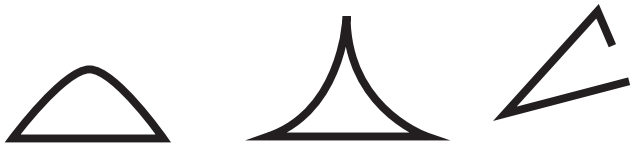
Grade-one students describe and classify shapes by geometric attributes, and they explain why a shape belongs to a given category (e.g., squares, triangles, circles, rectangles, rhombuses, hexagons, and trapezoids). Students differentiate between defining attributes (e.g., "hexagons have six straight sides") and non-defining attributes such as color, overall size, and orientation (**1.G.1**) (**MP.1, MP.3, MP.4, MP.7**) [adapted from UA Progressions Documents 2012c].

An *attribute* refers to any characteristic of a shape. Students learn to use attribute language to describe two-dimensional shapes (e.g., number of sides, number of vertices/points, straight sides, closed figures). A student might describe a triangle as "right side up" or "red," but students learn these are not defining attributes because they are not relevant to whether a shape is a triangle or not.

3. Students do not need to learn formal names such as "right rectangular prism."

Examples: Using Attributes to Name Shapes	1.G.1
Teacher: "Which figure is a triangle? How do you know?" **Student:** "I know that shape A has three sides and the shape is closed up, so it is a triangle. Shape B has too many sides, and shape C has an opening, so it's not closed."	
Teacher: "Are both figures presented here squares? Explain how you know." **Student:** "I know that a square has 4 sides and that each side has the same length. Even though figure E has a point facing down, it is still a square."	

Students are exposed to both regular and irregular shapes. In first grade, students use attribute language to describe why the following shapes are not triangles.



Students need opportunities to use appropriate language to describe a given three-dimensional shape (e.g., number of faces, number of vertices/points, and number of edges). For example, a cylinder is a three-dimensional shape that has two circular faces connected by a curved surface (which is not considered a face), but a grade-one student might say, "It looks like a can." Teachers can support learning by defining and using appropriate mathematical terms.

Students need opportunities to compare and contrast two- and three-dimensional figures using defining attributes. The following examples were adapted from ADE 2010:

- Students find two things that are the same and two things that are different between a rectangle and a cube.
- Given a circle and a sphere, students identify the sphere as three-dimensional and both shapes as round.

The ability to describe, use, and visualize the effect of composing and decomposing shapes is an important mathematical skill (**1.G.2**). It is not only relevant to geometry, but also to children's ability to compose and decompose numbers.

Students may use pattern blocks, plastic shapes, tangrams, or computer environments to make new shapes. Teachers can provide students with cutouts of shapes and ask them to combine the cutouts to make a particular shape. Composing with squares and rectangles and with pairs of right triangles that make squares and rectangles is especially important for future geometric thinking.

Students need experiences with different-sized circles and rectangles to recognize that when they cut something into two equal pieces, each piece will equal one half of its original whole (**1.G.3**). Children should recognize that the halves of two different wholes are not necessarily the same size. They should also reason that decomposing equal shares into more equal shares results in smaller equal shares.

Focus, Coherence, and Rigor

As grade-one students partition circles and rectangles into two and four equal shares and use related language (*halves, fourths* and *quarters* [**1.G.3**]), they build understanding of part–whole relationships and are introduced to fractional language. Fraction notation will first be introduced in grade three.

Essential Learning for the Next Grade

In kindergarten through grade five, the focus is on the addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals, with a balance of concepts, skills, and problem solving. Arithmetic is viewed as an important set of skills and also as a thinking subject that, done thoughtfully, prepares students for algebra. Measurement and geometry develop alongside number and operations and are tied specifically to arithmetic along the way.

In kindergarten through grade two, students focus on addition and subtraction and measurement using whole numbers. To be prepared for grade-two mathematics, students should be able to demonstrate that they have acquired particular mathematical concepts and procedural skills by the end of grade one and have met the fluency expectations for the grade. For grade-one students, the expected fluencies are to add and subtract within 10 (**1.OA.6▲**). These fluencies and the conceptual understandings that support them are foundational for work in later grades.

It is particularly important for students in grade one to attain the concepts, skills, and understandings necessary to represent and solve problems involving addition and subtraction (**1.OA.1–2▲**); understand and apply properties of operations and the relationship between addition and subtraction (**1.OA.3–4▲**); add and subtract within 20 (**1.OA.5–6▲**); work with addition and subtraction equations (**1.OA.7–8▲**); extend the counting sequence (**1.NBT.1▲**); understand place value and use place-value understanding and properties of operations to add and subtract (**1.NBT.2–6▲**); and measure lengths indirectly and by iterating length units (**1.MD.1–2▲**).

Place Value

By the end of grade one, students are expected to count to 120 (starting from any number), compare whole numbers (at least to 100), and read and write numerals in the same range. Students need to think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). Counting to 120 and reading and representing these numbers with numerals will prepare students to count, read, and write numbers within 1000 in grade two.

Addition and Subtraction

By the end of grade one, students are expected to add and subtract within 20 and demonstrate fluency with these operations within 10 (**1.OA.6▲**). Students can represent and solve word problems involving add-to, take-from, put-together, take-apart, and compare situations, including addend-unknown situations. They know how to apply properties of addition (associative and commutative) and strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems. Students use a variety of methods to add within 100, subtract multiples of 10 (using various strategies), and mentally find 10 more or 10 less without counting. Students understand how to solve addition and subtraction equations.

Addition and subtraction are major instructional foci for kindergarten through grade two. Students who have met the grade-one standards for addition and subtraction will be prepared to meet the grade-two standards of adding and subtracting within 1000 (using concrete models, drawings, and strategies); fluently adding and subtracting within 100 (using various strategies) and within 20 (using mental strategies); and knowing from memory all sums of two one-digit numbers.

Measurement of Lengths

By the end of grade one, students are expected to order three objects by length (using non-standard units). Students indirectly measure objects, comparing the lengths of two objects by using a third object as a measuring tool. Mastering grade-one measurement standards will prepare students to measure and estimate lengths (in standard units) as required in grade two.

Grade 1 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving addition and subtraction.
- Understand and apply properties of operations and the relationship between addition and subtraction.
- Add and subtract within 20.
- Work with addition and subtraction equations.

Number and Operations in Base Ten

- Extend the counting sequence.
- Understand place value.
- Use place-value understanding and properties of operations to add and subtract.

Measurement and Data

- Measure lengths indirectly and by iterating length units.
- Tell and write time.
- Represent and interpret data.

Geometry

- Reason with shapes and their attributes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

1 Grade 1

Operations and Algebraic Thinking 1.OA

Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.[4]

2. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Understand and apply properties of operations and the relationship between addition and subtraction.

3. Apply properties of operations as strategies to add and subtract.[5] *Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)*

4. Understand subtraction as an unknown-addend problem. *For example, subtract 10 – 8 by finding the number that makes 10 when added to 8.*

Add and subtract within 20.

5. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).

6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).

Work with addition and subtraction equations.

7. Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. *For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 – 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.*

8. Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = ☐ – 3, 6 + 6 = ☐.*

Number and Operations in Base Ten 1.NBT

Extend the counting sequence.

1. Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.

4. See glossary, table GL-4.

5. Students need not use formal terms for these properties.

Understand place value.

2. Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:
   a. 10 can be thought of as a bundle of ten ones—called a "ten."
   b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.
   c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).
3. Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.

Use place-value understanding and properties of operations to add and subtract.

4. Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.
5. Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.
6. Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Measurement and Data 1.MD

Measure lengths indirectly and by iterating length units.

1. Order three objects by length; compare the lengths of two objects indirectly by using a third object.
2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*

Tell and write time.

3. Tell and write time in hours and half hours using analog and digital clocks.

Represent and interpret data.

4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

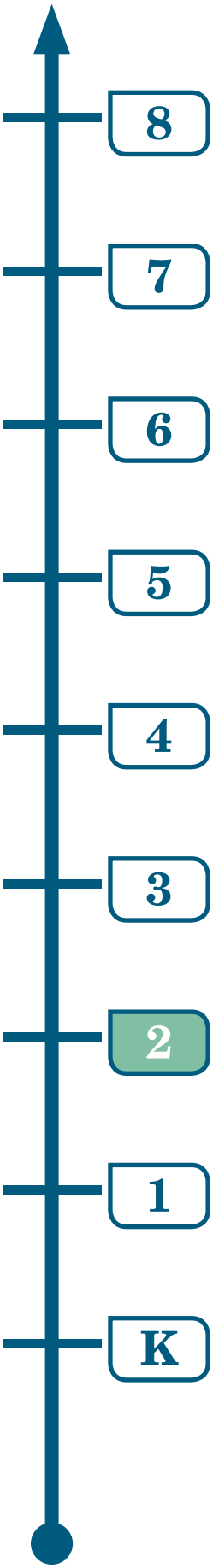
1 Grade 1

Geometry 1.G

Reason with shapes and their attributes.

1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.
2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[6]
3. Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves, fourths,* and *quarters,* and use the phrases *half of, fourth of,* and *quarter of.* Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

6. Students do not need to learn formal names such as "right rectangular prism."



Grade Two

In grade two, students further build a mathematical foundation that is critical to learning higher mathematics. In previous grades, students developed a foundation for understanding place value, including grouping in tens and ones. They built understanding of whole numbers to 120 and developed strategies to add, subtract, and compare numbers. They solved addition and subtraction word problems within 20 and developed fluency with these operations within 10. Students also worked with non-standard measurement and reasoned about attributes of geometric shapes (adapted from Charles A. Dana Center 2012).

Critical Areas of Instruction

In grade two, instructional time should focus on four critical areas: (1) extending understanding of base-ten notation; (2) building fluency with addition and subtraction; (3) using standard units of measure; and (4) describing and analyzing shapes (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010i). Students also work toward fluency with addition and subtraction within 20 using mental strategies and within 100 using strategies based on place value, properties of operations, and the relationship between addition and subtraction. They know from memory all sums of two one-digit numbers.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade-level examples of focus, coherence, and rigor are indicated throughout the chapter.

The standards do not give equal emphasis to all content for a particular grade level. Cluster headings can be viewed as the most effective way to communicate the **focus** and **coherence** of the standards. Some clusters of standards require a greater instructional emphasis than others based on the depth of the ideas, the time needed to master those clusters, and their importance to future mathematics or the later demands of preparing for college and careers.

Table 2-1 highlights the content emphases at the cluster level for the grade-two standards. The bulk of instructional time should be given to "Major" clusters and the standards within them, which are indicated throughout the text by a triangle symbol (▲). However, standards in the "Additional/Supporting" clusters should not be neglected; to do so would result in gaps in students' learning, including skills and understandings they may need in later grades. Instruction should reinforce topics in major clusters by using topics in the additional/supporting clusters and including problems and activities that support natural connections between clusters.

Teachers and administrators alike should note that the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences presented in a coherent manner (adapted from Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

Table 2-1. Grade Two Cluster-Level Emphases

Operations and Algebraic Thinking — 2.OA

Major Clusters

- Represent and solve problems involving addition and subtraction. (**2.OA.1** ▲)
- Add and subtract within 20. (**2.OA.2** ▲)

Additional/Supporting Clusters

- Work with equal groups of objects to gain foundations for multiplication. (**2.OA.3–4**)

Number and Operations in Base Ten — 2.NBT

Major Clusters

- Understand place value. (**2.NBT.1–4** ▲)
- Use place-value understanding and properties of operations to add and subtract. (**2.NBT.5–9** ▲)

Measurement and Data — 2.MD

Major Clusters

- Measure and estimate lengths in standard units. (**2.MD.1–4** ▲)
- Relate addition and subtraction to length. (**2.MD.5–6** ▲)

Additional/Supporting Clusters

- Work with time and money. (**2.MD.7–8**)
- Represent and interpret data. (**2.MD.9–10**)

Geometry — 2.G

Additional/Supporting Clusters

- Reason with shapes and their attributes. (**2.G.1–3**)

Explanations of Major and Additional/Supporting Cluster-Level Emphases

Major Clusters (▲) — Areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than others based on the depth of the ideas, the time needed to master them, and their importance to future mathematics or the demands of college and career readiness.

Additional Clusters — Expose students to other subjects; may not connect tightly or explicitly to the major work of the grade.

Supporting Clusters — Designed to support and strengthen areas of major emphasis.

Note of caution: Neglecting material, whether it is found in the major or additional/supporting clusters, will leave gaps in students' skills and understanding and will leave students unprepared for the challenges they face in later grades.

Adapted from Achieve the Core 2012.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (MP) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The MP standards represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students.

Although the description of the MP standards remains the same at all grade levels, the way these standards look as students engage with and master new and more advanced mathematical ideas does change. Table 2-2 presents examples of how the MP standards may be integrated into tasks appropriate for students in grade two. (Refer to the Overview of the Standards Chapters for a description of the MP standards.)

Table 2-2. Standards for Mathematical Practice—Explanation and Examples for Grade Two

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	In grade two, students realize that doing mathematics involves reasoning about and solving problems. Students explain to themselves the meaning of a problem and look for ways to solve it. They may use concrete objects or pictures to help them conceptualize and solve problems. They may check their thinking by asking themselves, "Does this make sense?" They make conjectures about the solution and plan out a problem-solving approach.
MP.2 Reason abstractly and quantitatively.	Younger students recognize that a number represents a specific quantity. They connect the quantity to written symbols. Quantitative reasoning entails creating a representation of a problem while attending to the meanings of the quantities. Students represent situations by decontextualizing tasks into numbers and symbols. For example, a task may be presented as follows: "There are 25 children in the cafeteria, and they are joined by 17 more children. How many students are in the cafeteria?" Students translate the situation into an equation (such as 25 + 17 = ___) and then solve the problem. Students also contextualize situations during the problem-solving process. To reinforce students' reasoning and understanding, teachers might ask, "How do you know?" or "What is the relationship of the quantities?"
MP.3 Construct viable arguments and critique the reasoning of others.	Grade-two students may construct arguments using concrete referents, such as objects, pictures, math drawings, and actions. They practice their mathematical communication skills as they participate in mathematical discussions involving questions such as "How did you get that?", "Explain your thinking," and "Why is that true?" They not only explain their own thinking, but also listen to others' explanations. They decide if the explanations make sense and ask appropriate questions. Students critique the strategies and reasoning of their classmates. For example, to solve 74 – 18, students might use a variety of strategies and discuss and critique each other's reasoning and strategies.
MP.4 Model with mathematics.	In early grades, students experiment with representing problem situations in multiple ways, including writing numbers, using words (mathematical language), drawing pictures, using objects, acting out, making a chart or list, or creating equations. Students need opportunities to connect the different representations and explain the connections.

Table 2-2 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
	Students model real-life mathematical situations with an equation and check to make sure that their equation accurately matches the problem context. They use concrete manipulatives or math drawings (or both) to explain the equation. They create an appropriate problem situation from an equation. For example, students create a story problem for the equation 43 +☐ = 82, such as "There were 43 mini-balls in the machine. Tom poured in some more mini-balls. There are 82 mini-balls in the machine now. How many balls did Tom pour in?" Students should be encouraged to answer questions, such as "What math drawing or diagram could you make and label to represent the problem?" or "What are some ways to represent the quantities?"
MP.5 Use appropriate tools strategically.	In second grade, students consider the available tools (including estimation) when solving a mathematical problem and decide when certain tools might be better suited than others. For instance, grade-two students may decide to solve a problem by making a math drawing rather than writing an equation. Students may use tools such as snap cubes, place-value (base-ten) blocks, hundreds number boards, number lines, rulers, virtual manipulatives, diagrams, and concrete geometric shapes (e.g., pattern blocks, three-dimensional solids). Students understand which tools are the most appropriate to use. For example, while measuring the length of the hallway, students are able to explain why a yardstick is more appropriate to use than a ruler. Students should be encouraged to answer questions such as, "Why was it helpful to use _____ ?"
MP.6 Attend to precision.	As children begin to develop their mathematical communication skills, they try to use clear and precise language in their discussions with others and when they explain their own reasoning. Students communicate clearly, using grade-level-appropriate vocabulary accurately and precise explanations and reasoning to explain their process and solutions. For example, when measuring an object, students carefully line up the tool correctly to get an accurate measurement. During tasks involving number sense, students consider if their answers are reasonable and check their work to ensure the accuracy of solutions.
MP.7 Look for and make use of structure.	Grade-two students look for patterns and structures in the number system. For example, students notice number patterns within the tens place as they connect counting by tens to corresponding numbers on a hundreds chart. Students see structure in the base-ten number system as they understand that 10 ones equal a ten, and 10 tens equal a hundred. Teachers might ask, "What do you notice when _____ ?" or "How do you know if something is a pattern?" Students adopt mental math strategies based on patterns (making ten, fact families, doubles). They use structure to understand subtraction as an unknown addend problem (e.g., 50 – 33 = __ can be written as 33 + __ = 50 and can be thought of as "How much more do I need to add to 33 to get to 50?").
MP.8 Look for and express regularity in repeated reasoning.	Second-grade students notice repetitive actions in counting and computation (e.g., number patterns to count by tens or hundreds). Students continually check for the reasonableness of their solutions during and after completion of a task by asking themselves, "Does this make sense?" Students should be encouraged to answer questions—such as "What is happening in this situation?" or "What predictions or generalizations can this pattern support?"

Adapted from Arizona Department of Education (ADE) 2010 and North Carolina Department of Public Instruction (NCDPI) 2013b.

Standards-Based Learning at Grade Two

The following narrative is organized by the domains in the Standards for Mathematical Content and highlights some necessary foundational skills from previous grade levels. It also provides exemplars to explain the content standards, highlight connections to Standards for Mathematical Practice (**MP**), and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. A triangle symbol (▲) indicates standards in the major clusters (see table 2-1).

Domain: Operations and Algebraic Thinking

In grade one, students solved addition and subtraction word problems within 20 and developed fluency with these operations within 10. A critical area of instruction in grade two is building fluency with addition and subtraction. Second-grade students fluently add and subtract within 20 and solve addition and subtraction word problems involving unknown quantities in all positions within 100. Grade-two students also work with equal groups of objects to gain the foundations for multiplication.

Operations and Algebraic Thinking	**2.OA**
Represent and solve problems involving addition and subtraction.	
1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]	

In grade two, students add and subtract numbers within 100 in the context of one- and two-step word problems (**2.OA.1▲**). By second grade, students have worked with various problem situations (add to, take from, put together, take apart, and compare) with unknowns in all positions (result unknown, change unknown, and start unknown). Grade-two students extend their work with addition and subtraction word problems in two significant ways:

- They represent and solve problems of all types involving addition and subtraction within 100, building upon their previous work within 20.
- They represent and solve two-step word problems of all types, extending their work with one-step word problems (adapted from ADE 2010; NCDPI 2013b; Georgia Department of Education [GaDOE] 2011; and Kansas Association of Teachers of Mathematics [KATM] 2012, 2nd Grade Flipbook).

Different types of addition and subtraction problems are presented in table 2-3.

Table 2-3. Types of Addition and Subtraction Problems (Grade Two)

	Result Unknown	Change Unknown	Start Unknown
Add to	There are 22 marbles in a bag. Thomas placed 23 more marbles in the bag. How many marbles are in the bag now? 22 + 23 = ☐	Bill had 25 baseball cards. His mom gave him some more. Now he has 73 baseball cards. How many baseball cards did his mom give him? In this problem, the starting quantity is provided (25 baseball cards), a second quantity is added to that amount (some baseball cards), and the result quantity is given (73 baseball cards). This question type is more algebraic and challenging than a "result unknown" problem and can be modeled by a situational equation (25 + ☐ = 73) that does not immediately lead to the answer. Students can write a related equation (73 – 25 = ☐)—called a solution equation—to solve the problem.	Some children were playing on the playground, and 5 more children joined them. Then there were 22 children. How many children were playing before? This problem can be represented by ☐ + 5 = 22. The "start unknown" problems are difficult for students to model because the initial quantity is unknown, and therefore some students do not know how to start a solution strategy. They can make a drawing, where it is crucial that they realize that the 5 is part of the 22 total children. This leads to more general solutions by subtracting the known addend or counting/adding on from the known addend to the total.
Take from	There were 45 apples on the table. I took 12 of those apples and placed them in the refrigerator. How many apples are on the table now? 45 – 12 = ☐	Andrea had 51 stickers. She gave away some stickers. Now she has 22 stickers. How many stickers did she give away? This question may be modeled by a situational equation (51 – ☐ = 22) or a solution equation (51 – 22 = ☐). Both the "take from" and "add to" questions involve actions.	Some children were lining up for lunch. After 4 children left, there were 26 children still waiting in line. How many children were there before? This problem can be modeled by ☐ – 4 = 26. Similar to the previous "add to (with start unknown)" problem, "take from (with start unknown)" problems require a high level of conceptual understanding. Students need to understand that the total is first in a subtraction equation and that this total is broken apart into the 4 and the 26.

Table 2-3 *(continued)*

	Total Unknown	Addend Unknown	Both Addends Unknown
Put together/ Take apart	There are 30 red apples and 20 green apples on the table. How many apples are on the table? 30 + 20 = ?	Roger puts 24 apples in a fruit basket. Nine (9) are red and the rest are green. How many are green?" There is no direct or implied action. The problem involves a set and its subsets. It may be modeled by 24 – 9 = ☐ or 9 + ☐ = 24. This type of problem provides students with opportunities to understand subtraction as an unknown-addend problem.	Grandma has 5 flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2

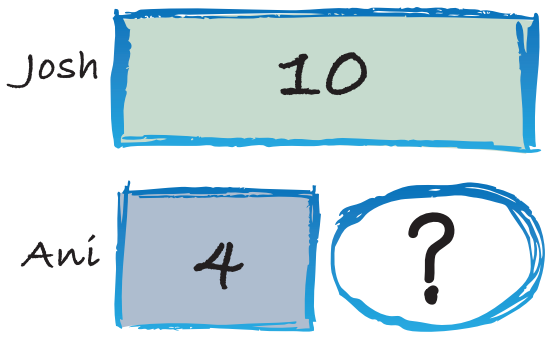
	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	Pat has 19 peaches. Lynda has 14 peaches. How many more peaches does Pat have than Lynda? "Compare" problems involve relationships between quantities. Although most adults might use subtraction to solve this type of problem (19 – 14 = ☐), students will often solve this problem as an unknown-addend problem (14 + ☐ = 19) by using a counting-up or matching strategy. In all mathematical problem solving, what matters is the explanation a student gives to relate a representation to a context—not the representation separated from its context.	(**"More" version**): Theo has 23 action figures. Rosa has 2 more action figures than Theo. How many action figures does Rosa have? This problem can be modeled by 23 + 2 = ☐. (**"Fewer" version**): Lucy has 28 apples. She has 2 fewer apples than Marcus. How many apples does Marcus have? This problem can be modeled as 28 + 2 = ☐. The misleading language form "fewer" may lead students to choose the wrong operation.	(**"More" version**): David has 27 more bunnies than Keisha. David has 28 bunnies. How many bunnies does Keisha have? This problem can be modeled by 28 – 27 = ☐. The misleading language form "more" may lead students to choose the wrong operation. (**"Fewer" version**): Bill has 24 stamps. Lisa has 2 fewer stamps than Bill. How many stamps does Lisa have? This problem can be modeled as 24 – 2 = ☐.

Note: Further examples are provided in table GL-4 of the glossary.

For these more complex grade-two problems, it is important for students to represent the problem situations with drawings and equations (**2.OA.1▲**). Drawings can be shown more easily to the whole class during explanations and can be related to equations. Students can also use manipulatives (e.g., snap cubes, place-value blocks), but drawing quantities is an exercise that can be used anywhere to solve problems and support students in describing their strategies. Second-grade students represent problems with equations and use boxes, blanks, or pictures for the unknown amount. For example, students can represent "compare" problems using comparison bars (see figure 2-1). Students can draw these bars, fill in numbers from the problem, and label the bars.

Figure 2-1. Comparison Bars

Josh has 10 markers, and Ani has 4 markers. How many more markers does Josh have than Ani?



One-step word problems use one operation. Two-step word problems (**2.OA.1▲**) are new for second-graders and require students to complete two operations, which may include the same operation or different operations.

Initially, two-step problems should not involve the most difficult subtypes of problems (e.g., "compare" and "start unknown" problems) and should be limited to single-digit addends. There are many problem-situation subtypes and various ways to combine such subtypes to devise two-step problems. Introducing easier problems first will provide support for second-grade students who are still developing proficiency with "compare" and "start unknown" problems (adapted from the University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2011a).

The following table presents examples of easy and moderately difficult two-step word problems that would be appropriate for grade-two students.

One-Step Word Problem One Operation	**Two-Step Word Problem** Two Operations, Same	**Two-Step Word Problem** Two Operations, Opposite
There are 15 stickers on the page. Brittany put some more stickers on the page and now there are 22. How many stickers did Brittany put on the page? $15 + \square = 22$ or $22 - 15 = \square$	There are 9 blue marbles and 6 red marbles in the bag. Maria put in 8 more marbles. How many marbles are in the bag now? $9 + 6 + 8 = \square$ or $(9 + 6) + 8 = \square$	There are 39 peas on the plate. Carlos ate 25 peas. Mother put 7 more peas on the plate. How many peas are on the plate now? $39 - 25 + 7 = \square$ or $(39 - 25) + 7 = \square$

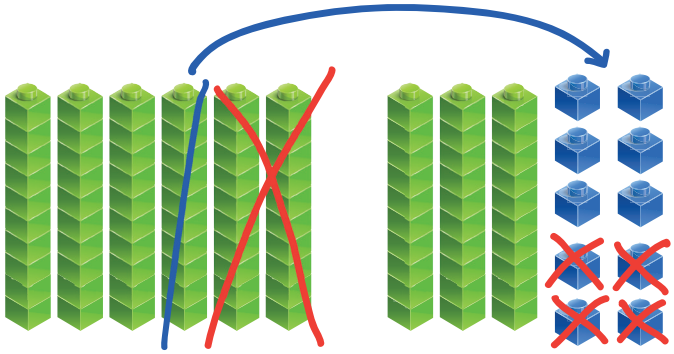
Adapted from NCDPI 2013b.

Grade-two students use a range of methods, often mastering more complex strategies such as making tens and doubles and near doubles that were introduced in grade one for problems involving single-digit addition and subtraction. Second-grade students also begin to apply their understanding of place value to solve problems, as shown in the following example.

One-Step Problem: Some students are in the cafeteria. Twenty-four (24) more students came in. Now there are 60 students in the cafeteria. How many students were in the cafeteria to start with? Use drawings and equations to show your thinking.

Student A: I read the problem and thought about how to write it with numbers. I thought, "What and 24 makes 60?" I used a math drawing to solve it. I started with 24. Then I added tens until I got close to 60; I added 3 tens. I stopped at 54. Then I added 6 more ones to get to 60. So, 10 + 10 + 10 + 6 = 36. So, there were 36 students in the cafeteria to start with. My equation for the problem is ☐ + 24 = 60. (**MP.2, MP.7, MP.8**)

Student B: I read the problem and thought about how to write it with numbers. I thought, "There are 60 total. I know about the 24. So, what is 60 – 24?" I used place-value blocks to solve it. I started with 60 and took 2 tens away. I needed to take 4 more away. So, I broke up a ten into 10 ones. Then I took 4 away. That left me with 36. So, 36 students were in the cafeteria at the beginning. 60 – 24 = 36. My equation for the problem is 60 – 24 = ☐. (**MP.2, MP.4, MP.5, MP.6**)



Adapted from ADE 2010, NCDPI 2013b, GaDOE 2011, and KATM 2012 (2nd Grade Flipbook).

As students solve addition and subtraction word problems, they use concrete manipulatives, pictorial representations, and mental mathematics to make sense of a problem (**MP.1**); they reason abstractly and quantitatively as they translate word problem situations into equations (**MP.2**); and they model with mathematics (**MP.4**).

Table 2-4 presents a sample classroom activity that connects the Standards for Mathematical Content and Standards for Mathematical Practice.

Table 2-4. Connecting to the Standards for Mathematical Practice—Grade Two

Standards Addressed	Explanation and Examples
Connections to Standards for Mathematical Practice **MP.1.** Students are challenged to think through how they would solve a potentially unfamiliar problem situation and to devise a strategy. The teacher can assess each student's starting point and move him or her forward from there. **MP.3.** When students are asked to explain to their peers how they solved the problems, they are essentially constructing a mathematical argument that justifies that they have performed the addition or subtraction correctly. **MP.7.** When students begin exchanging sticks and units to represent grouping and breaking apart tens and ones, they are making use of the structure of the base-ten number system to understand addition and subtraction. **Standards for Mathematical Content** **2.OA.1.** Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g. by using drawings and equations with a symbol for the unknown number to represent the problem. **2.NBT.5.** Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.	**Task: Base-Ten Block Activities.** This is a two-tiered approach to problem solving with basic operations within 100. The first task involves students seeing various strategies for adding two-digit numbers using base-ten blocks. The second is an extension that builds facility in adding and subtracting such numbers. 1. The teacher should present several problem situations that involve addition and subtraction in which students can use base-ten blocks to model their solution strategies. Such solutions are made public through an overhead display or by the teacher rephrasing and demonstrating student solutions. Four sample problems are provided below: • *Micah had 24 marbles. Sheila had 15. Micah and Sheila decided to put all of their marbles in a box. How many marbles were there altogether?* (This is an addition problem that does not require bundling ones into a ten.) • *There were 28 boys and 35 girls on the playground at recess. How many children were there on the playground at recess?* (This is an addition problem that requires bundling.) • *There were 48 cows on a pasture. Seventeen (17) of the cows went into the barn. How many cows are left on the pasture?* (This is a subtraction problem that does not require exchanging a ten for ones.) • *There were 54 erasers in a basket. Twenty-six (26) students were allowed to take one eraser each. How many erasers are left over after the children have taken theirs?* (This is a subtraction problem involving the exchange of a ten for 10 ones.) 2. Next, the teacher can play a game that reinforces understanding of addition, subtraction, and skill in doing addition and subtraction. Each student takes out base-ten blocks to represent a given number—for example, 45. The teacher then asks students how many more blocks are needed to make 80. Students represent the difference with base-ten blocks and justify how they know their answers are correct. The teacher can ask several variations of this same basic question; the task can be used repeatedly throughout the school year to reinforce concepts of operations. **Classroom Connections.** When students are given the opportunity to construct their own strategies for adding and subtracting numbers, they reinforce their understanding of place value and the base-ten number system. Activities such as those presented here help build this foundation in context and through modeling numbers with objects (e.g., with base-ten blocks).

To solve word problems, students learn to apply various computational methods. Kindergarten students generally use Level 1 methods, and students in first and second grade use Level 2 and Level 3 methods. The three levels are summarized in table 2-5 and explained more thoroughly in appendix C.

Table 2-5. Methods Used for Solving Single-Digit Addition and Subtraction Problems

Level 1: Direct Modeling by Counting All or Taking Away

Represent the situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Level 2: Counting On

Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total, but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. The count is tracked and monitored in some way (e.g., with fingers, objects, mental images of objects, body motions, or other count words).

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as the unknown addend).

Level 3: Converting to an Easier Equivalent Problem

Decompose an addend and compose a part with another addend.

Adapted from UA Progressions Documents 2011a.

In grade two, students extend their fluency with addition and subtraction from within 10 to within 20 (**2.OA.2▲**). The experiences students have had with addition and subtraction in kindergarten (within 5) and grade one (within 10) culminate in grade-two students becoming fluent in single-digit additions and related subtractions, using Level 2 and Level 3 methods and strategies as needed.

Operations and Algebraic Thinking 2.OA

Add and subtract within 20.

2. Fluently add and subtract within 20 using mental strategies.[2] By end of Grade 2, know from memory all sums of two one-digit numbers.

Students may still need to support the development of their fluency with math drawings when solving problems. Math drawings represent the number of objects counted (using dots and sticks) and do not need to represent the context of the problem. Thinking about numbers by using 10-frames or making drawings using groups of fives and tens may be helpful ways to understand single-digit additions and subtractions. The National Council of Teachers of Mathematics Illuminations project (NCTM Illuminations 2013a) offers examples of interactive games that students can play to develop counting and addition skills.

2. See Standard **1.OA.6** for a list of mental strategies.

FLUENCY

California's Common Core State Standards for Mathematics (K–6) set expectations for fluency in computation (e.g., "*Fluently* add and subtract within 20 . . .") [**2.OA.2▲**]. Such standards are culminations of progressions of learning, often spanning several grades, involving conceptual understanding, thoughtful practice, and extra support where necessary. The word *fluent* is used in the standards to mean "reasonably fast and accurate" and possessing the ability to use certain facts and procedures with enough facility that using such knowledge does not slow down or derail the problem solver as he or she works on more complex problems. Procedural fluency requires skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Developing fluency in each grade may involve a mixture of knowing some answers, knowing some answers from patterns, and knowing some answers through the use of strategies.

Adapted from UA Progressions Documents 2011a.

Mental strategies, such as those listed in table 2-6, help students develop fluency in adding and subtracting within 20 as they make sense of number relationships. Table 2-6 presents the mental strategies listed with standard **1.OA.6** as well as two additional strategies.

Table 2-6. Mental Strategies

- Counting on
- Making tens (9 + 7 = [9 + 1] + 6 = 10 + 6)
- Decomposing a number leading to a ten (14 – 6 = 14 – 4 – 2 = 10 – 2 = 8)
- Related facts (8 + 5 = 13 and 13 – 8 = 5)
- Doubles (1 + 1, 2 + 2, 3 + 3, and so on)
- Doubles plus one (7 + 8 = 7 + 7 + 1)
- Relationship between addition and subtraction (e.g., by knowing that 8 + 4 = 12, one also knows that 12 – 8 = 4)
- Equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13)

Adapted from NCDPI 2013b.

Grade-two students build important foundations for multiplication as they explore odd and even numbers in a variety of ways (**2.OA.3**). They use concrete objects (e.g., counters or place-value cubes) and move toward pictorial representations such as circles or arrays (**MP.1**). Through investigations, students realize that an even number of objects can be separated into two equal groups (without extra objects remaining), while an odd number of objects will have one object remaining (**MP.7 and MP.8**).

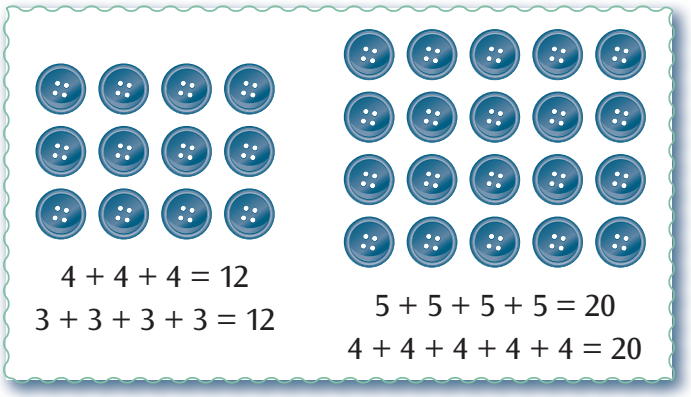
Operations and Algebraic Thinking 2.OA

Work with equal groups of objects to gain foundations for multiplication.

3. Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.
4. Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Students also apply their work with doubles addition facts and decomposition of numbers (breaking them apart) into two equal addends (e.g., 10 = 5 + 5) to understand the concept of even numbers. Students reinforce this concept as they write equations representing sums of two equal addends, such as 2 + 2 = 4, 3 + 3 = 6, 5 + 5 = 10, 6 + 6 = 12, or 8 + 8 = 16. Students are encouraged to explain how they determined if a number is odd or even and what strategies they used (**MP.3**).

With standard **2.OA.4**, second-grade students use rectangular arrays to work with repeated addition—a building block for multiplication in grade three—using concrete objects (e.g., counters, buttons, square tiles) as well as pictorial representations on grid paper or other drawings of arrays (**MP.1**). Using the commutative property of multiplication, students add either the rows or the columns and arrive at the same solution (**MP.2**). Students write equations that represent the total as the sum of equal addends, as shown in the examples at right.



The first example helps students to understand that 3 × 4 = 4 × 3; the second example supports the fact that 4 × 5 = 5 × 4 (ADE 2010).

Focus, Coherence, and Rigor

In the cluster "Work with equal groups of objects to gain foundations for multiplication," student work reinforces addition skills and understandings and is connected to work in the major clusters "Represent and solve problems involving addition and subtraction" (**2.OA.1▲**) and "Add and subtract within 20" (**2.OA.2▲**). Also, as students work with odd and even groups (**2.OA.3**) they build a conceptual understanding of equal groups, which supports their introduction to multiplication and division in grade three.

Domain: Number and Operations in Base Ten

In grade one, students viewed two-digit numbers as amounts of tens and ones. A critical area of instruction in grade two is to extend students' understanding of base-ten notation to include hundreds. Second-grade students understand multi-digit numbers (up to 1000). They add and subtract within 1000 and become fluent with addition and subtraction within 100 using place-value strategies (UA Progressions Documents 2012b).

Number and Operations in Base Ten — 2.NBT

Understand place value.

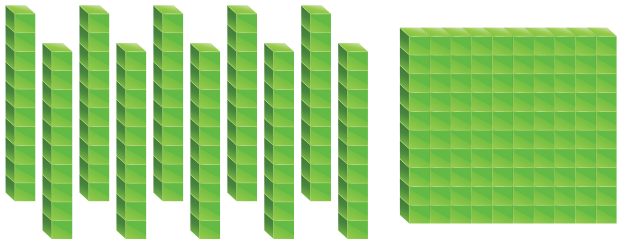
1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:
   a. 100 can be thought of as a bundle of 10 tens—called a "hundred."
   b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
2. Count within 1000; skip-count by **2s**, 5s, 10s, and 100s. **CA**
3. Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.
4. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.

Second-grade students build on their previous work with groups of tens to make bundles of hundreds, with or without leftovers, using base-ten blocks, cubes in towers of 10, 10-frames, and so forth, as well as math drawings that initially show the 10 tens within 1 hundred, but then move to a quick-hundred version that is a drawn square in which students visualize 10 tens; see figure 2-2 for examples. Bundling hundreds will support students' discovery of place-value patterns (**MP.7**). Students explore the idea that numbers such as 100, 200, 300, and so on are groups of hundreds that have "0" in the tens and ones places. Students might represent numbers using place-value (base-ten) blocks or math drawings (**MP.1**).

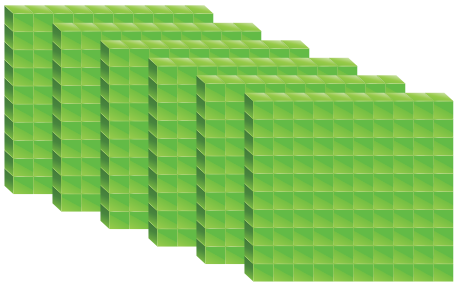
Figure 2-2. Recognizing 10 Tens as 1 Hundred

Using Base-Ten Blocks — **2.NBT.1▲**

These have the same value:

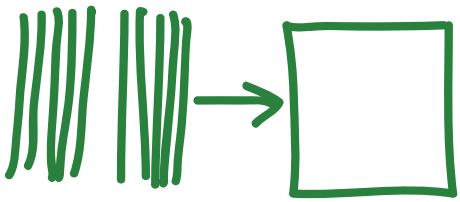


Six (6) hundreds is the same as 600:

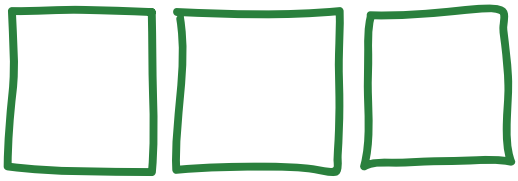


Using Math Drawings

When I bundle 10 "ten-sticks," I get 1 "hundred square."



The picture shows 3 hundreds, or 300.



Adapted from KATM 2012 (2nd Grade Flipbook).

As students represent various numbers, they associate number names with number quantities (**MP.2**). For example, 243 can be expressed as both "2 groups of hundred, 4 groups of ten, and 3 ones" and "24 tens and 3 ones." Students can read number names as well as place-value concepts to say a number. For example, 243 should be read as "two hundred forty-three" as well as "2 hundreds, 4 tens, and 3 ones." Flexibility with seeing a number like 240 as "2 hundreds and 4 tens" as well as "24 tens" is an important indicator of place-value understanding (KATM 2012, 2nd Grade Flipbook).

In kindergarten, students were introduced to counting by tens. In second grade they extend this to skip-count by twos, fives, tens, and hundreds (**2.NBT.2▲**). Exploring number patterns can help students skip-count. For example, when skip-counting by fives, the ones digit alternates between 5 and 0, and when skip-counting by tens and hundreds, only the tens and hundreds digits change, increasing by one each time. In this way, skip-counting can reinforce students' understanding of place value. Work with skip-counting lays a foundation for multiplication; however, because students do not keep track of the number of groups they have counted, they are not yet learning true multiplication. The ultimate goal is for grade-two students to count in multiple ways without visual support.

Focus, Coherence, and Rigor

As students explore number patterns by skip-counting, they also develop mathematical practices such as understanding the meaning of written quantities (**MP.2**) and recognizing number patterns and structures in the number system (**MP.7**).

Grade-two students need opportunities to read and represent numerals in various ways (**2.NBT.3▲**). An example adapted from KATM (2012, 2nd Grade Flipbook) illustrates different ways for second-graders to represent numerals:

- Standard form (e.g., 637)
- Base-ten numerals in standard form (e.g., 6 hundreds, 3 tens, and 7 ones)
- Number names in word form (e.g., six hundred thirty-seven)
- Expanded form (e.g., 600 + 30 + 7)
- Equivalent representations (e.g., 500 + 130 + 7; 600 + 20 + 17; 30 + 600 + 7)

When students read the expanded form for a number, they might say "6 hundreds plus 3 tens plus 7 ones" or "600 plus 30 plus 7." Understanding the expanded form is valuable when students use place-value strategies to add and subtract large numbers (see also **2.NBT.7**).

Second-grade students use the symbols for greater than (>), less than (<), and equal to (=) to compare numbers within 1000 (**2.NBT.4▲**). Students build on work in standards (**2.NBT.1▲** and **2.NBT.3▲**) by examining the amounts of hundreds, tens, and ones in each number. To compare numbers, students apply their understanding of place value. The goal is for students to understand that they look at the numerals in the hundreds place first, then the tens place, and if necessary, the ones place. Students should have experience communicating their comparisons in words before using only symbols to indicate greater than, less than, and equal to.

Example: Compare 452 and 455.	2.NBT.4▲
Student 1 explains that 452 has 4 hundreds, 5 tens, and 2 ones and that 455 has 4 hundreds, 5 tens, and 5 ones. "They have the same number of hundreds and the same number of tens, but 455 has 5 ones and 452 only has 2 ones. So, 452 is less than 455, or 452 < 455."	
Student 2 might think that 452 is less than 455. "I know this because when I count up, I say 452 before I say 455."	

Adapted from KATM 2012 (2nd Grade Flipbook).

As students compare numbers, they also develop mathematical practices such as making sense of quantities (**MP.2**), understanding the meaning of symbols (**MP.6**), and making use of number patterns and structures in the number system (**MP.7**).

Number and Operations in Base Ten — 2.NBT

Use place-value understanding and properties of operations to add and subtract.

5. Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
6. Add up to four two-digit numbers using strategies based on place value and properties of operations.
7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

7.1 Use estimation strategies to make reasonable estimates in problem solving. CA

8. Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.
9. Explain why addition and subtraction strategies work, using place value and the properties of operations.[3]

Place-value understanding is central to multi-digit computations. In grade two, students develop, discuss, and later use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation. While students become fluent in such methods within 100 at grade two, they also use these methods for sums and differences within 1000 (**2.NBT.5–7▲**).

General written methods for numbers within 1000 are discussed in the chapter first, as these strategies are merely extensions of those for numbers within 100. Of course, all methods for adding and subtracting two- and three-digit numbers should be based on place value and should be learned by students with an emphasis on understanding. Math drawings can support student understanding, and as students become familiar with math drawings, these drawings should accompany written methods.

3. Explanations may be supported by drawings or objects.

Written methods for recording addition and subtraction are based on two important features of the base-ten number system:

- When numbers are added or subtracted in the base-ten system, like units are added or subtracted (e.g., ones are added to ones, tens to tens, hundreds to hundreds).
- Adding and subtracting multi-digit numbers written in base-ten can be facilitated by composing and decomposing units appropriately, so as to reduce the calculations to adding and subtracting within 20 (e.g., 10 ones make 1 ten, 100 ones make 1 hundred, 1 hundred makes 10 tens).

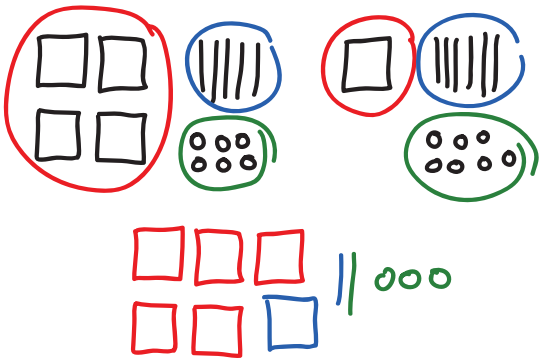
Addition

Figure 2-3 presents two written methods for addition, with accompanying illustrations (base-ten blocks can also be used to illustrate). Students initially work with math drawings or manipulatives alongside the written methods, but they will eventually use written methods exclusively, mentally constructing pictures as necessary and using other strategies. Teachers should note the importance of these written methods as students generalize to larger numbers and decimals and emphasize the regrouping nature of combining units. These two methods are given only as examples and are not meant to represent all such place-value methods.

Figure 2-3. Addition Methods Supported with Math Drawings

Examples	2.NBT.7

Addition Method 1: In this written addition method, all partial sums are recorded underneath the addition bar. Addition is performed from left to right in this example, but students can also work from right to left. In the accompanying drawing, it is clear that hundreds are added to hundreds, tens to tens, and ones to ones, which are eventually grouped into larger units where possible to represent the total, 623.



$$\begin{array}{r} 456 \\ +\ 167 \\ \hline \end{array}$$

$$\begin{array}{r} 456 \\ +\ 167 \\ \hline 500 \end{array}$$

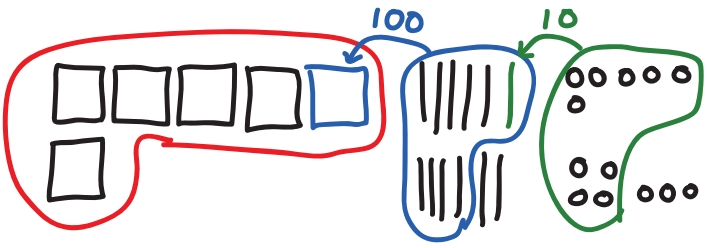
$$\begin{array}{r} 456 \\ +\ 167 \\ \hline 500 \\ 110 \end{array}$$

$$\begin{array}{r} 456 \\ +\ 167 \\ \hline 500 \\ 110 \\ 13 \\ \hline 623 \end{array}$$

Figure 2-3. *(continued)*

Examples **2.NBT.7**

Addition Method 2: In this written addition method, digits representing newly composed units are placed below the addends from which they were derived, to the right to indicate that they are represented as a larger, newly composed unit. The addition proceeds right to left. The advantage to placing the composed units as shown is that it is clearer where they came from—e.g., the 1 and 3 that came from the sum of the ones-place digits (6 + 7) are close to each other. This eliminates confusion that can arise from traditional methods involving "carrying,"which tends to separate the two digits that came from 13 and obscure the meaning of the numbers.



4 5 6 + 1 6 7	4 5 6 + 1 6 7 1 3	4 5 6 + 1 6 7 1 1 2 3	4 5 6 + 1 6 7 1 1 6 2 3
	Add the ones, 6 + 7, and record these as 13, with 3 in the ones place and a 1 underneath the tens column.	Add the tens, 5 + 6 + 1, and record these 12 tens with 2 in the tens place and 1 under the hundreds column.	Add the hundreds, 4 + 1 + 1, and record these 6 hundreds in the hundreds column.

Adapted from Fuson and Beckmann 2013 and UA Progressions Documents 2012b.

Subtraction

In grade one, students were not expected to compute differences of two-digit numbers other than multiples of 10. In grade two, students subtract two-digit numbers, with and without decomposing, which highlights the similarity between these two cases.

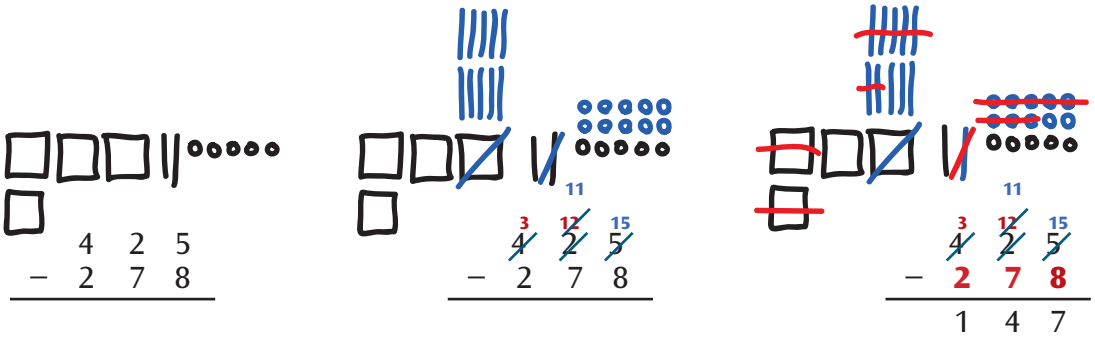
Figure 2-4 presents two methods for subtraction, one where all decomposing is done first, the other where decomposing is done as needed. Students will encounter situations in which they "don't have enough" to subtract. This is more precise than saying, "You can't subtract a larger number from a smaller number," or the like, as the latter assertion is a false mathematical statement. In later grades, students will subtract larger numbers from smaller ones, and that will result in negative numbers as answers (for example, 9 – 15 = –6).

Note that the accompanying illustrations show the decomposing steps in each written subtraction method. Again, these methods generalize to numbers of all sizes and are based on decomposing larger units into smaller units when necessary.

Figure 2-4. Subtraction Methods Supported with Math Drawings

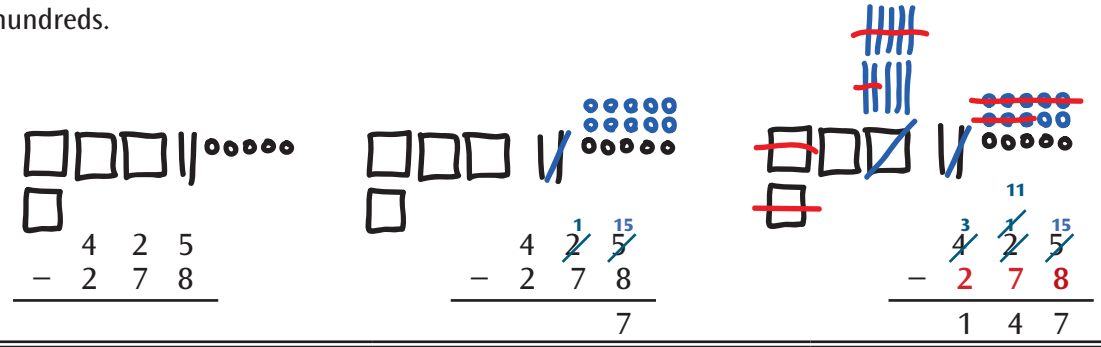
Examples	2.NBT.7

Subtraction Method 1: In this written subtraction method, all necessary decompositions are done first. Decomposing can start from the left or the right with this method. Students may be less likely to erroneously subtract the top number from the bottom in this method.



Decomposing left to right, 1 hundred, then 1 ten

Subtraction Method 2: In this written subtraction method, decomposing is done as needed. Students first ungroup a ten so they can subtract 8 from 15. They may erroneously try to subtract the tens as well, getting 7 – 1 = 6. Led to see their error, students find they need to ungroup hundreds first to subtract the tens, then the hundreds.



Adapted from Fuson and Beckmann 2013 and UA Progressions Documents 2012b.

When developing fluency with adding and subtracting within 100 (**2.NBT.5▲**), second-grade students use the methods just discussed, as well as other strategies, without the support of drawings.

Strategies for Addition and Subtraction

Examples of addition strategies based on place value for 48 + 37:

- Adding by place value: 40 + 30 = 70, 8 + 7 = 15, and 70 + 15 = 85
- Incremental adding (by tens and ones): 48 + 10 = 58, 58 + 10 = 68, 68 + 10 = 78, and 78 + 7 = 85
- Composing and decomposing (making a "friendly" number): 48 + 2 = 50, 37 – 2 = 35, and 50 + 35 = 85

Examples of subtraction strategies based on place value for 81 – 37:

- Adding up (from smaller number to larger number): 37 + 3 = 40, 40 + 40 = 80, 80 + 1 = 81, and 3 + 40 + 1 = 44
- Incremental subtracting: 81 – 10 = 71, 71 – 10 = 61, 61 – 10 = 51, 51 – 7 = 44
- Subtracting by place value: 81 – 30 = 51, 51 – 7 = 44

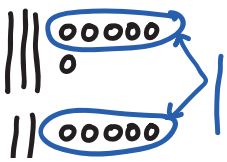
Adapted from ADE 2010.

As students develop fluency with adding and subtracting within 100, they also support mathematical practices such as making sense of quantities (**MP.2**), calculating accurately (**MP.6**), and making use of number patterns and structures in the number system (**MP.7**).

Example: Find the sum of 43 + 34 + 57 + 24.	**2.NBT.6**
Student A (Commutative and Associative Properties). "I saw the 43 and 57 and added them first. I know 3 plus 7 equals 10, so when I added them, 100 was my answer. Then I added 34 and had 134. Then I added 24 and had 158. So 43 + 57 + 34 + 24 = 158."	
Student B (Place-Value Strategies). "I broke up all of the numbers into tens and ones. First I added the tens: 40 + 30 + 50 + 20 = 140. Then I added the ones: 3 + 4 + 7 + 4 =18. That meant I had 1 ten and 8 ones. So, 140 + 10 is 150. 150 and 8 more is 158. So, 43 + 34 + 57 + 24 = 158."	
Student C (Place-Value Strategies and Commutative and Associative Property). "I broke up all the numbers into tens and ones. First I added up the tens: 40 + 30 + 50 + 20. I changed the order of the numbers to make adding easier. I know that 30 plus 20 equals 50, and 50 more equals 100. Then I added the 40 and got 140. Then I added up the ones: 3 + 4 + 7 + 4. I changed the order of the numbers to make adding easier. I know that 3 plus 7 equals 10 and 4 plus 4 equals 8. I also know that 10 plus 8 equals 18. I then combined my tens and my ones: 140 plus 18 (1 ten and 8 ones) equals 158."	

Adapted from NCDPI 2013b.

Finally, students explain why addition and subtraction strategies work, using place value and the properties of operations (**2.NBT.9▲**). Second-grade students need multiple opportunities to explain their addition and subtraction thinking (**MP.2**). For example, students use place-value understanding, properties of operations, number names, words (including mathematical language), math drawings, number lines, and physical objects to explain why and how they solve a problem (**MP.1, MP.6**). Students can also critique the work of other students (**MP.3**) to deepen their understanding of addition and subtraction strategies.

Example	**2.NBT.9**
There are 36 birds in the park. Suddenly, 25 more birds arrive. How many birds are there? Solve the problem and show your work.	
Student A. "I broke 36 and 25 into tens and ones (30 + 6) + (20 + 5). I can change the order of my numbers, since it doesn't change any amounts, so I added 30 + 20 and got 50. Then I added 5 and 5 to make 10 and added it to the 50. So, 50 and 10 more is 60. I added the one that was left over and got 61. So there are 61 birds in the park."	
Student B. "I used a math drawing and made a pile of 36 and a pile of 25. Altogether, I had 5 tens and 11 ones. 11 ones is the same as one ten and one left over. So, I really had 6 tens and 1 one. That makes 61."	

Adapted from NCDPI 2013b.

Focus, Coherence, and Rigor

When students explain why addition and subtraction strategies work (**2.NBT.9▲**), they reinforce foundations for solving one- and two-step word problems (**2.OA.1▲**) and extend their understanding and use of various strategies and models, drawings, and a written method to add and subtract (**2.NBT.5▲** and **2.NBT.7▲**).

Students are to fluently add and subtract within 100 in grade two (using place-value strategies, properties of operations, and/or the relationship between addition and subtraction) (**2.NBT.5▲**). In grade one, students added within 100 using concrete models or drawings and used at least one method that is generalizable to larger numbers (such as between 101 and 1000). In grade two, students extend addition to within 1000 using these generalizable concrete methods. This extension could be connected first to adding all two-digit numbers (e.g., 78 + 47) so that students can see and discuss composing both ones and tens without the complexity of hundreds in the drawings or numbers.

After solving addition problems that compose both ones and tens for all two-digit numbers and then three-digit numbers within 1000, the fluency problems for grade two seem easy: 28 + 47 requires composing only the ones. This is now easier to do without drawings: one just records the new ten before it is added in, or adds it in mentally. *Fluent adding* means adding without drawings.

The same approach may be taken for subtraction, first solving with concrete models or drawings of subtractions within 100 that involve decomposing 1 ten to make 10 ones and then solving subtraction problems that require two decompositions, of 1 hundred to make 10 tens and of 1 ten to make 10 ones. Spending a long time subtracting within 100 initially can stimulate students to count on or count down, methods that become considerably more difficult above 100. Problems with all possibilities of decompositions should be mixed in so that students solve problems requiring two, one, and no decompositions. Then students can spend time on subtractions that include multiple hundreds (totals from 201 to 1000). After this experience, focusing within 100 just on the two cases of one decomposition (e.g., 73 – 28) or no decomposition (e.g., 78 – 23) is relatively easy to do without drawings.

Mental math as an instructional tool. Many teachers incorporate an activity known as "mental math" into their classrooms. The teacher typically writes a problem on the board (such as 45 + 47) and asks students to solve the problem only through a mental process. The teacher then records all answers given by students, whether correct or incorrect, without judgment. A class discussion follows; students explain how they got their answers and decide which answer is correct. The class may agree or disagree with a particular method, find out where another student made an error, or compare different solution methods (e.g., how finding 45 + 45 + 2 is similar to finding 40 + 40 + 12). In mental math, multiple strategies often emerge naturally from the students, and opportunities to explore these strategies arise. When students do not have more than one strategy for solving a problem, this can be an indication to the teacher that students have a limited repertoire of such strategies, and therefore mental math can be used as a valuable instructional tool. Mental math supports several Mathematical Practice standards, including **MP.1, MP.2, MP.3, MP.7,** and **MP.8**. (Standard **2.NBT.8▲** calls for second-grade students to practice mental math by adding and subtracting multiples of 10 and 100 from a given number between 100 and 900.)

Domain: Measurement and Data

Grade-two students transition from measuring lengths with informal or non-standard units to measuring with standard units—inches, feet, centimeters, and meters—and using standard measurement tools (**2.MD.1▲**). Students learn the measure of length as a count of how many units are needed to match the length of the object or distance being measured. Using both customary units (inches and feet) and metric units (centimeters and meters), students measure the length of objects with rulers, yardsticks, meter sticks, and tape measures. Students become familiar with standard units (e.g., 12 inches in a foot, 3 feet in a yard, and 100 centimeters in a meter) and how to estimate lengths (adapted from KATM 2012, 2nd Grade Flipbook).

Measurement and Data	**2.MD**

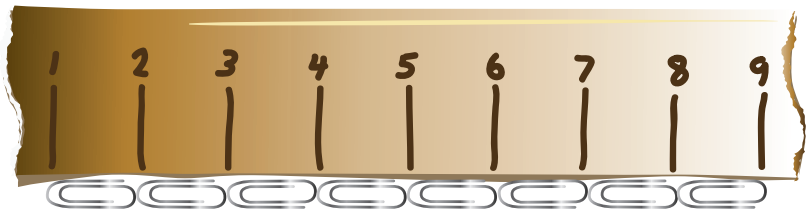
Measure and estimate lengths in standard units.

1. Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
3. Estimate lengths using units of inches, feet, centimeters, and meters.
4. Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

Students also can learn accurate measurement procedures and concepts by constructing simple unit rulers (see figure 2-5). Using copies of a standard unit, such as manipulatives that measure one inch, students mark off unit lengths on strips of paper, explicitly connecting the process of measuring with a ruler to measuring by iterating physical units.

Figure 2-5. Example of a Student-Created Ruler

Using a unit to draw a ruler



Students use a standard unit (shown below the ruler) to make rulers, helping them to understand the meaning of the marks on rulers.

Adapted from UA Progressions Documents 2012a.

Thus, students' first rulers are simple tools to help count the iteration of unit lengths. Frequently comparing results of measuring the same object with manipulatives of standard unit length (e.g., a block that is one inch long) and with student-created rulers can help students connect their experiences and ideas. As they build and use these tools, they develop the ideas of unit length iteration (unit lengths are all of equal size), correct alignment (with a ruler), measurement of the length *between* hashmarks on the ruler, and the zero-point concept (the idea that the zero of the ruler indicates one endpoint of a length).

Grade-two students learn the concept of the inverse relationship between the size of the unit of length and the number of units required to cover a definite length or distance—specifically, that the larger the unit, the fewer units are needed to measure something, and vice versa (**2.MD.2▲**). Students measure the length of the same object using units of different lengths (ruler with inches versus ruler with centimeters, or a foot ruler versus a yardstick) and discuss the relationship between the size of the units and the measurements.

Example	**2.MD.2▲**
A student measured the length of a desk in both feet and inches. The student found that the desk was 3 feet long and that it was 36 inches long. **Teacher:** "Why do you think you have two different measurements for the same desk?" **Student:** "It only took 3 feet because the feet are so big. It took 36 inches because an inch is much smaller than a foot."	

Students use this information to understand how to select appropriate tools for measuring a given object. For instance, a student might think, "The longer the unit, the fewer units I need." Measurement problems also support mathematical practices such as reasoning quantitatively (**MP.2**), justifying conclusions (**MP.3**), using appropriate tools (**MP.5**), attending to precision (**MP.6**), and making use of structure or patterns (**MP.7**).

After gaining experience with measurement, students learn to estimate lengths using units of inches, feet, centimeters, and meters (**2.MD.3▲**). Students estimate lengths before they measure. After measuring an object, students discuss their estimations, measurement procedures, and the differences between their estimates and the measurements. Students should begin by estimating measurements of familiar items (e.g., the length of a desk, pencil, favorite book, and so forth). Estimation helps students focus on the attribute to be measured, the length units, and the process. Students need many experiences with the use of measurement tools to develop their understanding of linear measurement; an example is provided below.

Example	**2.MD.3▲**
Teacher: "How many inches do you think this string is if you measure it with a ruler?" **Student:** "An inch is pretty small. I'm thinking it will be somewhere between 8 and 9 inches." **Teacher:** "Measure it and see." **Student:** "It is 9 inches. I thought that it would be somewhere around there."	

This example also supports mathematical practices such as making sense of quantities (**MP.2**) and using appropriate tools strategically (**MP.5**).

Students also measure to determine the difference in length between one object and another, expressing the difference in terms of a standard length unit (**2.MD.4▲**). Grade-two students use inches, feet, yards, centimeters, and meters to compare the lengths of two objects. They use comparative phrases such as "It is 2 inches longer" or "It is shorter by 5 centimeters" to describe the difference in length between the two objects. Students use both the quantity and the unit name to precisely compare length (ADE 2010 and NCDPI 2013b).

Focus, Coherence, and Rigor

As students compare objects by their lengths, they also reinforce skills and understanding related to solving "compare" problems in the major cluster "Represent and solve problems involving addition and subtraction." Drawing comparison bars to represent the different measurements helps make this link explicit (see standard **2.OA.1▲**).

Students apply the concept of length to solve addition and subtraction problems. Word problems should refer to the same unit of measure (**2.MD.5▲**).

Measurement and Data — 2.MD

Relate addition and subtraction to length.

5. Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.
6. Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . ., and represent whole-number sums and differences within 100 on a number line diagram.

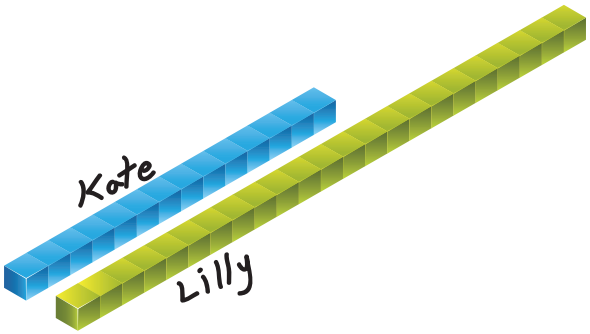
In grade two, students also connect the concept of the ruler to the concept of the number line. These understandings are essential to supporting work with number line diagrams.

Example **2.MD.5▲**

Kate jumped 14 inches in gym class. Lilly jumped 23 inches. How much farther did Lilly jump than Kate? Solve the problem and then write an equation.

Student A: My equation is 14 + ___ = 23. I thought, "14 and what makes 23?" I used cubes. I made a train of 14. Then I made a train of 23. When I put them side by side, I saw that Kate would need 9 more cubes to be the same as Lilly. So, Lilly jumped 9 more inches than Kate.

14 + 9 = 23. (**MP.1, MP.2, MP.4**)



Student B: My equation is 23 – 14 = ___. I thought about what the difference was between Kate and Lilly. I broke up 14 into 10 and 4. I know that 23 minus 10 is 13. Then, I broke up the 4 into 3 and 1. 13 minus 3 is 10. Then, I took one more away. That left me with 9. So, Lilly jumped 9 inches more than Kate. That seems to make sense, since 23 is almost 10 more than 14.

23 – 14 = 9. (**MP.2, MP.7, MP.8**)

Focus, Coherence, and Rigor

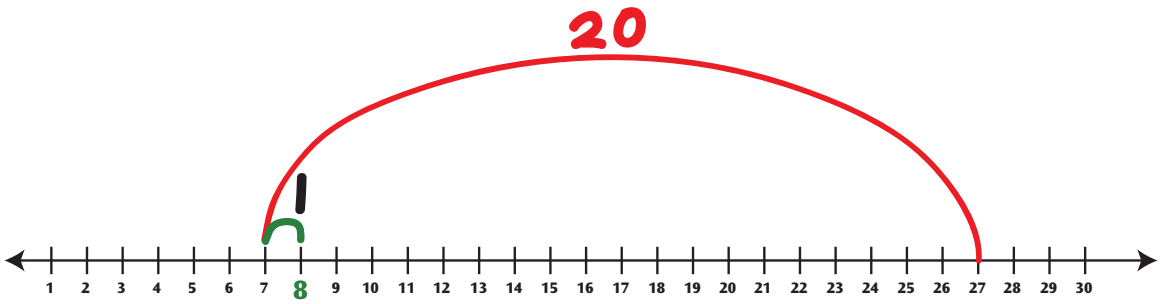
Addition and subtraction word problems involving lengths develop mathematical practices such as making sense of problems (**MP.1**), reasoning quantitatively (**MP.2**), justifying conclusions (**MP.3**), using appropriate tools strategically (**MP.5**), attending to precision (**MP.6**), and evaluating the reasonableness of results (**MP.8**). Similar word problems also support students' ability to fluently add and subtract, which is part of the major work at the grade (refer to fluency expectations in standards **2.OA.1▲** and **2.NBT.5▲**).

Using a number line diagram to understand number and number operations requires students to comprehend that number line diagrams have specific conventions: namely, that a single position is used to represent a whole number and that marks are used to indicate those positions. Students need to understand that a number line diagram is like a ruler in that consecutive whole numbers are one unit apart; thus, they need to consider the distances between positions and segments when identifying missing numbers. These understandings underlie the successful use of number line diagrams. Students think of a number line diagram as a measurement model and use strategies relating to distance, proximity of numbers, and reference points (UA Progressions Documents 2012a).

Example **2.MD.6▲**

There were 27 students on the bus. Nineteen (19) students got off the bus. How many students are on the bus?

Student: I used a number line. I saw that 19 is really close to 20. Since 20 is a lot easier to work with, I took a jump of 20. But, that was one too many. So, I took a jump of 1 to make up for the extra. I landed on 8. So, there are 8 students on the bus. What I did was 27 – 20 = 7, and then 7 + 1 = 8.



Adapted from ADE 2010 and NCDPI 2013b.

Teachers should ensure that students make the connection between problems involving measuring with a ruler and those involving a number line as a problem-solving strategy.

Focus, Coherence, and Rigor

Using addition and subtraction within 100 to solve word problems involving length (**2.MD.5**) and representing sums and differences on a number line (**2.MD.6**) reinforces the use of models to add and subtract and supports major work at grade two (see standards **2.OA.A.1▲** and **2.NBT.7▲**). Similar problems also develop mathematical practices such as making sense of problems (**MP.2**), justifying conclusions (**MP.3**), and modeling mathematics (**MP.4**).

In grade one, students learned to tell time to the nearest hour and half-hour. Second-grade students tell time to the nearest five minutes (**2.MD.7▲**).

Measurement and Data 2.MD

Work with time and money.

7. Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m. **Know relationships of time (e.g., minutes in an hour, days in a month, weeks in a year). CA**
8. Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using $ and ¢ symbols appropriately. *Example: If you have 2 dimes and 3 pennies, how many cents do you have?*

Students can make connections between skip-counting by fives (**2.NBT.2▲**) and five-minute intervals on the clock. Students work with both digital and analog clocks. They recognize time in both formats and communicate their understanding of time using both numbers and language.

Second-grade students also understand that there are two 12-hour cycles in a day—a.m. and p.m. A daily journal can help students make real-world connections and understand the differences between these two cycles.

Focus, Coherence, and Rigor

Students' understanding and use of skip-counting by fives and tens (**2.NBT.2▲**) can also support telling and writing time to the nearest five minutes (**2.MD.7▲**). Students notice the pattern of numbers and apply this understanding to time (**MP.7**).

In grade two, students solve word problems involving dollars or cents (**2.MD.8**). They identify, count, recognize, and use coins and bills in and out of context. Second-grade students should have opportunities to make equivalent amounts using both coins and bills. Dollar bills should include denominations up to one hundred ($1, $5, $10, $20, $50, $100). Note that students in second grade *do no*t express money amounts using decimal points.

Just as students learn that a number may be represented in different ways and still remain the same amount—e.g., 38 can be 3 tens and 8 ones or 2 tens and 18 ones—students can apply this understanding to money. For example, 25 cents could be represented as a quarter, two dimes and a nickel, or 25 pennies, all of which have the same value. Building the concept of equivalent worth takes time, and students will need numerous opportunities to create and count different sets of coins and to recognize the "purchasing power" of coins (e.g., a girl can buy the same things with a nickel that she can purchase with 5 pennies).

As teachers provide students with opportunities to explore coin values (e.g., 25 cents), actual coins (e.g., 2 dimes and 1 nickel), and drawings of circles that have values indicated, students gradually learn to mentally assign a value to each coin in a set, place a random set of coins in order, use mental math, add on to find differences, and skip-count to determine the total amount.

Examples	**2.MD.8**
Using pennies, nickels, dimes, and quarters, how many different ways can you make 37 cents?	
Using $1, $5, and $10 bills, how many different ways can you make $12?	

Adapted from ADE 2010 and NCDPI 2013b.

Measurement and Data 2.MD

Represent and interpret data.

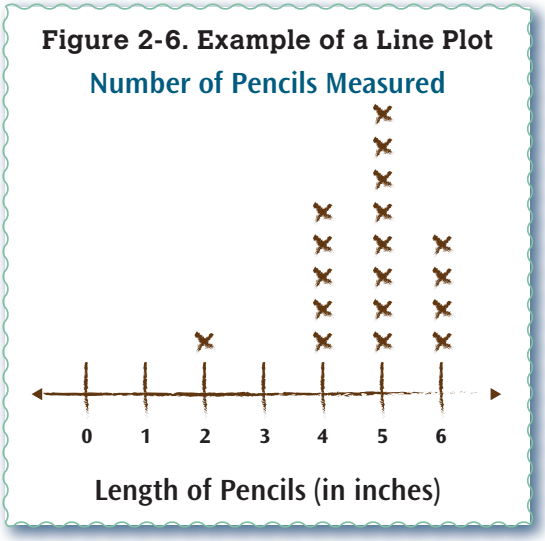
9. Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.

10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems[4] using information presented in a bar graph.

Students use the measurement skills described in previous standards (**2.MD.1–4▲**) to measure objects and create measurement data (**2.MD.9**). For example, they measure objects in their desk to the nearest inch, display the data collected on a line plot, and answer related questions. Line plots are first introduced in grade two. A line plot can be thought of as plotting data on a number line (see figure 2-6).

In grade one, students worked with three categories of data. In grade two, students work with data that have up to four categories. Students organize and represent data on a picture graph or bar graph (with single-unit scale) and interpret the results. They solve simple put-together, take-apart, and "compare" problems using information presented in a bar graph (**2.MD.10**). In grade two, picture graphs (pictographs) include symbols that represent single units. Pictographs should include a title, categories, category label, key, and data.

Figure 2-6. Example of a Line Plot



Focus, Coherence, and Rigor

Students use data to pose and solve simple one-step addition and subtraction problems. The use of picture graphs and bar graphs to represent a data set (**2.MD.10**) reinforces grade-level work in the major cluster **"Represent and solve problems involving addition and subtraction"** and provides a context for students to solve related addition and subtraction problems (**2.OA.1▲**).

4. See glossary, table GL-4.

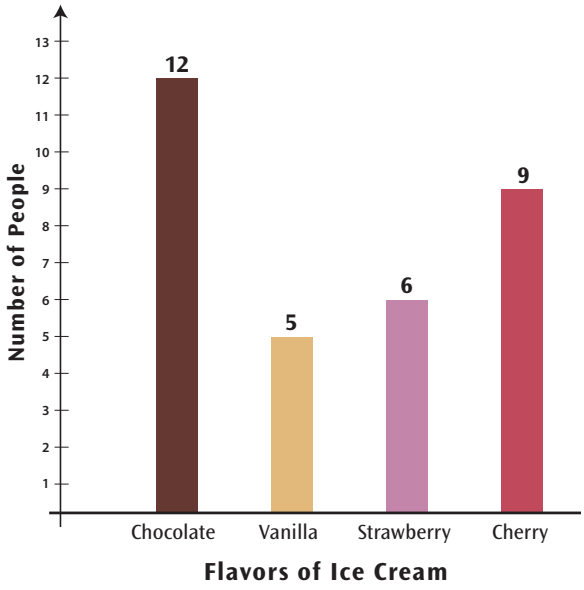
Example 2.MD.10

Students are responsible for purchasing ice cream for an event at school. They decide to collect data to determine which flavors to buy for the event. Students decide on the question to ask their peers—"What is your favorite flavor of ice cream?"—and four likely responses: chocolate, vanilla, strawberry, and cherry. Students form two teams and collect information from different classes in their school. Each team decides how to keep track of its data (e.g., with tally marks, check marks, or in a table). Each team selects either a picture graph or a bar graph to display its data. Graphs are created using paper or a computer.

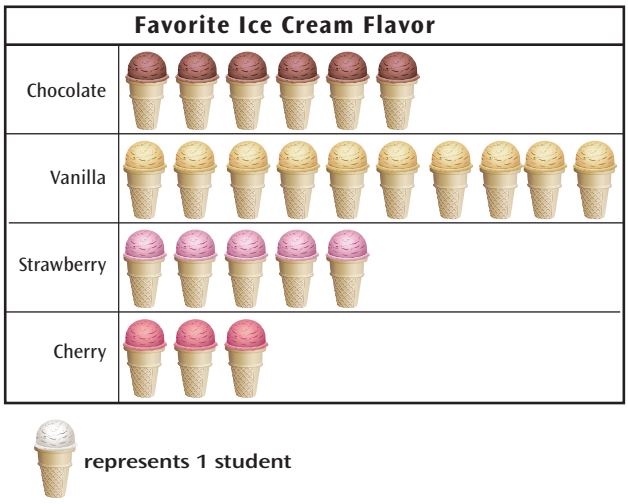
The teacher facilitates a discussion about the data collected, asking questions such as these:

- Based on the graph from Team A, how many students voted for cherry, strawberry, vanilla, or chocolate ice cream?
- Based on the graph from Team B, how many students voted for cherry, strawberry, vanilla, or chocolate ice cream?
- Based on the data from both teams, which flavor received the most votes? Which flavor received the fewest votes?
- What was the second-favorite flavor?
- Based on the data collected, what flavors of ice cream do you think we should purchase for our event, and why do you think that?

Team A: Bar Graph



Team B: Picture Graph



Adapted from NCDPI 2013b.

Representing and interpreting data to solve problems also develops mathematical practices such as making sense of problems (**MP.1**), reasoning quantitatively (**MP.2**), justifying conclusions (**MP.3**), using appropriate tools strategically (**MP.5**), attending to precision (**MP.6**), and evaluating the reasonableness of results (**MP.8**).

Domain: Geometry

In grade one, students reasoned about attributes of geometric shapes. A critical area of instruction in second grade is for students to describe and analyze shapes by examining their sides and angles. This work develops a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.

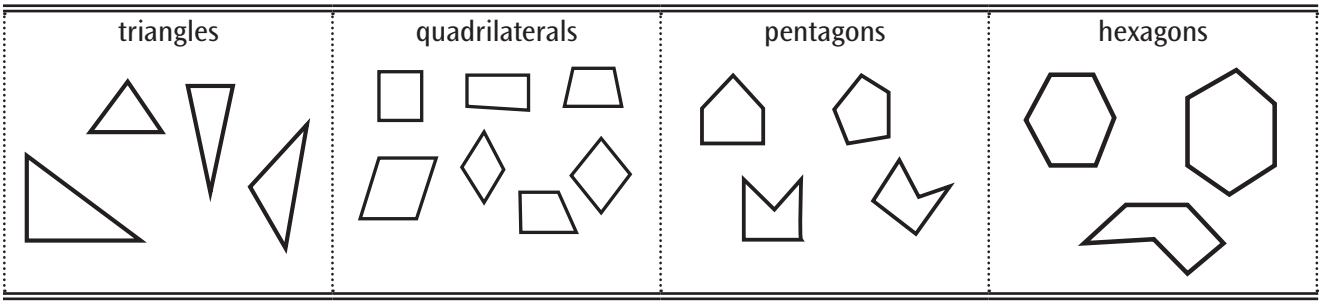
Geometry	2.G

Reason with shapes and their attributes.

1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[5] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.
2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.
3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves, thirds, half of, a third of,* etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Students identify, describe, and draw triangles, quadrilaterals (squares, rectangles and parallelograms, and trapezoids), pentagons, hexagons, and cubes (**2.G.1**); see figure 2-7. Pentagons, triangles, and hexagons should appear as both regular (having equal sides and equal angles) and irregular. Second-grade students recognize all four-sided shapes as quadrilaterals. They use the vocabulary word *angle* in place of *corner,* but they do not need to name angle types (e.g., right, acute, obtuse). Shapes should be presented in a variety of orientations and configurations.

Figure 2-7. Examples of the Presentation of Various Shapes



Source: ADE 2010.

As students use attributes to identify and describe shapes, they also develop mathematical practices such as analyzing givens and constraints (**MP.1**), justifying conclusions (**MP.3**), modeling with mathematics (**MP.4**), using appropriate tools strategically (**MP.5**), attending to precision (**MP.6**), and looking for a pattern or structure (**MP.7**).

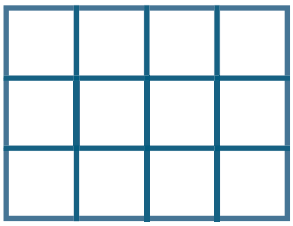
Students partition a rectangle into rows and columns of same-size squares and count to find the total number of squares (**2.G.2**). As students partition rectangles into rows and columns, they build a foundation for learning about the area of a rectangle and using arrays for multiplication.

5. Sizes are compared directly or visually, not by measuring.

Example **2.G.2**

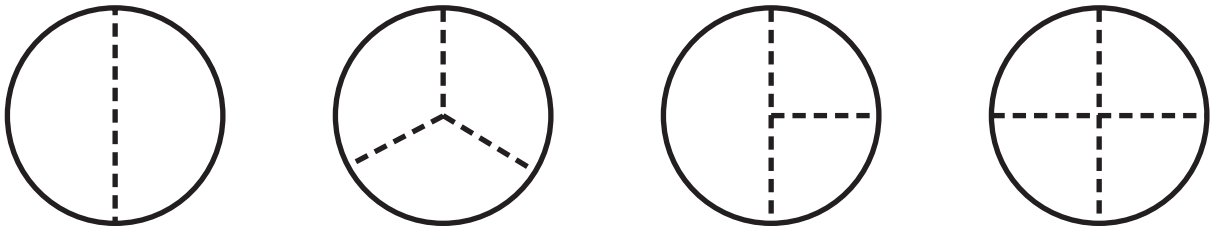
Teacher: Partition this rectangle into 3 equal rows and 4 equal columns. How can you partition into 3 equal rows? Then into 4 equal columns? Can you do it in the other order? How many small squares did you make?

Student: I counted 12 squares in the rectangle. This is a lot like when we counted arrays by counting 4 + 4 + 4 = 12.



An interactive whiteboard or manipulatives such as square tiles, cubes, or other square-shaped objects can be used to help students partition rectangles (**MP.5**).

In grade one, students partitioned shapes into halves, fourths, and quarters. Second-grade students partition circles and rectangles into two, three, or four equal shares (regions). Students explore this concept with paper strips and pictorial representations and work with the vocabulary terms *halves, thirds,* and *fourths* (**2.G.3**). Students recognize that when they cut a circle into three equal pieces, each piece will equal one-third of its original whole and the whole may be described as three-thirds. If a circle is cut into four equal pieces, each piece will equal one-fourth of its original whole, and the whole is described as four-fourths.



Circle cut into halves

Circle cut into thirds

Circle ***not*** cut into thirds

Circle cut into fourths

Students should see circles and rectangles partitioned in multiple ways so they learn to recognize that equal shares can be different shapes within the same whole.

As students partition circles and squares and explain their thinking, they develop mathematical practices such as making sense of quantities (**MP.2**), justifying conclusions (**MP.3**), attending to precision (**MP.6**), and evaluating the reasonableness of their results (**MP.7**). They also develop understandings that will support grade-three work in the major cluster "Develop understanding of fractions as numbers" (**3.NF.1–3▲**) [adapted from ADE 2010 and NCDPI 2013b].

Essential Learning for the Next Grade

In kindergarten through grade five, the focus is on the addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals, with a balance of concepts, skills, and problem solving. Arithmetic is viewed as an important set of skills and also as a thinking subject that, when done thoughtfully, prepares students for algebra. Measurement and geometry develop alongside number and operations and are tied specifically to arithmetic along the way.

In kindergarten through grade two, students focus on addition, subtraction, and measurement using whole numbers. To be prepared for grade-three mathematics, students should be able to demonstrate by the end of grade two that they have acquired specific mathematical concepts and procedural skills and have met the fluency expectations for the grade. For grade-two students, the expected fluencies are to add and subtract within 20 using mental strategies and know from memory all sums of two one-digit numbers (**2.OA.2▲**), and to add and subtract within 100 using various strategies (**2.NBT.5▲**). These fluencies and the conceptual understandings that support them are foundational for work in later grades.

Of particular importance at grade two are concepts, skills, and understandings of addition and subtraction within 20 and representing and solving problems involving addition and subtraction (**2.OA.1–2▲**); place value (**2.NBT.1–4▲**) and the use of place-value understanding and properties of operations to add and subtract (**2.NBT.5–9▲**); measuring and estimating lengths in standard units (**2.MD.1–4▲**); and relating addition and subtraction to length (**2.MD.5–6▲**).

Place Value

By the end of grade two, students are expected to read, write, and count to 1000, skip-counting by twos, fives, tens, and hundreds. Students need to understand that 100 can be thought of as a bundle of 10 tens and also understand three-digit whole numbers in terms of hundreds, tens, and ones.

Addition and Subtraction

Addition and subtraction are major instructional focuses in kindergarten through grade two. By the end of grade two, students are expected to add and subtract (using concrete models, drawings, and strategies) within 1000 (**2.NBT.7▲**). Students should add and subtract fluently within 100 using various strategies (**2.NBT.5▲**), and add and subtract fluently within 20 using mental strategies (**2.OA.2▲**). Students mentally add and subtract 10 or 100, within the range 100–900 (**2.NBT.8▲**). They are expected to know from memory all sums of two one-digit numbers (**2.OA.2▲**). Students should also know how to apply addition and subtraction to solve a variety of one- and two-step word problems (within 100) involving add-to, take-from, put-together, take-apart, and compare situations (**2.OA.1▲**); refer to table 2-3 for additional information.

Students who have met the grade-two standards for addition and subtraction will be prepared to fluently add and subtract within 1000 using strategies and algorithms, as required in the grade-three standards. These foundations will also prepare students for concepts, skills, and problem solving with multiplication and division, which are introduced in grade three.

Measurement

By the end of grade two, students can measure lengths using standard units—inches, feet, centimeters, and meters. Students need to know how to use addition and subtraction within 100 to solve word problems involving lengths (**2.MD.5▲**). Mastering grade-two measurement standards will prepare students to measure fractional amounts and to add, subtract, multiply, or divide to solve word problems involving mass or volume, as required in the grade-three standards.

Grade 2 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving addition and subtraction.
- Add and subtract within 20.
- Work with equal groups of objects to gain foundations for multiplication.

Number and Operations in Base Ten

- Understand place value.
- Use place-value understanding and properties of operations to add and subtract.

Measurement and Data

- Measure and estimate lengths in standard units.
- Relate addition and subtraction to length.
- Work with time and money.
- Represent and interpret data.

Geometry

- Reason with shapes and their attributes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

2 Grade 2

Operations and Algebraic Thinking 2.OA

Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[6]

Add and subtract within 20.

2. Fluently add and subtract within 20 using mental strategies.[7] By end of Grade 2, know from memory all sums of two one-digit numbers.

Work with equal groups of objects to gain foundations for multiplication.

3. Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.

4. Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Number and Operations in Base Ten 2.NBT

Understand place value.

1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:
   a. 100 can be thought of as a bundle of 10 tens—called a "hundred."
   b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).

2. Count within 1000; skip-count by **2s**, 5s, 10s, and 100s. **CA**

3. Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

4. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.

Use place-value understanding and properties of operations to add and subtract.

5. Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

6. Add up to four two-digit numbers using strategies based on place value and properties of operations.

7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

6. See glossary, table GL-4.
7. See standard 1.OA.6 for a list of mental strategies.

7.1 Use estimation strategies to make reasonable estimates in problem solving. CA

8. Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.

9. Explain why addition and subtraction strategies work, using place value and the properties of operations.[8]

Measurement and Data 2.MD

Measure and estimate lengths in standard units.

1. Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

3. Estimate lengths using units of inches, feet, centimeters, and meters.

4. Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

Relate addition and subtraction to length.

5. Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

6. Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

Work with time and money.

7. Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m. **Know relationships of time (e.g., minutes in an hour, days in a month, weeks in a year). CA**

8. Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using $ and ¢ symbols appropriately. *Example: If you have 2 dimes and 3 pennies, how many cents do you have?*

Represent and interpret data.

9. Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.

10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems[9] using information presented in a bar graph.

8. Explanations may be supported by drawings or objects.
9. See glossary, table GL-4.

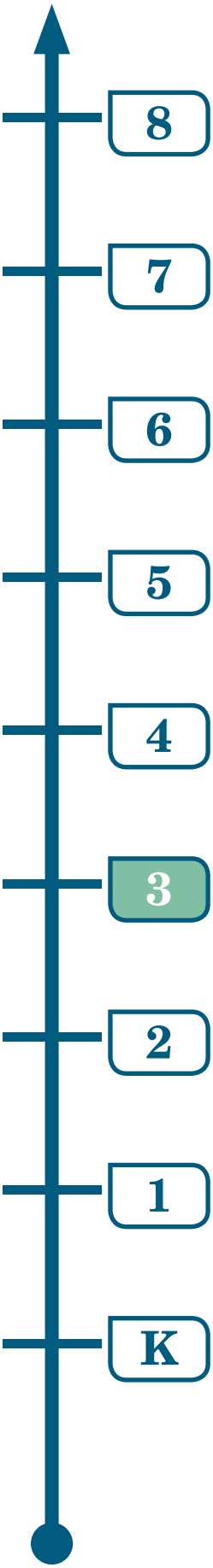
2 Grade 2

Geometry 2.G

Reason with shapes and their attributes.

1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[10] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.
2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.
3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves, thirds, half of, a third of,* etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

10. Sizes are compared directly or visually, not by measuring.



Grade Three

In grade three, students continue to build upon their mathematical foundation as they focus on the operations of multiplication and division and the concept of fractions as numbers. In previous grades, students developed an understanding of place value and used methods based on place value to add and subtract within 1000. They developed fluency with addition and subtraction within 100 and laid a foundation for understanding multiplication based on equal groups and the array model. Students also worked with standard units to measure length and described attributes of geometric shapes (adapted from Charles A. Dana Center 2012).

Critical Areas of Instruction

In grade three, instructional time should focus on four critical areas: (1) developing understanding of multiplication and division, as well as strategies for multiplication and division within 100; (2) developing understanding of fractions, especially unit fractions (fractions with a numerator of 1); (3) developing understanding of the structure of rectangular arrays and of area; and (4) describing and analyzing two-dimensional shapes (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010j). Students also work toward fluency with addition and subtraction within 1000 and multiplication and division within 100. By the end of grade three, students know all products of two one-digit numbers from memory.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade-level examples of focus, coherence, and rigor are indicated throughout the chapter.

The standards do not give equal emphasis to all content for a particular grade level. Cluster headings can be viewed as the most effective way to communicate the **focus** and **coherence** of the standards. Some clusters of standards require a greater instructional emphasis than others based on the depth of the ideas, the time needed to master those clusters, and their importance to future mathematics or the later demands of preparing for college and careers.

Table 3-1 highlights the content emphases at the cluster level for the grade-three standards. The bulk of instructional time should be given to "Major" clusters and the standards within them, which are indicated throughout the text by a triangle symbol (▲). However, standards in the "Additional/Supporting" clusters should not be neglected; to do so would result in gaps in students' learning, including skills and understandings they may need in later grades. Instruction should reinforce topics in major clusters by using topics in the additional/supporting clusters and including problems and activities that support natural connections between clusters.

Teachers and administrators alike should note that the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences presented in a coherent manner (adapted from Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

Table 3-1. Grade Three Cluster-Level Emphases

Operations and Algebraic Thinking — 3.OA

Major Clusters

- Represent and solve problems involving multiplication and division. (**3.OA.1–4▲**)
- Understand properties of multiplication and the relationship between multiplication and division. (**3.OA.5–6▲**)
- Multiply and divide within 100. (**3.OA.7▲**)
- Solve problems involving the four operations, and identify and explain patterns in arithmetic. (**3.OA.8–9▲**)

Number and Operations in Base Ten — 3.NBT

Additional/Supporting Clusters

- Use place-value understanding and properties of operations to perform multi-digit arithmetic. (**3.NBT.1–3**)

Number and Operations—Fractions — 3.NF

Major Clusters

- Develop understanding of fractions as numbers. (**3.NF.1–3▲**)

Measurement and Data — 3.MD

Major Clusters

- Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects. (**3.MD.1–2▲**)
- Geometric measurement: understand concepts of area and relate area to multiplication and to addition. (**3.MD.5–7▲**)

Additional/Supporting Clusters

- Represent and interpret data. (**3.MD.3–4**)
- Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures. (**3.MD.8**)

Geometry — 3.G

Additional/Supporting Clusters

- Reason with shapes and their attributes. (**3.G.1–2**)

Explanations of Major and Additional/Supporting Cluster-Level Emphases

Major Clusters (▲) — Areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than others based on the depth of the ideas, the time needed to master them, and their importance to future mathematics or the demands of college and career readiness.

Additional Clusters — Expose students to other subjects; may not connect tightly or explicitly to the major work of the grade.

Supporting Clusters — Designed to support and strengthen areas of major emphasis.

Note of caution: Neglecting material, whether it is found in the major or additional/supporting clusters, will leave gaps in students' skills and understanding and will leave students unprepared for the challenges they face in later grades.

Adapted from Smarter Balanced Assessment Consortium 2011, 83.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (MP) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The MP standards represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students.

Although the description of the MP standards remains the same at all grade levels, the way these standards look as students engage with and master new and more advanced mathematical ideas does change. Table 3-2 presents examples of how the MP standards may be integrated into tasks appropriate for students in grade three. (Refer to the Overview of the Standards Chapters for a description of the MP standards.)

Table 3-2. Standards for Mathematical Practice—Explanation and Examples for Grade Three

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	In third grade, mathematically proficient students know that doing mathematics involves solving problems and discussing how they solved them. Students explain to themselves the meaning of a problem and look for ways to solve it. Students may use concrete objects, pictures, or drawings to help them conceptualize and solve problems such as these: "Jim purchased 5 packages of muffins. Each package contained 3 muffins. How many muffins did Jim purchase?"; or "Describe another situation where there would be 5 groups of 3 or 5×3." Students may check their thinking by asking themselves, "Does this make sense?" Students listen to other students' strategies and are able to make connections between various methods for a given problem.
MP.2 Reason abstractly and quantitatively.	Students recognize that a number represents a specific quantity. They connect the quantity to written symbols and create a logical representation of the problem at hand, considering both the appropriate units involved and the meaning of quantities. For example, students apply their understanding of the meaning of the equal sign as "the same as" to interpret an equation with an unknown. When given $4 \times __ = 40$, they might think: • 4 groups of some number is the same as 40. • 4 times some number is the same as 40. • I know that 4 groups of 10 is 40, so the unknown number is 10. • The missing factor is 10, because 4 times 10 equals 40. To reinforce students' reasoning and understanding, teachers might ask, "How do you know?" or "What is the relationship between the quantities?"
MP.3 Construct viable arguments and critique the reasoning of others.	Students may construct arguments using concrete referents, such as objects, pictures, and drawings. They refine their mathematical communication skills as they participate in mathematical discussions that the teacher facilitates by asking questions such as "How did you get that?" and "Why is that true?" Students explain their thinking to others and respond to others' thinking. For example, after investigating patterns on the hundreds chart, students might explain why the pattern makes sense.

Table 3-2 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
MP.4 Model with mathematics.	Students represent problem situations in multiple ways using numbers, words (mathematical language), objects, and math drawings. They might also represent a problem by acting it out or by creating charts, lists, graphs, or equations. For example, students use various contexts and a variety of models (e.g., circles, squares, rectangles, fraction bars, and number lines) to represent and develop understanding of fractions. Students use models to represent both equations and story problems and can explain their thinking. They evaluate their results in the context of the situation and reflect on whether the results make sense. Students should be encouraged to answer questions such as "What math drawing or diagram could you make and label to represent the problem?" or "What are some ways to represent the quantities?"
MP.5 Use appropriate tools strategically.	Mathematically proficient students consider the available tools (including drawings or estimation) when solving a mathematical problem and decide when particular tools might be helpful. For instance, they may use graph paper to find all the possible rectangles that have a given perimeter. They compile the possibilities into an organized list or a table and determine whether they have all the possible rectangles. Students should be encouraged to answer questions (e.g., "Why was it helpful to use _____ ?").
MP.6 Attend to precision.	Students develop mathematical communication skills as they use clear and precise language in their discussions with others and in their own reasoning. They are careful to specify units of measure and to state the meaning of the symbols they choose. For instance, when calculating the area of a rectangle they record the answer in square units.
MP.7 Look for and make use of structure.	Students look closely to discover a pattern or structure. For instance, students use properties of operations (e.g., commutative and distributive properties) as strategies to multiply and divide. Teachers might ask, "What do you notice when _____ ?" or "How do you know if something is a pattern?"
MP.8 Look for and express regularity in repeated reasoning.	Students notice repetitive actions in computations and look for "shortcut" methods. For instance, students may use the distributive property as a strategy to work with products of numbers they know to solve products they do not know. For example, to find the product of 7×8, students might decompose 7 into 5 and 2 and then multiply 5×8 and 2×8 to arrive at $40 + 16$, or 56. Third-grade students continually evaluate their work by asking themselves, "Does this make sense?" Students should be encouraged to answer questions such as "What is happening in this situation?" or "What predictions or generalizations can this pattern support?"

Adapted from Arizona Department of Education (ADE) 2010 and North Carolina Department of Public Instruction (NCDPI) 2013b.

Standards-Based Learning at Grade Three

The following narrative is organized by the domains in the Standards for Mathematical Content and highlights some necessary foundational skills from previous grade levels. It also provides exemplars to explain the content standards, highlight connections to Standards for Mathematical Practice (**MP**), and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. A triangle symbol (▲) indicates standards in the major clusters (see table 3-1).

Domain: Operations and Algebraic Thinking

In kindergarten through grade two, students focused on developing an understanding of addition and subtraction. Beginning in grade three, students focus on concepts, skills, and problem solving for multiplication and division. Students develop multiplication strategies, make a shift from additive to multiplicative reasoning, and relate division to multiplication. Third-grade students become fluent with multiplication and division within 100. This work will continue in grades four and five, preparing the way for work with ratios and proportions in grades six and seven (adapted from the University of Arizona Progressions Documents for the Common Core Math Standards [UA Progressions Documents] 2011a and PARCC 2012).

Multiplication and division are new concepts in grade three, and meeting fluency is a major portion of students' work (see **3.OA.7▲**). Reaching fluency will take much of the year for many students. These skills and the understandings that support them are crucial; students will rely on them for years to come as they learn to multiply and divide with multi-digit whole numbers and to add, subtract, multiply, and divide with rational numbers.

There are many patterns to be discovered by exploring the multiples of numbers. Examining and articulating these patterns is an important part of the mathematical work on multiplication and division. Practice—and, if necessary, extra support—should continue all year for those students who need it to attain fluency. This practice can begin with the easier multiplication and division problems while the pattern work is occurring with more difficult numbers (adapted from PARCC 2012). Relating and practicing multiplication and division problems involving the same number (e.g., the 4s) may be helpful.

Operations and Algebraic Thinking — 3.OA

Represent and solve problems involving multiplication and division.

1. Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as* 5×7.
2. Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as* $56 \div 8$.
3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]
4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations* $8 \times ? = 48$, $5 = \square \div 3$, $6 \times 6 = ?$.

A critical area of instruction is to develop student understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models (NGA/CCSSO 2010c). Multiplication and division are new concepts in grade three. Initially,

1. See glossary, table GL-5.

students need opportunities to develop, discuss, and use efficient, accurate, and generalizable methods to compute. The goal is for students to use general written methods for multiplication and division that students can explain and understand (e.g., using visual models or place-value language). The general written methods should be variations of the standard algorithms. Reaching fluency with these operations requires students to use variations of the standard algorithms without visual models, and this could take much of the year for many students.

Students recognize multiplication as finding the total number of objects in a particular number of equal-sized groups (**3.OA.1▲**). Also, students recognize division in two different situations: *partitive division* (also referred to as *fair-share division*), which requires equal sharing (e.g., how many are in each group?); and *quotitive division* (or *measurement division*), which requires determining how many groups (e.g., how many groups can you make?) [**3.OA.2▲**]. These two interpretations of division have important uses later, when students study division of fractions, and both interpretations should be explored as representations of division. In grade three, teachers should use the terms *number of shares* or *number of groups* with students rather than *partitive* or *quotitive*.

Multiplication of Whole Numbers	**3.OA.1▲**
Note that the standards define multiplication of whole numbers $a \times b$ as finding the total number of objects in a groups of b objects. ***Example:*** There are 3 bags of apples on the table. There are 4 apples in each bag. How many apples are there altogether?	
Partitive Division (also known as Fair-Share or Group Size Unknown Division)	**3.OA.2▲**
The number of groups or shares to be made is known, but the number of objects in (or size of) each group or share is unknown. ***Example:*** There are 12 apples on the counter. If you are sharing the apples equally among 3 bags, how many apples will go in each bag?	
Quotitive Division (also known as Measurement or Number of Groups Unknown Division)	**3.OA.2▲**
The number of objects in (or size of) each group or share is known, but the number of groups or shares is unknown. ***Example:*** There are 12 apples on the counter. If you put 3 apples in each bag, how many bags will you fill?	

Students are exposed to related terminology for multiplication (*factor* and *product*) and division (*quotient, dividend, divisor,* and *factor*). They use multiplication and division within 100 to solve word problems (**3.OA.3▲**) in situations involving equal groups, arrays, and measurement quantities. Note that although "repeated addition" can be used in some cases as a strategy for finding whole-number products, repeated addition should not be misconstrued as the meaning of multiplication. The intention of the standards in grade three is to move students beyond additive thinking to multiplicative thinking.

The three most common types of multiplication and division word problems for this grade level are summarized in table 3-3.

Table 3-3. Types of Multiplication and Division Problems (Grade Three)

	Unknown Product	Group Size Unknown[2]	Number of Groups Unknown[3]
	$3 \times 6 = ?$	$3 \times ? = 18$ and $18 \div 3 = ?$	$? \times 6 = 18$ and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there altogether? **Measurement example** You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally and packed into 3 bags, then how many plums will be in each bag? **Measurement example** You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed, with 6 plums to a bag, then how many bags are needed? **Measurement example** You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays, Area	There are 3 rows of apples with 6 apples in each row. How many apples are there? **Area example** What is the area of a rectangle that measures 3 centimeters by 6 centimeters?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? **Area example** A rectangle has an area of 18 square centimeters. If one side is 3 centimeters long, how long is a side next to it?.	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? **Area example** A rectangle has an area of 18 square centimeters. If one side is 6 centimeters long, how long is a side next to it?
Compare	Grade-three students do not solve multiplicative "compare" problems; these problems are introduced in grade four (with whole-number values) and also appear in grade five (with unit fractions).		
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

Source: NGA/CCSSO 2010d. A nearly identical version of this table appears in the glossary (table GL-5).

In grade three, students focus on equal groups and array problems. Multiplicative-compare problems are introduced in grade four. The more difficult problem types include "Group Size Unknown" ($3 \times ? = 18$ or $18 \div 3 = 6$) or "Number of Groups Unknown" ($? \times 6 = 18$, $18 \div 6 = 3$). To solve problems, students determine the unknown whole number in a multiplication or division equation relating three whole numbers (**3.OA.4▲**). Students use numbers, words, pictures, physical objects, or equations to represent problems, explain their thinking, and show their work (**MP.1**, **MP.2**, **MP.4**, **MP.5**).

2. These problems ask the question, "How many in each group?" The problem type is an example of *partitive* or *fair-share* division.

3. These problems ask the question, "How many groups?" The problem type is an example of *quotitive* or *measurement* division.

Example: Number of Groups Unknown 3.OA.4▲

Molly the zookeeper has 24 bananas to feed the monkeys. Each monkey needs to eat 4 bananas. How many monkeys can Molly feed?

Solution: $? \times 4 = 24$

Students might draw on the remembered product $6 \times 4 = 24$ to say that the related quotient is 6. Alternatively, they might draw on other known products—for example, if $5 \times 4 = 20$ is known, then since $20 + 4 = 24$, one more group of 4 will give the desired factor $(5 + 1 = 6)$. Or, knowing that $3 \times 4 = 12$ and $12 + 12 = 24$, students might reason that the desired factor is $3 + 3 = 6$. Any of these methods (or others) might be supported by a representational drawing that shows the equal groups in the situation.

Operations and Algebraic Thinking **3.OA**

Understand properties of multiplication and the relationship between multiplication and division.

5. Apply properties of operations as strategies to multiply and divide.[4] *Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. (Commutative property of multiplication.) $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. (Associative property of multiplication.) Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find 8×7 as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. (Distributive property.)*
6. Understand division as an unknown-factor problem. *For example, find $32 \div 8$ by finding the number that makes 32 when multiplied by 8.*

In grade three, students apply properties of operations as strategies to multiply and divide (**3.OA.5▲**). Third-grade students do not need to use the formal terms for these properties. Students use increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn about the relationship between multiplication and division.

Focus, Coherence, and Rigor

Arrays can be seen as equal-sized groups where objects are arranged by rows and columns, and they form a major transition to understanding multiplication as finding area (connection to **3.MD.7▲**). For example, students can analyze the structure of multiplication and division (**MP.7**) through their work with arrays (**MP.2**) and work toward precisely expressing their understanding of the connections between area and multiplication (**MP.6**).

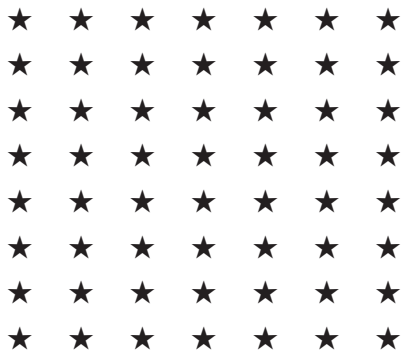
The distributive property is the basis for the standard multiplication algorithm that students can use to fluently multiply multi-digit whole numbers in grade five. Third-grade students are introduced to the distributive property of multiplication over addition as a strategy for using products they know to solve products they do not know (**MP.2**, **MP.7**).

4. Students need not use formal terms for these properties.

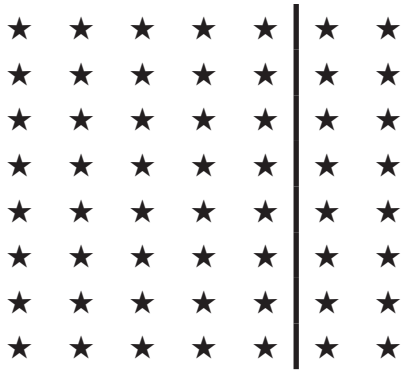
Example: Using the Distributive Property **3.OA.5▲**

Students can use the distributive property to discover new products of whole numbers (such as 7×8) based on products they can find more easily.

Strategy 1: By creating an array, I can find how many total stars there are in 7 columns of 8 stars.

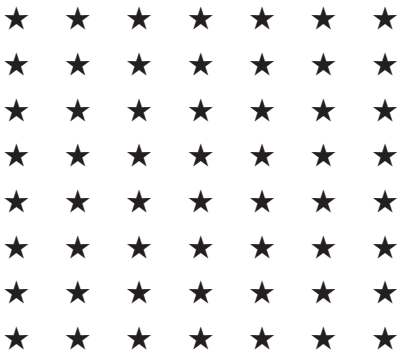


I see that I can arrange the 7 columns into a group of 5 columns and a group of 2 columns.

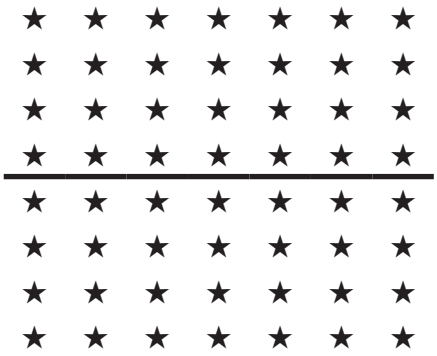


I know that the 5×8 array gives me 40 and the 2×8 array gives me 16. So altogether I have $5 \times 8 + 2 \times 8 = 40 + 16 = 56$ stars.

Strategy 2: By creating an array, I can find how many total stars there are in 8 rows of 7 stars.



I see that I can arrange the 8 rows of stars into 2 groups of 4 rows.



I know that each new 4×7 array gives me 28 stars, so altogether I have $4 \times 7 + 4 \times 7 = 28 + 28 = 56$ stars.

Adapted from ADE 2010.

The connection between multiplication and division should be introduced early in the year. Students understand division as an unknown-factor problem (**3.OA.6▲**). *For example, find $15 \div 3$ by finding the number that makes 15 when multiplied by 3.* Multiplication and division are inverse operations, and students use this inverse relationship to compute and check results.

Operations and Algebraic Thinking 3.OA

Multiply and divide within 100.

7. Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of grade 3, know from memory all products of two one-digit numbers.

Students in grade three use various strategies to fluently multiply and divide within 100 (**3.OA.7▲**). The following are some general strategies that can be used to help students know from memory all products of two one-digit numbers.

Strategies for Learning Multiplication Facts **3.OA.7▲**

Patterns

- Multiplication by zeros and ones
- Doubles (twos facts), doubling twice (fours), doubling three times (eights)
- Tens facts (relating to place value, 5×10 is 5 tens, or 50)
- Fives facts (knowing the fives facts are half of the tens facts)

General Strategies

- Use skip-counting (counting groups of specific numbers and knowing how many groups have been counted). For example, students count by twos, keeping track of how many groups (to multiply) and when they reach the known product (to divide). Students gradually abbreviate the "count by" list and are able to start within it.
- Decompose into known facts (e.g., 6×7 is 6×6 plus one more group of 6).
- Use "turn-around facts" (based on the commutative property—for example, knowing that 2×7 is the same as 7×2 reduces the total number of facts to memorize).

Other Strategies

- Know square numbers (e.g., 6×6).
- Use arithmetic patterns to multiply. Nines facts include several patterns. For example, using the fact that $9 = 10 - 1$, students can use the tens multiplication facts to help solve a nines multiplication problem.
  $9 \times 4 = 9$ fours = 10 fours – 1 four = $40 - 4 = 36$
  Students may also see this as:
  $4 \times 9 = 4$ nines = 4 tens – 4 ones = $40 - 4 = 36$

Strategies for Learning Division Facts

- Turn the division problem into an unknown-factor problem. Students can state a division problem as an unknown-factor problem (e.g., $24 \div 4 = ?$ becomes $4 \times ? = 24$). Knowing the related multiplication facts can help a student obtain the answer and vice versa, which is why studying multiplication and division involving a particular number can be helpful.
- Use related facts (e.g., $6 \times 4 = 24$; $24 \div 6 = 4$; $24 \div 4 = 6$; $4 \times 6 = 24$).

Adapted from ADE 2010.

Multiplication and division are new concepts in grade three, and reaching fluency with these operations within 100 represents a major portion of students' work. By the end of grade three, students also know all products of two one-digit numbers from memory (**3.OA.7▲**). Organizing practice to focus most heavily on products and unknown factors that are understood but not yet fluent in students can speed learning and support fluency with multiplication and division facts. Practice and extra support should continue all year for those who need it to attain fluency.

FLUENCY

California's Common Core State Standards for Mathematics (K–6) set expectations for fluency in computation (e.g., "*Fluently* multiply and divide within 100 . . ." [**3.OA.7▲**]). Such standards are culminations of progressions of learning, often spanning several grades, involving conceptual understanding, thoughtful practice, and extra support where necessary. The word *fluent* is used in the standards to mean "reasonably fast and accurate" and possessing the ability to use certain facts and procedures with enough facility that using such knowledge does not slow down or derail the problem solver as he or she works on more complex problems. Procedural fluency requires skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Developing fluency in each grade may involve a mixture of knowing some answers, knowing some answers from patterns, and knowing some answers through the use of strategies.

Adapted from UA Progressions Documents 2011a.

Students in third grade begin to take steps toward formal algebraic language by using a letter for the unknown quantity in expressions or equations when solving one- and two-step word problems (**3.OA.8▲**).

Operations and Algebraic Thinking 3.OA

Solve problems involving the four operations, and identify and explain patterns in arithmetic.

8. Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[5]
9. Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.*

Students do not formally solve algebraic equations at this grade level. Students know to perform operations in the conventional order when there are not parentheses to specify a particular order (order of operations). Students use estimation during problem solving and then revisit their estimates to check for reasonableness.

5. This standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

Example 1: Chicken Coop **3.OA.8▲**

There are 5 nests in a chicken coop and 2 eggs in each nest. If the farmer wants 25 eggs, how many more eggs does she need?

Solution: Students might create a picture representation of this situation using a tape diagram:

2	2	2	2	2	m
25					

Students might solve this by seeing that when they add up the 5 nests with 2 eggs, they have 10 eggs. Thus, to make 25 eggs the farmer would need $25-10=15$ more eggs. A simple equation that represents this situation could be $5\times 2+m=25$, where m is the number of additional eggs the farmer needs.

Example 2: Soccer Club **3.OA.8▲**

The soccer club is going on a trip to the water park. The cost of attending the trip is $63, which includes $13 for lunch and the price of 2 wristbands (one for the morning and one for the afternoon). Both wristbands are the same price. Find the price of one of the wristbands, and write an equation that represents this situation.

Solution: Students might solve the problem by seeing that the total cost of the two tickets must be $\$63-\$13=\$50$.

w	w	\$13
\$63		

Therefore, the cost of one wristband must be $\$50\div 2=\$25$. Equations that represent this situation are $w+w+13=63$ and $63=w+w+13$.

Adapted from Kansas Association of Teachers of Mathematics (KATM) 2012, 3rd Grade Flipbook, and NCDPI 2013b.

In grade three, students identify arithmetic patterns and explain them using properties of operations (**3.OA.9▲**). Students can investigate addition and multiplication tables in search of patterns (**MP.7**) and explain or discuss why these patterns make sense mathematically and how they are related to properties of operations (e.g., why is the multiplication table symmetric about its diagonal from the upper left to the lower right?) [**MP.3**].

Domain: Number and Operations in Base Ten

Number and Operations in Base Ten — 3.NBT

Use place-value understanding and properties of operations to perform multi-digit arithmetic.[6]

1. Use place-value understanding to round whole numbers to the nearest 10 or 100.
2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., $9\times 80, 5\times 60$) using strategies based on place value and properties of operations.

6. A range of algorithms may be used.

In grade three, students are introduced to the concept of rounding whole numbers to the nearest 10 or 100 (**3.NBT.1**), an important prerequisite for working with estimation problems. Students can use a number line or a hundreds chart as tools to support their work with rounding. They learn when and why to round numbers and extend their understanding of place value to include whole numbers with four digits.

Third-grade students continue to add and subtract within 1000 and achieve fluency with strategies and algorithms that are based on place value, properties of operations, and/or the relationship between addition and subtraction (**3.NBT.2**). They use addition and subtraction methods developed in grade two, where they began to add and subtract within 1000 without the expectation of full fluency and used at least one method that generalizes readily to larger numbers—so this is a relatively small and incremental expectation for third-graders. Such methods continue to be the focus in grade three, and thus the extension at grade four to generalize these methods to larger numbers (up to 1,000,000) should also be relatively easy and rapid.

Students in grade three also multiply one-digit whole numbers by multiples of 10 (**3.NBT.3**) in the range 10–90, using strategies based on place value and properties of operations (e.g., "I know $5 \times 90 = 450$ because $5 \times 9 = 45$, and so 5×90 should be 10 times as much"). Students also interpret 2×40 as 2 groups of 4 tens or 8 groups of ten. They understand that 5×60 is 5 groups of 6 tens or 30 tens, and they know 30 tens are 300. After developing this understanding, students begin to recognize the patterns in multiplying by multiples of 10 (ADE 2010). The ability to multiply one-digit numbers by multiples of 10 can support later student learning of standard algorithms for multiplication of multi-digit numbers.

Domain: Number and Operations—Fractions

In grade three, students develop an understanding of fractions as numbers. They begin with unit fractions by building on the idea of partitioning a whole into equal parts. Student proficiency with fractions is essential for success in more advanced mathematics such as percentages, ratios and proportions, and algebra.

Number and Operations—Fractions[7] 3.NF

Develop understanding of fractions as numbers.

1. Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.
   a. Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.
   b. Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

7. Grade-three expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

In grades one and two, students partitioned circles and rectangles into two, three, and four equal shares and used fraction language (e.g., *halves, thirds, half of, a third of*). In grade three, students begin to enlarge their concept of *number* by developing an understanding of fractions as numbers (adapted from PARCC 2012).

Grade-three students understand a fraction $\frac{1}{b}$ as the quantity formed by 1 part when a whole is partitioned into b equal parts and the fraction $\frac{a}{b}$ as the quantity formed by a parts of size $\frac{1}{b}$ (**3.NF.1▲**).

Focus, Coherence, and Rigor

When working with fractions, teachers should emphasize two main ideas:

- Specifying the whole
- Explaining what is meant by "equal parts"

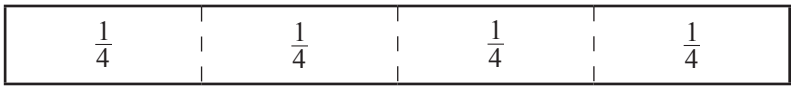
Student understanding of fractions hinges on understanding these ideas.

To understand fractions, students build on the idea of *partitioning* (dividing) a whole into equal parts. Students begin their study of fractions with *unit fractions* (fractions with the numerator 1), which are formed by partitioning a whole into equal parts (the number of equal parts becomes the denominator). One of those parts is a unit fraction. An important goal is for students to see unit fractions as the basic building blocks of all fractions, in the same sense that the number 1 is the basic building block of whole numbers. Students make the connection that, just as every whole number is obtained by combining a sufficient number of 1s, every fraction is obtained by combining a sufficient number of unit fractions (adapted from UA Progressions Documents 2013a). They explore fractions first, using concrete models such as fraction bars and geometric shapes, and this culminates in understanding fractions on the number line.

Examples 3.NF.1▲

Teacher: Show fourths by folding the piece of paper into equal parts.

Student: I know that when the number on the bottom is *4*, I need to make four equal parts. By folding the paper in half once and then again, I get four parts, and each part is equal. Each part is worth $\frac{1}{4}$.

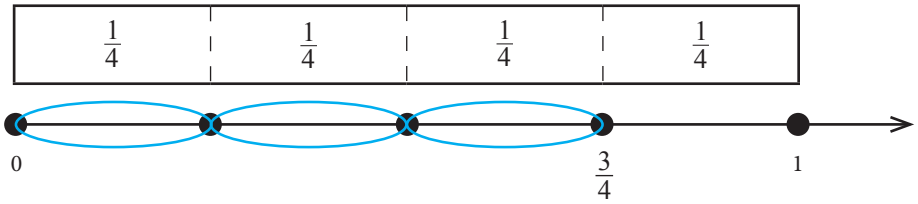


Teacher: Shade $\frac{3}{4}$ using the fraction bar you created.

Student: My fraction bar shows fourths. The *3* tells me I need three of them, so I'll shade them. I could have shaded any three of them and I would still have $\frac{3}{4}$.

Teacher: Explain how you know your mark is in the right place. 3.NF.2b▲

Student (Solution): When I use my fraction strip as a measuring tool, it shows me how to divide the unit interval into four equal parts (since the denominator is *4*). Then I start from the mark that has *0* and measure off three pieces of $\frac{1}{4}$ each. I circled the pieces to show that I marked three of them. This is how I know I have marked $\frac{3}{4}$.

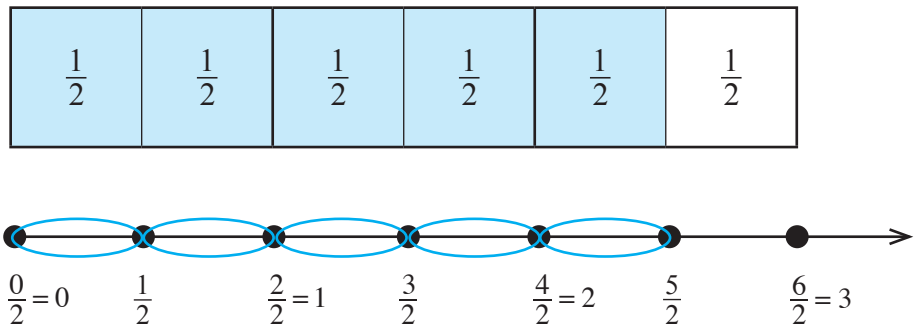


Third-grade students need opportunities to place fractions on a number line and understand fractions as a related component of the ever-expanding number system. The number line reinforces the analogy between fractions and whole numbers. Just as 5 is the point on the number line reached by marking off 5 times the length of the unit interval from 0, so is $\frac{5}{3}$ the point obtained by marking off 5 times the length of a different interval as the basic unit of length, namely the interval from 0 to $\frac{1}{3}$.

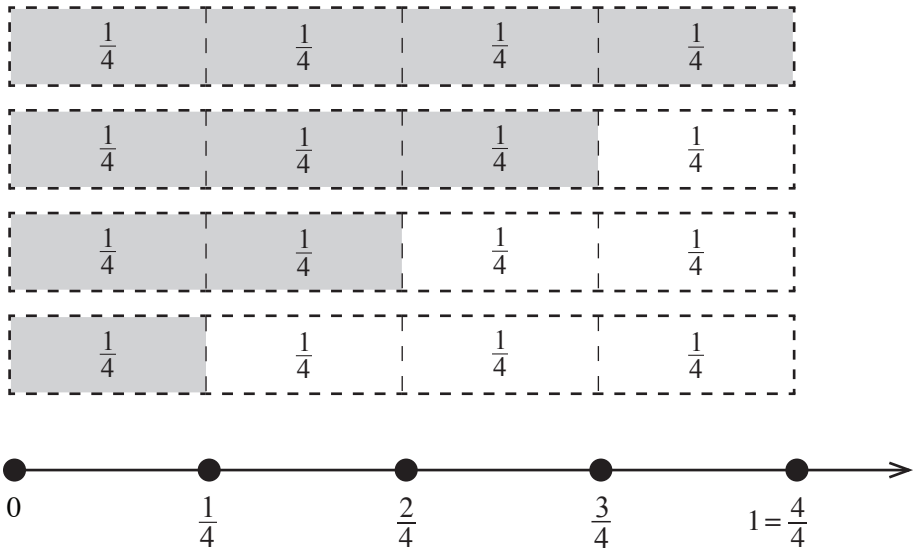
Fractions Greater Than 1

California's Common Core State Standards for Mathematics do not designate fractions greater than 1 as "improper fractions." Fractions greater than 1, such as $\frac{5}{2}$, are simply numbers in themselves and are constructed in the same way as other fractions.

Thus, to construct $\frac{5}{2}$ we might use a fraction strip as a measuring tool to mark off lengths of $\frac{1}{2}$. Then we count five of those halves to get $\frac{5}{2}$.



Students recognize that when examining fractions with common denominators, the wholes have been divided into the same number of equal parts, so the fraction with the larger numerator has the larger number of equal parts. Students develop an understanding of the numerator and denominator as they label each fractional part based on how far it is from 0 to the endpoint (**MP.7**).



Number and Operations—Fractions[8] 3.NF

Develop understanding of fractions as numbers.

3. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.
   a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.
   b. Recognize and generate simple equivalent fractions, e.g., $\frac{1}{2} = \frac{2}{4}$, $\frac{4}{6} = \frac{2}{3}$). Explain why the fractions are equivalent, e.g., by using a visual fraction model.
   c. Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form* $3 = \frac{3}{1}$*; recognize that* $\frac{6}{1} = 6$*; locate* $\frac{4}{4}$ *and 1 at the same point of a number line diagram.*
   d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Students develop an understanding of fractions as they use visual models and a number line to represent, explain, and compare unit fractions, equivalent fractions (e.g., $\frac{1}{2} = \frac{2}{4}$), whole numbers as fractions (e.g., $3 = \frac{3}{1}$), and fractions with the same numerator (e.g., $\frac{4}{3}$ and $\frac{4}{6}$) or the same denominator (e.g., $\frac{4}{8}$ and $\frac{5}{8}$) [**3.NF.2–3▲**].

Students develop an understanding of order in terms of position on a number line. Given two fractions—thus two points on the number line—students understand that the one to the left is said to be smaller and the one to the right is said to be larger (adapted from UA Progressions Documents 2013a).

Students learn that when comparing fractions, they need to look at the size of the parts and the number of the parts. For example, $\frac{1}{8}$ is smaller than $\frac{1}{2}$ because when 1 whole is cut into 8 pieces, the pieces are much smaller than when 1 whole of the same size is cut into 2 pieces.

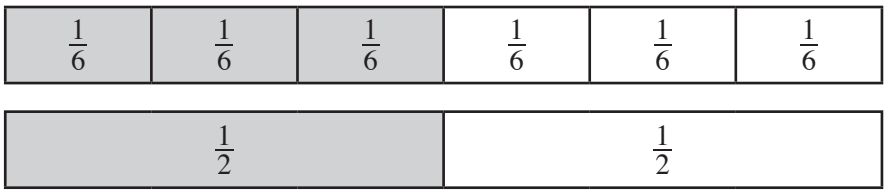
To compare fractions that have the same numerator but different denominators, students understand that each fraction has the same number of equal parts but the size of the parts is different. Students can infer that the same number of smaller pieces is less than the same number of bigger pieces (adapted from ADE 2010 and KATM 2012, 3rd Grade Flipbook).

Students develop an understanding of equivalent fractions as they compare fractions using a variety of visual fraction models and justify their conclusions (**MP.3**). Through opportunities to compare fraction models with the same whole divided into different numbers of pieces, students identify fractions that show the same amount or name the same number, learning that they are equal (or *equivalent*).

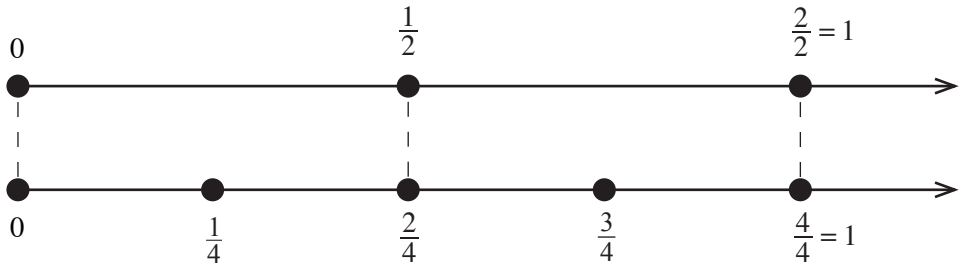
8. Grade-three expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

Using Models to Understand Basic Fraction Equivalence

Fraction bars



Number line



Adapted from UA Progressions Documents 2013a.

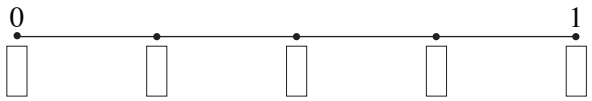
Important Concepts Related to Understanding Fractions

- Fractional parts must be the same size.
- The number of equal parts tells how many make a whole.
- As the number of equal pieces in the whole increases, the size of the fractional pieces decreases.
- The size of the fractional part is relative to the whole.
- When a shape is divided into equal parts, the denominator represents the number of equal parts in the whole and the numerator of a fraction is the count of the demarcated congruent or equal parts in a whole (e.g., $\frac{3}{4}$ means that there are 3 one-fourths or 3 out of 4 equal parts).
- Common benchmark numbers such as 0, $\frac{1}{2}$, $\frac{3}{4}$, and 1 can be used to determine if an unknown fraction is greater or smaller than a benchmark fraction.

Adapted from ADE 2010 and KATM 2012, 3rd Grade Flipbook.

Illustrative Mathematics offers a Fractions Progression Module (http://www.illustrativemathematics.org/pages/fractions_progression [Illustrative Mathematics 2013k]) that provides an overview of fractions. Table 3-4 presents a sample classroom activity that connects the Standards for Mathematical Content and Standards for Mathematical Practice.

Table 3-4. Connecting to the Standards for Mathematical Practice—Grade Three

Standards Addressed	Explanation and Examples
Connections to Standards for Mathematical Practice **MP.2.** Students reason quantitatively as they determine why a placement was correct or incorrect and assign a fractional value to a distance. **MP.4.** Students use the number line model for fractions. Although this is not an application of mathematics to a real-world situation in the true sense of *modeling*, it is an appropriate use of modeling for the grade level. **MP.8.** Students see repeated reasoning in dividing up the number line into equal parts (of varied sizes) and form the basis for how they would place fifths, tenths, and other fractions. **Standards for Mathematical Content** **3.NF.1.** Understand a fraction $\frac{1}{b}$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction $\frac{a}{b}$ as the quantity formed by a parts of size $\frac{1}{b}$. **3.NF.2.** Understand a fraction as a number on the number line; represent fractions on a number line diagram. a. Represent a fraction $\frac{1}{b}$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $\frac{1}{b}$ and that the endpoint of the part based at 0 locates the number $\frac{1}{b}$ on the number line. b. Represent a fraction $\frac{a}{b}$ on a number line diagram by marking off a lengths $\frac{1}{b}$ from 0. Recognize that the resulting interval has size $\frac{a}{b}$ and that its endpoint locates the number $\frac{a}{b}$ on the number line. **3.NF.3.** Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size. b. Recognize and generate simple equivalent fractions (e.g., $\frac{1}{2}=\frac{2}{4}$, $\frac{4}{6}=\frac{2}{3}$). Explain why the fractions are equivalent, e.g., by using a visual fraction model.	**Task: The Human Fraction Number Line Activity.** In this activity, the teacher posts a long sheet of paper on a wall of the classroom to act as a number line, with 0 marked at one end and 1 marked at the other. Gathered around the wall, groups of students are given cards with different-sized fractions indicated on them—for example, $\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$—and are asked to locate themselves along the number line according to the fractions assigned to them. Depending on the size of the class and the length of the number line, fractions with denominators 2, 3, 4, 6, and 8 may be used. The teacher can ask students to explain to each other why their placements are correct or incorrect, emphasizing that the students with cards marked in fourths, say, have divided the number line into four equal parts. Furthermore, a student with the card $\frac{a}{b}$ is standing in the correct place if he or she represents a lengths of size $\frac{1}{b}$ from 0 on the number line. As a follow-up activity, teachers can give students several unit number lines that are marked off into equal parts but that are unlabeled. Students are required to fill in the labels on the number lines. An example is shown here:  **Classroom Connections.** There are several big ideas included in this activity. One is that when talking about fractions as points on a number line, the whole is represented by the *length* or amount of distance from 0 to 1. By requiring students to physically line up in the correct places on the number line, the idea of partitioning this distance into equal parts is emphasized. In addition, other students can physically mark off the placement of fractions by starting from 0 and walking the required number of lengths $\frac{1}{b}$ from 0; for example, with students placed at the locations for sixths, another student can start at 0 and walk off a distance of $\frac{5}{6}$. As an extension, teachers can have students mark off equivalent fraction distances, such as $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{3}{6}$, and can discuss why those fractions represent the same amount.

Measurement and Data 3.MD

Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.

1. Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.
2. Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l).[9] Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[10]

Students have experience telling and writing time from analog and digital clocks to the hour and half hour in grade one and in five-minute intervals in grade two. In grade three, students write time to the nearest minute and measure time intervals in minutes. Students solve word problems involving addition and subtraction of time intervals in minutes and represent these problems on a number line (**3.MD.1▲**).

Students begin to understand the concept of continuous measurement quantities, and they add, subtract, multiply, or divide to solve one-step word problems involving such quantities. Multiple opportunities to weigh classroom objects and fill containers will help students develop a basic understanding of the size and weight of a liter, a gram, and a kilogram (**3.MD.2▲**).

Focus, Coherence, and Rigor

Students' understanding and work with measuring and estimating continuous measurement quantities, such as liquid volume and mass (**3.MD.2▲**), are an important context for the fraction arithmetic they will experience in later grade levels.

Measurement and Data 3.MD

Represent and interpret data.

3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.*
4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

9. Excludes compound units such as cm^3 and finding the geometric volume of a container.

10. Excludes multiplicative comparison problems (problems involving notions of "times as much"; see glossary, table GL-5).

In grade three, the most important development in data representation for categorical data is that students draw picture graphs in which each picture represents more than one object, and they draw bar graphs in which the scale uses multiples, so the height of a given bar in tick marks must be multiplied by the scale factor to yield the number of objects in the given category. These developments connect with the emphasis on multiplication in this grade (adapted from UA Progressions Documents 2011b).

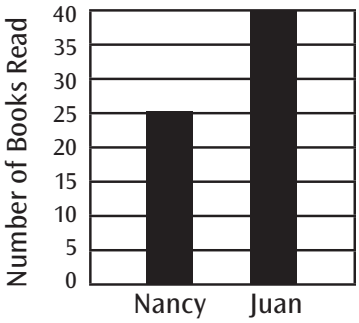
Students draw a scaled pictograph and a scaled bar graph to represent a data set and solve word problems (**3.MD.3**).

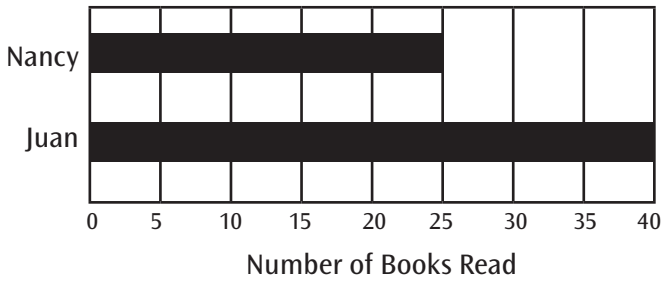
Examples **3.MD.3**

Students might draw or reference a pictograph with symbols that represent multiple units.

Number of Books Read	
Nancy	📖 📖 📖 📖 📖
Juan	📖 📖 📖 📖 📖 📖 📖 📖
Note: 📖 represents 5 books.	

Students might draw or reference bar graphs to solve related problems.





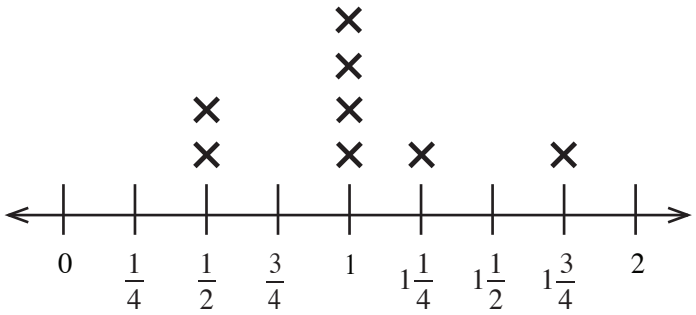
Adapted from KATM 2012, 3rd Grade Flipbook.

Focus, Coherence, and Rigor

Pictographs and scaled bar graphs offer a visually appealing context and support major work in the cluster "Represent and solve problems involving multiplication and division" as students solve multiplication and division word problems (**3.OA.3▲**).

Students use their knowledge of fractions and number lines to work with measurement data involving fractional measurement values. They generate data by measuring lengths using rulers marked with halves and fourths of an inch and create a line plot to display their findings (**3.MD.4**) [adapted from UA Progressions Documents 2011b].

For example, students might use a line plot to display data.



Adapted from NCDPI 2013b.

A critical area of instruction at grade three is for students to develop an understanding of the structure of rectangular arrays and of area measurement.

Measurement and Data — 3.MD

Geometric measurement: understand concepts of area and relate area to multiplication and to addition.

5. Recognize area as an attribute of plane figures and understand concepts of area measurement.
   a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.
   b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.
6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).
7. Relate area to the operations of multiplication and addition.
   a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
   b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real-world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.
   c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b+c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.
   d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real-world problems.

Students recognize area as an attribute of plane figures, and they develop an understanding of concepts of area measurement (**3.MD.5▲**). They discover that a square with a side length of 1 unit, called "a unit square," is said to have one square unit of area and can be used to measure area.

Students measure areas by counting unit squares (square centimeters, square meters, square inches, square feet, and improvised units) [**3.MD.6▲**]. Students develop an understanding of using square units to measure area by using different-sized square units, filling in an area with the same-sized square units, and then counting the number of square units.

Students relate the concept of area to the operations of multiplication and addition and show that the area of a rectangle can be found by multiplying the side lengths (**3.MD.7▲**). Students make sense of these quantities as they learn to interpret measurement of rectangular regions as a multiplicative relationship of the number of square units in a row and the number of rows. Students should understand and explain why multiplying the side lengths of a rectangle yields the same measurement of area as counting the number of tiles (with the same unit length) that fill the rectangle's interior. For example, students might explain that one length tells the number of unit squares in a row and the other length tells how many rows there are (adapted from UA Progressions Documents 2012a).

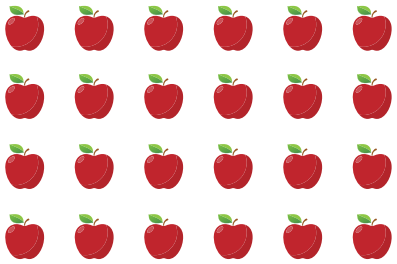
Students need opportunities to tile a rectangle with square units and then multiply the side lengths to show that they both give the area. For example, to find the area, a student could count the squares or multiply $4 \times 3 = 12$.

1	2	3	4
5	6	7	8
9	10	11	12

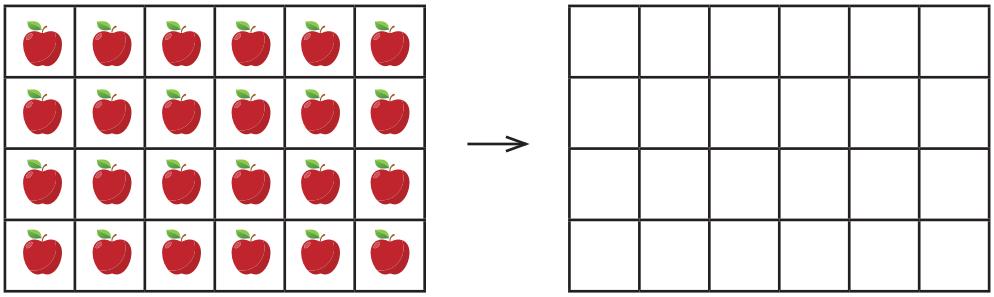
The transition from counting unit squares to multiplying side lengths to find area can be aided when students see the progression from multiplication as equal groups to multiplication as a total number of objects in an array, and then see the area of a rectangle as an array of unit squares. An example is presented below.

Students see multiplication as counting objects in equal groups—for example, 4×6 as 4 groups of 6 apples:

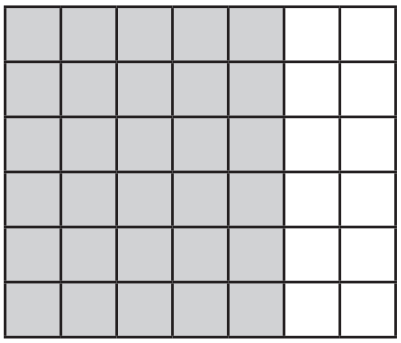
They see the objects arranged in arrays, as in a 4×6 array of the same apples:



They eventually see that finding area by counting unit squares is like counting an array of objects, where the objects are unit squares.



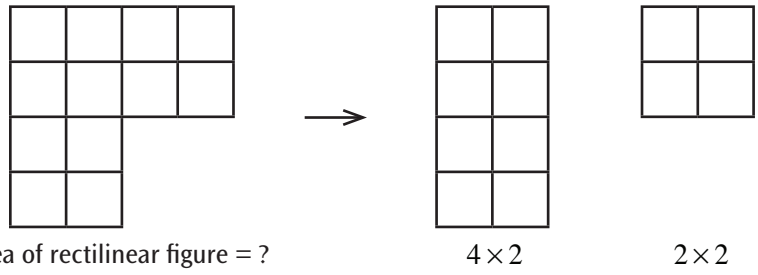
Students use area models to represent the distributive property in mathematical reasoning. For example, the area of a 6×7 figure can be determined by finding the area of a 6×5 figure and a 6×2 figure and adding the two products.



Students recognize area as additive and find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts.

Example **3.MD.7d▲**

The standards mention *rectilinear figures*. A *rectilinear figure* is a polygon whose every angle is a right angle. Such figures can be decomposed into rectangles to find their areas.



By breaking the figure into two pieces, it becomes easier to see that the area of the figure is $8 + 4 = 12$ square units.

Adapted from NCDPI 2013b.

Students apply these techniques and understandings to solve real-world problems.

Focus, Coherence, and Rigor

The use of area models (**3.MD.7▲**) also supports multiplicative reasoning, a major focus in grade three in the domain "Operations and Algebraic Thinking" (**3.OA.1–9▲**). Students must begin work with multiplication and division at or near the start of the school year to allow time for understanding and to develop fluency with these skills. Because area models for products are an important part of this process (**3.MD.7▲**), work on concepts of area (**3.MD.5–6▲**) should begin at or near the start of the year as well (adapted from PARCC 2012).

Measurement and Data — 3.MD

Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

8. Solve real-world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

In grade three, students solve real-world and mathematical problems involving perimeters of polygons (**3.MD.8**). Students can develop an understanding of the concept of *perimeter* as they walk around the perimeter of a room, use rubber bands to represent the perimeter of a plane figure with whole-number side lengths on a geoboard, or trace around a shape on an interactive whiteboard. They find the perimeter of objects, use addition to find perimeters, and recognize the patterns that exist when finding the sum of the lengths and widths of rectangles. They explain their reasoning to others. Given a perimeter and a length or width, students use objects or pictures to find the unknown length or width. They justify and communicate their solutions using words, diagrams, pictures, and numbers (adapted from ADE 2010).

Domain: Geometry

Geometry — 3.G

Reason with shapes and their attributes.

1. Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.
2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as $\frac{1}{4}$ of the area of the shape.*

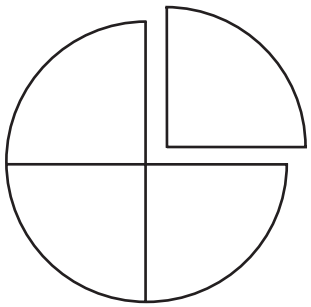
A critical area of instruction at grade three is for students to describe and analyze two-dimensional

shapes. Students compare common geometric shapes (e.g., rectangles and quadrilaterals) based on common attributes, such as four sides (**3.G.1**). In earlier grades, students informally reasoned about particular shapes through sorting and classifying based on geometric attributes. Students also built and drew shapes given the number of faces, number of angles, and number of sides. In grade three, students describe properties of two-dimensional shapes in more precise ways, referring to properties that are shared rather than the appearance of individual shapes. For example, students could start by identifying shapes with right angles, explain and discuss why the remaining shapes do not fit this category, and determine common characteristics of the remaining shapes.

Students relate their work with fractions to geometry as they partition shapes into parts with equal areas and represent each part as a unit fraction of the whole (**3.G.2**).

Example **3.G.2**

The figure below was partitioned (divided) into four equal parts. Each part is $\frac{1}{4}$ of the total area of the figure.



Adapted from NCDPI 2013b.

Focus, Coherence, and Rigor

As students partition shapes into parts with equal areas (**3.G.2**), they also reinforce concepts of area measurement and fractions that are part of the major work at the grade in the clusters "Geometric measurement: understand concepts of area and relate area to multiplication and to addition" (**3.MD.5–7▲**) and "Develop understanding of fractions as numbers" (**3.NF▲**).

Adapted from PARCC 2012.

Essential Learning for the Next Grade

In kindergarten through grade five, the focus is on the addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals, with a balance of concepts, procedural skills, and problem solving. Arithmetic is viewed as an important set of skills and also as a thinking subject that, when done thoughtfully, prepares students for algebra. Measurement and geometry develop alongside number and operations and are tied specifically to arithmetic along the way. Multiplication and division of whole numbers and fractions are an instructional focus in grades three through five.

To be prepared for grade-four mathematics, students should be able to demonstrate they have acquired certain mathematical concepts and procedural skills by the end of grade three and have met the fluency expectations for the grade. For third-graders, the expected fluencies are to add and subtract within 1000 using strategies and algorithms (**3.NBT.2**), multiply and divide within 100 using various strategies, and know all products of two one-digit numbers from memory (**3.OA.7▲**). These fluencies and the conceptual understandings that support them are foundational for work in later grades.

Of particular importance for grade four are concepts, skills, and understandings needed to represent and solve problems involving multiplication and division (**3.OA.1–4▲**); understand properties of multiplication and the relationship between multiplication and division (**3.OA.5–6▲**); multiply and divide within 100 (**3.OA.7▲**); solve problems involving the four operations and identify and explain patterns in arithmetic (**3.OA.8–9▲**); develop understanding of fractions as numbers (**3.NF.1–3▲**); solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects (**3.MD.1–2▲**); and geometric measurement—concepts of area and relating area to multiplication and to addition (**3.MD.5–7▲**).

Multiplication and Division

By the end of grade three, students develop both conceptual understanding and procedural skills of multiplication and division. Students are expected to fluently multiply and divide within 100 and to know from memory all of the products of two one-digit numbers (**3.OA.7▲**). Fluency in multiplication and division within 100 includes understanding and being able to apply strategies such as using mental math, understanding division as an unknown-factor problem, applying the properties of operations, and identifying arithmetic patterns. Students also need to understand the relationship between multiplication and division and apply that understanding by using inverse operations to verify the reasonableness of their answers. Students with a firm grasp of grade-three multiplication and division can apply their knowledge to interpret, solve, and even compose simple word problems, including problems involving equal groups, arrays, and measurement quantities. Fluency in multiplication and division ensures that when students know from memory all of the products of two one-digit numbers, they have an understanding of the two operations—and have not merely learned to produce answers through rote memorization.

Fractions

In grade three, students are formally introduced to fractions as numbers, thus broadening their understanding of the number system. Students must understand that fractions are composed of unit fractions; this is essential for their ongoing work with the number system. Students must be able to place fractions on a number line and use the number line as a tool to compare fractions and recognize equivalent fractions. They should be able to use other visual models to compare fractions. Students also must be able to express whole numbers as fractions and place them on a number line. It is essential for students to understand that the denominator determines the number of equally sized pieces that make up a whole and the numerator determines how many pieces of the whole are being referred to in the fraction.

Addition and Subtraction

By the end of grade three, students are expected to fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction (**3.NBT.2**). This fluency is both the culmination of work at previous grade levels and preparation for solving multi-step word problems using all four operations beginning in grade four. Students should be able to use more than one strategy to add or subtract and should also be able to relate the strategies they use to a written method.

Grade 3 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving multiplication and division.
- Understand properties of multiplication and the relationship between multiplication and division.
- Multiply and divide within 100.
- Solve problems involving the four operations, and identify and explain patterns in arithmetic.

Number and Operations in Base Ten

- Use place-value understanding and properties of operations to perform multi-digit arithmetic.

Number and Operations—Fractions

- Develop understanding of fractions as numbers.

Measurement and Data

- Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.
- Represent and interpret data.
- Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
- Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

Geometry

- Reason with shapes and their attributes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Operations and Algebraic Thinking 3.OA

Represent and solve problems involving multiplication and division.

1. Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as* 5×7.

2. Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as* $56 \div 8$.

3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[11]

4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations* $8 \times ? = 48,\ 5 = \square \div 3,\ 6 \times 6 = ?$.

Understand properties of multiplication and the relationship between multiplication and division.

5. Apply properties of operations as strategies to multiply and divide.[12] *Examples: If* $6 \times 4 = 24$ *is known, then* $4 \times 6 = 24$ *is also known. (Commutative property of multiplication.)* $3 \times 5 \times 2$ *can be found by* $3 \times 5 = 15$, *then* $15 \times 2 = 30$, *or by* $5 \times 2 = 10$, *then* $3 \times 10 = 30$. *(Associative property of multiplication.) Knowing that* $8 \times 5 = 40$ *and* $8 \times 2 = 16$, *one can find* 8×7 *as* $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. *(Distributive property.)*

6. Understand division as an unknown-factor problem. *For example, find* $32 \div 18$ *by finding the number that makes 32 when multiplied by 8.*

Multiply and divide within 100.

7. Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of grade 3, know from memory all products of two one-digit numbers.

Solve problems involving the four operations, and identify and explain patterns in arithmetic.

8. Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[13]

9. Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.*

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11. See glossary, table GL-5.

12. Students need not use formal terms for these properties.

13. This standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

Number and Operations in Base Ten — 3.NBT

Use place-value understanding and properties of operations to perform multi-digit arithmetic.[14]

1. Use place-value understanding to round whole numbers to the nearest 10 or 100.
2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations.

Number and Operations—Fractions[15] — 3.NF

Develop understanding of fractions as numbers.

1. Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.
2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.
   a. Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.
   b. Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.
3. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.
   a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.
   b. Recognize and generate simple equivalent fractions, e.g., $1/2 = 2/4$, $4/6 = 2/3$). Explain why the fractions are equivalent, e.g., by using a visual fraction model.
   c. Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form $3 = 3/1$; recognize that $6/1 = 6$; locate $4/4$ and 1 at the same point of a number line diagram.*
   d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

14. A range of algorithms may be used.

15. Grade-three expectations in this domain are limited to fractions with denominations 2, 3, 4, 6, and 8.

Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.

1. Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

2. Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l).[16] Add, subtract, multiply, or divide to solve one-step word involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[17]

Represent and interpret data.

3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.*

4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

Geometric measurement: understand concepts of area and relate area to multiplication and to addition.

5. Recognize area as an attribute of plane figures and understand concepts of area measurement.
   a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.
   b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

7. Relate area to the operations of multiplication and addition.
   a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
   b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real-world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.
   c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b+c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.
   d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real-world problems.

Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

8. Solve real-world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

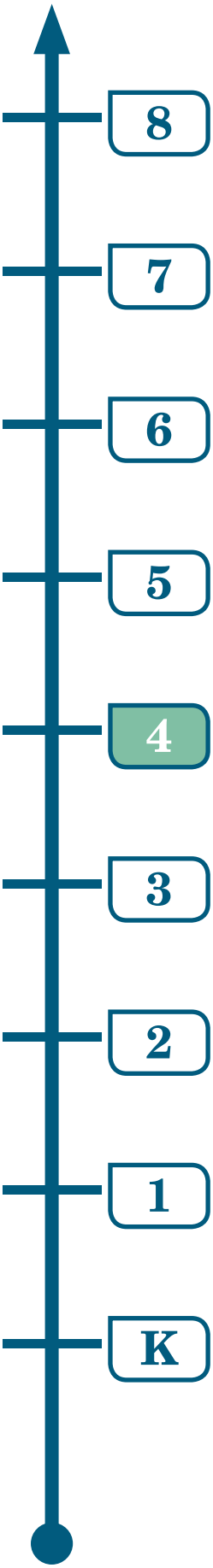
16. Excludes compound units such as cm^3 and finding the geometric volume of a container.

17. Excludes multiplicative comparison problems (problems involving notions of "times as much"; see glossary, table GL-5).

Reason with shapes and their attributes.

1. Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as* $\frac{1}{4}$ *of the area of the shape.*



Grade Four

In grade four, students continue to build a strong foundation for higher mathematics. In previous grades, students developed place-value understandings, generalized written methods for addition and subtraction, and added and subtracted fluently within 1000. They gained an understanding of single-digit multiplication and division and became fluent with such operations. They also developed an understanding of fractions built from unit fractions (adapted from Charles A. Dana Center 2012).

Critical Areas of Instruction

In grade four, instructional time should focus on three critical areas: (1) developing understanding and fluency with multi-digit multiplication and developing understanding of dividing to find quotients involving multi-digit dividends; (2) developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers; and (3) understanding that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, perpendicular sides, particular angle measures, and symmetry (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010k). Students also work toward fluency in addition and subtraction within 1,000,000 using the standard algorithm.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade-level examples of focus, coherence, and rigor are indicated throughout the chapter.

The standards do not give equal emphasis to all content for a particular grade level. Cluster headings can be viewed as the most effective way to communicate the **focus** and **coherence** of the standards. Some clusters of standards require a greater instructional emphasis than others based on the depth of the ideas, the time needed to master those clusters, and their importance to future mathematics or the later demands of preparing for college and careers.

Table 4-1 highlights the content emphases at the cluster level for the grade-four standards. The bulk of instructional time should be given to "Major" clusters and the standards within them, which are indicated throughout the text by a triangle symbol (▲). However, standards in the "Additional/Supporting" clusters should not be neglected; to do so would result in gaps in students' learning, including skills and understandings they may need in later grades. Instruction should reinforce topics in major clusters by using topics in the additional/supporting clusters and including problems and activities that support natural connections between clusters.

Teachers and administrators alike should note that the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences presented in a coherent manner (adapted from Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

Table 4-1. Grade Four Cluster-Level Emphases

Operations and Algebraic Thinking — 4.OA

Major Clusters

- Use the four operations with whole numbers to solve problems. (**4.OA.1–3▲**)

Additional/Supporting Clusters

- Gain familiarity with factors and multiples.[1] (**4.OA.4**)
- Generate and analyze patterns. (**4.OA.5**)

Number and Operations in Base Ten — 4.NBT

Major Clusters

- Generalize place-value understanding for multi-digit whole numbers. (**4.NBT.1–3▲**)
- Use place-value understanding and properties of operations to perform multi-digit arithmetic. (**4.NBT.4–6▲**)

Number and Operations—Fractions — 4.NF

Major Clusters

- Extend understanding of fraction equivalence and ordering. (**4.NF.1–2▲**)
- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers. (**4.NF.3–4▲**)
- Understand decimal notation for fractions, and compare decimal fractions. (**4.NF.5–7▲**)

Measurement and Data — 4.MD

Additional/Supporting Clusters

- Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.[2] (**4.MD.1–3**)
- Represent and interpret data. (**4.MD.4**)
- Geometric measurement: understand concepts of angle and measure angles. (**4.MD.5–7**)

Geometry — 4.G

Additional/Supporting Clusters

- Draw and identify lines and angles, and classify shapes by properties of their lines and angles. (**4.G.1–3**)

Table continues on next page

1. Supports students' work with multi-digit arithmetic as well as their work with fraction equivalence.

2 . Students use a line plot to display measurements in fractions of a unit and to solve problems involving addition and subtraction of fractions, connecting this work to the "Number and Operations—Fractions" clusters.

Table 4-1 *(continued)*

Explanations of Major and Additional/Supporting Cluster-Level Emphases

Major Clusters (▲) — Areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than others based on the depth of the ideas, the time needed to master them, and their importance to future mathematics or the demands of college and career readiness.

Additional Clusters — Expose students to other subjects; may not connect tightly or explicitly to the major work of the grade.

Supporting Clusters — Designed to support and strengthen areas of major emphasis.

Note of caution: Neglecting material, whether it is found in the major or additional/supporting clusters, will leave gaps in students' skills and understanding and will leave students unprepared for the challenges they face in later grades.

Adapted from Smarter Balanced Assessment Consortium 2011, 84.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (MP) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The MP standards represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students.

Although the description of the MP standards remains the same at all grade levels, the way these standards look as students engage with and master new and more advanced mathematical ideas does change. Table 4-2 presents examples of how the MP standards may be integrated into tasks appropriate for students in grade four. (Refer to the "Overview of the Standards Chapters" for a description of the MP standards.)

Table 4-2. Standards for Mathematical Practice—Explanation and Examples for Grade Four

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	In grade four, students know that doing mathematics involves solving problems and discussing how they solved them. Students explain to themselves the meaning of a problem and look for ways to solve it. Students might use an equation strategy to solve a word problem. For example: "Chris bought clothes for school. She bought 3 shirts for $12 each and a skirt for $15. How much money did Chris spend on her new school clothes?" Students could solve this problem with the equation $3 \times \$12 + \$15 = a$. Students may use visual models to help them conceptualize and solve problems. They may check their thinking by asking themselves, "Does this make sense?" They listen to the strategies of others and will try different approaches. They often use another method to check their answers.

Table 4-2 *(continued)*

MP.2 Reason abstractly and quantitatively.	Grade-four students recognize that a number represents a specific quantity. They connect the quantity to written symbols and create a logical representation of the problem at hand, considering both the appropriate units involved and the meaning of quantities. They extend this understanding from whole numbers to their work with fractions and decimals. Students write simple expressions, record calculations with numbers, and represent or round numbers using place-value concepts. Students might use array or area drawings to demonstrate and explain 154 × 6 as 154 added six times, and so they develop an understanding of the distributive property. For example: $154 \times 6 = (100 + 50 + 4) \times 6$ $= (100 \times 6) + (50 \times 6) + (4 \times 6)$ $= 600 + 300 + 24$ $= 924$ To reinforce students' reasoning and understanding, teachers might ask, "How do you know?" or "What is the relationship of the quantities?"
MP.3 Construct viable arguments and critique the reasoning of others.	Students may construct arguments using concrete referents, such as objects, pictures, drawings, and actions. They practice their mathematical communication skills as they participate in mathematical discussions involving questions such as "How did you get that?", "Explain your thinking," and "Why is that true?" They not only explain their own thinking, but also listen to others' explanations and ask questions. Students explain and defend their answers and solution strategies as they answer questions that require an explanation.
MP.4 Model with mathematics.	Students experiment with representing problem situations in multiple ways, including writing numbers; using words (mathematical language); creating math drawings; using objects; making a chart, list, or graph; and creating equations. Students need opportunities to connect the different representations and explain the connections. They should be able to use all of these representations as needed. Students should be encouraged to answer questions such as "What math drawing or diagram could you make and label to represent the problem?" or "What are some ways to represent the quantities?" Fourth-grade students evaluate their results in the context of the situation and reflect on whether the results make sense. For example, a student may use an area/array rectangle model to solve the following problem by extending from multiplication to division: "A fourth-grade teacher bought 4 new pencil boxes. She has 260 pencils. She wants to put the pencils in the boxes so that each box has the same number of pencils. How many pencils will there be in each box?"
MP.5 Use appropriate tools strategically.	Students consider the available tools (including estimation) when solving a mathematical problem and decide when certain tools might be helpful. For instance, they might use graph paper, a number line, or drawings of dimes and pennies to represent and compare decimals, or they might use protractors to measure angles. They use other measurement tools to understand the relative size of units within a system and express measurements given in larger units in terms of smaller units. Students should be encouraged to answer questions such as, "Why was it helpful to use _____ ?"

Table 4-2 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
MP.6 Attend to precision.	As fourth-grade students develop their mathematical communication skills, they try to use clear and precise language in their discussions with others and in their own reasoning. They are careful about specifying units of measure and state the meaning of the symbols they choose. For instance, they use appropriate labels when creating a line plot.
MP.7 Look for and make use of structure.	Students look closely to discover a pattern or structure. For instance, students use properties of operations to explain calculations (partial products model). They generate number or shape patterns that follow a given rule. Teachers might ask, "What do you notice when _____ ?" or "How do you know if something is a pattern?"
MP.8 Look for and express regularity in repeated reasoning.	In grade four, students notice repetitive actions in computation to make generalizations. Students use models to explain calculations and understand how algorithms work. They examine patterns and generate their own algorithms. For example, students use visual fraction models to write equivalent fractions. Students should be encouraged to answer questions such as "What is happening in this situation?" or "What predictions or generalizations can this pattern support?"

Adapted from Arizona Department of Education (ADE) 2010 and North Carolina Department of Public Instruction (NCDPI) 2013b.

Standards-Based Learning at Grade Four

The following narrative is organized by the domains in the Standards for Mathematical Content and highlights some necessary foundational skills from previous grade levels. It also provides exemplars to explain the content standards, highlight connections to the various Standards for Mathematical Practice (**MP**), and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. A triangle symbol (▲) indicates standards in the major clusters (see table 4-1).

Domain: Operations and Algebraic Thinking

In grade three, students focused on concepts, skills, and problem solving with single-digit multiplication and division (within 100). A critical area of instruction in grade four is developing understanding and fluency with multi-digit multiplication and developing understanding of division to find quotients involving multi-digit dividends.

Operations and Algebraic Thinking 4.OA

Use the four operations with whole numbers to solve problems.

1. Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.
2. Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.[3]
3. Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

In earlier grades, students focused on addition and subtraction of positive whole numbers and worked with additive comparison problems (e.g., what *amount* would be added to one quantity in order to result in the other?). In grade four, students compare quantities multiplicatively for the first time.

In a multiplicative comparison problem, the underlying structure is that a *factor* multiplies one quantity to result in another quantity (e.g., b is n times as much as a, represented by $b = n \times a$. Students interpret a multiplication equation as a comparison and solve word problems involving multiplicative comparison (**4.OA.1–2▲**) and should be able to identify and verbalize all three quantities involved: which quantity is being multiplied, which number tells how many times, and which number is the product. Teachers should be aware that students often have difficulty understanding the order and meaning of numbers in multiplicative comparison problems, and therefore special attention should be paid to understanding these types of problem situations (**MP.1**).

Example: Multiplicative Comparison Problems **4.OA.2▲**

Unknown Product: "Sally is 5 years old. Her mother is 8 times as old as Sally is. How old is Sally's mother?" This problem takes the form $a \times b = ?$, where the factors are known but the product is unknown.

Unknown Factor (**Group Size Unknown**): "Sally's mother is 40 years old. That is 8 times as old as Sally is. How old is Sally?" This problem takes the form $a \times ? = p$, where the product is known, but the quantity being multiplied is unknown.

Unknown Factor 2 (**Number of Groups Unknown**): "Sally's mother is 40 years old. Sally is 5 years old. How many times older than Sally is this?" This problem takes the form $? \times b = p$, where the product is known but the multiplicative factor, which does the enlarging in this case, is unknown.

Adapted from Kansas Association of Teachers of Mathematics (KATM) 2012, 4th Grade Flipbook.

In grade four, students solve various types of multiplication and division problems, which are summarized in table 4-3.

3. See glossary, table GL-5.

Table 4-3. Types of Multiplication and Division Problems (Grade Four)

	Unknown Product	Group Size Unknown[4]	Number of Groups Unknown[5]
	$3 \times 6 = ?$	$3 \times ? = 18$ and $18 \div 3 = ?$	$? \times 6 = 18$ and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there altogether? **Measurement example** You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally and packed inside 3 bags, then how many plums will be in each bag? **Measurement example** You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed, with 6 plums to a bag, then how many bags are needed? **Measurement example** You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays, Area	There are 3 rows of apples with 6 apples in each row. How many apples are there? **Area example** What is the area of a rectangle that measures 3 centimeters by 6 centimeters?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? **Area example** A rectangle has an area of 18 square centimeters. If one side is 3 centimeters long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? **Area example** A rectangle has an area of 18 square centimeters. If one side is 6 centimeters long, how long is a side next to it?
Compare	A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? **Measurement example** A rubber band is 6 centimeters long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs \$18, and that is three times as much as a blue hat costs. How much does a blue hat cost? **Measurement example** A rubber band is stretched to be 18 centimeters long and that is three times as long as it was at first. How long was the rubber band at first?	A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat? **Measurement example** A rubber band was 6 centimeters long at first. Now it is stretched to be 18 centimeters long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ *and* $p \div a = ?$	$? \times b = p$ *and* $p \div b = ?$

Source: NGA/CCSSO 2010d. A nearly identical version of this table appears in the glossary (table GL-5).

Students need many opportunities to solve contextual problems. A tape diagram or bar diagram can help students visualize and solve multiplication and division word problems. Tape diagrams are useful for connecting what is happening in the problem with an equation that represents the problem (**MP.2, MP.4, MP.5, MP.7**).

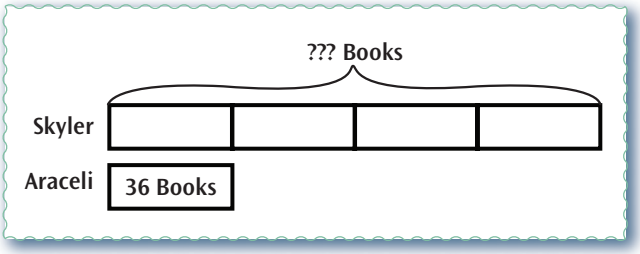
4. These problems ask the question, "How many in each group?" The problem type is an example of *partitive* or *fair-share* division.

5. These problems ask the question, "How many groups?" The problem type is an example of *quotitive* or *measurement* division.

Examples: Using Tape Diagrams to Represent Multiplication "Compare" Problems **4.OA.2▲**

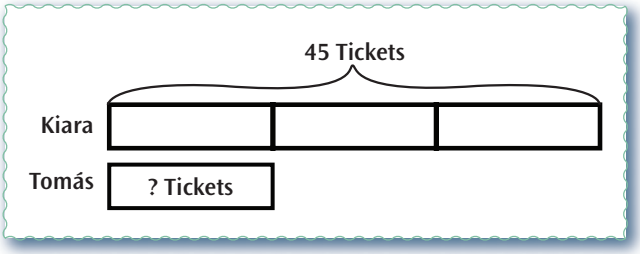
Unknown Product: "Skyler has 4 times as many books as Araceli. If Araceli has 36 books, how many books does Skyler have?"

Solution: If we represent the number of books that Araceli has with a piece of tape, then the number of books Skyler has is represented by 4 pieces of tape of the same size. Students can represent this as $4 \times 36 = \square$.



Unknown Factor (Group Size Unknown): "Kiara sold 45 tickets to the school play, which is 3 times as many as the number of tickets sold by Tomás. How many tickets did Tomás sell?"

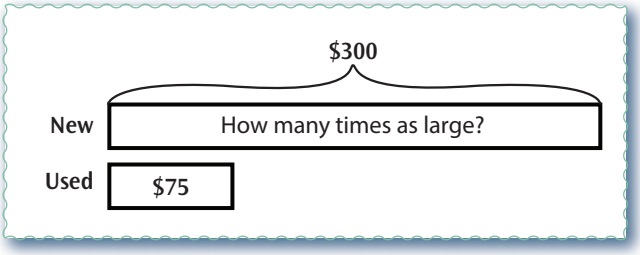
Solution: The number of tickets Kiara sold (the *product*) is known and is represented by 3 pieces of tape. The number of tickets Tomás sold would be represented by one piece of tape. This representation helps students see that the equations $3 \times \square = 45$ or $45 \div 3 = \square$ represent the problem.



Unknown Factor (Number of Groups Unknown): "A used bicycle costs $75; a new one costs $300. How many times as much does the new bike cost compared with the used bike?"

Solution: The student represents the cost of the used bike with a piece of tape and decides how many pieces of this tape will make up the cost of the new bike. The representation leads to the equations $\square \times 75 = 300$ and $300 \div 75 = \square$.

Adapted from KATM 2012, 4th Grade Flipbook.



Additionally, students solve multi-step word problems using the four operations, including problems in which remainders must be interpreted (**4.OA.3▲**). Students use estimation to assess the reasonableness of answers. They determine the level of accuracy needed to estimate the answer to a problem and select the appropriate method of estimation. This strategy gives rounding usefulness, instead of making it a separate topic that is covered arbitrarily.

Examples: Multi-Step Word Problems and Strategies Called for in Standard 4.OA.3▲

1. There are 146 students going on a field trip. If each bus holds 30 students, how many buses are needed?

   Solution: "Since $150 \div 30 = 5$, it seems like there should be around 5 buses. When we try to divide 146 by 30, we get 4 groups with 26 left over. This means that $146 = 4 \times 30 + 26$. There are 4 buses filled with 30 students, with a fifth bus holding only 26 students." (Given the context of the problem, one more than the quotient is the answer.)

2. Suppose that 250 colored pencils were distributed equally among 33 students for a geometry project. What is the largest number of colored pencils each student can receive?

   Solution: "Since $240 \div 30 = 8$, it seems like each student should receive close to 8 colored pencils. When we divide 250 by 33, we get 7 with a remainder of 19. This means that $250 = 33 \times 7 + 19$. This tells us that each student can have 7 colored pencils with 19 left over for the teacher."

3. Your class is collecting bottled water for a service project. The goal is to collect 300 bottles of water. On the first day, Max brings in 3 packs, with 6 bottles of water in each pack. Sarah wheels in 6 packs, each containing 6 bottles of water. About how many bottles of water still need to be collected?

   Solution: "First, I multiplied 3 packs by 6 bottles per pack, which equals 18 bottles. Then I multiplied 6 packs by 6 bottles per pack, which is 36 bottles. I added 18 and 36 and got 54. Then I subtracted 300 – 54 and got 246. I know 18 is close to 20, and 20 plus 36 is around 50. Since we're trying to get to 300, we'll need about 250 more bottles, so my answer of 246 seems reasonable."

Adapted from KATM 2012, 4th Grade Flipbook.

As students compute and interpret multi-step problems with remainders (**4.OA.3▲**), they also reinforce important mathematical practices as they make sense of the problem and reason about how the context is connected to the four operations (**MP.1, MP.2**).

Common Misconceptions

- Teachers may try to help their students by telling them that multiplying two numbers in a multiplicative comparison situation always makes the product *bigger.* While this is true with whole numbers greater than 1, it is *not* true when one of the factors is a fraction smaller than 1 (or when one of the factors is negative), something students will encounter in later grades. Teachers should be careful to emphasize that multiplying by a number *greater than 1* results in a product larger than the original number (**4.OA.1–2▲**).
- Students might be confused by the difference between 6 more than a number (additive) and 6 times a number (multiplicative). For example, using 18 and 6, a question could be "How much more is 18 than 6?" Thinking multiplicatively, the answer is 3; however, thinking additively, the answer is 12 (adapted from KATM 2012, 4th Grade Flipbook).
- It is common practice when dividing numbers to write, for example, $250 \div 33 = 7R19$. Although this notation has been used for quite some time, it obscures the relationship between the numbers in the problem. When students find fractional answers, the correct equation for the present example becomes $250 \div 33 = 7\frac{19}{33}$. It is more accurate to write the answer in words, such as by saying, "When we divide 250 by 33, the quotient is 7 with 19 left over," or to write the equation as $250 = 33 \times 7 + 19$ (see standard **4.NBT.6▲**).

At grade four, students find all factor pairs for whole numbers in the range 1–100 (**4.OA.4**). Knowing how to determine factors and multiples is the foundation for finding common multiples and factors in grade six.

Operations and Algebraic Thinking 4.OA

Gain familiarity with factors and multiples.

4. Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

Students extend the idea of decomposition to multiplication and learn to use the term *multiple*. Any whole number is a multiple of each of its factors. For example, 21 is a multiple of 3 and a multiple of 7 because $21 = 3 \times 7$. A number can be multiplicatively decomposed into equal groups (e.g., 3 equal groups of 7) and expressed as a product of these two factors (called *factor pairs*). The only factors for a *prime number* are 1 and the number itself. A *composite number* has two or more factor pairs. The number 1 is neither prime nor composite. To find all factor pairs for a given number, students need to search systematically—by checking if 2 is a factor, then 3, then 4, and so on, until they start to see a "reversal" in the pairs. For example, after finding the pair 6 and 9 for 54, students will next find the reverse pair, 9 and 6 (adapted from the University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2011a).

Common Misconceptions

- Students may think the number 1 is a prime number or that all prime numbers are odd numbers. (*Counterexample:* 2 has only two factors—1 and 2—and is therefore prime.)
- When listing multiples of numbers, students may omit the number itself. Students should be reminded that the smallest multiple is the number itself.
- Students may think larger numbers have more factors. (*Counterexample:* 98 has six factors: 1, 2, 7, 14, 49, and 98; 36 has nine factors: 1, 2, 3, 4, 6, 9, 12, 18, and 36.)

Having students share all factor pairs and explain how they found them will help students avoid some of these misconceptions.

Adapted from KATM 2012, 4th Grade Flipbook.

Focus, Coherence, and Rigor

The concepts and terms *prime* and *composite* are new at grade four. As students gain familiarity with factors and multiples (**4.OA.4**), they also reinforce and support major work at the grade, such as multi-digit arithmetic in the cluster "Use place-value understanding and properties of operations to perform multi-digit arithmetic" (**4.NBT.4–6▲**) and fraction equivalence in the cluster "Extend understanding of fraction equivalence and ordering" (**4.NF.1–2▲**).

Understanding patterns is fundamental to algebraic thinking. In grade four, students generate and analyze number and shape patterns that follow a given rule (**4.OA.5**).

Operations and Algebraic Thinking	**4.OA**
Generate and analyze patterns.	
5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.*	

Students begin by reasoning about patterns, connecting a rule for a given pattern with its sequence of numbers or shapes. A *pattern* is a sequence that repeats or evolves in a predictable process over and over. A *rule* dictates what that process will look like. Patterns that consist of repeated sequences of shapes or growing sequences of designs can be appropriate for the grade. For example, students could examine a sequence of dot designs in which each design has 4 more dots than the previous one and then reason about how the dots are organized in the design to determine the total number of dots in the 100th design (**MP.2, MP.4, MP.5, MP.7**) [adapted from UA Progressions Documents 2011a].

Illustrative Mathematics (2013a) offers two examples of problems that can help students understand patterns: "Double Plus One" and "Multiples of Nine" (https://www.illustrativemathematics.org/4 [accessed November 5, 2014]).

Focus, Coherence, and Rigor

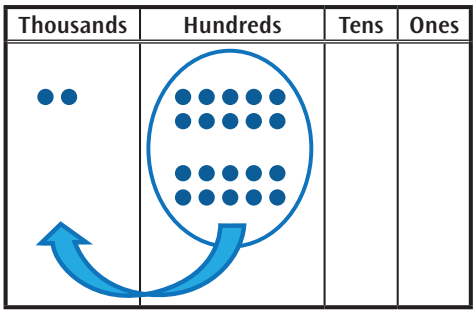
Numerical patterns (**4.OA.5**) allow students to reinforce facts and develop fluency with operations. They also support major work in grade four in the cluster "Use place-value understanding and properties of operations to perform multi-digit arithmetic" (**4.NBT.4–6▲**). This is an example of a standard in an additional/supporting cluster that reinforces standards in a major cluster.

Domain: Number and Operations in Base Ten

In grade four, students extend their work in the base-ten number system and generalize previous place-value understanding to multi-digit whole numbers (less than or equal to 1,000,000).

Number and Operations in Base Ten	**4.NBT**
Generalize place-value understanding for multi-digit whole numbers.	
1. Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. *For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division.*	
2. Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.	
3. Use place-value understanding to round multi-digit whole numbers to any place.	

Students read, write, and compare numbers based on the meaning of the digits in each place (**4.NBT.1–2▲**). In the base-ten system, the value of each place is 10 times the value of the place to the immediate right. Students can come to see and understand that multiplying by 10 yields a product in which each digit of the multiplicand is shifted one place to the left (adapted from UA Progressions Documents 2012b). Students can develop their understanding of millions by using a place-value chart to understand the pattern of *times ten* in the base-ten system; for example, 20 hundreds can be bundled into 2 thousands.



2 thousands = 20 hundreds

Students need multiple opportunities to use real-world contexts to read and write multi-digit whole numbers. As they extend their understanding of numbers to 1,000,000, students reason about the magnitude of digits in a number and analyze the relationships of numbers. They can build larger numbers by using graph paper and labeling examples of each place with digits and words (e.g., *10,000* and *ten thousand*).

Millions	Hundred-thousands	Ten-thousands	Thousands	Hundreds	Tens	Ones
						4
					4	0
				4	0	0
			4	0	0	0
		4	0	0	0	0
	4	0	0	0	0	0
	4	4	4	4	4	4

"Four hundred forty-four thousand, four hundred forty-four"

To read and write numerals between 1,000 and 1,000,000, students need to understand the role of commas. Each sequence of three digits made by commas is read as hundreds, tens, and ones, followed by the name of the appropriate base-thousand unit (e.g., *thousand, million*). Layered place-value cards such as those used in earlier grades can be put on a frame with the base-thousand units labeled below. Then cards that form hundreds, tens, and ones can be placed on each section and the name read off using the card values followed by the word *million,* then *thousand,* then the silent ones (**MP.2, MP.3, MP.8**).

Grade-four students build on the grade-three skill of rounding to the nearest 10 or 100 to round multi-digit numbers and to make reasonable estimates of numerical values (**4.NBT.3▲**).

Example: Rounding Numbers in Context **4.NBT.3▲ (MP.4)**

The population of the fictional Midtown, USA, was last recorded as 76,398. The city council wants to round the population to the nearest thousand for a business brochure. What number should they round the population to?

Solution: When students represent numbers stacked vertically, they can see the relationships between the numbers more clearly. Students might think: "I know the answer is either 76,000 or 77,000. If I write 76,000 below 76,398 and 77,000 above it, I can see that the midpoint is 76,500, which is *above* 76,398. This tells me they should round the population to 76,000."

77,000
76,398
76,000

Adapted from ADE 2010.

Number and Operations in Base Ten — 4.NBT

Use place-value understanding and properties of operations to perform multi-digit arithmetic.

4. Fluently add and subtract multi-digit whole numbers using the standard algorithm.
5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
6. Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

At grade four, students become fluent with addition and subtraction with multi-digit whole numbers to 1,000,000 using standard algorithms (**4.NBT.4▲**). A central theme in multi-digit arithmetic is to encourage students to develop methods they understand and can explain rather than merely following a sequence of directions, rules, or procedures they do not understand. In previous grades, students built a conceptual understanding of addition and subtraction with whole numbers as they applied multiple methods to compute and solve problems. The emphasis in grade four is on the power of the regular one-for-ten trades between adjacent places that let students extend a method they already know to many places. Because students in grades two and three have been using at least one method that will generalize to 1,000,000, this extension in grade four should not take a long time. Thus, students will also have sufficient time for the major new topics of multiplication and division (**4.NBT.5–6▲**).

FLUENCY

California's Common Core State Standards for Mathematics (K–6) set expectations for fluency in computations using the standard algorithm (e.g., "*Fluently* add and subtract multi-digit whole numbers using the standard algorithm" [**4.NBT.4▲**]). Such standards are culminations of progressions of learning, often spanning several grades, involving conceptual understanding, thoughtful practice, and extra support where necessary. The word *fluent* is used in the standards to mean "reasonably fast and accurate" and possessing the ability to use certain facts and procedures with enough facility that using such knowledge does not slow down or derail the problem solver as he or she works on more complex problems. Procedural fluency requires skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Developing fluency in each grade may involve a mixture of knowing some answers, knowing some answers from patterns, and knowing some answers through the use of strategies.

Adapted from UA Progressions Documents 2011a.

In grade four, students extend multiplication and division to include whole numbers greater than 100. Students should use methods they understand and can explain to multiply and divide. The standards (**4.NBT.5–6▲**) call for students to use visual representations such as area and array models that students

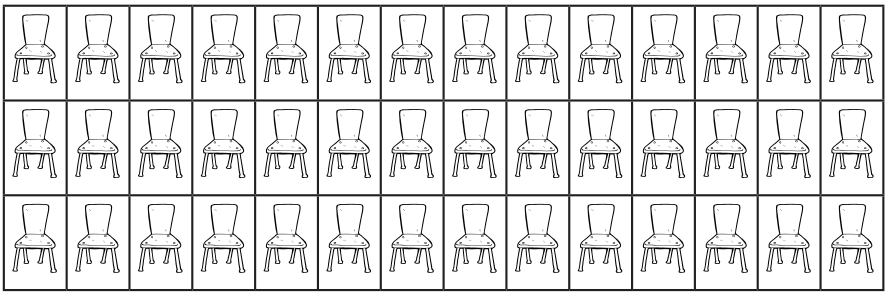
draw and connect to equations, as well as written numerical work, to support student reasoning and explanation of methods. By reasoning repeatedly about the connections between math drawings and written numerical work, students can come to see multiplication and division algorithms as abbreviations or summaries of their reasoning about quantities.

Students can use area models to represent various multiplication situations. The rows can represent the equal groups of objects in the situation, and students then imagine that the objects lie in the squares forming an array. With larger numbers, such array models become too difficult to draw, so students can make sketches of rectangles and then label the resulting product as the number of things or square units. When area models are used to represent an actual area situation, the two factors are expressed in length units (e.g., cm) while the product is in square units (e.g., cm^2).

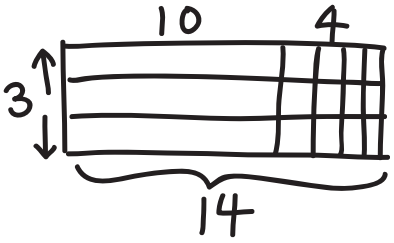
Example: Area Models and Strategies for Multi-Digit Multiplication with a Single-Digit Multiplier **4.NBT.5▲**

Chairs are being set up for a small play. There should be 3 rows of chairs and 14 chairs in each row. How many chairs will be needed?

Solution: As in grade three, when students first made the connection between array models and the area model, students might start by drawing a sketch of the situation. They can then be reminded to see the chairs as if surrounded by unit squares and hence a model of a rectangular region. With base-ten blocks or math drawings (**MP.2, MP.5**), students represent the problem and see it broken down into $3\times(10+4)$.



Making a sketch like the one above becomes cumbersome, so students move toward representing such drawings more abstractly, with rectangles, as shown to the right. This builds on the work begun in grade three. Such diagrams help children see the distributive property: "3×14 can be written as $3\times(10+4)$, and I can do the multiplications separately and add the results: $3\times(10+4)=3\times10+3\times4$. The answer is $30+12=42$, or 42 chairs."



In grade three, students multiplied single-digit numbers by multiples of 10 (**3.NBT.3**). This idea is extended in grade four. For example, since $6\times7=42$, the following equations and statements must be true:

- $6\times70=420$, since this is "6 times 7 tens," which is 42 tens.
- $6\times700=4200$, since this is "6 times 7 hundreds," which is 42 hundreds.
- $6\times7000=42{,}000$, since this is "6 times 7 thousands," which is 42 thousands.
- $60\times70=4200$, since this is "60 times 7 tens," which is 420 tens, or 4200.

Math drawings and base-ten blocks support the development of these *extended multiplication facts.* The ability to find products such as these is important when variations of the standard algorithm are used for multi-digit multiplication, as described in the following examples.

Examples: Developing Written Methods for Multi-Digit Multiplication **4.NBT.5▲**

Find the product: 6×729

Solution: Sufficient practice with drawing rectangles (or constructing them with base-ten blocks) will help students understand that the problem can be represented with a rectangle such as the one shown. The product is given by the total area: $6 \times 729 = 6 \times 700 + 6 \times 20 + 6 \times 9$. Understanding extended multiplication facts allows students to find the *partial products* quickly. Students can record the multiplication in several ways:

	729 = **700**	**+ 20**	**+ 9**
6	6 × 700 = 6 groups of 7 hundreds = 42 hundreds = 4200	6 × 20 = 6 groups of 2 tens = 12 tens = 120	6 × 9 = 54

Left to right, showing the partial products

729 × 6	Thinking:
4200	6 × 7 hundreds
120	6 × 2 tens
54	6 × 9
4374	

Right to left, showing the partial products

729 × 6	
54	6 × 9
120	6 × 2 tens
4200	6 × 7 hundreds
4374	

Right to left, recording the newly composed tens and hundreds below the line

729
× 6
15
4224
4374

Find the product: 27×65

Solution: This time, a rectangle is drawn, and like base-ten units (i.e., tens and ones) are represented by subregions of the rectangle. Repeated use of the distributive property shows that:

$$27 \times 65 = (20 + 7) \times 65 = 20 \times 65 + 7 \times 65$$
$$= 20 \times (60 + 5) + 7 \times (60 + 5)$$
$$= 20 \times 60 + 20 \times 5 + 7 \times 60 + 7 \times 5.$$

The product is again given by the total area:

$1200 + 100 + 420 + 35 = 1755$

	60	**+ 5**
20	20 × 60 = 2 tens times 6 tens = 12 hundreds = 1200	20 × 5 = 2 tens × 5 = 10 tens = 100
+		
7	7 × 60 = 7 × 6 tens = 42 tens = 420	7 × 5 = 35

Below are two written methods for recording the steps of the multiplication.

Showing the partial products

65 × 27	Thinking:
35	7 × 5
420	7 × 6 tens
100	2 tens × 5
1200	2 tens × 6 tens
1755	

Recording the newly composed tens and hundreds below the line for correct place-value position

$$\begin{array}{r} 65 \\ \times\ 27 \\ \hline {\scriptstyle 43}\ 25 \\ {\scriptstyle 11}\ \mathbf{200} \\ \hline 1755 \end{array}$$

In this example, digits that represent newly composed tens and hundreds are written below the line instead of above 65. The digit 3 from $7 \times 5 = 35$ is placed correctly in the tens place, and the 5 is correctly placed in the ones. Similarly, the digit 4 from $7 \times 6 = 42$ is correctly placed in the hundreds place and the 2 in the tens place. When these digits are placed above the 65, it becomes harder to see where the digits came from and what their true place value is.

Notice that the boldface **0** is included in the second method, indicating that we are multiplying not just by 2 in this row, but by 2 *tens*.

General methods for computing quotients of multi-digit numbers and one-digit numbers (**4.NBT.6▲**) rely on the same understandings as for multiplication, but these are cast in terms of division. For example, students may see division problems as knowing the area of a rectangle but not one side length (the quotient), or as finding the size of a group when the number of groups is known (measurement division).

Example: Using the Area Model to Develop Division Strategies **4.NBT.6▲**

Find the quotient: $750 \div 6$

Solution: "Just as with multiplication, I can set this up as a rectangle, but with one side unknown since this is the same as $? \times 6 = 750$. I find out what the number of hundreds would be for the unknown side length; that's 1 hundred or 100, since $100 \times 6 = 600$, and that's as large as I can go. Then, I have $750 - 600 = 150$ square units left, so I find the number of tens that are in the other side. That's 2 tens, or 20, since $20 \times 6 = 120$. Last, there are $150 - 120 = 30$ square units left, so the number of ones on the other side must be 5, since $5 \times 6 = 30$."

	? hundreds + ? tens + ? ones
6	750

	100	+ 20	+ 5
6	750	150	30
	−600	−120	−30
	150	30	0

One way students can record this is shown at right: *partial quotients* are stacked atop one another, with zeros included to indicate place value and as a reminder of how students obtained the numbers. The full quotient is the sum of these stacked numbers.

$$\begin{array}{r|l} & 5 \\ & 20 \\ & 100 \\ \hline 6 & 750 \\ & -600 \\ \hline & 150 \\ & -120 \\ \hline & 30 \\ & -30 \\ \hline & 0 \end{array} \quad 125$$

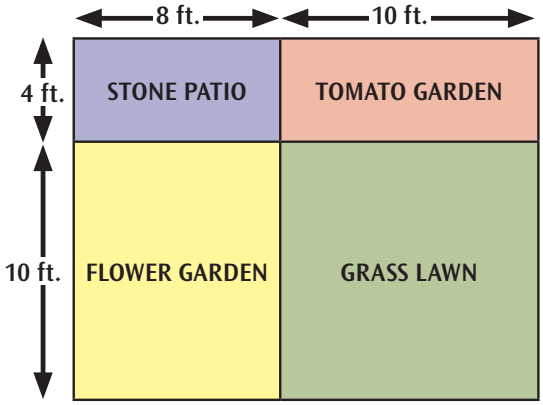
General methods for multi-digit division computation include decomposing the dividend into like base-ten units and finding the quotient unit by unit, starting with the largest unit and continuing on to smaller units. As with multiplication, this method relies on the distributive property. This work continues in grade five and culminates in fluency with the standard algorithm in grade six (adapted from PARCC 2012).

In grade four, students also find whole-number quotients with remainders (**4.NBT.6▲**) and learn the appropriate way to write the result. For instance, students divide and find that $195 \div 9 = 21$, with 6 left over. This can be written as $195 = 21(9) + 6$. When put into a context, the latter equation makes sense. For instance, if 195 books are distributed equally among 9 classrooms, then each classroom gets 21 books, and 6 books will be left over. The equation $195 = 21(9) + 6$ is closely related to the equation $195 \div 9 = 21\frac{6}{9}$, which students will write in later grades. It is best to avoid the notation $195 \div 9 = 21R6$.

As students decompose numbers to solve division problems, they also reinforce important mathematical practices such as seeing and making use of structure (**MP.7**). As they illustrate and explain calculations, they model (**MP.4**), strategically use appropriate drawings as tools (**MP.5**), and attend to precision (**MP.6**) using base-ten units.

Table 4-4 presents a sample classroom activity that connects the Standards for Mathematical Content and Standards for Mathematical Practice.

Table 4-4. Connecting to the Standards for Mathematical Practice—Grade Four

Standards Addressed	Explanation and Examples
Connections to Standards for Mathematical Practice **MP.1.** Students make sense of the problem when they see that the measurements on the side and top of the diagram persist and yield the measurements of the smaller areas. **MP.2.** Students reason abstractly as they represent the areas of the yard as multiplication problems to be solved. **MP.5.** Students use appropriate tools strategically when they apply the formula for the area of a rectangle to solve the problem. They organize their work in a way that makes sense to them. **MP.7.** Teachers can use this problem and similar problems to illustrate the distributive property of multiplication. In this case, we find that $18\times14 = (10\times14) + (8\times14) = (10\times10) + (10\times4) + (8\times10) + (8\times4)$.	**Sample Problem:** What are the areas of the four sections of Mr. Griffin's backyard? The yard has a stone patio, a tomato garden, a flower garden, and a grass lawn. What is the area of his entire backyard? How did you find your answer?  18 ×14 Area of Stone Patio → 32 (4×8) Area of Tomato Garden → 40 (4×10) Area of Flower Garden → 80 (10×8) Area of Grass Lawn → 100 (10×10) Area of Entire Backyard → 252
Standards for Mathematical Content **4.NBT.5.** Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and properties of operations. Illustrate and explain the calculation using equations, rectangular arrays, and/or area models. **4.MD.3.** Apply the area and perimeter formulas for rectangles in real-world and mathematical problems. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*	**Solution.** The areas of the four sections are 32 square feet, 40 square feet, 80 square feet, and 100 square feet, respectively. The area of the entire backyard is the sum of these areas: (32+40+80+100), or 252 square feet. This is the same as finding the product of $18\times14 = 252$ square feet. **Classroom Connections.** The purpose of this task is to illuminate the connection between the area of a rectangle as representing the product of two numbers and the partial products algorithm for multiplying multi-digit numbers. In this algorithm, which is shown beneath the area model, each digit of one number is multiplied by each digit of the other number, and the "partial products" are written down. The sum of these partial products is the product of the original numbers. Place value can be emphasized by specifically reminding students that if we multiply the 2 tens together, since each represents 1 ten, the product is 100. Finally, the area model provides a visual justification for how the algorithm works.

Domain: Number and Operations—Fractions

Student proficiency with fractions is essential to success in algebra. In grade three, students developed an understanding of fractions as built from unit fractions. A critical area of instruction in grade four is fractions, including developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers. In grade four, fractions include those with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.

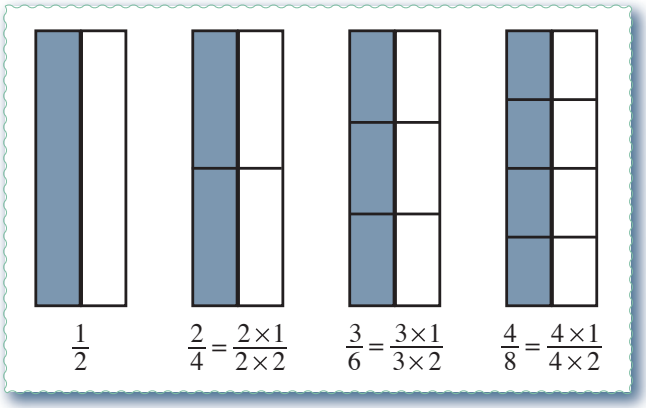
Number and Operations—Fractions 4.NF

Extend understanding of fraction equivalence and ordering.

1. Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.
2. Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as $1/2$. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Grade-four students learn a fundamental property of equivalent fractions: multiplying the numerator and denominator of a fraction by the same non-zero whole number results in a fraction that represents the same number as the original fraction (e.g., $\frac{a}{b} = \frac{n \times a}{n \times b}$, for $n \neq 0$). Students use visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size (**4.NF.1▲**). This property forms the basis for much of the work with fractions in grade four, including comparing, adding, and subtracting fractions and the introduction of finite decimals.

Students use visual models to reason about and explain why fractions are equivalent. For example, the area models below all show fractions equivalent to $\frac{1}{2}$, and although students in grade three justified that all the models represent the same amount visually, fourth-grade students reason about *why* it is true that $\frac{1}{2} = \frac{2 \times 1}{2 \times 2} = \frac{3 \times 1}{3 \times 2} = \frac{4 \times 1}{4 \times 2}$, and so on.



Adapted from ADE 2010.

Students use reasoning such as this: when a horizontal line is drawn through the center of the first model to obtain the second, both the number of equal parts and the number of those parts being counted are doubled ($2 \times 2 = 4$ in the denominator, $2 \times 1 = 2$ in the numerator, respectively), but even though there are more parts counted, they are smaller parts. Students notice connections between the models and the fractions represented by the models in the way both the parts and wholes are counted, and they begin to generate a rule for writing equivalent fractions. Students also emphasize the inversely related changes: the number of unit fractions becomes larger, but the size of the unit fraction becomes smaller.

Students should have repeated opportunities to use math drawings such as these (and the ones that follow in this chapter) to understand the general method for finding equivalent fractions. Of course, students may also come to see that the rule works both ways. For example:

$$\frac{28}{35}=\frac{7\times4}{7\times5}=\frac{4}{5}$$

Teachers must be careful to avoid overemphasizing this "simplifying" of fractions, as there is no mathematical reason for doing so—although, depending on the problem context, one form (renamed or not renamed) may be more desirable than another. In particular, teachers should avoid using the term *reducing* fractions for this process, as the value of the fraction itself is *not* being reduced. A more neutral term, such as *renaming* (which hints at these fractions being different names for the same amount), allows teachers to refer to this strategy with less potential for student misunderstanding.

Focus, Coherence, and Rigor

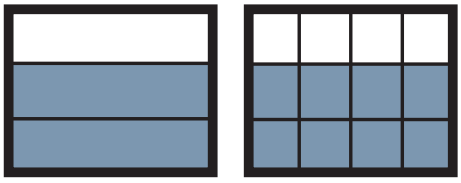
It is true that one can justify that $\frac{a}{b}=\frac{n\times a}{n\times b}$ by arguing as follows:

$$\frac{n\times a}{n\times b}=\frac{n}{n}\times\frac{a}{b}=1\times\frac{a}{b}=\frac{a}{b}$$

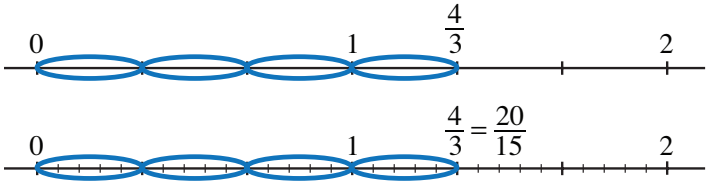
This is simply multiplying by 1 in the form of $\frac{n}{n}$. Since students in grade four have not yet encountered the general notion of fraction multiplication, this argument should be avoided in favor of developing an understanding with diagrams and reasoning about the size and number of parts that are created in this process. In grade five, students will learn the general rule that $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

Examples: Reasoning with Diagrams That $\frac{a}{b}=\frac{n\times a}{n\times b}$ 4.NF.1▲

Using an Area Model. The area of the rectangle represents one whole. In the illustrations provided, the rectangle on the left shows the area divided into three rectangles of equal area (thirds), with two of them shaded (2 pieces of size $\frac{1}{3}$), representing $\frac{2}{3}$. In the figure on the right, the vertical lines divide the parts (the thirds) into smaller parts. There are now 4×3 smaller rectangles of equal area, and the shaded area now comprises 4×2 of them, so it represents $\frac{4\times2}{4\times3}=\frac{8}{12}$.



Using a Number Line. The top number line shown below indicates $\frac{4}{3}$. Each unit length is divided into three equal parts. When each $\frac{1}{3}$ is further divided into 5 equal parts, there are now 5×3 of these new equal parts. Since 4 of the $\frac{1}{3}$ parts were circled before, and each of these has been subdivided into 5 parts, there are now 5×4 of these new small parts. Therefore, $\frac{4}{3}=\frac{5\times4}{5\times3}=\frac{20}{15}$.



Adapted from UA Progressions Documents 2013a.

Creating equivalent fractions by dividing and shading squares or circles and matching each fraction to its location on the number line can reinforce students' understanding of fractions. The National Council of Teachers of Mathematics (NCTM) provides an activity on creating equivalent fractions at http://illuminations.nctm.org/Activity.aspx?id=3510 (NCTM Illuminations 2013b).

Students apply their new understanding of equivalent fractions to compare two fractions with different numerators and different denominators (**4.NF.2▲**). They compare fractions by using benchmark fractions and finding common denominators or common numerators. Students explain their reasoning and record their results using the symbols >, =, and <.

Examples: Comparing Fractions	**4.NF.2▲**
1. Students might compare fractions to benchmark fractions—for example, comparing to $\frac{1}{2}$ when comparing $\frac{3}{8}$ and $\frac{2}{3}$. Students see that $\frac{3}{8} < \frac{4}{8} = \frac{1}{2}$, and that since $\frac{2}{3} = \frac{4}{6}$ and $\frac{4}{6} > \frac{3}{6} = \frac{1}{2}$, it must be true that $\frac{3}{8} < \frac{2}{3}$.	
2. Students compare $\frac{5}{8}$ and $\frac{7}{12}$ by writing them with a common denominator. They find that $\frac{5}{8} = \frac{5 \times 12}{8 \times 12} = \frac{60}{96}$ and $\frac{7}{12} = \frac{7 \times 8}{12 \times 8} = \frac{56}{96}$ and reason therefore that $\frac{5}{8} > \frac{7}{12}$. Notice that students do not need to find the **smallest** common denominator for two fractions; any common denominator will work.	
3. Students can also find a common numerator to compare $\frac{5}{8}$ and $\frac{7}{12}$. They find that $\frac{5}{8} = \frac{5 \times 7}{8 \times 7} = \frac{35}{56}$ and $\frac{7}{12} = \frac{7 \times 5}{12 \times 5} = \frac{35}{60}$. Then they reason that, since parts of size $\frac{1}{56}$ are larger than parts of size $\frac{1}{60}$ when the whole is the same, $\frac{5}{8} > \frac{7}{12}$.	
Adapted from ADE 2010.	

In grade four, students extend previous understanding of addition and subtraction of whole numbers to add and subtract fractions with like denominators (**4.NF.3▲**).

Number and Operations—Fractions 4.NF

Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.

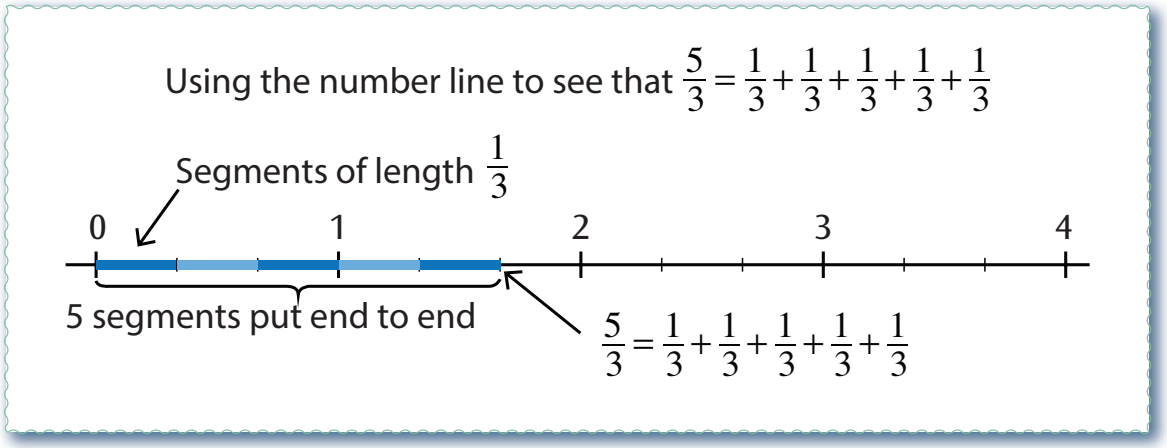
3. Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.
   a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.
   b. Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. *Examples:* $3/8 = 1/8 + 1/8 + 1/8$; $3/8 = 1/8 + 2/8$; $2\,1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8$.
   c. Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.
   d. Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

Students begin by understanding a fraction $\frac{a}{b}$ as a sum of the unit fractions $\frac{1}{b}$. In grade three, students learned that the fraction $\frac{a}{b}$ represents a parts when a whole is broken into b equal parts (i.e., parts of size $\frac{1}{b}$.) However, in grade four, students connect this understanding of a fraction with the operation of addition; for instance, they see now that if a whole is broken into 4 equal parts and 5 of them are taken, then this is represented by both $\frac{5}{4}$ and the expression $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$ (**4.NF.3b▲**). They experience composing fractions from and decomposing fractions into sums of unit fractions and non-unit fractions in this general way—for example, by seeing $\frac{5}{4}$ also as:

$$\boxed{\frac{1}{4}+\frac{1}{4}+\frac{3}{4}} \quad \text{or} \quad \boxed{\frac{2}{4}+\frac{3}{4}} \quad \text{or} \quad \boxed{\frac{1}{4}+\frac{3}{4}+\frac{1}{4}}$$

Students write and use unit fractions while working with standard **4.NF.3b▲**, which supports their conceptual understanding of adding fractions and solving problems (**4.NF.3a▲, 4.NF.3d▲**). Students write and use unit fractions while decomposing fractions in several ways (**4.NF.3b▲**). This work helps students understand addition and subtraction of fractions (**4.NF.3a▲**) and how to solve word problems involving fractions with the same denominator (**4.NF.3d▲**). Writing and using unit fractions also helps students avoid the common misconception of adding two fractions by adding their numerators and denominators—for example, erroneously writing $\frac{1}{2}+\frac{5}{6}=\frac{6}{8}$. In general, the meaning of addition is the same for both fractions and whole numbers. Students understand addition as "putting together" like units, and they visualize how fractions are built from unit fractions and that a fraction is a sum of unit fractions.

Students may use visual models to support this understanding—for example, showing that $\frac{5}{3}=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$ by using a number line model (**MP.1, MP.2, MP.4, MP.6, MP.7**).



Source: UA Progressions Documents 2013a.

Students add or subtract fractions with like denominators, including mixed numbers (**4.NF.3a, c▲**), and solve word problems involving fractions (**4.NF.3d▲**). They use their understanding that every fraction is composed of unit fractions to make connections such as this:

$$\frac{7}{5}+\frac{4}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}\ +\ \frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{7+4}{5}$$

This quickly allows students to develop a general principle that $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$. Using similar reasoning, students understand that $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$.

Students also compute sums of whole numbers and fractions, realizing that any whole number can be written as an equivalent number of unit fractions of a given size. For example, they find the sum $3+\frac{7}{2}$ in the following way:

$$3+\frac{7}{2}=\frac{6}{2}+\frac{7}{2}=\frac{13}{2}$$

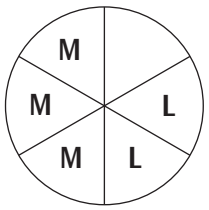
Understanding this method of adding a whole number and a fraction allows students to accurately convert mixed numbers into fractions, as in this example:

$$4\frac{5}{8}=4+\frac{5}{8}=\frac{32}{8}+\frac{5}{8}=\frac{37}{8}$$

Students should develop a firm understanding that a mixed number indicates the sum of a whole number and a fraction (i.e., $a\frac{b}{c}=a+\frac{b}{c}$). They should also learn a method for converting mixed numbers to fractions that is connected to the meaning of fractions (such as the one demonstrated above), rather than typical rote methods.

Examples: Reasoning with Addition and Subtraction of Fractions **4.NF.3a–d▲**

1. Mary and Lacey share a pizza. Mary ate $\frac{3}{6}$ of the pizza and Lacey ate $\frac{2}{6}$ of the pizza. How much of the pizza did the girls eat altogether? (**MP.3, MP.4**). Use the picture of a pizza to explain your answer.

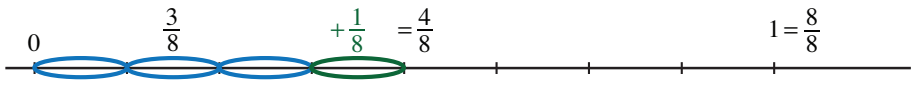


Solution: "I labeled three of the one-sixth pieces for Mary and two of the one-sixth pieces for Lacey. I can see that altogether, they've eaten five of the one-sixth pieces, or $\frac{5}{6}$ of the pizza. Also, I know that $\frac{3}{6}+\frac{2}{6}=\frac{2+3}{6}=\frac{5}{6}$."

Adapted from KATM 2012, 4th Grade Flipbook.

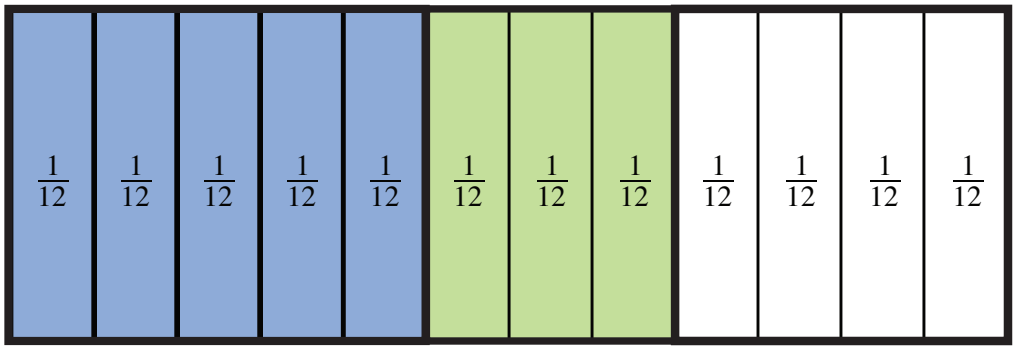
2. Susan and Maria need $8\frac{3}{8}$ feet of ribbon to package gift baskets. Susan has $3\frac{1}{8}$ feet of ribbon and Maria has $5\frac{3}{8}$ feet of ribbon. How much ribbon do they have altogether? Is it enough to complete the packaging?

Solution: "I know I need to find $5\frac{3}{8}+3\frac{1}{8}$ to find out how much they have altogether. I know that Susan and Maria have $3+5=8$ feet of ribbon plus the other $\frac{3}{8}+\frac{1}{8}$ feet of ribbon. Altogether, this is $8\frac{4}{8}$ feet of ribbon, which means they have enough ribbon to do their packaging. They even have $\frac{1}{8}$ feet of ribbon left."



Adapted from KATM 2012, 4th Grade Flipbook.

3. Elena, Matthew, and Kevin painted a wall. Elena painted $\frac{5}{12}$ of the wall and Matthew painted $\frac{3}{12}$ of the wall. Kevin painted the rest. How much of the wall did Kevin paint? Use the picture below to help find your answer.



Solution: "By shading what Elena and Matthew painted, I can show in the picture that Elena and Matthew painted $\frac{8}{12}$ of the wall. The remaining part that Kevin painted was $\frac{4}{12}$ of the wall. I can write this as $\frac{12}{12}-\frac{8}{12}=\frac{4}{12}$, or $1-\frac{8}{12}=\frac{4}{12}$, or even $1-\frac{5}{12}-\frac{3}{12}=\frac{4}{12}$."

Adapted from New York State Education Department (NYSED) 2012.

Number and Operations—Fractions 4.NF

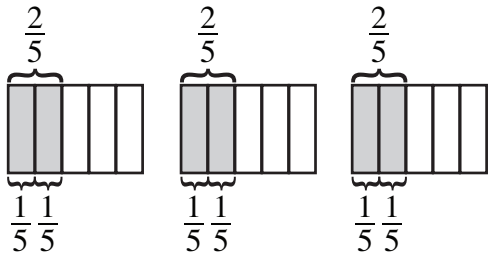
Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.

4. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.
   a. Understand a fraction a/b as a multiple of $1/b$. *For example, use a visual fraction model to represent $5/4$ as the product $5\times(1/4)$, recording the conclusion by the equation $5/4=5\times(1/4)$.*
   b. Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number. *For example, use a visual fraction model to express $3\times(2/5)$ as $6\times(1/5)$, recognizing this product as $6/5$. (In general, $n\times(a/b)=(n\times a)/b$.)*
   c. Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. *For example, if each person at a party will eat $3/8$ of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?*

In grade three, students learned that 3×7 can be represented as the total number of objects in 3 groups of 7 objects and that they could solve this by adding $7+7+7$. Fourth-grade students apply this concept to fractions, understanding a fraction $\frac{a}{b}$ as a multiple of $\frac{1}{b}$ (**4.NF.4a▲**). This understanding is connected with standard **4.NF.3,** and students make the shift to see $\frac{5}{3}$ as $5\times\frac{1}{3}$. For example, they see the following:

$$\frac{5}{3}=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=5\times\frac{1}{3}$$

Students then extend this understanding to make meaning of the product of a whole number and a fraction (**4.NF.4b▲**)—for example, seeing $3 \times \frac{2}{5}$ in the following ways:



$$\frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{3 \times 2}{5} = \frac{6}{5}$$

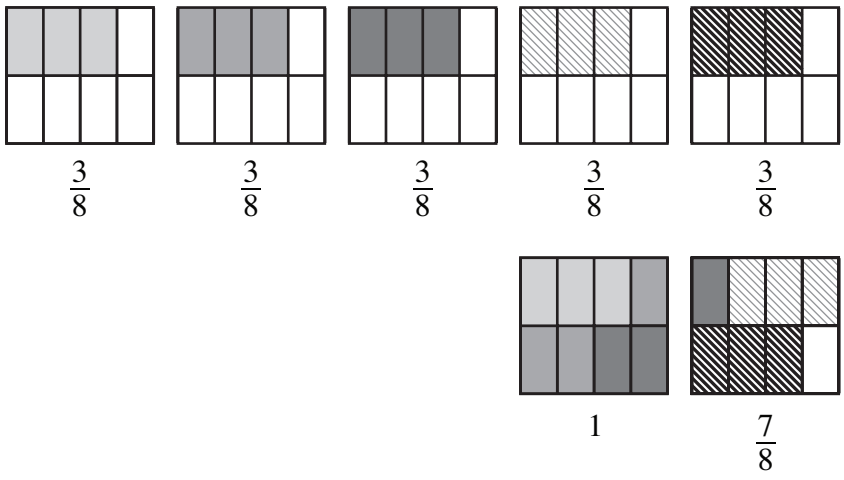
Source: UA Progressions Documents 2013a.

Students are also presented with opportunities to work with word problems involving multiplication of a fraction by a whole number to relate situations, models, and corresponding equations (**4.NF.4c▲**).

Example: Multiplying a Fraction by a Whole Number **4.NF.4c▲**

Each person at a dinner party eats $\frac{3}{8}$ of a pound of pasta. There are 5 people at the party. How many pounds of pasta are needed? Pasta comes in 1-pound boxes. How many boxes of pasta should be purchased? (**MP.1, MP.2, MP.7**)

Solution: If 5 rectangles are drawn, with $\frac{3}{8}$ of a pound shaded in each rectangle, then students see that they are finding $5 \times \frac{3}{8} = \frac{15}{8}$.



The separate eighths can be collected together to illustrate that a total of $1\frac{7}{8}$ pounds of pasta will be needed for the party. This means that 2 boxes of pasta should be purchased.

Adapted from ADE 2010 and NCDPI 2013b.

Number and Operations—Fractions 4.NF

Understand decimal notation for fractions, and compare decimal fractions.

5. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[6] *For example, express* $3/10$ *as* $30/100$*, and add* $3/10 + 4/100 = 34/100$.
6. Use decimal notation for fractions with denominators 10 or 100. *For example, rewrite 0.62 as* $62/100$*; describe a length as 0.62 meters; locate 0.62 on a number line diagram.*
7. Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using **the number line or another** visual model. **CA**

Fourth-grade students develop an understanding of decimal notation for fractions and compare *decimal fraction*s (fractions with a denominator of 10 or 100). This work lays the foundation for performing operations with decimal numbers in grade five. Students learn to add decimal fractions by converting them to fractions with the same denominator (**4.NF.5▲**). For example, students express $\frac{3}{10}$ as $\frac{30}{100}$ before they add $\frac{30}{100} + \frac{4}{100} = \frac{34}{100}$. Students can use graph paper, base-ten blocks, and other place-value models to explore the relationship between fractions with denominators of 10 and 100 (adapted from UA Progressions Documents 2013a).

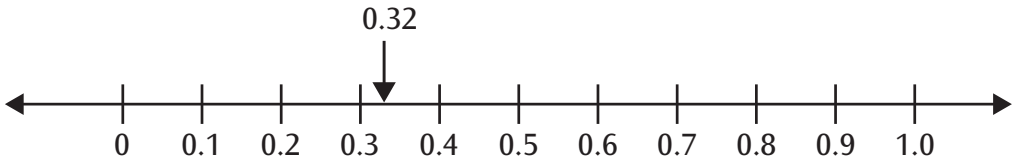
In grade four, students first use decimal notation for fractions with denominators 10 or 100 (**4.NF.6▲**), understanding that the number of digits to the right of the decimal point indicates the number of zeros in the denominator. Students make connections between fractions with denominators of 10 and 100 and place value. They read and write decimal fractions; for example, students say 0.32 as "thirty-two hundredths" and learn to flexibly write this as both 0.32 and $\frac{32}{100}$.

Focus, Coherence, and Rigor

To reinforce student understanding, teachers are urged to consistently use place-value-based language when naming decimals—for example, by saying "four-tenths" rather than "point four" when referring to 0.4, and by saying "sixty-eight hundredths" as opposed to "point sixty-eight" or "point six eight" when referring to 0.68.

6. Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators in general is not a requirement at this grade.

Students represent values such as 0.32 or $\frac{32}{100}$ on a number line. They reason that $\frac{32}{100}$ is a little more than $\frac{30}{100}$ (or $\frac{3}{10}$) and less than $\frac{40}{100}$ (or $\frac{4}{10}$). It is closer to $\frac{30}{100}$, so it would be placed on the number line near that value (**MP.2, MP.4, MP.5, MP.7**).



Students compare two decimals to hundredths by reasoning about their size (**4.NF.7▲**). They relate their understanding of the place-value system for whole numbers to fractional parts represented as decimals. Students compare decimals using the meaning of a decimal as a fraction, making sure to compare fractions with the same denominator and ensuring that the "wholes" are the same.

Common Misconceptions

- Students sometimes treat decimals as whole numbers when making comparisons of two decimals, ignoring place value. For example, they may think that $0.2 < 0.07$ simply because $2 < 7$.
- Students sometimes think, *"The longer the decimal number, the greater the value."* For example, they may think that 0.03 is greater than 0.3.

Domain: Measurement and Data

Measurement and Data — 4.MD

Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.

1. Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. *For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), . . .*
2. Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

Students will need ample opportunities to become familiar with new units of measure. In prior years, work with units was limited to units such as pounds, ounces, grams, kilograms, and liters, and students did not convert measurements.

Students may use two-column tables to convert from larger to smaller units and record equivalent measurements. For example:

kg	g
1	1000
2	2000
3	3000

ft	in
1	12
2	24
3	36

lb	oz
1	16
2	32
3	48

Students in grade four begin using the four operations to solve word problems involving measurement quantities such as liquid volume, mass, and time (**4.MD.2**), including problems involving simple fractions or decimals.

Examples: Word Problems Involving Measures **4.MD.2**

1. **Division/Fractions:** Susan has 2 feet of ribbon. She wants to give her ribbon to her 3 best friends so each friend gets the same amount. How much ribbon will each friend get? Students may record their solutions by using fractions or inches.

   Solution: The answer would be $\frac{2}{3}$ of a foot, or 8 inches. Students are able to express the answer in inches because they understand that $\frac{1}{3}$ of a foot is 4 inches and $\frac{2}{3}$ of a foot is 2 groups of $\frac{1}{3}$.

2. **Addition:** Mason ran for an hour and 15 minutes on Monday, 25 minutes on Tuesday, and 40 minutes on Wednesday. What was the total number of minutes Mason ran?

   Solution: Students know that 60 minutes make up one hour. We know Mason ran one hour, which is 60 minutes. He also ran 15 + 25 + 40 = 80 minutes more, which makes 140 total minutes.

3. **Multiplication:** Mario and his two brothers are selling lemonade. Mario brought one and a half liters, Javier brought 2 liters, and Ernesto brought 450 milliliters. How many total milliliters of lemonade did the boys have?

   Solution: Students know that 1 liter is 1000 milliliters (ml), so Mario brought 1000 + 500 = 1500 ml, and Javier brought 2 × 1000 = 2000 ml. This means the three brothers had a total of 1500 + 2000 + 450 = 3950 ml.

Adapted from ADE 2010.

Focus, Coherence, and Rigor

In grade four, students use the four operations to solve word problems involving measurement quantities such as liquid volume, mass, and time (**4.MD.1–2**). Measurement provides a context for solving problems using the four operations and connects to and supports major grade-level work in the cluster "Use the four operations with whole numbers to solve problems" (**4.OA.1–3▲**) and clusters in the domain "Number and Operations—Fractions" (**4.NF.1–4▲**). For example, students use whole-number multiplication to express measurements given in a larger unit in terms of a smaller unit, and they solve word problems involving addition and subtraction of fractions or multiplication of a fraction by a whole number.

Adapted from PARCC 2012.

Measurement and Data 4.MD

Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.

3. Apply the area and perimeter formulas for rectangles in real-world and mathematical problems. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*

In grade three, students developed an understanding of area and perimeter by using visual models. Students in grade four are expected to use formulas to calculate area and perimeter of rectangles; however, they still need to understand and be able to communicate their understanding of why the formulas work. It is still important for students to draw length units or square units inside a small rectangle to keep the distinction fresh and visual, and some students may still need to write the lengths of all four sides before finding the perimeter. Students know that answers for the area formula ($\ell \times w$) will be in square units and that answers for the perimeter formula ($2\ell \times 2w$ or $2(\ell \times w)$) will be in linear units (adapted from ADE 2010).

Example: Area and Perimeter of Rectangles (MP.2, MP.4) **4.MD.3**

Sally wants to build a pen for her dog, Callie. Her parents give her $200 to buy the fencing material, but they want Sally to design the pen. Her parents suggest that she consider different plans. Her parents also remind her that Callie needs as much room as possible to run and play, that the pen can be placed anywhere in the yard, and that the wall of the house could be used as one side of the pen. Sally decides to buy fencing material that costs $8.50 per foot. She also needs at least one three-foot-wide gate for the pen that costs $15.

- Design a pen for Callie. Experiment with different pen designs and consider the advice from Sally's parents. Sally's house can be any configuration.
- Write a letter to Sally with your diagrams and calculations. Explain why some designs are better for Callie than others.

Measurement and Data 4.MD

Represent and interpret data.

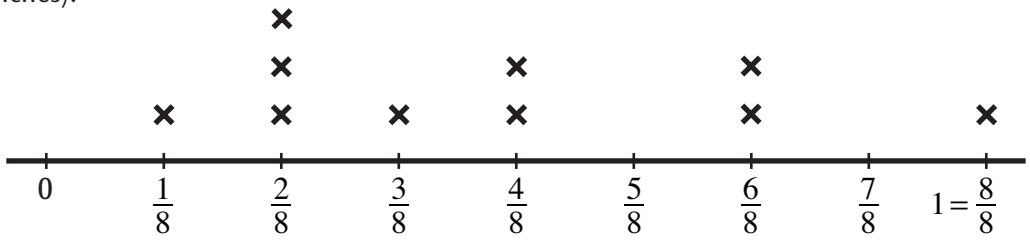
4. Make a line plot to display a data set of measurements in fractions of a unit ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$). Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*

As students work with data in kindergarten through grade five, they build foundations for the study of statistics and probability in grades six and beyond, and they strengthen and apply what they learn in arithmetic.

Fourth-grade students make a line plot to display a data set of measurements in fractions of a unit $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$, and they solve problems involving addition and subtraction of fractions by using information presented in line plots (**4.MD.4**).

Example: Interpreting Line Plots **4.MD.4**

Ten students measure objects in their desk to the nearest $\frac{1}{8}$ inch. They record their results on the line plot below (in inches).



Possible related questions:

- How many objects measured $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ inch?
- If you put the objects end to end, what would the total length be?
- If five $\frac{1}{8}$-inch pencils are placed end to end, what would the total length of the pencils be?

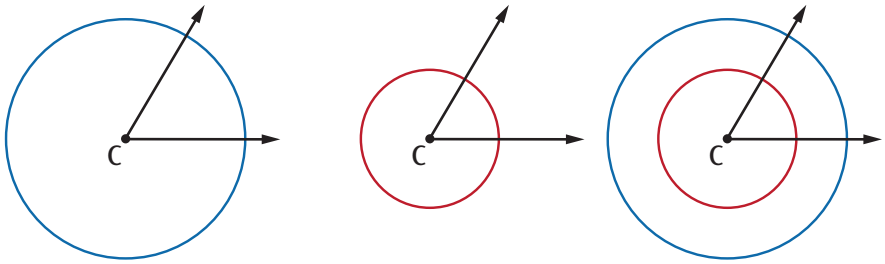
Adapted from ADE 2010.

Measurement and Data 4.MD

Geometric measurement: understand concepts of angle and measure angles.

5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
   a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through $1/360$ of a circle is called a "one-degree angle," and can be used to measure angles.
   b. An angle that turns through n one-degree angles is said to have an angle measure of n degrees.
6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.
7. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real-world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Students in grade four learn that angles are geometric shapes formed by two rays that share a common endpoint (**4.MD.5**). They understand angle measure as being that portion of a circular arc that is formed by the angle when a circle is centered at their shared vertex. The following figure helps students see that an angle is determined by the arc it creates relative to the size of the entire circle, evidenced by the picture showing two angles of the same measure (though their circles are not the same).

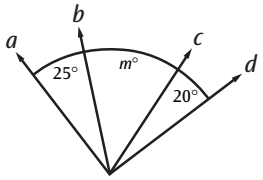


However, the pie-shaped pieces formed by each angle are not the same size; this shows that angle measure is not defined in terms of these areas. The angle in each case is 60°, since it measures an arc that is $\frac{1}{6}$ the total circumference of the circle in both the larger and smaller circles—but the pie-shaped pieces formed by the angle have different areas.

Before students begin measuring angles with protractors (**4.MD.6**), they need to have some experience with benchmark angles. They transfer their understanding that a 360° rotation about a point makes a complete circle to recognize and sketch angles that measure approximately 90° and 180°. They extend this understanding and recognize and sketch angles that measure approximately 45° and 30°. Students use appropriate terminology (*acute, right,* and *obtuse*) to describe angles and rays (*perpendicular*). Students recognize angle measure as additive and use this to solve addition and subtraction problems to find unknown angles on a diagram.

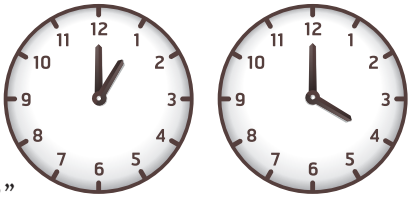
Examples: Angle measure is additive. **4.MD.7 (MP.1, MP.2, MP.4, MP.7)**

1. If ray a is perpendicular to ray d (see 4.G.1), what is the value of m?



Solution: "Since perpendicular lines make an angle that measures 90°, I know that $25 + m + 20 = 90$. This means that $m = 90 - 45 = 45$."

2. Joey knows that when a clock's hands are exactly on 12 and 1, the angle formed by the clock's hands measures 30°. What is the measure of the angle formed when a clock's hands are exactly on the 12 and 4?



Solution: "This looks like it is four times as much, so it is $4 \times 30° = 120°$."

Adapted from ADE 2010.

Focus, Coherence, and Rigor

Students' work with concepts of angle measures (**4.MD.5a, 4.MD.7**) also connects to and supports the addition of fractions, which is major work at the grade in the cluster "Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers" (**4.NF.3–4▲**). For example, a 1° measure is a fraction of an entire rotation, and adding angle measures together is the same as adding fractions with a denominator of 360.

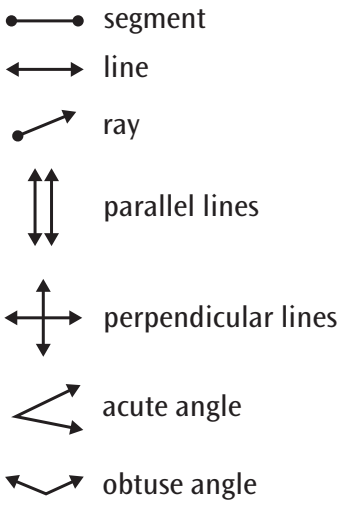
Before students solve word problems involving unknown angle measures (**4.MD.7**), they need to understand concepts of angle measure (**4.MD.5**) and, presumably, gain some experience measuring angles (**4.MD.6**). Students also need some familiarity with the geometric terms that are used to define angles as geometric shapes (**4.G.1**) [adapted from PARCC 2012].

Domain: Geometry

A critical area of instruction in grade four is for students to understand that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, perpendicular sides, particular angle measures, and symmetry.

Geometry	**4.G**
Draw and identify lines and angles, and classify shapes by properties of their lines and angles.	
1. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.	
2. Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles. (**Two-dimensional shapes should include special triangles, e.g., equilateral, isosceles, scalene, and special quadrilaterals, e.g., rhombus, square, rectangle, parallelogram, trapezoid.**) **CA**	
3. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.	

In grade four, students are exposed to the concepts of rays, angles, and perpendicular and parallel lines (**4.G.1**) for the first time. In addition, students classify figures based on the presence and absence of parallel or perpendicular lines and angles (**4.G.2**). It is helpful to provide students with a visual reminder of examples of points, line segments, lines, angles, parallelism, and perpendicularity. For example, a wall chart with the images shown at right could be displayed in the classroom.



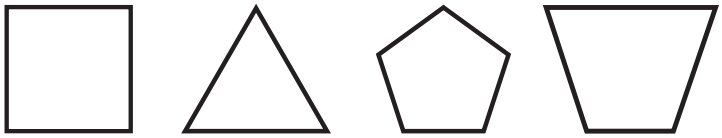
Students need to see all of these representations in different orientations. Students could draw these in different orientations and decide if all of the drawings are correct. They also need to see and draw the range of angles that are acute and obtuse.

Two-dimensional figures may be classified according to characteristics, such as the presence of parallel or perpendicular lines or by angle measurements. Students may use transparencies with lines drawn on them to arrange two lines in different ways to determine that the two lines might intersect at one point or might never intersect, thereby understanding the notion of parallel lines. Further investigations may be initiated with geometry software. These types of explorations may lead to a discussion on angles.

Students' prior experience with drawing and identifying right, acute, and obtuse angles helps them classify two-dimensional figures based on specified angle measurements. They use the benchmark angles of 90°, 180°, and 360° to approximate the measurement of angles. Right triangles (triangles with one right angle) can be a category for classification, with subcategories—for example, an isosceles right triangle has two or more congruent sides and a scalene right triangle has no congruent sides.

Examples: Classifying Shapes According to Attributes **4.G.2 (MP.3)**

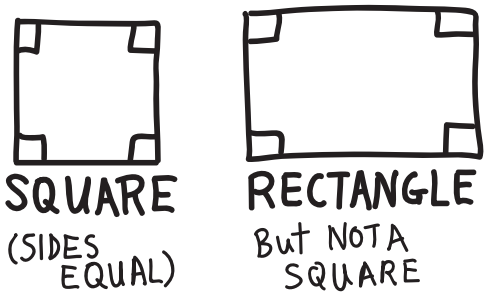
1. Identify which of the following shapes have perpendicular or parallel sides, and justify your selection(s).



Solution: "I know that pairs of lines are perpendicular if they cross to form square corners or right angles. Lines are parallel if they are always the same distance apart and never cross each other. I compared pairs of line segments in each of the four figures and found the first figure includes both parallel and perpendicular lines, and the last figure includes one pair of parallel lines. The other two figures do not include either perpendicular or parallel lines."

2. Explain why a square is considered a rectangle, but a rectangle is not necessarily a square.

Solution: "I know that rectangles are four-sided shapes that have four right angles. This makes any square a rectangle, since a square has four sides and four right angles also. But a square is a *special* kind of rectangle. What I mean is that you can have a rectangle that has its sides not all equal, and then it isn't a square. I drew examples to show what I mean."



Finally, students recognize a line of symmetry for a two-dimensional figure as a line across the figure, such that the figure can be folded along the line into matching parts (adapted from ADE 2010).

Essential Learning for the Next Grade

In kindergarten through grade five, the focus is on the addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals, with a balance of concepts, procedural skills, and problem solving. Arithmetic is viewed as an important set of skills and also as a thinking subject that, done thoughtfully, prepares students for algebra. Measurement and geometry develop alongside number and operations and are tied specifically to arithmetic along the way. Multiplication and division of whole numbers and fractions are instructional foci in grades three through five.

To be prepared for grade-five mathematics, students should be able to demonstrate that they have learned certain mathematical concepts and acquired procedural skills by the end of grade four and have met the fluency expectations for the grade level. For students in grade four, the expected fluencies are to add and subtract multi-digit whole numbers using the standard algorithm within 1,000,000 (**4.NBT.4▲**). These fluencies and the conceptual understandings that support them are foundational for work in later grades.

Of particular importance at grade four are concepts, skills, and understandings needed to use the four operations with whole numbers to solve problems (**4.OA.1–3▲**); generalize place-value understanding for multi-digit whole numbers (**4.NBT.1–3▲**); use place-value understanding and properties of operations to perform multi-digit arithmetic (**4.NBT.4–6▲**); extend understanding of fraction equivalence and ordering (**4.NF.1–2▲**); build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers (**4.NF.3–4▲**); and understand decimal notation for fractions and compare decimal fractions (**4.NF.5–7▲**).

Fractions

Fraction equivalence is an important theme in the standards. Understanding fraction equivalence is necessary to extend arithmetic from whole numbers to fractions and decimals. Students need to understand fraction equivalence and that $\frac{a}{b} = \frac{n \times a}{n \times b}$. They should be able to represent equivalent common fractions and apply this understanding to compare fractions and express their relationships using the symbols >, =, or <. Students understand how to represent and read fractions and mixed numbers in multiple ways.

Grade-four students should understand addition and subtraction with fractions having like denominators. This understanding represents a multi-grade progression, as students add and subtract fractions in grade four with like denominators by thinking of adding or subtracting so many unit fractions. Students should be able to solve word problems involving addition and subtraction of fractions that refer to the same whole and have like denominators (e.g., by using visual fraction models and equations to represent the problem). Students should understand how to add and subtract fractions and mixed numbers with like denominators.

Students further extend their understanding of multiplication to multiply fractions by whole numbers. To support their understanding, students should understand a fraction as the numerator times the unit fraction with the same denominator. Students should be able to rewrite fractions as multiples of the unit fraction of the same denominator, use a visual model to multiply a fraction by a whole number, and use equations to represent problems involving the multiplication of a fraction by a whole number by multiplying the whole number times the numerator.

Four Operations with Whole Numbers

By the end of grade four, students should fluently add and subtract multi-digit whole numbers to 1,000,000 using the standard algorithm. Students should also be able to use the four operations to solve multi-step word problems with whole-number remainders.

In grade four, students develop their understanding and skills with multiplication and division. They combine their understanding of the meanings and properties of multiplication and division with their understanding of base-ten units to begin to multiply and divide multi-digit numbers. Fourth-grade students should know how to express the product of two multi-digit numbers as another multi-digit number. They also should know how to find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors. Using a rectangular area model to represent multiplication and division helps students visualize these operations. This work will develop further in grade five and culminates in fluency with the standard algorithms in grade six.

Grade 4 Overview

Operations and Algebraic Thinking

- Use the four operations with whole numbers to solve problems.
- Gain familiarity with factors and multiples.
- Generate and analyze patterns.

Number and Operations in Base Ten

- Generalize place-value understanding for multi-digit whole numbers.
- Use place-value understanding and properties of operations to perform multi-digit arithmetic.

Number and Operations—Fractions

- Extend understanding of fraction equivalence and ordering.
- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
- Understand decimal notation for fractions, and compare decimal fractions.

Measurement and Data

- Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
- Represent and interpret data.
- Geometric measurement: understand concepts of angle and measure angles.

Geometry

- Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

4 Grade 4

Operations and Algebraic Thinking 4.OA

Use the four operations with whole numbers to solve problems.

1. Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.
2. Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.[7]
3. Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Gain familiarity with factors and multiples.

4. Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

Generate and analyze patterns.

5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.*

Number and Operations in Base Ten[8] 4.NBT

Generalize place-value understanding for multi-digit whole numbers.

1. Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. *For example, recognize that $700 \div 70 = 10$ by applying concepts of place value and division.*
2. Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.
3. Use place-value understanding to round multi-digit whole numbers to any place.

Use place-value understanding and properties of operations to perform multi-digit arithmetic.

4. Fluently add and subtract multi-digit whole numbers using the standard algorithm.

7. See glossary, table GL-5.

8. Grade-four expectations in this domain are limited to whole numbers less than or equal to 1,000,000.

5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

6. Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Number and Operations—Fractions[9] 4.NF

Extend understanding of fraction equivalence and ordering.

1. Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

2. Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as $1/2$. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.

3. Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

   a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

   b. Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. *Examples:* $3/8 = 1/8 + 1/8 + 1/8$; $3/8 = 1/8 + 2/8$; $2\,1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8$.

   c. Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.

   d. Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

4. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

   a. Understand a fraction a/b as a multiple of $1/b$. *For example, use a visual fraction model to represent $5/4$ as the product $5 \times (1/4)$, recording the conclusion by the equation $5/4 = 5 \times (1/4)$.*

   b. Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number. *For example, use a visual fraction model to express $3 \times (2/5)$ as $6 \times (1/5)$, recognizing this product as $6/5$. (In general, $n \times (a/b) = (n \times a)/b$.)*

9. Grade-four expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.

c. Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. *For example, if each person at a party will eat $\frac{3}{8}$ of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?*

Understand decimal notation for fractions, and compare decimal fractions.

5. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[10] *For example, express $\frac{3}{10}$ as $\frac{30}{100}$, and add $\frac{3}{10} + \frac{4}{100} = \frac{34}{100}$.*

6. Use decimal notation for fractions with denominators 10 or 100. *For example, rewrite 0.62 as $\frac{62}{100}$; describe a length as 0.62 meters; locate 0.62 on a number line diagram.*

7. Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using **the number line or another** visual model. **CA**

Measurement and Data 4.MD

Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.

1. Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. *For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), . . .*

2. Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

3. Apply the area and perimeter formulas for rectangles in real-world and mathematical problems. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*

Represent and interpret data.

4. Make a line plot to display a data set of measurements in fractions of a unit ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$). Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*

10. Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators in general is not a requirement at this grade.

Geometric measurement: understand concepts of angle and measure angles.

5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

   a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through $\frac{1}{360}$ of a circle is called a "one-degree angle," and can be used to measure angles.

   b. An angle that turns through n one-degree angles is said to have an angle measure of n degrees.

6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

7. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real-world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Geometry 4.G

Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

1. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

2. Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles. **(Two-dimensional shapes should include special triangles, e.g., equilateral, isosceles, scalene, and special quadrilaterals, e.g., rhombus, square, rectangle, parallelogram, trapezoid.) CA**

3. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

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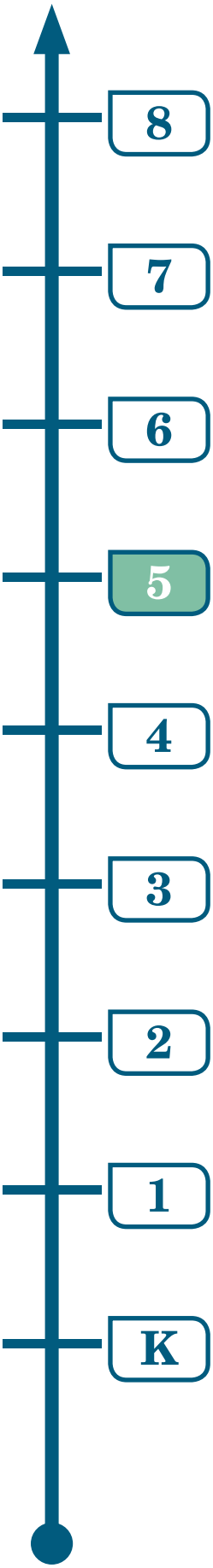


Grade Five

In the years prior to grade five, students learned strategies for multiplication and division, developed an understanding of the structure of the place-value system, and applied understanding of fractions to addition and subtraction with like denominators and to multiplying a whole number times a fraction. They gained understanding that geometric figures can be analyzed and classified based on the properties of the figures and focused on different measurements, including angle measures. Students also learned to fluently add and subtract whole numbers within 1,000,000 using the standard algorithm (adapted from Charles A. Dana Center 2012).

Critical Areas of Instruction

In grade five, instructional time should focus on three critical areas: (1) developing fluency with addition and subtraction of fractions and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); (2) extending division to two-digit divisors, integrating decimal fractions into the place-value system, developing understanding of operations with decimals to hundredths, and developing fluency with whole-number and decimal operations; and (3) developing understanding of volume (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010l). Students also fluently multiply multi-digit whole numbers using the standard algorithm.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade-level examples of focus, coherence, and rigor are indicated throughout the chapter.

The standards do not give equal emphasis to all content for a particular grade level. Cluster headings can be viewed as the most effective way to communicate the **focus** and **coherence** of the standards. Some clusters of standards require a greater instructional emphasis than others based on the depth of the ideas, the time needed to master those clusters, and their importance to future mathematics or the later demands of preparing for college and careers.

Table 5-1 highlights the content emphases at the cluster level for the grade-five standards. The bulk of instructional time should be given to "Major" clusters and the standards within them, which are indicated throughout the text by a triangle symbol (▲). However, standards in the "Additional/Supporting" clusters should not be neglected; to do so would result in gaps in students' learning, including skills and understandings they may need in later grades. Instruction should reinforce topics in major clusters by using topics in the additional/supporting clusters and including problems and activities that support natural connections between clusters.

Teachers and administrators alike should note that the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences presented in a coherent manner (adapted from Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

Table 5-1. Grade Five Cluster-Level Emphases

Operations and Algebraic Thinking — 5.OA

Additional/Supporting Clusters

- Write and interpret numerical expressions. (**5.OA.1–2**)
- Analyze patterns and relationships. (**5.OA.3**)

Number and Operations in Base Ten — 5.NBT

Major Clusters

- Understand the place-value system. (**5.NBT.1–4▲**)
- Perform operations with multi-digit whole numbers and with decimals to hundredths. (**5.NBT.5–7▲**)

Number and Operations—Fractions — 5.NF

Major Clusters

- Use equivalent fractions as a strategy to add and subtract fractions. (**5.NF.1–2▲**)
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions. (**5.NF.3–7▲**)

Measurement and Data — 5.MD

Major Clusters

- Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition. (**5.MD.3–5▲**)

Additional/Supporting Clusters

- Convert like measurement units within a given measurement system. (**5.MD.1**)
- Represent and interpret data. (**5.MD.2**)

Geometry — 5.G

Additional/Supporting Clusters

- Graph points on the coordinate plane to solve real-world and mathematical problems. (**5.G.1–2**)
- Classify two-dimensional figures into categories based on their properties. (**5.G.3–4**)

Explanations of Major and Additional/Supporting Cluster-Level Emphases

Major Clusters (▲) — Areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than others based on the depth of the ideas, the time needed to master them, and their importance to future mathematics or the demands of college and career readiness.

Additional Clusters — Expose students to other subjects; may not connect tightly or explicitly to the major work of the grade.

Supporting Clusters — Designed to support and strengthen areas of major emphasis.

Note of caution: Neglecting material, whether it is found in the major or additional/supporting clusters, will leave gaps in students' skills and understanding and will leave students unprepared for the challenges they face in later grades.

Adapted from Smarter Balanced Assessment Consortium 2011, 85.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (MP) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The MP standards represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students.

Although the description of the MP standards remains the same at all grades, the way these standards look as students engage with and master new and more advanced mathematical ideas does change. Table 5-2 presents examples of how the MP standards may be integrated into tasks appropriate for students in grade five. (Refer to the Overview of the Standards Chapters for a description of the MP standards.)

Table 5-2. Standards for Mathematical Practice—Explanation and Examples for Grade Five

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	In grade five, students solve problems by applying their understanding of operations with whole numbers, decimals, and fractions that include mixed numbers. They solve problems related to volume and measurement conversions. Students seek the meaning of a problem and look for efficient ways to represent and solve it. For example, "Sonia had $2\frac{1}{3}$ sticks of gum. She promised her brother that she would give him $\frac{1}{2}$ of a stick of gum. How much will she have left after she gives her brother the amount she promised?" Teachers can encourage students to check their thinking by having students ask themselves questions such as these: "What is the most efficient way to solve the problem?""Does this make sense?""Can I solve the problem in a different way?"
MP.2 Reason abstractly and quantitatively.	Students recognize that a number represents a specific quantity. They connect quantities to written symbols and create logical representations of problems, considering appropriate units and the meaning of quantities. They extend this understanding from whole numbers to their work with fractions and decimals. Teachers can support student reasoning by asking questions such as these: "What do the numbers in the problem represent?""What is the relationship of the quantities?" Students write simple expressions that record calculations with numbers and represent or round numbers using place-value concepts. For example, students use abstract and quantitative thinking to recognize, without calculating the quotient, that $0.5 \times (300 \div 15)$ is $\frac{1}{2}$ of $(300 \div 15)$.
MP.3 Construct viable arguments and critique the reasoning of others.	In grade five, students may construct arguments by using visual models such as objects and drawings. They explain calculations based upon models, properties of operations, and rules that generate patterns. They demonstrate and explain the relationship between volume and multiplication. They refine their mathematical communication skills as they participate in mathematical discussions involving questions such as "How did you get that?" and "Why is that true?" They explain their thinking to others and respond to others' thinking. Students use various strategies to solve problems, and they defend and justify their work to others. For example: "Two after-school clubs are having pizza parties. The teacher will order 3 pizzas for every 5 students in the math club and 5 equally sized pizzas for every 8 students on the student council. How much pizza will each student get at the respective parties? If a student wants to attend the party where she will get the most pizza (assuming the pizza is divided equally among the students at the parties), which party should she attend?"

Table 5-2 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
MP.4 Model with mathematics.	Fifth-grade students experiment with representing problem situations in multiple ways—for example, by using numbers, mathematical language, drawings, pictures, objects, charts, lists, graphs, and equations. Teachers might ask, "How would it help to create a diagram, chart, or table?" or "What are some ways to represent the quantities?" Students need opportunities to represent problems in various ways and explain the connections. Students in grade five evaluate their results in the context of the situation and explain whether answers to problems make sense. They evaluate the utility of models they see and draw and can determine which models are the most useful and efficient for solving particular problems.
MP.5 Use appropriate tools strategically.	Students consider available tools, including estimation, and decide which tools might help them solve mathematical problems. For instance, students may use unit cubes to fill a rectangular prism and then use a ruler to measure the dimensions to find a pattern for volume using the lengths of the sides. They use graph paper to accurately create graphs, solve problems, or make predictions from real-world data.
MP.6 Attend to precision.	Students continue to refine their mathematical communication skills by using clear and precise language in their discussions with others and in their own reasoning. Teachers might ask, "How do you know your solution is reasonable?" Students use appropriate terminology when they refer to expressions, fractions, geometric figures, and coordinate grids. Teachers might ask, "What symbols or mathematical notations are important in this problem?" Students are careful to specify units of measure and state the meaning of the symbols they choose. For instance, to determine the volume of a rectangular prism, students record their answers in cubic units.
MP.7 Look for and make use of structure.	Students look closely to discover a pattern or structure. For instance, they use properties of operations as strategies to add, subtract, multiply, and divide with whole numbers, fractions, and decimals. They examine numerical patterns and relate them to a rule or a graphical representation. Teachers might ask, "How do you know if something is a pattern?" or "What do you notice when _____?"
MP.8 Look for and express regularity in repeated reasoning.	Grade-five students use repeated reasoning to understand algorithms and make generalizations about patterns. Students connect place value and their prior work with operations to understand and use algorithms to extend multi-digit division from one-digit to two-digit divisors and to fluently multiply multi-digit whole numbers. They use various strategies to perform all operations with decimals to hundredths, and they explore operations with fractions with visual models and begin to formulate generalizations. Teachers might ask, "Can you explain how this strategy works in other situations?" or "Is this always true, sometimes true, or never true?"

Adapted from Arizona Department of Education (ADE) 2010 and North Carolina Department of Public Instruction 2013b.

Standards-Based Learning at Grade Five

The following narrative is organized by the domains in the Standards for Mathematical Content and highlights some necessary foundational skills from previous grade levels. It also provides exemplars to explain the content standards, highlight connections to Standards for Mathematical Practice (**MP**), and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. A triangle symbol (▲) indicates standards in the major clusters (see table 5-1).

Domain: Operations and Algebraic Thinking

To prepare for the progression of expressions and equations that occurs in the standards in grades six through eight, students in grade five begin working more formally with expressions.

Operations and Algebraic Thinking 5.OA

Write and interpret numerical expressions.

1. Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.
2. Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. *For example, express the calculation "add 8 and 7, then multiply by 2" as* $2 \times (8 + 7)$. *Recognize that* $3 \times (18932 + 921)$ *is three times as large as* $18932 + 921$, *without having to calculate the indicated sum or product.*

2.1 Express a whole number in the range 2-50 as a product of its prime factors. For example, find the prime factors of 24 and express 24 as $2 \times 2 \times 2 \times 3$. CA

In grade three, students began to use the conventional order of operations (i.e., multiplication and division are done before addition and subtraction). In grade five, students build on this work to write, interpret, and evaluate simple numerical expressions, including those that contain parentheses, brackets, or braces (ordering symbols) [**5.OA.1–2**]. Students need opportunities to describe numerical expressions without evaluating them. For example, they express the calculation "add 8 and 7, then multiply by 2" as $(8 + 7) \times 2$. Without calculating a sum or product, they recognize that $3 \times (18932 + 921)$ is three times as large as $18932 + 921$. Students begin to think about numerical expressions in anticipation of their later work with variable expressions—for example, three times an unknown length is $3 \times L$ (adapted from ADE 2010 and Kansas Association of Teachers of Mathematics [KATM] 2012, 5th Grade Flipbook).

Students need experiences with multiple expressions to understand when and how to use ordering symbols. Instruction in the order of operations should be carefully sequenced from simple to more complex problems. In grade five, this work should be viewed as exploratory rather than for attaining mastery; for example, expressions should not contain nested grouping symbols, and they should be no more complex than the expressions found in an application of the associative or distributive property, such as $(8 + 27) + 2$ or $(6 \times 30) + (6 \times 7)$ [adapted from the University of Arizona (UA) Progressions Documents for the Common Core Math Standards 2011a].

Students can begin by using these symbols with whole numbers and then expand the use to decimals and fractions.

Examples: Order of Operations—Use of Grouping Symbols	5.OA.1
Problems	***Answers***
$(28+16)\div 4$	The answer is 11. *Note:* If students arrive at 32 as their answer, they may have found $28+(16\div 4)$.
$12-(2\times 0.4)$	The answer is 11.2. *Note:* If students arrive at 4 as their answer, they may have found $(12-2)\times 0.4$.
$(2+3)\times(1.5-0.5)$	The answer is 5. *Note:* If students arrive at 6 as their answer, they may have found $2+3\times 1.5-0.5$, which yields 6 (based on order of operations without the parentheses).
$6-\left(\frac{1}{2}+\frac{1}{3}\right)$	The answer is $5\frac{1}{6}$. *Note:* If students arrive at $5\frac{5}{6}$ as their answer, they may have found $6-\frac{1}{2}+\frac{1}{3}$ (based on order of operations without the parentheses).

To further develop their understanding of grouping symbols and facility with operations, students place grouping symbols in equations to make the equations true or compare expressions that are grouped differently.

Examples	5.OA.1
Problems	***Answers***
Use grouping symbols to make the equation true: $15-7-2=10$	$15-(7-2)=10$
Use grouping symbols to make the equation true: $3\times 125\div 25+7=22$	$3\times(125\div 25)+7=22$
Compare $3\times 2+5$ and $3\times(2+5)$.	$3\times 2+5=11$ $3\times(2+5)=21$
Compare $15-6+7$ and $15-(6+7)$.	$15-6+7=16$ $15-(6+7)=2$

Common Misconceptions

- Students may believe that the order in which a problem with mixed operations is written is the correct order for solving the problem. The use of the mnemonic phrase "Please Excuse My Dear Aunt Sally" to remember the order of operations (**P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition, **S**ubtraction) may mislead students to always perform multiplication before division and addition before subtraction. To correct this thinking, students need to understand that they should work with the innermost grouping symbols first and that some operations are done before others, even if grouping symbols are not included. Multiplication and division are done at the same time (in order, from left to right). Addition and subtraction are also done at the same time (in order, from left to right).
- Students need a lot of experience with writing multiplication in different ways. Multiplication may be indicated with a raised dot (e.g., $4\cdot 5$), a raised cross symbol (e.g., 4×5), or parentheses (e.g., 4(5) or (4)(5)). Note that the raised cross symbol is not the same as the letter x and may be confused with the variable "x," so care should be taken when writing or typing this symbol. Students need to be exposed to all three notations and should be challenged to understand that all are useful. However, teachers are encouraged to use a consistent notation for instruction. Students also need help and practice remembering the convention that we write a rather than $1\times a$ or $1a$, especially in expressions such as $a+3a$.

Adapted from ADE 2010 and KATM 2012, 5th Grade Flipbook.

Understanding patterns is fundamental to algebraic thinking. Students extend their grade-four pattern work to include two numerical patterns that can be related, and they examine these relationships within sequences of ordered pairs.

Operations and Algebraic Thinking 5.OA

Analyze patterns and relationships.

3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. *For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.*

Students graph the ordered pairs to further examine the resulting pattern(s) [**5.OA.3**]. This work prepares students for studying proportional relationships and functions in middle school and is a precursor to work with slope and linear relationships (**5.G.1–2**).

Example **5.OA.3**

Create two sequences of numbers, both starting from 0, but one generated with a "+ 3" pattern, and the other with a "+ 6" pattern.

a. How are the sequences related to each other?

b. Graph the sequences together as ordered pairs, with the numbers in the first sequence (A) as the x-coordinate and the numbers in the second sequence (B) as the y-coordinate.

c. How are the sequences related based on the graph?

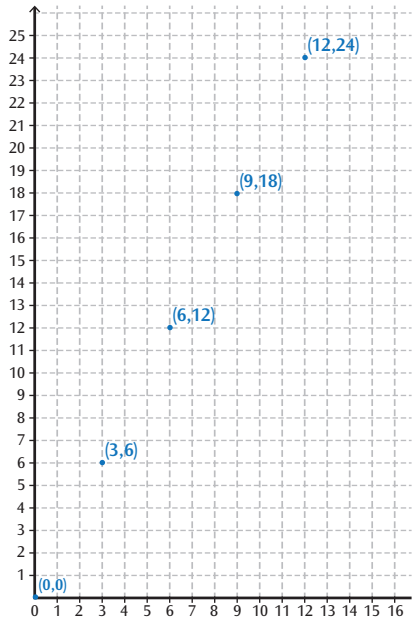
Solution:

Starting with 0, students create two sequences of numbers.

Sequence A:	0	3	6	9	12	15	…
Sequence B:	0	6	12	18	24	30	…

a. Students may notice that each term in sequence B is two times the corresponding term in sequence A. Organizing the sequences in a table (as shown above) can help students see the pattern more clearly. Students can explain the relationship between the sequences in several ways—for instance, by using the distributive property: $6+6+6=2\times(3+3+3)$

b. The ordered pairs come easily from the table layout: (0,0); (3,6); (6,12); (9,18); and so on. The graph is shown.



c. Students may see that the second coordinate of each point is two times the first coordinate—a natural observation based on the way the sequences were created. They may also see other features of the graph, such as the "+ 3" pattern moving in the x direction and the "+ 6" pattern moving in the y direction. (This is fully explored in grades six through eight.)

Adapted from ADE 2010 and KATM 2012, 5th Grade Flipbook.

Common Misconceptions

Students often reverse the order of the x, y pair when plotting them on a coordinate plane: they mistakenly count up first on the y-axis and then count over on the x-axis.

Domain: Number and Operations in Base Ten

In grade five, critical areas of instruction include integrating decimal fractions into the place-value system, developing an understanding of operations with decimals to hundredths, and working toward fluency with whole-number and decimal operations.

Number and Operations in Base Ten — 5.NBT

Understand the place-value system.

1. Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and $1/10$ of what it represents in the place to its left.
2. Explain patterns in the number of zeros in the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.
3. Read, write, and compare decimals to thousandths.
   a. Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., $347.392 = 3\times100 + 4\times10 + 7\times1 + 3\times\left(1/10\right) + 9\times\left(1/100\right) + 2\times\left(1/1000\right)$.
   b. Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.
4. Use place-value understanding to round decimals to any place.

Students extend their understanding of the base-ten system from whole numbers to decimals, focusing on the relationship between adjacent place values, how numbers compare, and how numbers round for decimals to thousandths. Before considering the relationship of decimal fractions, students reason that in multi-digit whole numbers, a digit in one place represents 10 times what it represents in the place to its right and $\frac{1}{10}$ of what it represents in the place to its left (**5.NBT.1▲**) [adapted from UA Progressions Documents 2012b].

Example **5.OA.3**

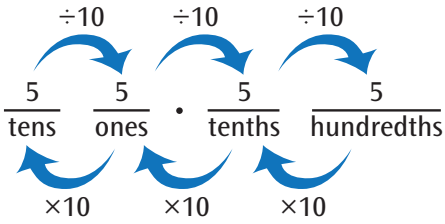
Through exploration with base-ten blocks or snap cubes, students can concretely explore the relationship between place values. They may be able to name place values, but this is not an indication that they understand the relationship between them. For example, a student may know the difference between the two 5s in the number 4554 (i.e., that they represent 500 and 50, respectively), but the further relationship that $500 = 50 \times 10$ and $50 = 500 \times \left(\frac{1}{10}\right)$ needs to be explored and made explicit.

To extend this understanding of place value to their work with decimals, students could use a model of one unit and cut it into 10 equal pieces, shade in, or describe $\frac{1}{10}$ of that model using fractional language:

"This is 1 out of 10 equal parts. So it is $\frac{1}{10}$. I can write this using $\frac{1}{10}$ or 0.1."

Students repeat the process by finding $\frac{1}{10}$ of $\frac{1}{10}$ (i.e., dividing $\frac{1}{10}$ into 10 equal parts to arrive at $\frac{1}{100}$ or 0.01) and explain their reasoning: "0.01 is $\frac{1}{10}$ of $\frac{1}{10}$ and therefore is $\frac{1}{100}$ of the whole unit."

Simple 10×10 grids can be very useful for exploring these ideas. Also, since the metric system is a base-ten system of measurement, working with simple metric length measurements and rulers can support this understanding (see standard **5.MD.1**). In general, students are led to recognize the following pattern in a multi-digit number:



Adapted from ADE 2010 and KATM 2012, 5th Grade Flipbook.

Students use place value to understand that multiplying a decimal by 10 results in the decimal point appearing one place to the right (e.g., $10 \times 4.2 = 42$), since the result is 10 times larger than the original number; similarly, multiplying a decimal by 100 results in the decimal point appearing two places to the right, because the number is 100 times larger. Students also make the connection that dividing by 10 results in the decimal point appearing one place to the left (e.g., $4 \div 10 = 0.4$), since the number is 10 times smaller (or $\frac{1}{10}$ of the original), and dividing a number by 100 results in the decimal point appearing two places to the left because the number is 100 times smaller (or $\frac{1}{100}$ of the original).

Focus, Coherence, and Rigor

The extension of the place-value system from whole numbers to decimals is a major accomplishment involving understanding and skill with base-ten units and fractions (**5.NBT.1▲**). As students understand that in a multi-digit number, a digit in one place represents $\frac{1}{10}$ of what it represents in the place to its left (**5.NBT.1▲**), they also reinforce their understanding of multiplying a quantity by a fraction (**5.NF.4▲**) [adapted from PARCC 2012].

Powers of 10 is a fundamental aspect of the base-ten system. Students extend their understanding of place value to explain patterns in the number of zeros of the product when multiplying a number by powers of 10, including the placement of the decimal point. The use of whole-number exponents to denote powers of 10 (**5.NBT.2▲**) is new to fifth-grade students.

Example	**5.NBT.2▲**
Students might write: $36 \times 10 = 36 \times 10^1 = 360$ $36 \times 10 \times 10 = 36 \times 10^2 = 3600$ $36 \times 10 \times 10 \times 10 = 36 \times 10^3 = 36{,}000$ $36 \times 10 \times 10 \times 10 \times 10 = 36 \times 10^4 = 360{,}000$	
Students might think or say: "I noticed that every time I multiplied by 10, I placed a zero at the end of the number. That makes sense because each digit's value became 10 times larger. To make a digit 10 times larger, I have to move it one place value to the left. When I multiplied 36 by 10, the 30 became 300. The 6 became 60 (or the 36 became 360)." Adapted from ADE 2010.	

Focus, Coherence, and Rigor

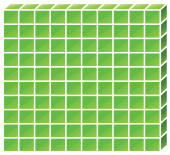
Students can use their understanding of the structure of whole numbers to generalize this understanding to decimals (**MP.7**) and explain the relationship between the numerals (**MP.6**) [adapted from Charles A. Dana Center 2012].

Students build on understandings from grade four to read, write, and compare decimals to thousandths (**5.NBT.3▲**). They connect this work with prior understanding of decimal notations for fractions and addition of fractions with denominators of 10 and 100. Students use concrete models or drawings and number lines to extend this understanding of decimals to the thousandths place. Models may include base-ten blocks, place-value charts, grids, pictures, math drawings, manipulatives, and examples created through technology. They read decimals using fractional language and write decimals in fractional form, as well as in expanded notation. This investigation leads them to understand the equivalence of decimals (e.g., 0.8 = 0.80 = 0.800).

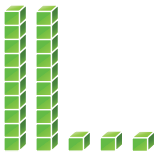
Example: Equivalent Forms of 0.72	**5.NBT.3a▲**
$\frac{72}{100}$	$\frac{70}{100}+\frac{2}{100}$
$\frac{7}{10}+\frac{2}{100}$	0.720
$7\times\frac{1}{10}+2\times\frac{1}{100}$	$7\times\frac{1}{10}+2\times\frac{1}{100}+0\times\frac{1}{1000}$
$0.70+0.02+0.000$	$\frac{720}{1000}$

Adapted from KATM 2012, 5th Grade Flipbook.

Base-ten blocks can be a powerful tool for seeing equivalent representations. For instance, if a "flat" is used to represent 1 (the whole or unit), then a "stick" represents $\frac{1}{10}$, and a small "cube" represents $\frac{1}{100}$. As shown below, students can be challenged to make sense of a number like 0.23 as being represented by both $\frac{2}{10}+\frac{3}{100}$ and $\frac{23}{100}$:

If  represents 1, then represents $\frac{1}{10}$ and represents $\frac{1}{100}$.

Teacher: "Explain why the following both represent the number 0.23."



Student: "Well, I see that the 20 hundredths in the picture on the right can be grouped into 2 sets of 10 hundredths. That means these 2 groups represent 2 tenths, or $\frac{2}{10}$. There are 3 hundredths left, so altogether there are $\frac{2}{10}+\frac{3}{100}$."

Students need to understand the size of decimal numbers and relate them to common benchmarks such as 0, 0.5 (0.50 and 0.500), and 1. Comparing tenths to tenths, hundredths to hundredths, and thousandths to thousandths is simplified if students use their understanding of fractions to compare decimals.

Example **5.NBT.3b▲**

Comparing 0.207 to 0.26, a student might think, "Both numbers have 2 tenths, so I need to compare the hundredths. The second number has 6 hundredths and the first number has no hundredths, so the second number must be larger."

While writing fractions, another student might think, "I know that 0.207 is 207 thousandths [and may write $\frac{207}{1000}$], and 0.26 is 26 hundredths [and may write $\frac{26}{100}$], but I can also think of it as 260 thousandths $\left(\frac{260}{1000}\right)$. So, 260 thousandths is more than 207 thousandths."

For students who are not able to read, write, and represent multi-digit numbers, working with decimals will be challenging. Teachers can use base-ten blocks and money to provide meaning for decimals. For example, dimes can represent tenths, pennies represent hundredths, and a penny circle with a $\frac{1}{10}$ sliver in it can represent thousandths.

Some students may be confused when reading decimals because whole numbers are read based on the place value of the digit farthest to the left of the decimal (e.g., 462 is read as *four hundred*

sixty-two). However, decimal numbers are read as whole numbers based on the place value of the digit farthest to the right of the decimal (e.g., 0.246 is read as *two hundred forty-six thousandths*). Decimals are read as fractions: the number is read as the numerator and then the denominator is expressed.

Common Misconceptions

Some students relate comparing decimals with the idea "the longer the number, the greater the number." With whole numbers, a five-digit number is always greater than a one-, two-, three-, or four-digit number. However, when comparing decimals, a number with one decimal place may be greater than a number with two or three decimal places.

Adapted from KATM 2012, 5th Grade Flipbook.

Students use place-value understanding to round decimals to any place (**5.NBT.4▲**). When rounding a decimal to a given place, students may identify two possible answers and use their understanding of place value to compare the given number to the possible answers.

Example: Round 14.235 to the nearest tenth. **5.NBT.4▲**

Students can read 14.235 as "14 and 235 thousandths." Since they are rounding to the nearest tenth, they are most likely rounding to either 14.2 or 14.3—that is, "14 and 200 thousandths" and "14 and 300 thousandths" (14.200 and 14.300). Students then see that they can momentarily disregard the 14 and focus on rounding 235 (thousandths) to the nearest hundred. In that case, since 235 would round down to 200, we would get 14.200 or 14.2 rounded to the nearest tenth.

14.2 14.3

Students can use benchmark numbers (e.g., 0, 0.5, 1, and 1.5) to support similar work.

Adapted from KATM 2012, 5th Grade Flipbook.

In grades three and four, students used various strategies to multiply. In grade five, students fluently multiply multi-digit whole numbers using the standard algorithm (**5.NBT.5▲**).

Number and Operations in Base Ten — 5.NBT

Perform operations with multi-digit whole numbers and with decimals to hundredths.

5. Fluently multiply multi-digit whole numbers using the standard algorithm.
6. Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties or operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Generally, the California Common Core State Standards for Mathematics distinguish between strategies and algorithms. In the present discussion, the *standard algorithm* refers to multiplying numbers digit by digit and recording the products piece by piece. Note that the method of recording the algorithm is not the same as the algorithm itself, in the sense that the "partial products" method, which lists every digit-by-digit product separately, is a completely valid recording method for the standard algorithm. Ultimately, the standards call for *understanding* the standard algorithm in terms of place value, and this should be the most important goal for instruction.

FLUENCY

California's Common Core State Standards for Mathematics (K–6) set expectations for fluency in computation (e.g., "*Fluently* multiply multi-digit whole numbers using the standard algorithm" [**5.NBT.5▲**]). Such standards are culminations of progressions of learning, often spanning several grades, involving conceptual understanding, thoughtful practice, and extra support where necessary. The word *fluent* is used in the standards to mean "reasonably fast and accurate" and possessing the ability to use certain facts and procedures with enough facility that using such knowledge does not slow down or derail the problem solver as he or she works on more complex problems. Procedural fluency requires skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Developing fluency in each grade may involve a mixture of knowing some answers, knowing some answers from patterns, and knowing some answers through the use of strategies.

Adapted from UA Progressions Documents 2011a.

In previous grades, students built a conceptual understanding of multiplication with whole numbers as they applied multiple strategies to compute and solve problems. Students can continue to use different strategies and methods learned previously—as long as the methods are efficient—but they must also understand and be able to use the standard algorithm.

Example: Find the product of 123×34. **5.NBT.5▲**

When students apply the standard algorithm, they decompose 34 into 30 + 4. Then they multiply 123 by 4, the value of the number in the ones place, and multiply 123 by 30, the value of the 3 in the tens place, and add the two products. The ways in which students are taught to record this method may vary, but all methods should emphasize the place-value nature of the algorithm. For example, a student might write:

$$\begin{array}{rl} 123 & \\ \times\ 34 & \\ \hline 492 & \leftarrow \text{this is the product of 4 and 123} \\ 3690 & \leftarrow \text{this is the product of 30 and 123} \\ \hline 4182 & \leftarrow \text{this is the sum of the two partial products} \end{array}$$

Note that a further decomposition of 123 into 100 + 20 + 3 and recording of the partial products would also be acceptable.

Adapted from ADE 2010.

In grade five, students use various strategies to extend division to include quotients of whole numbers with up to four-digit dividends and two-digit divisors, and they illustrate and explain calculations by using equations, rectangular arrays, and/or area models (**5.NBT.6▲**). When the two-digit divisor is a familiar number, students might use strategies based on place-value understanding.

Example 1: Find the quotient $2682 \div 25$. **5.NBT.6▲**

- Using expanded notation: $2682 \div 25 = (2000 + 600 + 80 + 2) \div 25$
- Using an understanding of the relationship between 100 and 25, a student might think:
  - "I know that 100 divided by 25 is 4, so 200 divided by 25 is 8 and 2000 divided by 25 is 80.
  - 600 divided by 25 has to be 24.
  - Since 3×25 is 75, I know that 80 divided by 25 is 3, with 5 left over. [Note that a student might divide into 82 and not 80.]
  - I can't divide 2 by 25, so 2 plus the 5 leaves a remainder of 7.
  - $80 + 24 + 3 = 107$. So, the answer is 107, with a remainder of 7."
- Using an equation that relates division to multiplication, $25 \times n = 2682$, a student might estimate the answer to be slightly larger than 100 by recognizing that $25 \times 100 = 2500$.

Adapted from ADE 2010.

To help students understand the use of place value when dividing with two-digit divisors, teachers can begin with simpler examples, such as having students divide 150 by 30; clearly, the answer is 5, since this is 15 tens divided by 3 tens. However, when dividing 1500 by 30, students need to think of this as 150 tens divided by 3 tens, which is 50. This illustrates why the 5 would go in the tens place of the quotient when using the division algorithm.

When the divisor is less familiar, students can use strategies based on area (as shown in the following example).

Example 2: Find the quotient $9984 \div 64$. **5.NBT.6▲**

An area model for division is shown below. As the student uses the area model, he or she keeps track of how much of the 9984 is left to divide.

Area model:

	64
100	6400
50	3200
5	320
1	64

Recording:

64	9984	
	−6400	(100×64)
	3584	
	−3200	(50×64)
	384	
	−320	(5×64)
	64	
	−64	(1×64)
	0	

Therefore the quotient is 100 + 50 + 5 + 1 = 156.

Adapted from ADE 2010.

The extension from one-digit divisors to two-digit divisors is a major milestone along the way to reaching fluency with the standard algorithm in grade six (**5.NBT.6▲**). Division strategies in grade five extend the methods learned in grade four to two-digit divisors. Students continue to break the dividend into base-ten units and find the quotient place by place, starting from the highest place. They illustrate and explain their calculations by using equations, rectangular arrays, and/or area models. Estimating the quotients is a difficult new aspect of dividing by a two-digit number. Even if students round appropriately, the resulting estimate may need to be adjusted up or down.

Focus, Coherence, and Rigor

When students break divisors and dividends into sums of multiples of base-ten units (**5.NBT.6▲**), they also develop important mathematical practices such as how to see and make use of structure (**MP.7**) and attend to precision (**MP.6**) [PARCC 2012].

In grade five, students expand on their grade-four work of comparing decimals and begin to add, subtract, multiply, and divide decimals to hundredths (**5.NBT.7▲**). They focus on reasoning about operations with decimals by using concrete models, math drawings, various strategies, and explanations. They also extend to decimal values the concrete models and written methods they developed for whole numbers in grades one through four. Students might estimate answers based on their understanding of operations and the value of the numbers (**MP.7, MP.8**).

Examples: Estimating	**5.NBT.7▲**
$3.6+1.7$ A student can make good use of rounding to estimate that since 3.6 rounds up to 4 and 1.7 rounds up to 2, the answer should be close to $4+2=6$.	
$5.4-0.8$ Students can again round and argue that since 5.4 rounds down to 5 and 0.8 rounds up to 1, the answer should be close to $5-1=4$.	
6×2.4 A student might estimate an answer between 12 and 18, since 6×2 is 12 and 6×3 is 18. Adapted from ADE 2010.	

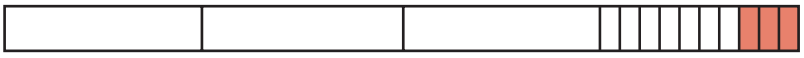
Students must understand and be able to explain that when adding decimals, they add tenths to tenths and hundredths to hundredths. When students add in a vertical format (numbers below each other), it is important that they write numbers with the same place value below each other. Students reinforce their understanding of adding decimals by connecting to prior understanding of adding fractions with denominators of 10 and 100 from grade four. They understand that when they add and subtract a whole number, the decimal point is at the end of the whole number. Students use various models to support their understanding of decimal operations.

Examples **5.NBT.7▲**

1. Model for decimal subtraction.

Solve 4 − 0.3. Explain how you found your solution.

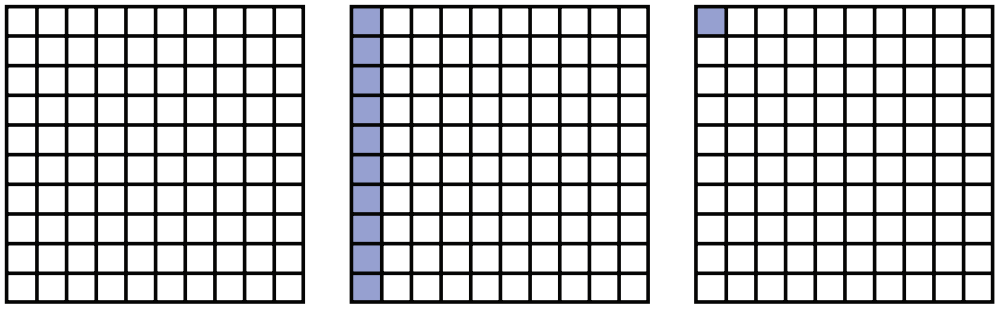
Solution: "Since I'm subtracting 3 tenths from 4 wholes, it would help to divide one of the wholes into tenths. The other 3 wholes don't need to be divided up. I can see there are 3 wholes and 7 tenths left over, or 3.7."



2. Use an area model to multiply unit fractions.

Demonstrate that $\frac{1}{10}$ of $\frac{1}{10}$ is $\frac{1}{100}$.

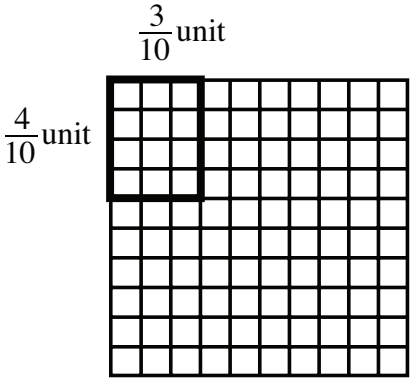
Solution: "If I use my 10×10 grid and set the whole grid equal to 1 square unit, then I can see that when each length of the grid is divided into 10 equal parts, each small square must represent a $\frac{1}{10}\times\frac{1}{10}$ square. But there are 100 of these small squares in the whole, so each little square must have an area of $\frac{1}{100}$ square units."



3. Use an area model to multiply fractions.

Demonstrate that $\frac{3}{10}\times\frac{4}{10}=\frac{12}{100}$.

Solution: "Just like in the previous problem, I use my 10×10 grid to represent 1 whole, with dimensions 1 unit by 1 unit. If I break up each side length into 10 equal parts, then I can create a smaller rectangle of dimensions 3 tenths of a unit by 4 tenths of a unit. It looks something like this:

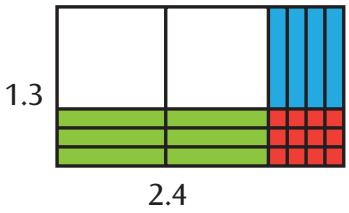


I know that each of the small squares is $\frac{1}{100}$ of a square unit, and I can see there are $3\times4=12$ of these small squares in the rectangle I outlined. This shows the answer is $\frac{12}{100}$." [See also **5.NF.4.**]

4. Use an area model to multiply decimals.

Show that 2.4 × 1.3 = 3.12.

Solution: "I drew a picture that shows a rectangle with dimensions of 1.3 units by 2.4 units. I know how to break up and keep track of smaller units, like tenths and hundredths. The partial products appear in my picture."

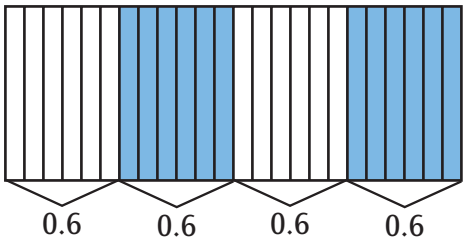


$$\begin{array}{r} 2.4 \\ \times\ 1.3 \\ \hline .12 \\ .60 \\ .40 \\ +\ 2.00 \\ \hline 3.12 \end{array}$$

5. Partitive ("fair-share") division model applied to decimals.

Solve 2.4 ÷ 4. Justify your answer.

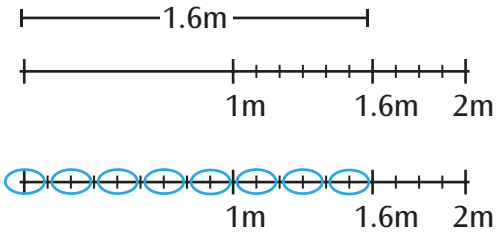
Solution: "My partner and I decided to think of this as fair-share division. We drew 2 wholes and 4 tenths and decided to break the wholes into tenths as well, since it would be easier to share them. When we tried to divide the total number of tenths into 4 equal parts, we got 0.6 in each part."



6. Quotitive ("measurement") division model applied to decimals.

Joe has 1.6 meters of rope. He needs to cut pieces of rope that are 0.2 meters long. How many pieces can he cut?

Solution: "We decided to draw a number line segment 2 units long and marked it to show 1.6 meters of rope—1 whole meter and 6 tenths of a meter. Since we need to count smaller ropes that are 0.2 meters in length, we decided to divide the 1 whole into tenths as well. Then it wasn't too hard to count that there are 8 pieces of 0.2-meter-long rope in his 1.6-meter rope."



Adapted from ADE 2010 and KATM 2012, 5th Grade Flipbook.

Domain: Number and Operations—Fractions

Student proficiency with fractions is essential to success in algebra in later grade levels. In grade five, a critical area of instruction is developing fluency with addition and subtraction of fractions, including adding and subtracting fractions with unlike denominators. Students also build an understanding of multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions).

Number and Operations—Fractions 5.NF

Use equivalent fractions as a strategy to add and subtract fractions.

1. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example,* $2/3 + 5/4 = 8/12 + 15/12 = 23/12$. *(In general,* $a/b + c/d = (ad + bc)/bd$.

2. Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases with unlike denominators, e.g., by using visual fraction models or equations to represent the problems. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. *For example, recognize an incorrect result* $2/5 + 1/2 = 3/7$, *by observing that* $3/7 < 1/2$.

In grade four, students learned to calculate sums of fractions with different denominators, where one denominator is a divisor of the other, so that only one fraction has to be changed. In grade five, students extend work with fractions to add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions with like denominators (**5.NF.1▲**) [adapted from UA Progressions Documents 2013a].

Students find a common denominator by finding the product of both denominators. For $\frac{1}{3} + \frac{1}{6}$, a common denominator is 18, which is the product of 3 and 6. This process should be introduced by using visual fraction models (area models, number lines, and so on) to build understanding before moving into the standard algorithm. Students should first solve problems that require changing one of the fractions (as in grade four) and progress to changing both fractions. Students understand that multiplying the denominators will always give a common denominator but may not result in the smallest denominator; however, it is not necessary to find a *least* common denominator to calculate sums and differences of fractions.

To add or subtract fractions with unlike denominators, students need to understand how to create equivalent fractions with the same denominators before adding or subtracting, a concept learned in grade four. In general, they understand that for any whole numbers a, b, and n, $\frac{a}{b} = \frac{n \times a}{n \times b}$ (given that n and b are non-zero).

Examples **5.NF.1▲**

$\frac{2}{5}+\frac{7}{8}=\frac{2}{5}\cdot\frac{8}{8}+\frac{7}{8}\cdot\frac{5}{5}=\frac{16}{40}+\frac{35}{40}=\frac{51}{40}$

$3\frac{1}{4}-\frac{1}{6}=3+\frac{1}{4}\cdot\frac{6}{6}-\frac{1}{6}\cdot\frac{4}{4}=3\frac{6}{24}-\frac{4}{24}=3\frac{2}{24}$ or $3\frac{1}{12}$

Adapted from UA Progressions Documents 2013a.

Using a variety of strategies, students make sense of fractional quantities when solving word problems involving addition and subtraction of fractions referring to the same whole (**5.NF.2▲**).

Example **5.NF.2▲**

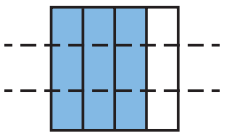
Jerry was making two different types of cookies. One recipe called for $\frac{3}{4}$ cup of sugar and the other called for $\frac{2}{3}$ cup of sugar. How much sugar did he need to make both recipes?

Solutions:

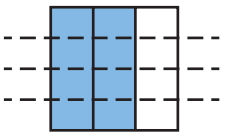
Mental estimation (MP.2). A student may say that Jerry needs more than 1 cup of sugar but less than 2 cups, because each fraction is larger than $\frac{1}{2}$ but less than 1.

Area model to show equivalence (MP.5). A student may choose to represent each partial cup of sugar with an area model, find equivalent fractions, and then add:

"I see that $\frac{3}{4}$ of a cup of sugar is equivalent to $\frac{9}{12}$ of a cup, while $\frac{2}{3}$ of a cup is equivalent to $\frac{8}{12}$ of a cup. Altogether, I have $\frac{17}{12}$ of a cup. This is more than one cup, since $\frac{17}{12}=\frac{12}{12}+\frac{5}{12}=1\frac{5}{12}$."



$\frac{3}{4}$ cup of sugar



$\frac{2}{3}$ cup of sugar

Adapted from ADE 2010.

Focus, Coherence, and Rigor

When students meet standard **5.NF.2▲**, they bring together the threads of fraction equivalence (learned in grades three through five) and addition and subtraction (learned in kindergarten through grade four) to fully extend addition and subtraction to fractions (adapted from PARCC 2012).

In grade four, students multiply a fraction by a whole number. In grade five, students build on this understanding to multiply and divide fractions by fractions.

Number and Operations—Fractions 5.NF

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

3. Interpret a fraction as division of the numerator by the denominator $\left(\frac{a}{b} = a \div b\right)$. Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. *For example, interpret $\frac{3}{4}$ as the result of dividing 3 by 4, noting that $\frac{3}{4}$ multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size $\frac{3}{4}$. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?*

4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

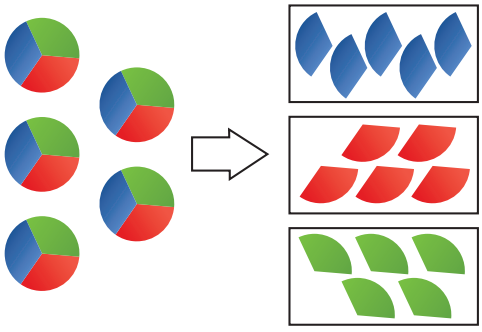
   a. Interpret the product $\left(\frac{a}{b}\right) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$. *For example, use a visual fraction model to show $\left(\frac{2}{3}\right) \times 4 = \frac{8}{3}$, and create a story context for this equation. Do the same with $\left(\frac{2}{3}\right) \times \left(\frac{4}{5}\right) = \frac{8}{15}$. (In general, $\left(\frac{a}{b}\right) \times \left(\frac{c}{d}\right) = \frac{ac}{bd}$.)*

   b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

In grade five, students connect fractions with division, understanding that $5 \div 3 = \frac{5}{3}$ or, more generally, that $a \div b = \frac{a}{b}$ for whole numbers a and b, with $b \neq 0$ (**5.NF.3▲**). Students can explain this by working with their understanding of division as equal sharing (e.g., Marissa has 5 carrots that she will share with three people. $5 \div 3 = \frac{5}{3}$, or $1\frac{2}{3}$ carrots).

Example **5.NF.3▲**

Divide 5 objects into three equal shares, showing that $5 \div 3 = 5 \times \frac{1}{3} = \frac{5}{3}$.



Solution: "If you divide 5 objects into 3 equal shares, each of the 5 objects should contribute $\frac{1}{3}$ of itself to each share. Thus each share consists of 5 pieces, and each of those pieces is $\frac{1}{3}$ of an object—so each share is $5 \times \frac{1}{3} = \frac{5}{3}$ of an object."

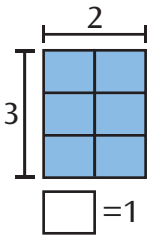
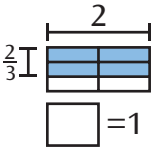
Adapted from UA Progressions Documents 2013a.

Students solve related word problems and demonstrate their understanding by using concrete materials, drawing models, and explaining their thinking when working with fractions in multiple contexts. Students read $\frac{3}{5}$ as *three-fifths* and, after experiences with sharing problems, they generalize that dividing 3 into 5 equal parts ($3 \div 5$ also written as $\frac{3}{5}$) results in the fraction $\frac{3}{5}$ (3 of 5 equal parts).

Students apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction (**5.NF.4▲**). They multiply fractions efficiently and accurately and solve problems in both contextual and non-contextual situations. Students reason about how to multiply fractions using fraction strips and number line diagrams. Using an understanding of multiplication by a fraction, students develop an understanding of a general formula for the product of two fractions: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

Examples	**5.NF.4a▲**
When students multiply fractions, such as in the problem $\frac{3}{5} \times 35$, they can think of the operation in more than one way: • As $3 \times (35 \div 5)$, or $3 \times \frac{35}{5}$. (This is equivalent to $3 \times \left(\frac{1}{5} \times 35\right)$ and expresses the idea in standard **5.NF.4.b▲**). • As $(3 \times 35) \div 5$, or $105 \div 5$. (This is equivalent to $\frac{105}{5}$.)	
Teachers may challenge students to write a story problem for this operation: "Mark's mother said he could have $\frac{3}{5}$ of the peanuts she bought for him and his younger brother to share. If she bought a bag of 35 peanuts, how many peanuts does Mark receive?"	

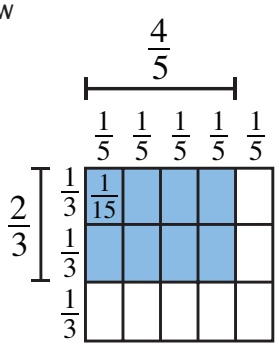
Building on previous understandings of multiplication, students find the area of a rectangle with fractional side lengths and represent fraction products as areas.

Examples of the Reasoning Called for in Standard 5.NF.4b▲	
Prior to grade five, students worked with examples of finding products as finding areas. In general, the factors in a multiplication problem represent the lengths of a rectangle and the product represents the area.	**Student:** "By counting the side lengths of this rectangle and the number of square units, I see that $2 \times 3 = 6$." 
When students move to examples such as $2 \times \frac{2}{3}$, they recognize that one side of a rectangle is less than a unit length (in this case, some sides have lengths that are mixed numbers). The idea of the picture is the same, but finding the area of the rectangle can be a little more challenging and requires reasoning about unit areas and the number of parts into which the unit areas are being divided.	**Student:** "I made a rectangle with sides of 2 units and $\frac{2}{3}$ of a unit. I can see that the 2-unit squares in the pictures are each divided into 3 equal parts (representing $\frac{1}{3}$), with two shaded in each unit square (4 total). That means that the total area of the shaded rectangle is $\frac{4}{3}$ square units." 

Examples of the Reasoning Called for in Standard 5.NF.4b▲ *(continued)*

Finally, when students move to examples such as $\frac{2}{3}\times\frac{4}{5}$, they see that the division of the side lengths into fractional parts creates a division of the unit area into fractional parts as well. Students will discover that the fractional parts of the unit area are related to the denominators of the original fractions. At right, a 1×1 square is divided into thirds in one direction and fifths in another. This results in the unit square itself being divided into fifteenths. This reasoning shows why $\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$.

Student: "I created a unit square and divided it into fifths in one direction and thirds in the other. This allows me to shade a rectangle of dimensions $\frac{2}{3}$ and $\frac{4}{5}$. I noticed that 15 of the new little rectangles make up the entire unit square, so they must be fifteenths $\left(\frac{1}{15}\right)$. Altogether, I had 2×4 of those fifteenths. So my answer is $\frac{8}{15}$."



Adapted from ADE 2010.

Focus, Coherence, and Rigor

When students meet standard **5.NF.4▲**, they fully extend multiplication to fractions, making division of fractions in grade six (**6.NS.1**) a near target.

Table 5-3 presents a sample classroom activity that connects the Standards for Mathematical Content and Standards for Mathematical Practice.

Table 5-3. Connecting to the Standards for Mathematical Practice—Grade Five

Standards Addressed	Explanation and Examples
Connections to Standards for Mathematical Practice **MP.1.** Students can be challenged to make sense out of each problem situation and to use their prior knowledge of fractions to try to model the situation and persevere in solving each problem. **MP.4.** Students use fractional representations to model simple, real-world situations. The real-world problems drive the mathematical concepts, which is the opposite approach of learning algorithms and later applying them.	**Task:** The following sequence of problems can be presented to students along with tools such as colored counters, rectangular and circular fraction pieces, fraction strips or rods, graph paper, and so forth. Students should be encouraged to use their tools to solve each problem before presenting algorithms for the computations involved. 1. There are 18 marbles in a box. Two-thirds of the marbles are red. How many red marbles are there? ***Solution:*** By seeing the 18 marbles as 3 sets of 6, we see that 2 × 6 = 12 marbles are red. Notice that we found thirds of 18 first $\left(\frac{1}{3}\times 18=6\right)$ and then decided we wanted two-thirds. Similar examples can be used to show that $\left(\frac{a}{b}\right)\times q=a\times\left(\frac{q}{b}\right)$.

Table continues on next page

Table 5-3 *(continued)*

MP.5. Students should have some familiarity with fraction models and have the opportunity to use them to solve problems and develop a conceptual understanding of fraction operations.

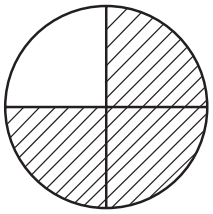
Standards for Mathematical Content

5.NF.4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

a. Interpret the product $\left(\frac{a}{b}\right) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$. *For example, use a visual fraction model to show* $\left(\frac{2}{3}\right) \times 4 = \frac{8}{3}$, *and create a story context for this equation. Do the same with* $\left(\frac{2}{3}\right) \times \left(\frac{4}{5}\right) = \frac{8}{15}$. *(In general,* $\left(\frac{a}{b}\right) \times \left(\frac{c}{d}\right) = \frac{ac}{bd}$.*)*

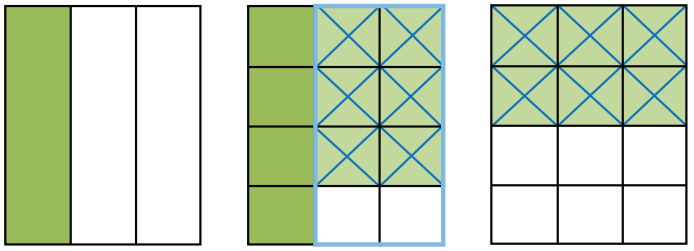
b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

2. Roberto had $\frac{3}{4}$ of a pizza left. He gave $\frac{1}{3}$ of the leftover pizza to his younger sister. How much of the whole pizza did his sister get?



Solution: Using the same reasoning as discussed above, and using pictures to support the reasoning, we can see that one-third of three-fourths is one-fourth, so that Roberto's sister got $\frac{1}{4}$ of a whole pizza.

3. Mr. Jones was mowing his lawn and had $\frac{2}{3}$ of the lawn left to cut before he had to answer a phone call. After the call, he finished $\frac{3}{4}$ of what he had left. How much of the lawn did Mr. Jones cut after the phone call?



Originally, he mowed $\frac{1}{3}$ of the yard.

Then he mowed $\frac{3}{4}$ of what was left.

After the call, what fraction of the whole lawn did he mow?

Solution: Here, we add the complication of finding fourths of thirds, which yields twelfths. In total, Mr. Jones has cut 6 of those twelfths, so the answer is $\frac{6}{12} = \frac{1}{2}$ of the lawn. This can be illustrated with rectangular fraction pictures (as shown). The lawn is first divided into thirds, one of which is shaded. Then the lawn is divided into fourths, and we notice that each of the small rectangular pieces represents $\frac{1}{12}$ of the entire lawn. Six of those are outlined in the illustration.

Classroom Connections. By using different fraction models to build students' understanding of fraction operations, teachers can help students lay a foundation for the algorithms that will follow. For example, eventually, students can attempt to justify the algorithm for multiplying fractions, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, by understanding that first $\frac{c}{d}$ can be divided into b equal parts; then, a of those parts are taken. In total, ac total parts of size $\frac{1}{bd}$ are taken.

Number and Operations—Fractions 5.NF

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

5. Interpret multiplication as scaling (resizing), by:
   a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.
   b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.
6. Solve real-world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

In preparation for grade-six work with ratios and proportional reasoning, fifth-grade students interpret multiplication as scaling (resizing) [**5.NF.5▲**] by examining how numbers change as the numbers are multiplied by fractions. Students should have ample opportunities to examine the following cases: (a) that when multiplying a number greater than 1 by a fraction greater than 1, the number increases; and (b) that when multiplying a number greater than 1by a fraction less than one, the number decreases. This is a new interpretation of multiplication that needs extensive exploration, discussion, and explanation by students.

Examples 5.NF.5▲
Student 1: "I know $\frac{3}{4} \times 7$ is less than 7, because I make 4 equal shares from 7, but I only take 3 of them ($\frac{3}{4}$ is a fractional part less than 1). If I'm taking a fractional part of 7 that is less than 1, the answer should be less than 7."
Student 2: "I know that $2\frac{3}{8} \times 8$ should be more than 16, because 2 groups of 8 are 16, and $2\frac{3}{8} > 2$. Also, I know the answer should be less than 3 × 8 or 24, since $2\frac{3}{8} < 3$."
Student 3: "I can show by equivalent fractions that $\frac{3}{4} = \frac{3 \times 5}{4 \times 5}$. I see that $\frac{5}{5} = 1$, so the result should still be equal to $\frac{3}{4}$."
Adapted from ADE 2010 and KATM 2012, 5th Grade Flipbook.

Students apply their understanding of multiplication of fractions and mixed numbers to solve real-world problems by using visual models or equations (**5.NF.6▲**).

Number and Operations—Fractions 5.NF

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[1]

   a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. *For example, create a story context for* $\left(\frac{1}{3}\right) \div 4$, *and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that* $\left(\frac{1}{3}\right) \div 4 = \frac{1}{12}$ *because* $\left(\frac{1}{12}\right) \times 4 = \frac{1}{3}$.

   b. Interpret division of a whole number by a unit fraction, and compute such quotients. *For example, create a story context for* $4 \div \left(\frac{1}{5}\right)$, *and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that* $4 \div \left(\frac{1}{5}\right) = 20$ *because* $20 \times \frac{1}{5} = 4$.

   c. Solve real-world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, how much chocolate will each person get if 3 people share* $\frac{1}{2}$ *lb of chocolate equally? How many* $\frac{1}{3}$*-cup servings are in 2 cups of raisins?*

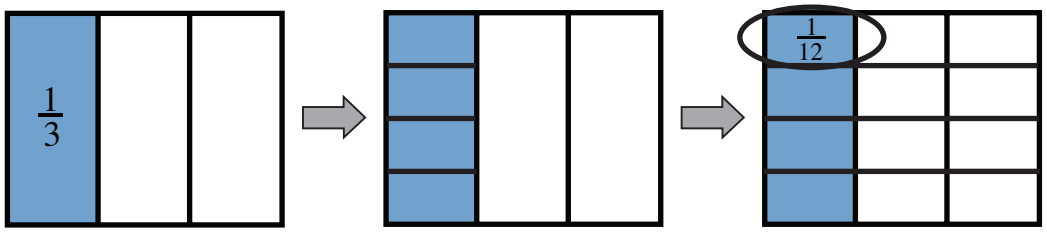
Students apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions (**5.NF.7▲**), a new concept at grade five. In grade six, students will extend their grade-five learning about division of fractions in simpler cases to the general case; division of a fraction by a fraction is not a requirement at grade five. Students in grade five use visual fraction models to show the quotient and solve related real-world problems.

Examples of the Reasoning Called for in Standard 5.NF.7▲

Partitive (fair-share) division for dividing a unit fraction by a whole number:

Four students sitting at a table were given $\frac{1}{3}$ of a pan of cornbread to share equally. What fraction of the whole pan of cornbread will each student get if they share the remaining cornbread equally?

Solution: The diagram shows the $\frac{1}{3}$ of a pan of cornbread divided into four equal shares. When replicated to fill out the entire pan, it becomes clear that each piece is $\frac{1}{12}$ of an entire pan. (If the $\frac{1}{3}$-sized pieces are each divided into 4 equal pieces, this makes a total of 12 equal pieces of the original whole.)



Students express their problem with an equation and relate it to their visual model: $\frac{1}{3} \div 4 = \frac{1}{12}$, which is the same as $\frac{1}{3} \times \frac{1}{4}$ (**MP.2, MP.4**).

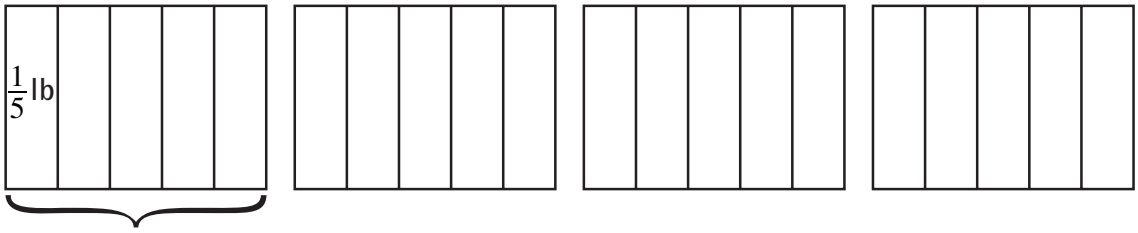
1. Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.

Examples of the Reasoning Called for in Standard 5.NF.7▲ *(continued)*

Quotitive (measurement) division for dividing a whole number by a unit fraction:

Angelo has 4 pounds of peanuts. He wants to give each of his friends $\frac{1}{5}$ of a pound. How many friends can receive $\frac{1}{5}$ of a pound of peanuts?

Solution: The question is asking how many $\frac{1}{5}$-pound groups are found in 4 (whole) pounds. This leads us to draw 4 wholes, divide each of them into pieces that are $\frac{1}{5}$ of a pound each, and count how many of these pieces are shown.



We see that there are 20 (twenty) $\frac{1}{5}$-pound portions in the original 4 pounds.

(Alternatively, a student may reason that since there are 5 [five] $\frac{1}{5}$-pound portions in each individual pound, there are $5 \times 4 = 20$ total. This reasoning lends itself to proportional reasoning in grades six and seven.)

Adapted from ADE 2010 and KATM 2012, 5th Grade Flipbook.

Domain: Measurement and Data

In grade five, another critical area of instruction is to develop an understanding of volume. Students recognize volume as an attribute of three-dimensional space. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume.

Measurement and Data	**5.MD**
Convert like measurement units within a given measurement system.	
1. Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real-world problems.	

Students in grade five build on prior knowledge from grade four to express measurements in larger or smaller units within a measurement system (**5.MD.1**). This provides an opportunity to reinforce notions of place value for whole numbers and decimals and connections between fractions and decimals (e.g., $2\frac{1}{2}$ meters may be expressed as 2.5 meters or 250 centimeters). Students use these conversions in solving multi-step, real-world problems (adapted from UA Progressions Documents 2012a).

Focus, Coherence, and Rigor

As fifth-grade students work with conversions in the metric system (**5.MD.1**), they experience practical applications of place-value understanding and reinforce major grade-level work in the cluster "Understand the place-value system" (**5.NBT.1▲**).

Measurement and Data 5.MD

Represent and interpret data.

2. Make a line plot to display a data set of measurements in fractions of a unit ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$). Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*

Students continue to extend their understanding of how to represent data, including fractional quantities from data in real-world situations.

Example **5.MD.2**

The line plot below shows the amount of liquid, in liters, in 10 beakers. If the liquid is redistributed equally, how much liquid would each beaker have? (This amount is the *mean*.)

Students apply their understanding of operations with fractions and use addition and/or multiplication to determine the total number of liters in the beakers. Then the sum of the liters is shared evenly among the 10 beakers. The graph shows the following as the total amount of liquid (in liters):

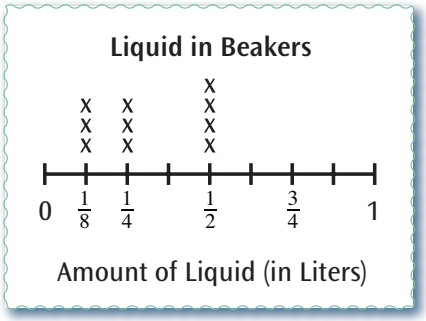
$$3\times\frac{1}{8}+3\times\frac{1}{4}+4\times\frac{1}{2}=\frac{3}{8}+\frac{3}{4}+\frac{4}{2}=\frac{25}{8}$$

If this $\frac{25}{8}$ liters of liquid is distributed among the 10 beakers, then there must be $\frac{25}{8}\div 10$ liters in each beaker.

Since $\frac{25}{8}\div 10=\frac{25}{8}\times\frac{1}{10}=\frac{25}{80}=\frac{5}{16}$, we see that each beaker would contain $\frac{5}{16}$ liters of liquid.

We can also represent the number of liters in each beaker with a decimal number: $\frac{25}{8}=3\frac{1}{8}=3.125$

$3.125\div 10=0.3125$ liter in each beaker.



Adapted from ADE 2010 and KATM 2012, 5th Grade Flipbook.

Focus, Coherence, and Rigor

As students solve real-world problems using operations on fractions based on information presented in line plots, they reinforce and support major grade-level work in the domain "Number and Operations—Fractions" (**5.NF**).

Measurement and Data 5.MD

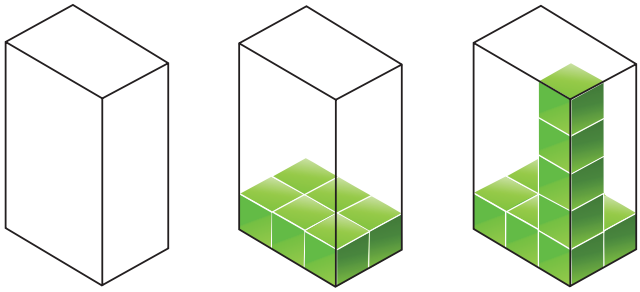
Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.

3. Recognize volume as an attribute of solid figures and understand concepts of volume measurement.
   a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.
   b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.
4. Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.
5. Relate volume to the operations of multiplication and addition and solve real-world and mathematical problems involving volume.
   a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.
   b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real-world and mathematical problems.
   c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding volumes of the non-overlapping parts, applying this technique to solve real-world problems.

Students develop an understanding of volume and relate volume to multiplication and addition. Volume introduces a third dimension, a significant challenge to some students' spatial structuring and also a complexity in the nature of the materials measured (**5.MD.3▲**). Solid units are "packed," such as cubes in a three-dimensional array, whereas a liquid "fills" three-dimensional space, taking the shape of the container. "Packing" volume is more difficult than area concepts in early grades. It may be simpler for students to think of volume as the number of cubes in n layers with a given area than to think of all three dimensions (adapted from PARCC 2012 and UA Progressions Documents 2012a).

Students learn about a unit of volume, such as a cube with a side length of 1 unit, called a *unit cube* (**5.MD.3▲**). They pack cubes (without gaps) into right rectangular prisms and count the cubes to determine the volume or build right rectangular prisms from cubes and see the layers as they build (**5.MD.4▲**). Students may also build up a rectangular prism with cubes to see the volume; it is easier to see the cubes in this method.

In grade three, students measured and estimated liquid volume and worked with area measurement. In grade five, the concept of volume can be developed by having students extend their prior work with area by covering the bottom of a cube with a layer of unit cubes and then adding layers of unit cubes on top of the bottom layer. For example:



- (3×2) represents the first layer
- $(3 \times 2) \times 5$ represents the number of 3×2 layers
- $(3 \times 2) + (3 \times 2) + (3 \times 2) + (3 \times 2) + (3 \times 2) = 6 + 6 + 6 + 6 + 6 = 30$ (6 represents the area of one layer)
- 30 represents the volume of the prism in cubic units

Adapted from KATM 2012, 5th Grade Flipbook.

Students can explore the concept of volume by filling containers with cubic units (cubes) to find the volume or by building up stacks of cubes without the containers. Students may also use drawings or interactive computer software to simulate this filling process. It is helpful for students to use concrete manipulatives before moving to pictorial representations.

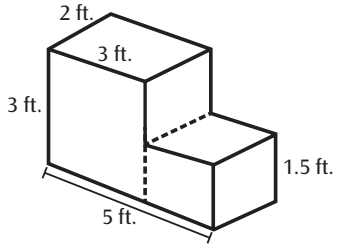
Students measure volume by filling rectangular prisms with cubes and looking at the relationship between the total volume and the area of the base. They derive the volume formula (volume equals the area of the base times the height) and explore how this idea would apply to other prisms. Students use the associative property of multiplication and decomposition of numbers using factors to investigate rectangular prisms with a given number of cubic units (**5.MD.5▲**).

Examples **5.MD.5▲**

Teachers give 24 "unit" cubes to students and ask them to make as many rectangular prisms as possible. Students build the prisms and record the dimensions as they build. It is important to note that there is a constant volume in this activity and that the product of the length, width, and height of each prism will always be 24.

Length	Width	Height
1	2	12
2	2	6
4	2	3
8	3	1

Teachers ask students to determine the volume of concrete needed to build the steps shown in the diagram at right (**5.MD.5c**).



Adapted from ADE 2010 and KATM 2012, 5th Grade Flipbook.

Focus, Coherence, and Rigor

When students show that the volume of a right rectangular prism is the same as would be found by multiplying the side lengths (**5.MD.5▲**), they also develop important mathematical practices such as looking for and expressing regularity in repeated reasoning (**MP.8**). They attend to precision (**MP.6**) as they use correct length or volume units, and they use appropriate tools strategically (**MP.5**) as they understand or make drawings to show these units.

Domain: Geometry

In grade five, students build on their previous work with number lines to use two perpendicular number lines to define a coordinate system (**5.G.1**). Students gain an understanding of the structure of the coordinate system. They learn that the two axes make it possible to locate points on a coordinate plane and that the names of the two axes and the coordinates correspond (i.e., x-axis and x-coordinate, y-axis and y-coordinate). This is the first time students work with coordinate planes, and at grade five this work is limited to the first quadrant.

Geometry 5.G

Graph points on the coordinate plane to solve real-world and mathematical problems.

1. Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate).
2. Represent real-world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

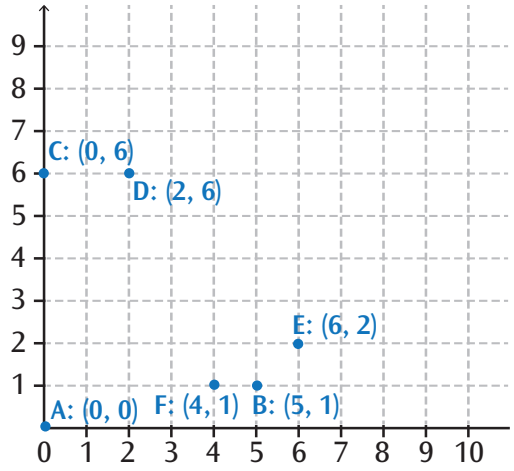
Students need opportunities to create a coordinate grid, connect ordered pairs of coordinates to points on the grid, and describe how to get to the location. For example, initially, the ordered pair (2, 3) could be described as a distance "2 from the origin along the x-axis and then 3 units up from the y-axis" or "right 2 and up 3." Another example follows.

Example **5.G.1**

Students might use a classroom-size coordinate system to physically locate coordinate points. For example, to locate the ordered pair (5, 3), students start at the origin point (0,0), then walk 5 units along the x-axis to find the first number in the pair (5), and then walk up 3 units for the second number in the pair (3). They continue this process to locate all the points in the following graph. Students recognize that ordered pairs name points in the plane.

Students graph and label the points below in a coordinate system.

A (0, 0)
B (5, 1)
C (0, 6)
D (2, 6)
E (6, 2)
F (4, 1)

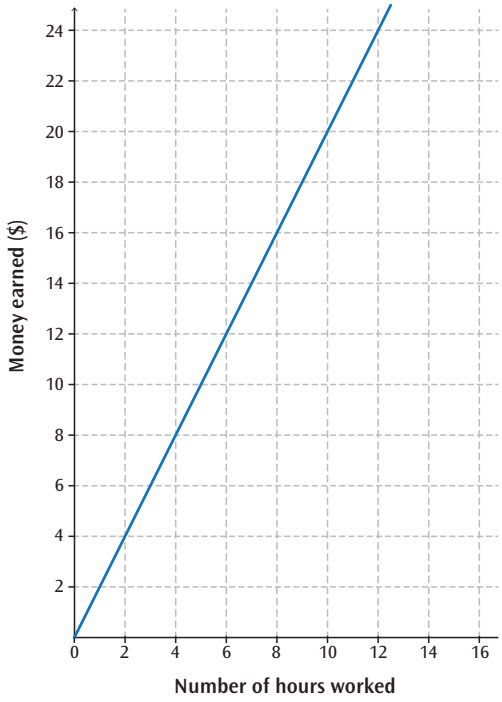


Students represent real-world and mathematical problems by graphing points in the first quadrant of the coordinate plane (**5.G.2**).

Example **5.G.2**

Use the following graph to determine how much allowance Jack makes after doing chores for exactly 10 hours.

Solution: "I can see that when I look up from the x-coordinate on the horizontal axis, the y-coordinate that matches up to it is 20. So Jack makes $20 if he does 10 hours of chores."



Adapted from ADE 2010 and KATM 2012, 5th Grade Flipbook.

Focus, Coherence, and Rigor

Students can connect their work with numerical patterns (**5.OA.3**) to form ordered pairs, graph these ordered pairs in the coordinate plane (**5.G.1–2**), and then use this model to make sense of and explain the relationships in the numerical patterns they generate. This work can help prepare students for future work with functions and proportional relations in the middle grades (adapted from Charles A. Dana Center 2012).

Common Misconceptions

Students may think the order in plotting a coordinate point is unimportant. To address this misconception, teachers can ask students to plot points with the coordinates switched. For example, referring to the graph from the previous example about Jack's allowance, students might locate points (4, 6) and (6, 4) and then discuss the order they used to locate the points and how the order might change the amount of earnings on the graph. Teachers should provide opportunities for students to realize the importance of direction and distance—for example, by having a student create directions for other students to follow as they plot points.

In prior years, students described and compared properties of two-dimensional shapes and built, drew, and analyzed these shapes. Fifth-grade students broaden their understanding to reason about the attributes *(properties)* of two-dimensional shapes and to classify these shapes in a hierarchy based on properties (**5.G.4**).

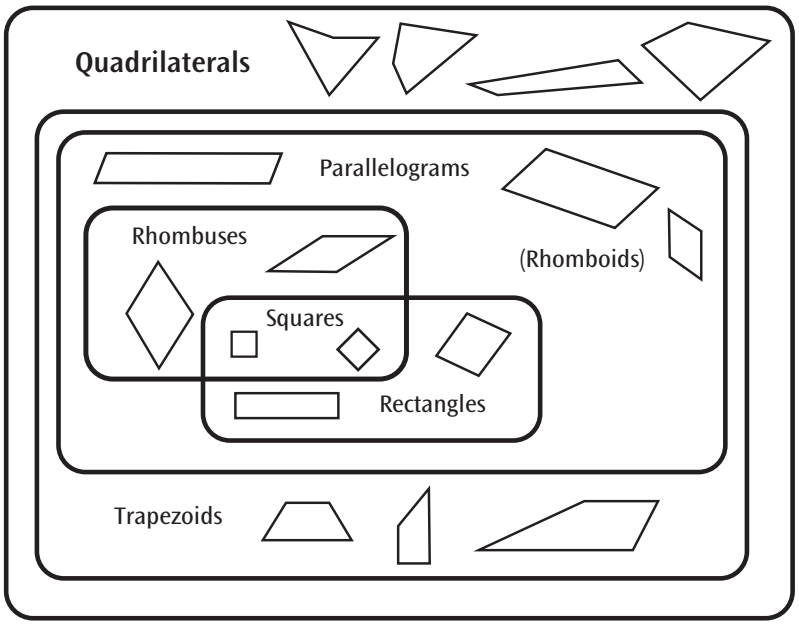
Geometry 5.G

Classify two-dimensional figures into categories based on their properties.

3. Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. *For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.*
4. Classify two-dimensional figures in a hierarchy based on properties.

Geometric properties include properties of sides (*parallel, perpendicular, congruent*), properties of angles (*type, measurement, congruent*), and properties of symmetry (*point, line*). For example, students conclude that all rectangles are parallelograms, because all rectangles are quadrilaterals with two pairs of opposite sides that are parallel and of equal length. In this way, students relate particular categories of shapes as subclasses of other categories (**5.G.3**); see figure 5-1.

Figure 5-1. Classification of Quadrilaterals



Source: UA Progressions Documents 2012c and KATM 2012, 5th Grade Flipbook.

Essential Learning for the Next Grade

In kindergarten through grade five, the focus is on the addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals, with a balance of concepts, procedural skills, and problem solving. Arithmetic is viewed as an important set of skills and also as a thinking subject that, done thoughtfully, prepares students for algebra. Measurement and geometry develop alongside number and operations and are tied specifically to arithmetic along the way. Multiplication and division of whole numbers and fractions are an instructional focus in grades three through five.

To be prepared for grade-six mathematics, students should be able to demonstrate they have acquired certain mathematical concepts and procedural skills by the end of grade five and have met the fluency expectations for the grade. For students in grade five, the expected fluency is to multiply multi-digit whole numbers (with up to four digits) using the standard algorithm (**5.NBT.5▲**). These fluencies and the conceptual understandings that support them are foundational for work in later grades.

Of particular importance at grade five are concepts, skills, and understandings needed to understand the place-value system (**5.NBT.1–4▲**); perform operations with multi-digit whole numbers and with decimals to hundredths (**5.NBT.5–7▲**); use equivalent fractions as a strategy to add and subtract fractions (**5.NF.1–2▲**); apply and extend previous understandings of multiplication and division to multiply and divide fractions (**5.NF.3–7▲**); and understand geometric measurement, including concepts of volume and how to relate volume to multiplication and addition (**5.MD.3–5▲**). In addition, graphing points on the coordinate plane to solve real-world and mathematical problems (**5.G.1–2**) is an important part of a student's progress toward algebra.

Fractions

Student proficiency with fractions is essential to success in later grades. By the end of grade five, students should be able to add, subtract, and multiply any two fractions and understand how to divide fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions).

Students should understand fraction equivalence and use their skills to generate equivalent fractions as a strategy to add and subtract fractions that have unlike denominators, including mixed fractions. Students should use these skills to solve related word problems. This understanding brings together the threads of fraction equivalence (emphasized in grades three through five) and addition and subtraction (emphasized in kindergarten through grade four) to fully extend addition and subtraction to fractions.

By the end of grade five, students know how to multiply a fraction or whole number by a fraction. Based on their understanding of the relationship between fractions and division, students divide any whole number by any non-zero whole number and express the answer in the form of a fraction or mixed number. Work with multiplying fractions extends from students' understanding of the operation of multiplication. For example, to multiply $\frac{a}{b} \times q$ (where q is a whole number or a fraction), students can interpret $\frac{a}{b} \times q$ as meaning a parts of a partition of q into b equal parts. This interpretation leads to a product that is less than, equal to, or greater than q, depending on whether $\frac{a}{b} < 1$, $\frac{a}{b} = 1$, or $\frac{a}{b} > 1$, respectively. In cases where $\frac{a}{b} < 1$, the result of multiplying contradicts earlier student experience with whole numbers, so this result needs to be explored, discussed, explained, and emphasized.

Fifth-grade students divide a unit fraction by a whole number or a whole number by a unit fraction. By the end of grade five, students should know how to multiply fractions to be prepared for division of a fraction by a fraction in grade six.

Decimals

In grade five, students integrate decimal fractions more fully into the place-value system as they learn to read, write, compare, and round decimals. By thinking about decimals as sums of multiples of base-ten units, students extend algorithms for multi-digit operations to decimals. By the end of grade five, students understand operations with decimals to hundredths. Students should understand how to add, subtract, multiply, and divide decimals to hundredths by using models, drawings, and various methods, including methods that extend from whole numbers and are explained by place-value meanings. The extension of the place-value system from whole numbers to decimals is a major accomplishment for a student that involves both understanding and skill with base-ten units and fractions. Skill and understanding with adding, subtracting, multiplying, and dividing multi-digit decimals will culminate in fluency with the standard algorithm in grade six.

Fluency with Whole-Number Operations

In grade five, the fluency expectation is to multiply multi-digit whole numbers using the standard algorithm: one-digit numbers multiplied by a number with up to four digits and two-digit numbers multiplied by two-digit numbers. Students also extend their grade-four work in finding whole-number

quotients and remainders to the case of two-digit divisors. Skill and understanding of division with multi-digit whole numbers will culminate in fluency with the standard algorithm in grade six.

Volume

Students in grade five work with volume as an attribute of a solid figure and as a measurement quantity. They also relate volume to multiplication and addition. Students' understanding and skill with this work support a learning progression that leads to valuable skills in geometric measurement in middle school.

Grade 5 Overview

Operations and Algebraic Thinking

- Write and interpret numerical expressions.
- Analyze patterns and relationships.

Number and Operations in Base Ten

- Understand the place-value system.
- Perform operations with multi-digit whole numbers and with decimals to hundredths.

Number and Operations—Fractions

- Use equivalent fractions as a strategy to add and subtract fractions.
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Measurement and Data

- Convert like measurement units within a given measurement system.
- Represent and interpret data.
- Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.

Geometry

- Graph points on the coordinate plane to solve real-world and mathematical problems.
- Classify two-dimensional figures into categories based on their properties.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

5 Grade 5

Operations and Algebraic Thinking 5.0A

Write and interpret numerical expressions.

1. Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.
2. Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. *For example, express the calculation "add 8 and 7, then multiply by 2" as* $2 \times (8 + 7)$. *Recognize that* $3 \times (18932 + 921)$ *is three times as large as* $18932 + 921$, *without having to calculate the indicated sum or product.*

2.1 Express a whole number in the range 2–50 as a product of its prime factors. For example, find the prime factors of 24 and express 24 as $2 \times 2 \times 2 \times 3$. CA

Analyze patterns and relationships.

3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. *For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.*

Number and Operations in Base Ten 5.NBT

Understand the place-value system.

1. Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and $1/10$ of what it represents in the place to its left.
2. Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.
3. Read, write, and compare decimals to thousandths.
   a. Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., $347.392 = 3 \times 100 + 4 \times 10 + 7 \times 1 + 3 \times \left(1/10\right) + 9 \times \left(1/100\right) + 2 \times \left(1/1000\right)$.
   b. Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.
4. Use place-value understanding to round decimals to any place.

Perform operations with multi-digit whole numbers and with decimals to hundredths.

5. Fluently multiply multi-digit whole numbers using the standard algorithm.
6. Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; *relate* the strategy to a written method and explain the reasoning used.

Number and Operations—Fractions 5.NF

Use equivalent fractions as a strategy to add and subtract fractions.

1. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example,* $\frac{2}{3}+\frac{5}{4}=\frac{8}{12}+\frac{15}{12}=\frac{23}{12}$. *(In general,* $\frac{a}{b}+\frac{c}{d}=\frac{(ad+bc)}{bd}$.*)*

2. Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. *For example, recognize an incorrect result* $\frac{2}{5}+\frac{1}{2}=\frac{3}{7}$*, by observing that* $\frac{3}{7}<\frac{1}{2}$.

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

3. Interpret a fraction as division of the numerator by the denominator $\left(\frac{a}{b}=a\div b\right)$. Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. *For example, interpret* $\frac{3}{4}$ *as the result of dividing 3 by 4, noting that* $\frac{3}{4}$ *multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size* $\frac{3}{4}$*. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?*

4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

   a. Interpret the product $\left(\frac{a}{b}\right)\times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a\times q\div b$. *For example, use a visual fraction model to show* $\left(\frac{2}{3}\right)\times 4=\frac{8}{3}$*, and create a story context for this equation. Do the same with* $\left(\frac{2}{3}\right)\times\left(\frac{4}{5}\right)=\frac{8}{15}$*. (In general,* $\left(\frac{a}{b}\right)\times\left(\frac{c}{d}\right)=\frac{ac}{bd}$.*)*

   b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

5. Interpret multiplication as scaling (resizing), by:

   a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

   b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $\frac{a}{b}=\frac{(n\times a)}{(n\times b)}$ to the effect of multiplying $\frac{a}{b}$ by 1.

6. Solve real-world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[2]

2. Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.

a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. *For example, create a story context for* $\left(\frac{1}{3}\right) \div 4$, *and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that* $\left(\frac{1}{3}\right) \div 4 = \frac{1}{12}$ *because* $\left(\frac{1}{12}\right) \times 4 = \frac{1}{3}$.

b. Interpret division of a whole number by a unit fraction, and compute such quotients. *For example, create a story context for* $4 \div \left(\frac{1}{5}\right)$, *and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that* $4 \div \left(\frac{1}{5}\right) = 20$ *because* $20 \times \frac{1}{5} = 4$.

c. Solve real-world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, how much chocolate will each person get if 3 people share* $\frac{1}{2}$ *lb of chocolate equally? How many* $\frac{1}{3}$*-cup servings are in 2 cups of raisins?*

Measurement and Data 5.MD

Convert like measurement units within a given measurement system.

1. Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real-world problems.

Represent and interpret data.

2. Make a line plot to display a data set of measurements in fractions of a unit ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$). Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*

Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.

3. Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

   a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.

   b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.

4. Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

5. Relate volume to the operations of multiplication and addition and solve real-world and mathematical problems involving volume.

   a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

   b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real-world and mathematical problems.

c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real-world problems.

Geometry 5.G

Graph points on the coordinate plane to solve real-world and mathematical problems.

1. Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate).

2. Represent real-world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

Classify two-dimensional figures into categories based on their properties.

3. Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. *For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.*

4. Classify two-dimensional figures in a hierarchy based on properties.

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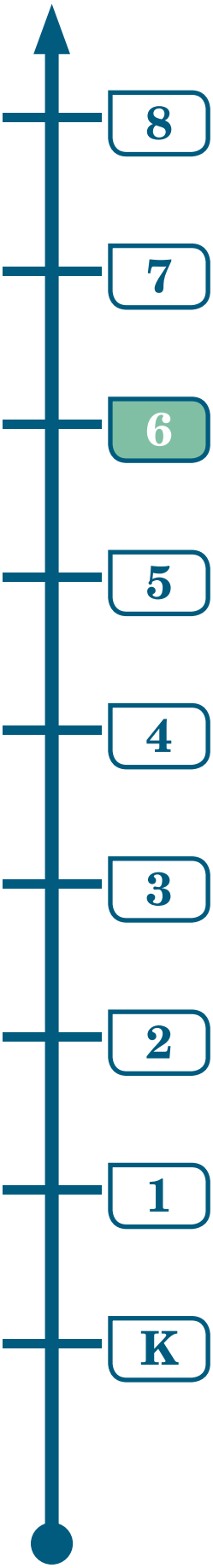


Grade Six

Students in grade six build on a strong foundation to prepare for higher mathematics. Grade six is an especially important year for bridging the concrete concepts of arithmetic and the abstract thinking of algebra (Arizona Department of Education [ADE] 2010). In previous grades, students built a foundation in number and operations, geometry, and measurement and data. When students enter grade six, they are fluent in addition, subtraction, and multiplication with multi-digit whole numbers and have a solid conceptual understanding of all four operations with positive rational numbers, including fractions. Students at this grade level have begun to understand measurement concepts (e.g., length, area, volume, and angles), and their knowledge of how to represent and interpret data is emerging (adapted from Charles A. Dana Center 2012).

Critical Areas of Instruction

In grade six, instructional time should focus on four critical areas: (1) connecting ratio, rate, and percentage to whole-number multiplication and division and using concepts of ratio and rate to solve problems; (2) completing understanding of division of fractions and extending the notion of *number* to the system of rational numbers, which includes negative numbers; (3) writing, interpreting, and using expressions and equations; and (4) developing understanding of statistical thinking (National Governors Association Center for Best Practices, Council of Chief State School Officers 2010m). Students also work toward fluency with multi-digit division and multi-digit decimal operations.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade-level examples of focus, coherence, and rigor are indicated throughout the chapter.

The standards do not give equal emphasis to all content for a particular grade level. Cluster headings can be viewed as the most effective way to communicate the **focus** and **coherence** of the standards. Some clusters of standards require a greater instructional emphasis than others based on the depth of the ideas, the time needed to master those clusters, and their importance to future mathematics or the later demands of preparing for college and careers.

Table 6-1 highlights the content emphases at the cluster level for the grade-six standards. The bulk of instructional time should be given to "Major" clusters and the standards within them, which are indicated throughout the text by a triangle symbol (▲). However, standards in the "Additional/Supporting" clusters should not be neglected; to do so would result in gaps in students' learning, including skills and understandings they may need in later grades. Instruction should reinforce topics in major clusters by using topics in the additional/supporting clusters and including problems and activities that support natural connections between clusters.

Teachers and administrators alike should note that the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences presented in a coherent manner (adapted from Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

Table 6-1. Grade Six Cluster-Level Emphases

Ratios and Proportional Relationships — 6.RP

Major Clusters

- Understand ratio concepts and use ratio reasoning to solve problems. (**6.RP.1–3▲**)

The Number System — 6.NS

Major Clusters

- Apply and extend previous understandings of multiplication and division to divide fractions by fractions. (**6.NS.1▲**)
- Apply and extend previous understandings of numbers to the system of rational numbers. (**6.NS.5–8▲**)

Additional/Supporting Clusters

- Compute fluently with multi-digit numbers and find common factors and multiples. (**6.NS.2–4**)

Expressions and Equations — 6.EE

Major Clusters

- Apply and extend previous understandings of arithmetic to algebraic expressions. (**6.EE.1–4▲**)
- Reason about and solve one-variable equations and inequalities. (**6.EE.5–8▲**)
- Represent and analyze quantitative relationships between dependent and independent variables. (**6.EE.9▲**)

Geometry — 6.G

Additional/Supporting Clusters

- Solve real-world and mathematical problems involving area, surface area, and volume. (**6.G.1–4**)

Statistics and Probability — 6.SP

Additional/Supporting Clusters

- Develop understanding of statistical variability. (**6.SP.1–3**)
- Summarize and describe distributions. (**6.SP.4–5**)

Explanations of Major and Additional/Supporting Cluster-Level Emphases

Major Clusters (▲) — Areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than others based on the depth of the ideas, the time needed to master them, and their importance to future mathematics or the demands of college and career readiness.

Additional Clusters — Expose students to other subjects; may not connect tightly or explicitly to the major work of the grade.

Supporting Clusters — Designed to support and strengthen areas of major emphasis.

Note of caution: Neglecting material, whether it is found in the major or additional/supporting clusters, will leave gaps in students' skills and understanding and will leave students unprepared for the challenges they face in later grades.

Adapted from Smarter Balanced Assessment Consortium 2012b.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (MP) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The MP standards represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students.

Although the description of the MP standards remains the same at all grade levels, the way these standards look as students engage with and master new and more advanced mathematical ideas does change. Table 6-2 presents examples of how the MP standards may be integrated into tasks appropriate for students in grade six. (Refer to the Overview of the Standards Chapters for a description of the MP standards.)

Table 6-2. Standards for Mathematical Practice—Explanation and Examples for Grade Six

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	In grade six, students solve real-world problems through the application of algebraic and geometric concepts. These problems involve ratio, rate, area, and statistics. Students seek the meaning of a problem and look for efficient ways to represent and solve it. They may check their thinking by asking themselves questions such as these: "What is the most efficient way to solve the problem?" "Does this make sense?" "Can I solve the problem in a different way?" Students can explain the relationships between equations, verbal descriptions, and tables and graphs. Mathematically proficient students check their answers to problems using a different method.
MP.2 Reason abstractly and quantitatively.	Students represent a wide variety of real-world contexts by using rational numbers and variables in mathematical expressions, equations, and inequalities. Students contextualize to understand the meaning of the number or variable as related to the problem and decontextualize to operate with symbolic representations by applying properties of operations or other meaningful moves. To reinforce students' reasoning and understanding, teachers might ask, "How do you know?" or "What is the relationship of the quantities?"
MP.3 Construct viable arguments and critique the reasoning of others.	Students construct arguments with verbal or written explanations accompanied by expressions, equations, inequalities, models, graphs, tables, and other data displays (e.g., box plots, dot plots, histograms). They further refine their mathematical communication skills through mathematical discussions in which they critically evaluate their own thinking and the thinking of other students. They pose questions such as these: "How did you get that?" "Why is that true?" "Does that always work?" They explain their thinking to others and respond to others' thinking.
MP.4 Model with mathematics	In grade six, students model problem situations symbolically, graphically, in tables, contextually, and with drawings of quantities as needed. Students form expressions, equations, or inequalities from real-world contexts and connect symbolic and graphical representations. They begin to explore covariance and represent two quantities simultaneously. Students use number lines to compare numbers and represent inequalities. They use measures of center and variability and data displays (e.g., box plots and histograms) to draw inferences about and make comparisons between data sets. Students need many opportunities to make sense of and explain the connections between the different representations. They should be able to use any of these representations, as appropriate, and apply them to a problem context. Students should be encouraged to answer questions such as "What are some ways to represent the quantities?" or "What formula might apply in this situation?"

Table 6-2 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
MP.5 Use appropriate tools strategically.	When solving a mathematical problem, students consider available tools (including estimation and technology) and decide when particular tools might be helpful. For instance, students in grade six may decide to represent figures on the coordinate plane to calculate area. Number lines are used to create dot plots, histograms, and box plots to visually compare the center and variability of the data. Visual fraction models can be used to represent situations involving division of fractions. Additionally, students might use physical objects or applets to construct nets and calculate the surface area of three-dimensional figures. Students should be encouraged to answer questions such as "What approach did you try first?" or "Why was it helpful to use _____?"
MP.6 Attend to precision.	Students continue to refine their mathematical communication skills by using clear and precise language in their discussions with others and in their own reasoning. Students use appropriate terminology when referring to rates, ratios, geometric figures, data displays, and components of expressions, equations, or inequalities. When using ratio reasoning in solving problems, students are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem. Students also learn to express numerical answers with an appropriate degree of precision when working with rational numbers in a situational problem. Teachers might ask, "What mathematical language, definitions, or properties can you use to explain _____?"
MP.7 Look for and make use of structure.	Students routinely seek patterns or structures to model and solve problems. For instance, students notice patterns that exist in ratio tables, recognizing both the additive and multiplicative properties. Students apply properties to generate equivalent expressions (e.g., $6 + 3x = 3(2 + x)$ by the distributive property) and solve equations (e.g., $2c + 3 = 15$, $2c = 12$ by the subtraction property of equality, $c = 6$ by the division property of equality). Students compose and decompose two- and three-dimensional figures to solve real-world problems involving area and volume. Teachers might ask, "What do you notice when _____?" or "What parts of the problem might you eliminate, simplify, or _____?"
MP.8 Look for and express regularity in repeated reasoning.	In grade six, students use repeated reasoning to understand algorithms and make generalizations about patterns. During opportunities to solve and model problems designed to support generalizing, they notice that $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ and construct other examples and models that confirm their generalization. Students connect place value and their prior work with operations to understand algorithms to fluently divide multi-digit numbers and perform all operations with multi-digit decimals. Students informally begin to make connections between covariance, rates, and representations that show the relationships between quantities. Students should be encouraged to answer questions such as, "How would we prove that _____?" or "How is this situation like and different from other situations?"

Adapted from ADE 2010, North Carolina Department of Public Instruction (NCDPI) 2013b, and Georgia Department of Education (GaDOE) 2011.

Standards-Based Learning at Grade Six

The following narrative is organized by the domains in the Standards for Mathematical Content and highlights some necessary foundational skills from previous grade levels. It also provides exemplars to explain the content standards, highlight connections to Standards for Mathematical Practice (**MP**), and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. A triangle symbol (▲) indicates standards in the major clusters (see table 6-1).

Domain: Ratios and Proportional Relationships

A critical area of instruction in grade six is to connect ratio, rate, and percentage to whole-number multiplication and division and use concepts of ratio and rate to solve problems. Students' prior understanding of and skill with multiplication, division, and fractions contribute to their study of ratios, proportional relationships, unit rates, and percentage in grade six. In grade seven, these concepts will extend to include scale drawings, slope, and real-world percent problems.

Ratios and Proportional Relationships — 6.RP

Understand ratio concepts and use ratio reasoning to solve problems.

1. Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*
2. Understand the concept of a unit rate a/b associated with a ratio $a:b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is $3/4$ cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."*[1]

A ratio is a pair of non-negative numbers, $A:B$, in a multiplicative relationship. The quantities A and B are related by a *rate* r, where $A = rB$. The number r is called a *unit rate* and is computed by $r = \frac{A}{B}$ as long as $B \neq 0$ (**6.RP. 1▲**). Although the introduction of ratios in grade six involves only non-negative numbers, ratios involving negative numbers are important in algebra and calculus. For example, if the slope of a line is −2, that means the ratio of rise to run is −2: the y-coordinate decreases by 2 when the x-coordinate increases by 1. In calculus, a negative rate of change means a function is decreasing.

Students work with models to develop their understanding of ratios (**MP.2, MP.6**). Initially, students do not express ratios using fraction notation; this is to allow students to differentiate ratios from fractions and rates. In grade six, students also learn that ratios can be expressed in fraction notation but are different from fractions in several ways. For example, in a litter of 7 puppies, 3 of them are white and 4 of them are black. The ratio of white puppies to black puppies is $3:4$. But the fraction of white puppies is not $\frac{3}{4}$; it is $\frac{3}{7}$. A fraction compares a part to the whole, while a ratio can compare either a part to a part or a part to a whole.

1. Expectations for unit rates in this grade are limited to non-complex fractions.

Ratios have associated *rates*. For example, in the ratio 3 cups of orange juice to 2 cups of fizzy water, the rate is $\frac{3}{2}$ cups of orange juice per 1 cup of fizzy water. The term *unit rate* refers to the numerical part of the rate; in the previous example, the unit rate is the number $\frac{3}{2} = 1.5$. (The word *unit* is used to highlight the 1 in "per 1 unit of the second quantity.") Students understand the concept of a unit rate $\frac{a}{b}$ associated with a ratio $a:b$ (with $a, b \neq 0$), and use rate language in the context of a ratio relationship (**6.RP.2▲**).

Examples of Ratio Language **6.RP.2▲**

1. If a recipe calls for a ratio of 3 cups of flour to 4 cups of sugar, then the *ratio* of flour to sugar is $3:4$. This can also be expressed with units included, as in "3 cups flour to 4 cups sugar." The associated *rate* is "$\frac{3}{4}$ cup of flour per cup of sugar." The *unit rate* is the number $\frac{3}{4} = .75$.
2. If the soccer team paid \$75 for 15 hamburgers, then this is a *ratio* of \$75 to 15 hamburgers or $75:15$. The associated *rate* is \$5 per hamburger. The *unit rate* is the number $\frac{75}{15} = 5$.

Students understand that rates always have units associated with them that are reflective of the quantities being divided. Common unit rates are cost per item or distance per time. In grade six, the expectation is that student work with unit rates is limited to fractions in which both the numerator and denominator are whole numbers. Grade-six students use models and reasoning to find rates and unit rates.

Students understand ratios and their associated rates by building on their prior knowledge of division concepts.

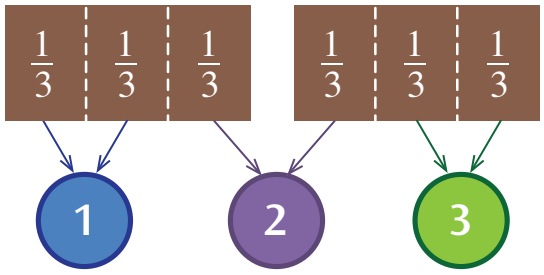
Why must b not be equal to 0? **6.RP.1▲**

For a unit rate, or any rational number $\frac{a}{b}$, the denominator b must not equal 0 because division by 0 is *undefined* in mathematics. To see that division by zero cannot be defined in a meaningful way, we relate division to multiplication. That is, if $a \neq 0$ and if $\frac{a}{0} = x$ for some number x, then it must be true that $a = 0 \cdot x$. But since $0 \cdot x = 0$ for any x, there is no x that makes the equation $a = 0 \cdot x$ true. For a different reason, $\frac{0}{0}$ is undefined because it cannot be assigned a unique value. Indeed, if $\frac{0}{0} = x$, then $0 = 0 \cdot x$, which is true for any value of x. So what would x be?

Example **6.RP.2▲ (MP.2, MP.6)**

There are 2 brownies for 3 students. What is the amount of brownie that each student receives? What is the unit rate?

Solution: This can be modeled to show that there are $\frac{2}{3}$ of a brownie for each student. The unit rate in this case is $\frac{2}{3}$. In the illustration at right, each student is counted as he or she receives a portion of brownie, and it is clear that each student receives $\frac{2}{3}$ of a brownie.



Adapted from Kansas Association of Teachers of Mathematics (KATM) 2012, 6th Grade Flipbook.

In general, students should be able to identify and describe any ratio using language such as, "For every _____, there are _____." For example, for every three students, there are two brownies (adapted from NCDPI 2013b).

Ratios and Proportional Relationships 6.RP

Understand ratio concepts and use ratio reasoning to solve problems.

3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.
   a. Make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.
   b. Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?*
   c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means $\frac{30}{100}$ times the quantity); solve problems involving finding the whole, given a part and the percent.
   d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

Students make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. They use tables to compare ratios (**6.RP.3a▲**). Grade-six students work with tables of quantities in equivalent ratios (also called *ratio tables*) and practice using ratio and rate language to deepen their understanding of what a ratio describes. As students generate equivalent ratios and record ratios in tables, they should notice the role of multiplication and division in how entries are related to each other. Students also understand that equivalent ratios have the same unit rate. Tables that are arranged vertically may help students to see the multiplicative relationship between equivalent ratios and help them avoid confusing ratios with fractions (adapted from the University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2011c).

Example: Representing Ratios in Different Ways **6.RP.3a▲**

A juice recipe calls for 5 cups of grape juice for every 2 cups of peach juice. How many cups of grape juice are needed for a batch that uses 8 cups of peach juice?

Using Ratio Reasoning: "For every 2 cups of peach juice, there are 5 cups of grape juice, so I can draw groups of the mixture to figure out how much grape juice I would need." [In the illustrations below, 🍇 represents 1 cup of grape juice and 🍊 represents 1 cup of peach juice.]

"It's easy to see that when you have $4 \times 2 = 8$ cups of peach juice, you need $4 \times 5 = 20$ cups of grape juice."

Using a Table: "I can set up a table. That way it's easy to see that every time I add 2 more cups of peach juice, I need to add 5 cups of grape juice."

Cups of Grape Juice	Cups of Peach Juice
5	2
10	4
15	6
20	8
25	10

+5 (Cups of Grape Juice) +2 (Cups of Peach Juice)

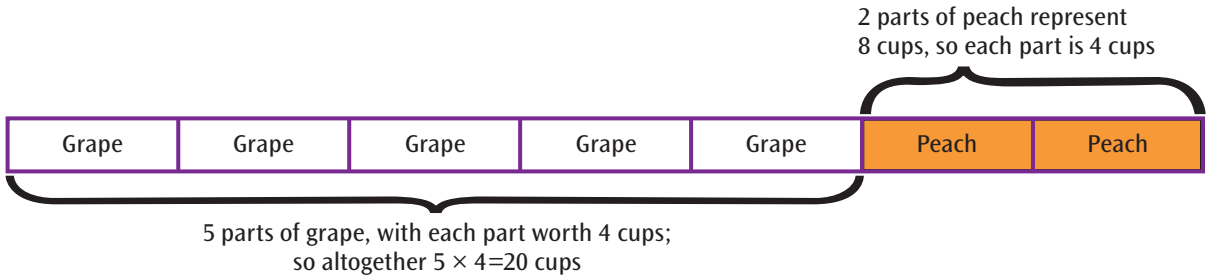
Tape diagrams and double number line diagrams (**6.RP.3▲**) are new to many grade-six teachers. A *tape diagram* (a drawing that looks like a segment of tape) can be used to illustrate a ratio. Tape diagrams are best used when the quantities in a ratio have the same units. A *double number line diagram* sets up two number lines with zeros connected. The same tick marks are used on each line, but the number lines have different units, which is central to how double number lines exhibit a ratio. Double number lines are best used when the quantities in a ratio have different units. The following examples show how tape diagrams and double number lines can be used to solve the problem from the previous example (adapted from UA Progressions Documents 2011c).

Representing Ratios with Tape Diagrams and Double Number Line Diagrams **6.RP.3▲**

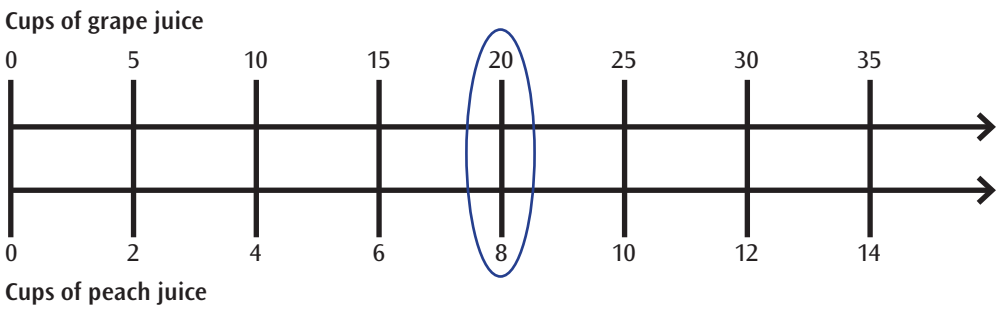
Using a Tape Diagram (Beginning Method): "I set up a tape diagram. I used pieces of tape to represent 1 cup of liquid. Then I copied the diagram until I had 8 cups of peach juice."

1 cup grape	1 cup grape	1 cup grape	1 cup grape	1 cup grape	1 cup peach	1 cup peach
1 cup grape	1 cup grape	1 cup grape	1 cup grape	1 cup grape	1 cup peach	1 cup peach
1 cup grape	1 cup grape	1 cup grape	1 cup grape	1 cup grape	1 cup peach	1 cup peach
1 cup grape	1 cup grape	1 cup grape	1 cup grape	1 cup grape	1 cup peach	1 cup peach

Using a Tape Diagram (Advanced Method): "I set up a tape diagram in a ratio of 5:2. Since I know there should be 8 cups of peach juice, each section of tape is worth 4 cups. That means there are $5 \times 4 = 20$ cups of grape juice."



Using a Double Number Line Diagram: "I set up a double number line, with cups of grape juice on the top and cups of peach juice on the bottom. When I count up to 8 cups of peach juice, I see that this brings me to 20 cups of grape juice."



Representing ratios in various ways can help students see the additive and multiplicative structure of ratios (**MP.7**). Standard **6.RP.3a▲** calls for students to create tables of equivalent ratios and represent the resulting data on a coordinate grid. Eventually, students see this additive and multiplicative structure in the graphs of ratios, which will be useful later when studying slopes and linear functions. (Refer to standard **6.EE.9▲** as well.)

Making Use of Structure in Tables and Graphs of Ratios **6.RP.3a▲**

The additive and multiplicative structure of ratios can be explained to students with tables as well as graphs (**6.RP.3a▲**).

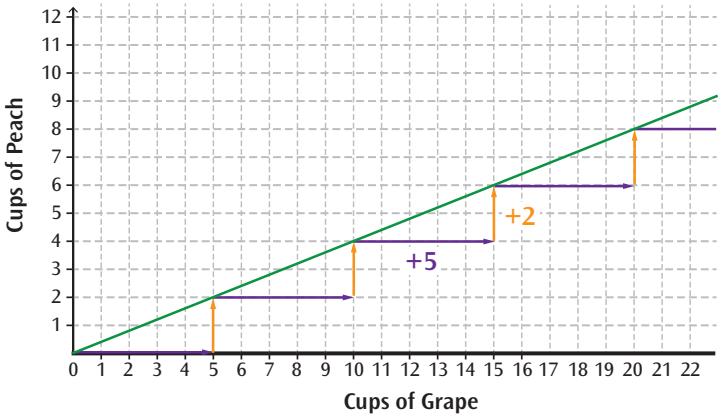
Additive Structure

Table

Cups of Grape	Cups of Peach
5	2
10	4
15	6
20	8
25	10

+5 +5 +5 +5 +2 +2 +2 +2

Graph



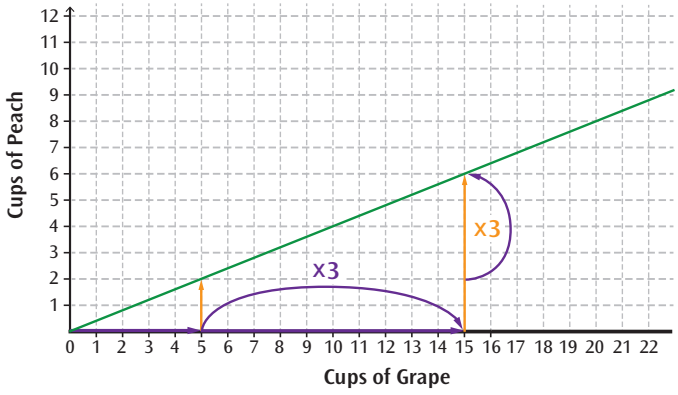
Multiplicative Structure

Table

Cups of Grape	Cups of Peach
5	2
10	4
15	6
20	8
100	40

×2 ×3 ×20 ×2 ×3 ×20

Graph

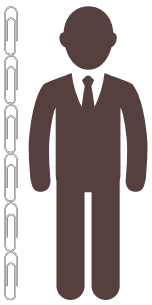


Adapted from UA Progressions Documents 2011c.

As students solve similar problems, they develop their skills in several mathematical practice standards, reasoning abstractly and quantitatively (**MP.2**), abstracting information from the problem, creating a mathematical representation of the problem, and correctly working with both part–part and part–whole situations. Students model with mathematics (**MP.4**) as they solve these problems by using tables and/or ratios. They attend to precision (**MP.6**) as they properly use ratio notation, symbolism, and label quantities.

Table 6-3 presents a sample classroom activity that connects the Standards for Mathematical Content and Standards for Mathematical Practice. The activity is appropriate for students who have already been introduced to ratios and associated rates.

Table 6-3. Connecting to the Standards for Mathematical Practice—Grade Six

Standards Addressed	Explanation and Examples
Connections to Standards for Mathematical Practice **MP.1.** Students who have little background in ratios can be challenged to solve the problem and to try to discover a relationship between paper clips and buttons. Students make sense of the problem as they create a simple illustration or try to picture how buttons are related to paper clips. **MP.3.** Students can be challenged to explain their reasoning for finding out how tall Suzy Short is in paper clips. They can be asked to share with a partner or the whole class how they found their answer. **MP.6.** Teachers can challenge students to use new vocabulary precisely when discussing solution strategies. Students are encouraged to explain why a ratio of $3:2$ is equivalent to a unit ratio of $1.5:1$. They include the units of paper clips and buttons in their solutions. **Standards for Mathematical Content** **6.RP.1.** Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."* **6.RP.2.** Understand the concept of a unit rate $\frac{a}{b}$ associated with a ratio $a:b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is $\frac{3}{4}$ cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."*	**Sample Problem.** When Mr. Short is measured with paper clips, he is found to be 6 paper clips tall. When he is measured with buttons, he is found to be 4 buttons tall. Mr. Short has a daughter named Suzy Short. When Suzy Short is measured with buttons, she is found to be 2 buttons tall. How many paper clips tall is Suzy Short?  **Solution.** Since Mr. Short is both 6 paper clips tall and 4 buttons tall, it must be true that 1.5 paper clips is the same height as 1 button. Therefore, since Suzy Short is 2 buttons tall, she is $2 \times 1.5 = 3$ paper clips tall. Also, since Suzy Short is half the number of buttons tall as her father, she must be half the number of paper clips tall. **Classroom Connections.** The purpose of this problem is to introduce students to the concepts of ***ratio*** and ***unit rate.*** Students can attempt to solve the problem and explain to other students how they arrived at an answer. Students should be encouraged to use diagrams if they have trouble beginning. Students arrive at the correct answer (3 paper clips) and discuss the commonly found incorrect answer (4 paper clips). A wrong answer of 8 paper clips typically appears when students think *additively* instead of *multiplicatively.* A simple diagram shows that for every 3 paper clips, there are 2 buttons, and in this way the notion of *ratio* is introduced. The language of a *ratio of 3:2* can be introduced here. Pictures can also help illustrate the concept of an *associated rate:* that there are $\frac{3}{2} = 1.5$ paper clips for every 1 button. Possible follow-up problems: 1. Mr. Short's car is 15 paper clips long. How long is his car when measured with buttons? 2. Mr. Short's car is 7.5 paper clips wide. How wide is his car when measured with buttons? 3. Mr. Short's house is 12 buttons tall. How tall is his house when measured with paper clips? 4. Make a table that compares the number of buttons and number of paper clips. How does your table show the ratio of $3:2$?

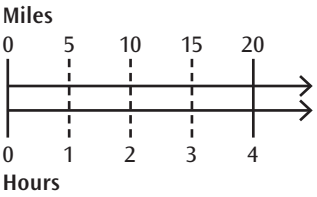
Adapted from Lamon 2012.

Standard **6.RP.3b–d▲** calls for students to apply their newfound ratio reasoning to various problems in which ratios appear, including problems involving unit price, constant speed, percent, and the conversion of measurement units. In grade six, generally only whole-number ratios are considered. The basic idea of *percent* is a particularly relevant and important topic for young students to learn, as they will use this concept throughout their lives (**MP.4**). Percent is discussed in a separate section that follows. Below are several more examples of ratios and the reasoning expected in the **6.RP** domain.

Examples of Problems Involving Ratio Reasoning

1. On a bicycle you can travel 20 miles in 4 hours. At the same rate, what distance can you travel in 1 hour? (**6.RP.3b▲, MP.2**)

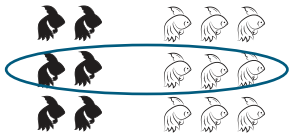
Solution: Students might use a double number line diagram to represent the relationship between miles ridden and hours elapsed. They build on fraction reasoning from earlier grades to divide the double number line into 4 equal parts and mark the double number line accordingly. It becomes clear that in 1 hour, a person can ride 5 miles, which is a rate of 5 miles per hour.



2. At the pet store, a fish tank has guppies and goldfish in a ratio of $6:9$. Show that this is the same as a ratio of $2:3$ (**6.RP.3b▲**).

Solution: Students should be able to find equivalent ratios by drawing pictures or using ratio tables. A ratio of $6:9$ might be represented in the following way, with black fish as guppies and white fish as goldfish:

This picture can be rearranged to show 3 sets of 2 guppies and 3 sets of 3 goldfish, for a ratio of $2:3$.



3. Use the information in the following table to find the number of yards that equals 24 feet (**6.RP.3d▲**).

Feet	3	6	9	15	24
Yards	1	2	3	5	?

Solution: Students can solve this in several ways.

1. They can observe the associated rate from the table, 3 feet per yard, and they can use multiplication to see that $24 \text{ feet} = 8 \times 3 \text{ feet}$, so the answer is 8×1 yard, or 8 yards.
2. They can notice that $24 \text{ feet} = 4 \times (6 \text{ feet})$, so the answer is $4 \times (2 \text{ yards}) = 8 \text{ yards}$.
3. They can see that with ratios, you can add entries in a table because of the distributive property:

   $9 \text{ feet} + 15 \text{ feet} = 24 \text{ feet}$
   $3(3 \text{ feet}) + 5(3 \text{ feet}) = 8(3 \text{ feet})$

   And since $3 \text{ feet} = 1$ yard, the correct answer is 8 yards.

4. The cost of 3 cans of pineapple at Superway Store is $2.25, and the cost of 6 cans of the same kind of pineapple is $4.80 at Grocery Giant. Which store has the better price for the pineapple? (**6.RP.3b▲**)

Solution: Students can solve this in several ways.

1. They can make a table that lists prices for different numbers of cans and compare the price for the same number of cans.
2. They can multiply the number of cans and their price at Superway Store by 2 to see that 6 cans there cost $4.50, so the same number of cans cost less at Superway Store than at Grocery Giant (where 6 cans cost $4.80).
3. Finally, they can find the unit price at each store:

   $\frac{\$2.25}{3} = \$0.75$ per can at Superway Store

   $\frac{\$4.80}{6} = \$0.80$ per can at Grocery Giant

Percent: A Special Type of Rate

Standard **6.RP.3c▲** calls for grade-six students to understand percent as a special type of rate, and students use models and tables to solve percent problems. This is students' first formal introduction to percent. Students understand that percentages represent a rate per 100; for example, to find 75% of a quantity means to multiply the quantity by $\frac{75}{100}$ or, equivalently, by the fraction $\frac{3}{4}$. They come to understand this concept as they represent percent problems with tables, tape diagrams, and double number line diagrams. Understanding of percent is related to students' understanding of fractions and decimals. A thorough understanding of place value helps students see the connection between decimals and percent (for example, students understand that 0.30 represents $\frac{30}{100}$, which is the same as 30%).

Students can use simple "benchmark percentages" (e.g., 1%, 10%, 25%, 50%, 75%, or 100%) as one strategy for solving percent problems. By using the distributive property to reason about rates, students see that percentages can be combined to find other percentages, and thus benchmark percentages become a very useful tool when learning about percent (**MP.5**).

Benchmark Percentages **6.RP.3c▲ (MP.7)**

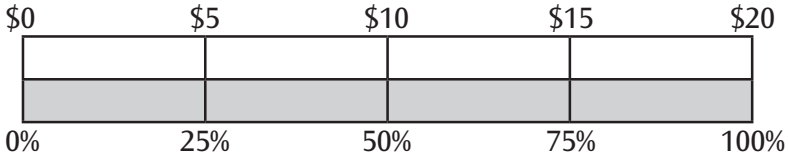
- 100% of a quantity is the entire quantity, or "1 times" the quantity.
- 50% of a quantity is half the quantity (since $50\% = \frac{50}{100} = \frac{1}{2}$), and 25% is one-quarter of a quantity (since $25\% = \frac{25}{100} = \frac{1}{4}$).
- 10% of a quantity is $\frac{1}{10}$ of the quantity (since $10\% = \frac{10}{100} = \frac{1}{10}$), so to find 10% of a quantity, students can divide the quantity by 10. Similarly, 1% is $\frac{1}{100}$ of a quantity.
- 200% of a quantity is twice the quantity (since $200\% = \frac{200}{100} = 2$).
- 75% of a quantity is $\frac{3}{4}$ of the quantity. Students also find that $75\% = 50\% + 25\%$, or $75\% = 3 \times 25\%$. Tape diagrams and double number lines can be useful for seeing this relationship.

A *percent bar* is a visual model, similar to a combined double number line and tape diagram, which can be used to solve percent problems. Students can fold the bar to represent benchmark percentages such as 50% (half), 25% and 75% (quarters), and 10% (tenths). Teachers should connect percent to ratios so that students see percent as a useful application of ratios and rates.

Examples: Connecting Percent to Ratio Reasoning **6.RP.3c▲**

1. Andrew was given an allowance of $20. He used 75% of his allowance to go to the movies. How much money was spent at the movies?

Solution: "By setting up a percent bar, I can divide the $20 into four equal parts. I see that he spent $15 at the movies."



2. What percent is 12 out of 25?

Part	12	24	48
Whole	25	50	100

Solutions: (a) "I set up a simple table and found that 12 out of 25 is the same as 24 out of 50, which is the same as 48 out of 100. So 12 out of 25 is 48%."
(b) "I saw that 4×25 is 100, so I found $4 \times 12 = 48$. So 12 out of 25 is the same as 48 out of 100, or 48%."
(c) "I know that I can divide 12 by 25, since $\frac{12}{25} = 12 \div 25$. I got 0.48, which is the same as $\frac{48}{100}$, or 48%."

Adapted from ADE 2010 and NCDPI 2013b.

There are several types of percent problems that students should become familiar with, including finding the percentage represented by a part of a whole, finding the unknown part when given a percentage and whole, and finding an unknown whole when a percentage and part are given. The following examples illustrate these problem types, as well as how to use tables, tape diagrams, and double number lines to solve them. (Students in grade six are not responsible for solving multi-step percent problems such as finding sales tax, markups and discounts, or percent change.)

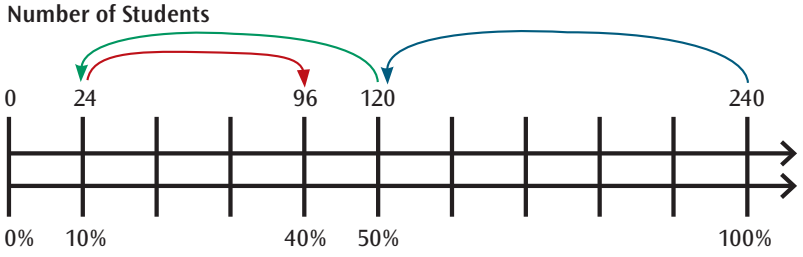
More Examples of Percent Problems **6.RP.3c▲**

Finding an Unknown Part. Last year, Mr. Christian's class had 30 students. This year, the number of students in his class is 150% of the number of students he had in his class last year. How many students does he have this year?

Solution: "Since 100% is 30 students, I know that 50% is $30 \div 2 = 15$ students. This means that 150% is $3 \times 15 = 45$ students, since $150\% = 3 \times 50\%$. His class is made up of 45 students this year."

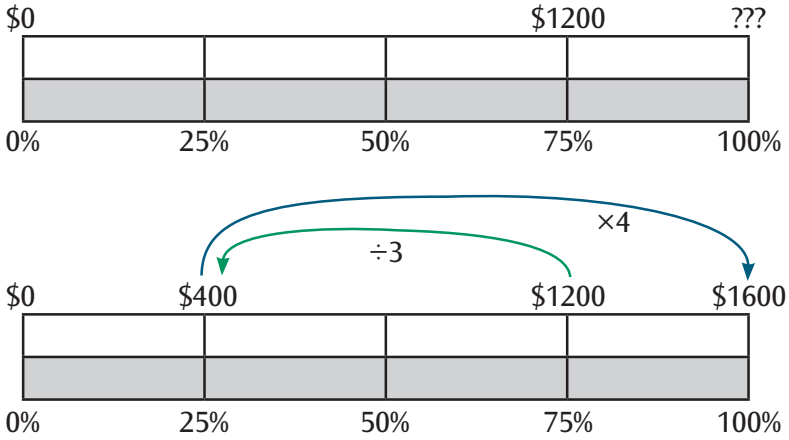
Finding an Unknown Percentage. When all 240 sixth-grade students were polled, results showed that 96 students were dissatisfied with the music played at a school dance. What percentage of sixth-grade students does this represent?

Solution: "I set up a double number line diagram. It was easy to find that 50% was 120 students. This meant that 10% was $120 \div 5 = 24$ students. I noticed that $96 \div 24$ is 4. Reading my double number line, this means that 40% of the students were dissatisfied $(40\% = 4 \times 10\%)$."



Finding an Unknown Whole. If 75% of the budget is $1200, what is the full budget?

Solution: "By setting up a fraction bar, I can find 25%, since I know 75% is $1200. Then, I multiply by 4 to give me 100%. Since 25% is $400, I see that 100% is $1600."



In problems such as this one, teachers can use scaffolding questions such as these:

- If you know 75% of the budget, how can we determine 25% of the budget?
- If you know 25% of the budget, how can this help you find 100% of the budget?

Source: UA Progressions Documents 2011c.

When students have had sufficient practice solving percent problems with tables and diagrams, they can be led to represent percentages as decimals to solve problems. For instance, the previous three problems can be solved using methods such as those shown below.

Examples **6.RP.3c▲**

If the class has 30 students, then 150% can be found by finding the fraction:

$$\frac{150}{100} = \frac{15}{10} = 1.5$$

$$1.5 \times 30 = 45$$

So the answer is 45 students.

Since 96 out of 240 students were dissatisfied with the music at the dance, this means that:

$$\frac{96}{240} = 0.4$$

$$0.4 = 40\%$$

40% were dissatisfied with the music.

Since the budget is unknown, let's call it B. Then we know that 75% of the budget is $1200, which means that $0.75B = 1200$. This can be solved by finding $B = 1200 \div 0.75$.

Alternatively, students may see that:

$\frac{75}{100} = \frac{1200}{B}$, which can be rewritten as $\frac{3}{4} = \frac{1200}{B}$

By reasoning with equivalent fractions, since $\frac{3}{4} = \frac{3 \times 400}{4 \times 400} = \frac{1200}{1600}$, we see that $B = \$1600$.

Percent problems give students opportunities to develop mathematical practices as they use a variety of strategies to solve problems, use tables and diagrams to represent problems (**MP.4**), and reason about percent (**MP.1, MP.2**).

Common Misconceptions: Ratios and Fractions

- Although ratios can be represented as fractions, the connection between ratios and fractions is subtle. Fractions express a part-to-whole comparison, but ratios can express part-to-whole or part-to-part comparisons. Care should be taken if teachers choose to represent ratios as fractions at this grade level.
- Proportional situations can have several ratios associated with them. For instance, in a mixture involving 1 part juice to 2 parts water, there is a ratio of 1 part juice to 3 *total parts* (1 : 3), as well as the more obvious ratio of 1 : 2.
- Students must carefully reason about why they can add ratios. For instance, in a mixture with lemon drink and fizzy water in a ratio of 2 : 3, mixtures made with ratios 2 : 3 and 4 : 6 can be added to give a mixture of ratio 6 : 9, equivalent to 2 : 3. This is because the following are true:

  2 (parts lemon drink) + 4 (parts lemon drink) = 6 (parts lemon drink)

  3 (parts fizzy water) + 6 (parts fizzy water) = 9 (parts fizzy water)

  However, one would never add fractions by adding numerators and denominators:

$$\frac{2}{3}+\frac{4}{6}\neq\frac{6}{9}$$

A detailed discussion of ratios and proportional relationships is provided online at http://commoncoretools.files.wordpress.com/2012/02/ccss_progression_rp_67_2011_11_12_corrected.pdf (accessed November 14, 2014) [UA Progressions Documents 2011c].

Domain: The Number System

In grade six, students complete their understanding of division of fractions and extend the notion of *number* to the system of rational numbers, which includes negative numbers. Students also work toward fluency with multi-digit division and multi-digit decimal operations.

The Number System — 6.NS

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, create a story context for* $\left(\frac{2}{3}\right)\div\left(\frac{3}{4}\right)$ *and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that* $\left(\frac{2}{3}\right)\div\left(\frac{3}{4}\right)=\frac{8}{9}$ *because* $\frac{3}{4}$ *of* $\frac{8}{9}$ *is* $\frac{2}{3}$. *(In general,* $\left(\frac{a}{b}\right)\div\left(\frac{c}{d}\right)=\frac{ad}{bc}$.*) How much chocolate will each person get if 3 people share* $\frac{1}{2}$ *lb of chocolate equally? How many* $\frac{3}{4}$*-cup servings are in* $\frac{2}{3}$ *of a cup of yogurt? How wide is a rectangular strip of land with length* $\frac{3}{4}$ *mi and area* $\frac{1}{2}$ *square mi?*

In grade five, students learned to divide whole numbers by unit fractions and unit fractions by whole numbers. These experiences lay the conceptual foundation for understanding general methods of division of fractions in sixth grade. Grade-six students continue to develop division by using visual models and equations to divide fractions by fractions to solve word problems (**6.NS.1▲**). Student understanding of the meaning of operations with fractions builds upon the familiar understandings of these meanings with whole numbers and can be supported with visual representations. To help students make this connection, teachers might have students think about a simpler problem with whole numbers and then use the same operation to solve with fractions.

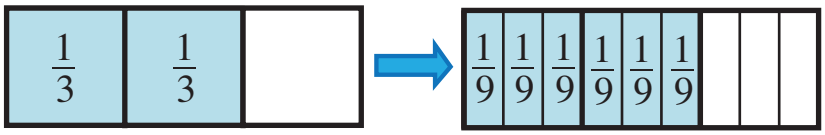
Looking at a problem through the lens of "How many groups?" or "How many in each group?" helps students visualize what is being sought. Encourage students to explain their thinking and to recognize division in two different situations: measurement division, which requires finding how many groups (e.g., how many groups can you make?); and fair-share division, which requires equal sharing (e.g., finding how many are in each group). In fifth grade, students represented division problems like $4 \div \frac{1}{2}$ with diagrams and reasoned why the answer is 8 (e.g., how many halves are in 4?). They may have discovered that $4 \div \frac{1}{2}$ can be found by multiplying 4×2 (i.e., each whole gives 2 halves, so there are 8 halves altogether). Similarly, students may have found that $\frac{1}{3} \div 5 = \frac{1}{3} \times \frac{1}{5}$. These generalizations will be exploited when students develop general methods for dividing fractions. Teachers should be aware that making visual models for general division of fractions can be difficult; it may be simpler to discuss general methods for dividing fractions and use these methods to solve problems.

The following examples illustrate how reasoning about division can help students understand fraction division before they move on to general methods.

Examples: Division Reasoning with Fractions **6.NS.1▲**

1. Three people share $\frac{2}{3}$ of a pound of watermelon. How much watermelon does each person get?

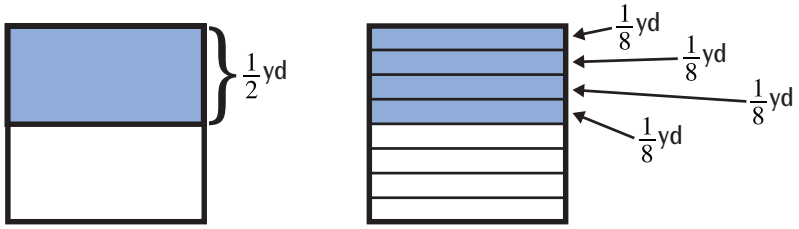
Solution: This problem can be represented by $\frac{2}{3} \div 3$. To solve it, students might represent the watermelon with a diagram such as the one below. There are two $\frac{1}{3}$-pound pieces represented in the picture. Students can see that $\frac{1}{3}$ divided among three people is $\frac{1}{9}$. Since there are 2 such pieces, each person receives $\frac{2}{9}$ of a pound of watermelon.



Problems like this one can be used to support the fact that, in general, $\frac{a}{b} \div c = \frac{a}{b} \times \frac{1}{c}$.

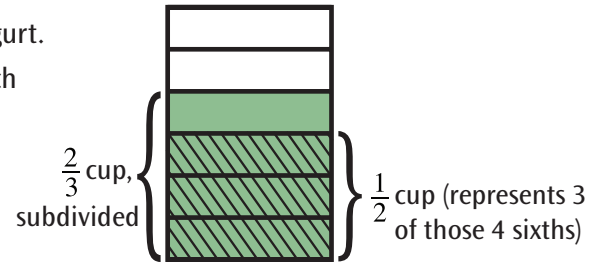
2. Manny has $\frac{1}{2}$ of a yard of fabric with which he intends to make bookmarks. Each bookmark is made from $\frac{1}{8}$ of a yard of fabric. How many bookmarks can Manny make?

Solution: Students can think, "How many $\frac{1}{8}$-yard pieces can I make from $\frac{1}{2}$ of a yard of fabric?" By subdividing the $\frac{1}{2}$ of a yard of fabric into eighths (of a yard), students can see that there are 4 such pieces.



Problems like this one can be used to support the fact that, in general, $\frac{1}{b} \div \frac{1}{d} = \frac{1}{b} \times d$.

3. You are making a recipe that calls for $\frac{2}{3}$ of a cup of yogurt. You have $\frac{1}{2}$ cup of yogurt from a snack pack. How much of the recipe can you make?

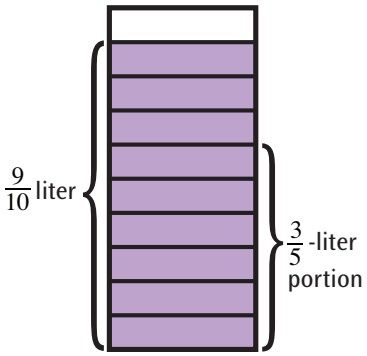


Solutions: Students can think, "How many $\frac{2}{3}$-cup portions can be made from $\frac{1}{2}$ cup?"

Students can reason that the answer will be less than 1, as there is not enough yogurt to make 1 full recipe. The difficulty with this problem is that it is not immediately apparent how to find thirds from halves. Students can convert the fractions into ones with common denominators to make the problem more accessible. Since $\frac{2}{3} = \frac{4}{6}$ and $\frac{1}{2} = \frac{3}{6}$, it makes sense to represent the $\frac{2}{3}$ cup required for the recipe divided into $\frac{1}{6}$-cup portions. As the diagram shows, the recipe calls for $\frac{4}{6}$ cup, but there are only 3 of the 4 sixths that are needed. Each sixth is $\frac{1}{4}$ of a recipe, and we have 3 of them, so we can make $\frac{3}{4}$ of a recipe.

Problems like this one can be used to support the division-by-common-denominators strategy.

4. A certain type of water bottle holds $\frac{3}{5}$ of a liter of liquid. How many of these bottles could be filled from $\frac{9}{10}$ of a liter of juice?



Solution: The picture shows $\frac{9}{10}$ of a liter of juice. Since 6 tenths make $\frac{3}{5}$ of a liter, it is clear that one bottle can be filled. The remaining $\frac{3}{10}$ of a liter represents $\frac{1}{2}$ of a bottle, so it makes sense to say that $1\frac{1}{2}$ bottles could be filled.Notice that $\frac{1}{10} \div \frac{1}{5} = \frac{1}{2}$, meaning that there is one-half of $\frac{1}{5}$ in each $\frac{1}{10}$. This means that in 9 tenths, there are 9 halves of $\frac{1}{5}$. But since the capacity of a bottle is 3 of these fifths, there are $\frac{9}{2} \div 3 = \frac{3}{2}$ of these bottles. This line of reasoning supports the idea that numerators and denominators can be divided — that is, $\frac{9}{10} \div \frac{3}{5} = \frac{9 \div 3}{10 \div 5} = \frac{3}{2}$.

Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Common Misconceptions

Students may confuse dividing a quantity **by** $\frac{1}{2}$ with dividing a quantity **in half.** Dividing **by** $\frac{1}{2}$ is finding how many $\frac{1}{2}$-sized portions there are, as in "dividing 7 **by** $\frac{1}{2}$," which is $7 \div \frac{1}{2} = 14$. On the other hand, to divide a quantity **in half** is to divide the quantity into two parts equally, as in "dividing 7 **in half**" yields $\frac{7}{2} = 3.5$. Students should understand that dividing **in half** is the same as dividing by 2.

Adapted from KATM 2012, 6th Grade Flipbook.

Students should also connect division of fractions with multiplication. For example, in the problems above, students should reason that $\frac{2}{3} \div 3 = \frac{2}{9}$, since $3 \times \frac{2}{9} = \frac{2}{3}$. Also, it makes sense that $\frac{1}{2} \div \frac{1}{8} = 4$, since $\frac{1}{8} \times 4 = \frac{1}{2}$, and that $\frac{1}{2} \div \frac{2}{3} = \frac{3}{4}$ because $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$. The relationship between division and multiplication is used to develop general methods for dividing fractions.

General Methods for Dividing Fractions

1. ***Finding common denominators.*** Interpreting division as measurement division allows one to divide fractions by finding common denominators (i.e., common *denominations*). For example, to divide $\frac{7}{8} \div \frac{2}{5}$, students need to find a common denominator, so $\frac{7}{8}$ is rewritten as $\frac{35}{40}$ and $\frac{2}{5}$ is rewritten as $\frac{16}{40}$. Now the problem becomes, "How many groups of 16 fortieths can we get out of 35 fortieths?" That is, the problem becomes $35 \div 16 = \frac{35}{16}$. This approach of finding common denominators reinforces the linguistic connection between *denominator* and *denomination.*

2. ***Dividing numerators and denominators (special case).*** By thinking about the relationship between division and multiplication, students can reason that a problem like $\frac{8}{15} \div \frac{2}{5} = ?$ is the same as finding $\frac{2}{5} \times ? = \frac{8}{15}$. Students can see that the fraction $\frac{4}{3}$ represents the missing factor, but this is the same result as if one simply divided numerators and denominators: $\frac{8}{15} \div \frac{2}{5} = \frac{8 \div 2}{15 \div 5} = \frac{4}{3}$. Although this strategy works in general, it is particularly useful when the numerator and denominator of the divisor are factors of the numerator and denominator of the dividend, respectively.

3. ***Dividing numerators and denominators (leading to the general case).*** By rewriting fractions as equivalent fractions, students can use the previous strategy in other cases—for instance, when the denominator of the divisor is not a factor of the denominator of the dividend. For example, when finding $\frac{2}{3} \div \frac{2}{7}$, students can rewrite $\frac{2}{3}$ as $\frac{14}{21} = \frac{2 \times 7}{3 \times 7}$, to arrive at:

$$\frac{2}{3} \div \frac{2}{7} = \frac{14}{21} \div \frac{2}{7} = \frac{14 \div 2}{21 \div 7} = \frac{7}{3}$$

4. ***Dividing numerators and denominators (general case).*** When neither the numerator nor the denominator of the divisor is a factor of those of the dividend, equivalent fractions can be used again to develop a strategy. For instance, with a problem like $\frac{3}{4} \div \frac{5}{7}$, the fraction $\frac{3}{4}$ can be rewritten as $\frac{3 \times 5 \times 7}{4 \times 5 \times 7}$ and then the division can be performed. When the fraction is left in this form, students can see that the following is true:

$$\frac{3}{4} \div \frac{5}{7} = \frac{3 \times 5 \times 7}{4 \times 5 \times 7} \div \frac{5}{7} = \frac{(3 \times 5 \times 7) \div 5}{(4 \times 5 \times 7) \div 7} = \frac{3 \times 7}{4 \times 5} = \frac{3}{4} \times \frac{7}{5}$$

This line of reasoning shows why it makes sense to find the reciprocal of the divisor and multiply to find the result.

Teaching the "multiply-by-the-reciprocal" method for dividing fractions without having students develop an understanding of why it works may confuse students and interfere with their ability to apply division of fractions to solve word problems. Teachers can gradually develop strategies (such as those described above) to help students see that, in general, fractions can be divided in two ways:

- Divide the first fraction (dividend) by the top and bottom numbers (numerator and denominator) of the second fraction (divisor).
- Find the reciprocal of the second fraction (divisor) and then multiply the first fraction (dividend) by it.

The following is an algebraic argument that $\frac{a}{b} \div \frac{c}{d} = x$ precisely when $x = \frac{a}{b} \times \frac{d}{c}$. Starting with $\frac{a}{b} = x \bullet \frac{c}{d}$, it can be argued that if both sides of the equation are multiplied by the multiplicative inverse of $\frac{c}{d}$, x can be isolated on the right. Thus, students examine $\frac{a}{b} \bullet \frac{d}{c} = \left(x \bullet \frac{c}{d}\right) \bullet \frac{d}{c}$. Continuing the computation on the right, students can see that $\left(x \bullet \frac{c}{d}\right) \bullet \frac{d}{c} = x \bullet \left(\frac{c}{d} \bullet \frac{d}{c}\right) = x \bullet 1 = x$. Since $\frac{a}{b} \div \frac{c}{d} = x$ as well, we have $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \bullet \frac{d}{c}$.

The Number System **6.NS**

Compute fluently with multi-digit numbers and find common factors and multiples.

2. Fluently divide multi-digit numbers using the standard algorithm.
3. Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.
4. Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. *For example, express 36 + 8 as 4(9 + 2).*

In previous grades, students built a conceptual understanding of operations with whole numbers and became fluent in multi-digit addition, subtraction, and multiplication. In grade six, students work toward fluency with multi-digit division and multi-digit decimal operations (**6.NS.2–3**). Fluency with the standard algorithms is expected, but an algorithm is defined by its steps, not by the way those steps are recorded in writing, so minor variations in written methods are acceptable.

FLUENCY

California's Common Core State Standards for Mathematics (K–6) set expectations for fluency in computation (e.g., "*Fluently* divide multi-digit numbers" [**6.NS.2**] and "*Fluently* add, subtract, multiply, and divide multi-digit decimals" [**6.NS.3**] using the standard algorithm). Such standards are culminations of progressions of learning, often spanning several grades, involving conceptual understanding, thoughtful practice, and extra support where necessary. The word *fluent* is used in the standards to mean "reasonably fast and accurate" and possessing the ability to use certain facts and procedures with enough facility that using such knowledge does not slow down or derail the problem solver as he or she works on more complex problems. Procedural fluency requires skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Developing fluency in each grade may involve a mixture of knowing some answers, knowing some answers from patterns, and knowing some answers through the use of strategies.

Adapted from UA Progressions Documents 2011a.

Focus, Coherence, and Rigor

In grade three, division was introduced conceptually as the inverse of multiplication. In grade four, students continued using place-value strategies, properties of operations, the relationship between multiplication and division, area models, and rectangular arrays to solve problems with one-digit divisors and develop and explain written methods. This work was extended in grade five to include two-digit divisors and all operations with decimals to hundredths. In grade six, fluency with the algorithms for division is reached (**6.NS.2**).

Grade-six students fluently divide using the standard algorithm (**6.NS.2**). Students should examine several methods for recording division of multi-digit numbers and focus on a variation of the standard algorithm that is efficient and makes sense to them. They can compare variations to understand how the same step can be written differently but still have the same place-value meaning. All such discussions should include place-value terms. Students should see examples of standard algorithm division that can be easily connected to place-value meanings.

Example: Scaffold Division **6.NS.2**

Scaffold division is a variation of the standard algorithm in which partial quotients are written to the right of the division steps rather than above them.

To find the quotient $3440 \div 16$, students can begin by asking, "How many groups of 16 are in 3440?" This is a measurement interpretation of division and can form the basis of the standard algorithm. Students estimate that there are at least 200 groups of 16, since $2 \times 16 = 32$, and therefore $200 \times 16 = 3200$. They would then ask, "How many groups of 16 are in the remaining 240?" Clearly, there are at least 10. The next remainder is then $80 = 240 - 160$, and we see that there are 5 more groups of 16 in this remaining 80. The quotient in this strategy is then found to be $200 + 10 + 5 = 215$.

Divisor	Dividend	Quotient
16	3440 –3200	200
	240 –160	10
	80 –80	5
	0	**215**
	Remainder	**Quotient**

As shown in the next example, the partial quotients may also be written above each other over the dividend. Students may also consider writing single digits instead of totals, provided they can explain why they do so with place-value reasoning, dropping all of the zeros in the quotients and subtractions in the dividend; in the example that follows, students would write "215" step by step above the dividend. In both cases, students use place-value reasoning.

Example: Division Using Single Digits Instead of Totals **6.NS.2**

To ensure that students understand and apply place-value reasoning when writing single digits, teachers can ask, "How many groups of 16 are in 34 hundreds?" Since there are two groups of 16 in 34, there are 2 hundred groups of 16 in 34 hundreds, so we record this with a 2 in the hundreds place above the dividend. The product of 2 and 16 is recorded, and we subtract 32 from 34, understanding that we are subtracting 32 hundreds from 34 hundreds, yielding 2 remaining hundreds. Next, when we "bring the 4 down to write 24," we understand this as moving to the digit in the dividend necessary to obtain a number larger than the divisor. Again, we focus on the fact that there are 24 (tens) remaining, and so the question becomes, "How many groups of 16 are in 24 tens?" The algorithm continues, and the quotient is found.

```
        2  1  5
    ┌──────────
16  │ 3  4  4  0
  − 3  2
    ──────
       2  4
     − 1  6
     ──────
          8  0
        − 8  0
        ──────
             0
```

Students should have experience with many examples similar to the two discussed above. Teachers should be prepared to support discussions involving place value if misunderstanding arises. There may be other effective ways for teachers to include place-value concepts when explaining a variation of the standard algorithm for division. Teachers are encouraged to find a method that works for them and their students. The standards support coherence of learning and conceptual understanding, and it is crucial for instruction to build on students' previous mathematical experiences. Refer to the following example and to the chapter on grade five for further explanation of division strategies.

Connecting Division Algorithms and Place Value			6.NS.2
Algorithm 1	**Explanation**	**Algorithm 2**	**Explanation**
200 32 ⟌ 8456	There are 200 groups of 32 in 8456.	2 32 ⟌ 8456	There are 2 (hundred) groups of 32 in 84 (hundred).
200 32 ⟌ 8456 −6400 2056	200 times 32 is 6400, so we subtract and find there is 2056 left to divide.	2 32 ⟌ 8456 −64 205	2 times 32 is 64, so there are 64 (hundreds) to subtract from 84 (hundreds). We include the 5 with what is left over, since the dividend (205) must be larger than the divisor.
60 200 32 ⟌ 8456 −6400 2056	There are 60 groups of 32 in 2056.	26 32 ⟌ 8456 −64 205	Now we see that there are 6 (tens) groups of 32 in 205 (tens).
60 200 32 ⟌ 8456 −6400 2056 −1920 136	60 times 32 is 1920, so we subtract and find there is 136 left to divide.	26 32 ⟌ 8456 −64 205 −192 136	6 times 32 is 192, so there are 192 (tens) to subtract from 205 (tens). Again, we include the 6 with what is left over since the dividend must be greater than the divisor.
4 60 200 32 ⟌ 8456 −6400 2056 −1920 136	There are 4 groups of 32 in 136.	264 32 ⟌ 8456 −64 205 −192 136	Now we see that there are 4 groups of 32 in 136.
4 60 200 32 ⟌ 8456 −6400 2056 −1920 136 −128 8	4 times 32 is 128, so we subtract and find 8 left to divide. But since 8 is smaller than the divisor, this is the remainder. So the quotient is $200+60+4=264$, with a remainder of 8, or $264\frac{8}{32}=264\frac{1}{4}$. Another way to say this is $8456=32(264)+8$.	264 32 ⟌ 8456 −64 205 −192 136 −128 8	4 times 32 is 128, so we subtract and find 8 left to divide. But since 8 is smaller than the divisor, this is the remainder. So the quotient is 264 with a remainder of 8, or $264\frac{8}{32}=264\frac{1}{4}$. Another way to say this is $8456=32(264)+8$.

Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Standard **6.NS.3** requires grade-six students to fluently apply standard algorithms when working with operations with decimals. In grades four and five, students learned to add, subtract, multiply, and divide decimals (to hundredths) with concrete models, drawings, and strategies and used place value to explain written methods for these operations. In grade six, students become fluent in the use of some written variation of the standard algorithms of each of these operations.

The notation for decimals depends upon the regularity of the place-value system across all places to the left and right of the ones place. This understanding explains why addition and subtraction of decimals can be accomplished with the same algorithms as for whole numbers; like values or units (such as tens or thousandths) are combined. To make sure students add or subtract like places, teachers should provide students with opportunities to solve problems that include zeros in various places and problems in which they might add zeros at the end of a decimal number. For adding and subtracting decimals, a conceptual approach that supports consistent student understanding of place-value ideas might instruct students to line up place values rather than "lining up the decimal point."

Focus, Coherence, and Rigor

Students should discuss how addition and subtraction of all quantities have the same basis: adding or subtracting like place-value units (whole numbers and decimal numbers), adding or subtracting like unit fractions, or adding or subtracting like measures. Thus, addition and subtraction are consistent concepts across grade levels and number systems.

In grade five, students multiplied decimals to hundredths. They understood that multiplying decimals by a power of 10 "moves" the decimal point as many places to the right as there are zeros in the multiplying power of 10 (see the discussion of standards **5.NBT.1–2** in the chapter on grade five). In grade six, students extend and apply their place-value understanding to fluently multiply multi-digit decimals (**6.NS.3**). Writing decimals as fractions whose denominator is a power of 10 can be used to explain the "decimal point rule" in multiplication. For example:

$$2.4 \times 0.37 = \frac{24}{10} \times \frac{37}{100} = \frac{24 \times 37}{10 \times 100} = \frac{888}{1000} = 0.888$$

This logical reasoning based on place value and decimal fractions justifies the typical rule, "Count the decimal places in the numbers and insert the decimal point to make that many places in the product."

The general methods used for computing quotients of whole numbers extend to decimals with the additional concern of where to place the decimal point in the quotient. Students divided decimals to hundredths in grade five, but in grade six they move to using standard algorithms for doing so. In simpler cases, such as $16.8 \div 8$, students can simply apply the typical division algorithm, paying particular attention to place value. When problems get more difficult (e.g., when the divisor also has a decimal point), then students may need to use strategies involving rewriting the problem through changing place values. Reasoning similar to that for multiplication can be used to explain the rule that "When the decimal point in the divisor is moved to make a whole number, the decimal point in the div-

idend should be moved the same number of places." For example, for a problem like $4.2 \div 0.35$, a student might give a rote recipe: "Move the decimal point two places to the right in 0.35 and also in 4.2." Teachers can instead appeal to the idea that a simpler but equivalent division problem can be formed by multiplying both numbers by 100 and still yield the same quotient. That is:

$$4.2 \div 0.35 = (4.2 \times 100) \div (0.35 \times 100) = 420 \div 35 = 12$$

It is vitally important for teachers to pay attention to students' understanding of place value. There is no conceptual understanding gained by referring to this only as "moving the decimal point." Teachers can refer to this more meaningfully as "multiplying by $\frac{n}{n}$ in the form of $\frac{100}{100}$."

Examples of Decimal Operations **6.NS.3**

1. Maria had 3 kilograms of sand for a science experiment. She had to measure out exactly 1.625 kilograms for a sample. How much sand will be left after she measures out the sample?

Solution: Student thinks, "I know that 1.625 is a little more than 1.5, so I should have about 1.5 kilograms remaining. I need to subtract like place values from each other, and I notice that 1.625 has three place values to the right of the ones place, so if I make zeros in the tenths, hundredths, and thousandths places of 3 to make 3.000, then the numbers have the same number of place values. Then it's easier to subtract: $3.000 - 1.625 = 1.375$. So there are 1.375 kilograms left."

2. How many ribbons 1.5 meters long can Victor cut from a cloth that is 15.75 meters long?

Solution: Student thinks, "This looks like a division problem, and since I can multiply both numbers by the same amount and get the same answer, I'll just multiply both numbers by 100. So now I need to find $1575 \div 150$ and this will give me the same answer. I did the division and got 10.5, which means that Victor can make 10 full ribbons, and he has enough left over to make half a ribbon."

In grade four, students identified prime numbers, composite numbers, and factor pairs. In grade six, students build on prior knowledge and find the greatest common factor (GCF) of two whole numbers less than or equal to 100 and find the least common multiple (LCM) of two whole numbers less than or equal to 12 (**6.NS.4**). Teachers might employ compact methods for finding the LCM and GCF of two numbers, such as the ladder method discussed below and other methods.

Example: Ladder Method for Finding the GCF and LCM **6.NS.4**

To find the LCM and GCF of 120 and 48, one can use the "ladder method" to systematically find common factors of 120 and 48 and identify the factors that 120 and 48 do not have in common. The GCF becomes the product of all those factors that 120 and 48 share, and the LCM is the product of the GCF and the remaining uncommon factors of 120 and 48.

With the ladder method, common factors (3, 4, 2 in this case) are divided from the starting and remaining numbers until there are no more common factors to divide (5, 2). The GCF is then $3 \cdot 4 \cdot 2 = 24$, and the LCM is $24 \cdot 5 \cdot 2 = 240$.

Note: The grade-six standard requires only that students find the GCF of numbers less than or equal to 100 and the LCM of numbers less than or equal to 12.

Common Factors	Remaining Numbers	
3	120	48
4	40	16
2	10	4
	5	2

The Number System 6.NS

Apply and extend previous understandings of numbers to the system of rational numbers.

5. Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

6. Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

   a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite.

   b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

   c. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

In grade six, students begin the formal study of negative numbers, their relationship to positive numbers, and the meaning and uses of absolute value. Students use rational numbers (expressed as fractions, decimals, and integers) to represent real-world contexts and understand the meaning of 0 in each situation (**6.NS.5▲**). Where the context gives rise to the basic meaning of $0 = (+n) + (-n)$, count models (e.g., positive or negative electric charge, credits and debits) will help develop student understanding of the relationship between a number and its opposite. In addition, measurement contexts such as temperature and elevation can contribute to student understanding of these ideas (**MP.1, MP.2, MP.4**).

Note that the standards do not specifically mention the set of *integers* (consisting of the whole numbers and their opposites) as a distinct set of numbers. Rather, the standards are focused on student understanding of the set of *rational numbers* in general (consisting of whole numbers, fractions, and their opposites). Thus, although early instruction in positive and negative numbers will likely start with examining whole numbers and their opposites, students must also work with negative fractions (and decimals) at this grade level. Ultimately, students learn that all numbers have an "opposite."

Examples of Rational Numbers in Context **6.NS.5▲**

1. All substances are made up of atoms, and atoms have protons and electrons. A proton has a positive charge, represented by "+1," and an electron has a negative charge, represented by "−1." A group of 5 protons has a total charge of +5, and a group of 8 electrons has a total charge of −8. One positive charge combines with one negative charge to result in a "neutral charge," which can be represented by $(+1)+(-1)=0$. So, for example, a group of 4 protons and 4 electrons together would have a neutral charge, since there are 4 positive charges to combine with 4 negative charges. We could write this as $(+4)+(-4)=0$.
   a. What is the overall charge of a group of 3 protons and 3 electrons?
   b. What is the overall charge of a group of 5 protons with no electrons?
   c. What is the overall charge of a group of 4 electrons with no protons?

2. In a checking account, *credits* to the account are recorded as positive numbers (since they add money to the account), and *debits* to the account are recorded as negative numbers (since they take money away from the account).
   a. Explain the meaning of an account statement that shows a total balance of −$100.15.
   b. Explain the meaning of an account statement that shows a total balance of $225.78.
   c. If a person's bank statement shows −$45.67, then explain how he or she can get to a $0 balance.

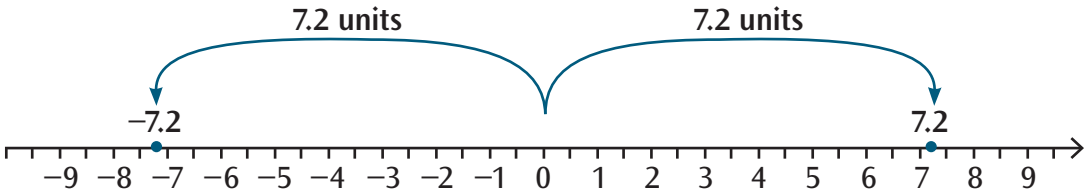
3. At any place on Earth, the elevation of the ground on which you are standing is measured by how far above or below the average level of water in the ocean (called *sea level*) the ground is.
   a. Discuss with a partner what an elevation of 0 means. Sketch a picture of what you think this means.
   b. Death Valley's Badwater Basin, located in California, is the point of lowest elevation in North America, at 282 feet below sea level. Explain why we would use a negative rational number to express this elevation.
   c. Mount Whitney is the highest mountain in California, at a height of 14,505 feet above sea level. Explain why we would use a positive number to express this elevation.

In prior grades, students worked with positive fractions, decimals, and whole numbers on the number line and in the first quadrant of the coordinate plane. In grade six, students extend the number line to represent all rational numbers, focusing on the relationship between a number and its opposite—namely, that they are equidistant from 0 on a number line (**6.NS.6▲**). Number lines may be either horizontal or vertical (such as on a thermometer); experiencing both types will facilitate students' movement from number lines to coordinate grids.

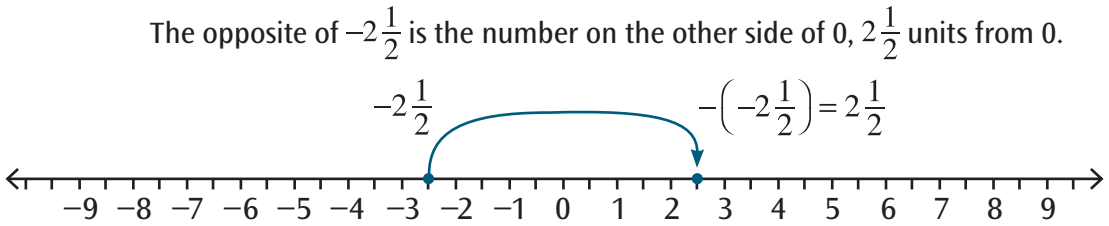
The Minus Sign

The minus sign (−) has several uses in mathematics. Since kindergarten, students have used this symbol to represent subtraction. In grade six they are responsible for understanding that the same symbol can be used to mean ***negative***, as in −5. (Negative numbers have also been represented with a "raised" minus sign, such as in −5; however, this practice is not consistent, and therefore teachers should use the more common minus sign.) However, students must also learn that the minus sign represents ***the opposite of***, as in, "−5 is the opposite of 5 since they are both the same distance from 0." This latter use is probably the most important, as it can be applied to cases such as, "$-(-9)$ is the opposite of the opposite of 9, which is 9." When viewing a stand-alone expression such as $-k$, students might erroneously think the expression represents a negative number. However, if the value of k itself is a negative number (that is, if $k = -3$), then $-k = -(-3) = 3$. Thus, reading $-k$ as "the opposite of k" is a more accurate way of reading this expression. Teachers should be consistent in using the word *minus* only when referring to subtraction and should use the word *negative* when referring to numbers like "−6" (that is, as opposed to saying "minus six").

In grade seven, students will explore operations with positive and negative rational numbers, so it is important that they develop a firm understanding of the relationship between positive and negative numbers and their opposites in grade six. Students recognize that a number and its opposite are the same distance from 0 on a number line, as in 7.2 and −7.2 being the same distance from 0:



In addition, students recognize the minus sign as meaning *the opposite of,* and that in general, the opposite of a number is the number on the other side of 0 at the same distance from 0 as the original number. For example, $-\left(-2\frac{1}{2}\right)$ is "the opposite of the opposite of $2\frac{1}{2}$," which is $2\frac{1}{2}$:



This understanding will help students develop the notion of *absolute value,* as the absolute value of a number is defined as its distance from 0 on a number line.

Students' previous work in the first quadrant grid helps them recognize the point where the x-axis and y-axis intersect as the origin. Grade-six students identify the four quadrants and the appropriate quadrant for an ordered pair based on the signs of the coordinates (**6.NS.6▲**). For example, students recognize that in Quadrant II, the signs of all ordered pairs would be (–, +). Students understand the relationship between two ordered pairs differing only by signs as reflections across one or both axes. For example, in the ordered pairs (–2, 4) and (–2, –4), the y-coordinates differ only by signs, which

represents a reflection across the x-axis. A change in the x-coordinate from (–2, 4) to (2, 4), represents a reflection across the y-axis. When the signs of both coordinates change—for example, when (2, –4) changes to (–2, 4)—the ordered pair is reflected across both axes.

The Number System	6.NS

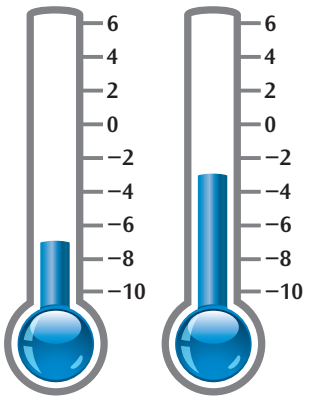
Apply and extend previous understandings of numbers to the system of rational numbers.

7. Understand ordering and absolute value of rational numbers.
   a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. *For example, interpret –3 > –7 as a statement that –3 is located to the right of –7 on a number line oriented from left to right.*
   b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. *For example, write –3°C > –7°C to express the fact that –3°C is warmer than –7°C.*
   c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. *For example, for an account balance of –30 dollars, write $|-30| = 30$ to describe the size of the debt in dollars.*
   d. Distinguish comparisons of absolute value from statements about order. *For example, recognize that an account balance less than –30 dollars represents a debt greater than 30 dollars.*
8. Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

In grade six, students reason about the order and absolute value of rational numbers (**6.NS.7▲**) and solve real-world and mathematical problems by graphing in all four quadrants of the coordinate plane (**6.NS.8▲**). Students use inequalities to express the relationship between two rational numbers. Working with number line models helps students internalize the order of the numbers—larger numbers on the right or top of the number line and smaller numbers to the left or bottom of the number line. Students correctly locate rational numbers on the number line, write inequalities, and explain the relationships between numbers. Students understand the absolute value of a rational number as its distance from zero on the number line and correctly use the absolute value symbol (e.g., $|3| = 3$, $|-2| = 2$). They distinguish comparisons of absolute value from statements about order (**6.NS.7▲**).

Example: Comparing Rational Numbers **6.NS.7b▲**

One of the thermometers at right shows –3°C, and the other shows –7°C. Which thermometer shows which temperature? Which is the colder temperature? How much colder? Write an inequality to show the relationship between the temperatures and explain how the model shows this relationship.



Solution: On a vertical number line, negative numbers get "more negative" as we go down the line, so it appears that the thermometer on the left must read –7° and the thermometer on the right must read –3°. By counting spaces, the thermometer on the left reads a temperature colder by 4 degrees. Related inequalities are $-7 < -3$ and $-3 > -7$.

Adapted from ADE 2010.

Common Misconceptions

With positive numbers, the absolute value (distance from zero) of the number and the value of the number are the same. However, students might be confused when they work with the absolute values of negative numbers. For negative numbers, as the value of the number decreases, the absolute value increases. For example, –24 is less than –14 because –24 is located to the left of –14 on the number line. However, the absolute value of –24 is greater than the absolute value of –14 because it is farther from zero. Students may also erroneously think that taking the absolute value means to "change the sign of a number," which is true for negative numbers but not for positive numbers or 0.

Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Domain: Expressions and Equations

A critical area of instruction at grade six is writing, interpreting, and using expressions and equations. In previous grades, students wrote numerical equations and simple equations involving one operation with a variable. In grade six, students start the systematic study of equations and inequalities and methods to solve them.

Students understand that mathematical expressions represent calculations with numbers. Some numbers, such as 2 or $\frac{3}{4}$, might be given explicitly. Other numbers are represented by letters, such as x, y, P, or n. The calculation represented by an expression might use a single operation, as in $4+3$ or $3x$, or a series of nested or parallel operations, as in $3(a+9)-\frac{b}{9}$. An expression may consist of a single number, even 0.

Students understand an equation as a statement that two expressions are equal. An important aspect of equations is that the two expressions on either side of the equal sign may not always be equal; that is, the equation might be a true statement for some values of the variable(s) and a false statement for others (adapted from UA Progressions Documents 2011d).

Expressions and Equations — 6.EE

Apply and extend previous understandings of arithmetic to algebraic expressions.

1. Write and evaluate numerical expressions involving whole-number exponents.
2. Write, read, and evaluate expressions in which letters stand for numbers.
   a. Write expressions that record operations with numbers and with letters standing for numbers. *For example, express the calculation "Subtract y from 5" as* $5-y$.
   b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. *For example, describe the expression* $2(8+7)$ *as a product of two factors; view* $(8+7)$ *as both a single entity and a sum of two terms.*
   c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). *For example, use the formulas* $V=s^3$ *and* $A=6s^2$ *to find the volume and surface area of a cube with sides of length* $s=1/2$.

Students demonstrate understanding of the meaning of exponents by writing and evaluating numerical expressions with whole-number exponents. The base can be a whole number, positive decimal, or a positive fraction (**6.EE.1▲**). Students should work with a variety of expressions and problem situations to practice and deepen their skills. They can start with simple expressions to evaluate and move to more complex expressions. For example, they begin with simple whole numbers and move to fractions and decimal numbers (**MP.2, MP.6**).

Examples **6.EE.1▲**

- What is the side length of a cube of volume 5^3 cubic cm? (Answer: 5 cm)
- Write $10{,}000 = 10 \times 10 \times 10 \times 10$ with an exponent. (Answer: 10^4)
- Andrea had half a pizza. She gave half of it to Marcus. Then Marcus gave half of what he had to Roger. Use exponents to write the amount of pizza Roger has.

  $\left(\text{Answer}: \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2} \right)^3 \right)$

Evaluate the following:

- 4^3 (Answer: $4 \times 4 \times 4 = 64$)
- $5 + 2^4 \cdot 6$ (Answer: $5 + (2 \cdot 2 \cdot 2 \cdot 2) \cdot 6 = 5 + 16 \cdot 6 = 5 + 96 = 101$)
- $7^2 - 24 \div 3 + 26$ (Answer: $7 \cdot 7 - 8 + 26 = 49 - 8 + 26 = 67$)

Grade six marks a foundational year for building the bridge between concrete concepts of arithmetic and the abstract thinking of algebra. Visual representations and concrete models (such as algebra tiles, counters, and cubes) can help students develop understanding as they move toward using abstract symbolic representations.

Common Misconceptions

Students in grade six may not understand how to read the operations referenced with notations (e.g., x^3, $4x$, $3(x + 2y)$, $a + 3a$). Students are learning the following:

- x^3 means $x \cdot x \cdot x$, not $3x$ or 3 times x.
- $4x$ means 4 times x or $x + x + x + x$, not forty-something.
- When $4x$ is evaluated where $x = 7$, substitution does not result in the expression meaning 47.
- For expressions like $a + 3a$, students need to understand a as $1a$ to know that $a + 3a = 4a$, not $3a^2$.

The use of the "x" notation as both the variable and the operation of multiplication may also be a source of confusion for students. In addition, students may need an explanation for why $x^0 = 1$ for all non-zero numbers x. Full explanations of this and other rules of working with exponents appear in the chapter on grade eight.

Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Students write, read, and evaluate expressions in which letters (called *variables*) stand for numbers (**6.EE.2▲**). Grade-six students write expressions that record operations with numbers and variables. Students need opportunities to read algebraic expressions to reinforce that the variable represents a number and therefore behaves according to the same rules for operations that numbers do (e.g., the distributive property).

Examples of Interpreting Expressions **6.EE.2a▲**

- Some (unknown) number plus 21 is represented by the expression $r + 21$.
- Six (6) times some number n is represented by the expression $6 \cdot n$.
- The variable s divided by 4, as well as one-quarter of s, is represented by the expression $\frac{s}{4}$.
- The variable r minus 4.5, or 4.5 less than r, is represented by the expression $r - 4.5$.
- Three (3) times the sum of a number and 5 is represented by the expression $3(x + 5)$.

Adapted from NCDPI 2013.

Students identify the parts of an algebraic expression using mathematical vocabulary such as *variable, coefficient, constant, term, factor, sum, difference, product,* and *quotient* (**6.EE.2b▲**). They should understand terms are the parts of a sum, and when a term is an explicit number, it is called a *constant*. When the term is a product of a number and a variable, the number is called the *coefficient* of the variable. *Variables* are letters that represent numbers. Development of this common mathematical vocabulary helps students understand the structure of expressions and explain their process for evaluating expressions.

As students move from numerical to algebraic work, the multiplication and division symbols × and ÷ are replaced by the conventions of algebraic notation. Students learn to use either a dot for multiplication (e.g., $1 \cdot 2 \cdot 3$ instead of $1 \times 2 \times 3$) or simple juxtaposition (e.g., $3x$ instead of $3 \times x$), which is potentially confusing. During the transition, students may indicate all multiplications with a dot, writing $3 \cdot x$ for $3x$. Students also learn that $x \div 2$ can be written as $\frac{x}{2}$ (adapted from UA Progressions Documents 2011d).

Examples of Expression Language **6.EE.2b▲**

In the expression $x^2 + 5y + 3x + 6$, the variables are x and y.

- There are 4 terms: x^2, $5y$, $3x$, and 6.
- There are 3 variable terms: x^2, $5y$, and $3x$. These have coefficients of 1, 5, and 3, respectively.
- The coefficient of x^2 is 1, since $x^2 = 1 \cdot x^2$.
- The term $5y$ represents $y + y + y + y + y$ or $5 \cdot y$.
- There is one constant term: 6.
- The expression shows a sum of all four terms.

Grade-six students evaluate various expressions at specific values of their variables, including expressions that arise from formulas used in real-world problems. Examples where students evaluate the same expression at several different values of a variable are important for the later development of the concept of a *function,* and these should be experienced more frequently than problems wherein the values of the variables stay the same and the expression continues to change (**MP.1, MP.2, MP.3, MP.4, MP.6**).

Examples of Evaluating Expressions **6.EE.2c▲**

1. Evaluate the two expressions $5(n+3)+7n$ and $12n+15$ for $n=-2$, 0, $\frac{1}{2}$, and 7.5. What do you notice?
2. The expression $c+0.07c$ can be used to find the total cost of an item with 7% sales tax added, where c is the cost of the item before taxes are added. Use this expression to find the total cost of an item that costs \$25, then an item that costs \$250, and finally an item that costs \$25,000.
3. The perimeter of a parallelogram is found using the formula $P=2l+2w$. What is the perimeter of a rectangular picture frame with dimensions of 8.5 inches by 11 inches?

Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Expressions and Equations 6.EE

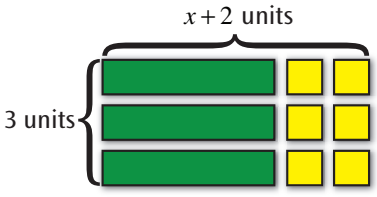
Apply and extend previous understandings of arithmetic to algebraic expressions.

3. Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression $3(2+x)$ to produce the equivalent expression $6+3x$; apply the distributive property to the expression $24x+18y$ to produce the equivalent expression $6(4x+3y)$; apply properties of operations to $y+y+y$ to produce the equivalent expression $3y$.*
4. Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions $y+y+y$ and $3y$ are equivalent because they name the same number regardless of which number y stands for.*

Students use their understanding of multiplication to interpret $3(2+x)$ as 3 groups of $(2+x)$ or the area of a rectangle of lengths 3 units and $(2+x)$ units (**MP.2, MP.3, MP.4, MP.6, MP.7**). They use a model to represent x and make an array to show the meaning of $3(2+x)$. They can explain why it makes sense that $3(2+x)$ is equal to $6+3x$. Manipulatives such as algebra tiles, which make use of the area model to represent quantities, may be used to show why this is true. Note that with algebra tiles, a 1-by-1 square represents a unit (the number 1), while the variable x is represented by a rectangle of dimensions 1 by x (that is, the longer side of the x-tile is not commensurate with a whole number of unit tiles, and therefore it represents an unknown length).

Example of Basic Reasoning with Algebra Tiles **6.EE.3▲**

Students can recognize $3(x+2)$ as representing the area of a rectangle of lengths 3 units and $(x+2)$ units. Using the appropriate number of tiles (or a sketch), students can see that there are $3\cdot 2=6$ and $3\cdot x=3x$ units altogether, so that $3(x+2)=3x+6$.



Standards **6.EE.3–4▲** highlight the importance of understanding the distributive property, which is the basis for combining like terms in an expression or equation. For instance, students understand that $4a+7a=11a$, because $4a+7a=(4+7)a=11a$.

It is important for students to develop the ability to use the distributive property flexibly—for example, to see that $3(2x+5)$ is the same as $(2x+5)3$. Students generate equivalent expressions using the associative, commutative, and distributive properties and can prove the expressions are equivalent (**MP.1, MP.2, MP.3, MP.4, MP.6**).

Example: Equivalent Expression **6.EE.4▲**

Show that the two expressions $5(n+3)+7n$ and $12n+15$ are equivalent.

Solution: "By applying the distributive property, I know that $5(n+3)+7n$ can be rewritten as $5n+15+7n$. Also, since $5n+7n=(5+7)n=12n$, I can write the expression as $12n+15$."

Expressions and Equations 6.EE

Reason about and solve one-variable equations and inequalities.

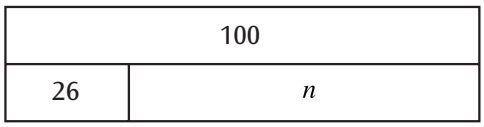
5. Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.
6. Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.
7. Solve real-world and mathematical problems by writing and solving equations of the form $x+p=q$ and $px=q$ for cases in which p, q and x are all non-negative rational numbers.
8. Write an inequality of the form $x>c$ or $x<c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x>c$ or $x<c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

In previous grade levels, students explored the concept of equality. In grade six, students explore equations as one expression being set equal to a specific value. A solution is a value of the variable that makes the equation true. Students use various processes to identify such values that, when substituted for the variable, will make the equation true (**6.EE.5▲**). Students can use manipulatives and pictures (e.g., tape-like diagrams) to represent equations and their solution strategies. When writing equations, students learn to be precise in their definition of a variable—for example, writing "n equals John's age in years" as opposed to writing only that "n is John" (**6.EE.6▲**) [**MP.6**].

Examples: Solving Equations of the Form $p+x=q$ and $px=q$ **6.EE.7▲**

1. Joey had 26 game cards. His friend Richard gave him some more, and now Joey has 100 cards. How many cards did Richard give to Joey? Write an equation and solve your equation.

Solution: Since Richard gave him some more cards, we let n represent the number of cards that Richard gave Joey. This means he now has $26+n$ cards. But the number of cards Joey has is 100, so we get the equation $26+n=100$. Using the relationship between addition and subtraction, we see that $n=100-26=74$, which means that his friend gave him 74 cards. This equation can be represented with a tape-like diagram:



2. A book of tickets for rides at an amusement park costs $30.00. Each ticket costs $2.50. How many tickets come in each book? Write and solve an equation that represents this situation.

Solution: If s represents the number of tickets in one booklet, then $(2.50)s$ is the cost of s tickets in dollars. Since the cost of one book is $30.00, solving the equation $(2.50)s=30.00$ would result in the number of tickets. To solve this equation, we realize that if $2.50\times s=30.00$, then $s=30.00\div 2.50=12$. This means there are 12 tickets in each book.

Examples: 6.EE.7 ***(continued)***

3. Meagan spent \$56.58 on three pairs of jeans. If each pair of jeans costs the same amount, write an equation that represents this situation and solve it to determine the price of one pair of jeans.

Solution: If J represents the cost of one pair of jeans in dollars, then the equation becomes $3J = 56.58$. If we solve this for J, we find $J = 56.58 \div 3 = 18.86$. This means each pair of jeans cost \$18.86.

\$56.58		
J	J	J

4. Julio was paid \$20.00 for babysitting. He spent \$1.99 on a package of trading cards and \$6.50 on lunch. Write and solve an equation to show how much money Julio has left.

Solution: One equation might be $1.99 + 6.50 + x = 20.00$, where x represents the amount of money (in dollars) that Julio has left. We find that $x = 11.51$, so Julio has \$11.51 left.

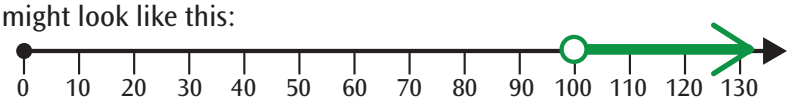
Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Many real-world situations are represented by inequalities. In grade six, students write simple inequalities involving < or > to represent real-world and mathematical situations, and they use the number line to model the solutions (**6.EE.8▲**). Students learn that when representing inequalities of these forms on a number line, the common practice is to draw an arrow on or above the number line with an open circle on or above the number in the inequality. The arrow indicates the numbers greater than or less than the number in question, and the solutions extend indefinitely. The arrow is a solid line indicating that even fractional and decimal amounts (i.e., points between marked values on the line) are included in the solution set.

Examples: Inequalities of the Form $x < c$ and $x > c$ **6.EE.8▲**

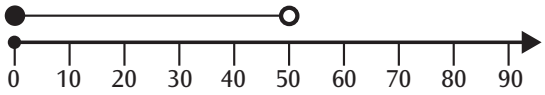
1. A class must raise more than \$100 to go on a field trip. Let m represent the amount of money in dollars that the class raises. Write an inequality that describes how much the class needs to raise. Represent this inequality on a number line.

Solution: Since the amount of money, m, needs to be greater than 100, the inequality is $m > 100$. A number line diagram for this might look like this:



2. The Flores family spent less than \$50 on groceries last week. Write an inequality that describes this situation, and graph the solution on a number line.

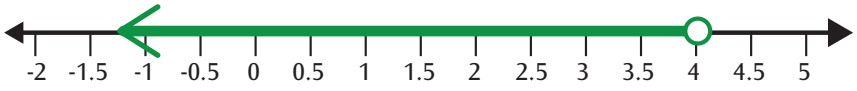
Solution: If we let g represent the amount of money (in dollars) that the family spent on groceries last week, then the inequality becomes $g < 50$. We might represent this in the following way:



(In this example, the Flores family could not have spent a negative amount of money on groceries, so the arrow would stop precisely at \$0; typically, this would be represented with a dot over 0 rather than the arrow.)

3. Graph $x < 4$.

Solution: This graph represents all numbers less than 4:



Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Expressions and Equations 6.EE

Represent and analyze quantitative relationships between dependent and independent variables.

9. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d = 65t$ to represent the relationship between distance and time.*

In grade six, students investigate the relationship between two variables, beginning with the distinction between dependent and independent variables (**6.EE.9▲**). The *independent variable* is the variable that can be changed; the *dependent variable* is the variable that is affected by the change in the independent variable. Students recognize that the independent variable is graphed on the x-axis and the dependent variable is graphed on the y-axis. They also understand that not all data should be graphed with a line. Discrete data would be graphed only with coordinates.

Students show relationships between quantities with multiple representations, using language, a table, an equation, or a graph. Translating between multiple representations helps students understand that each form represents the same relationship and provides a different perspective on the relationship.

Example: Exploring Dependent and Independent Variables **6.EE.9▲**

Stephanie is helping her band raise money to fund a field trip. The band decided to sell school pennants, which come in boxes of 20. Each pennant sells for $1.50. The partially completed table at right shows money collected for different numbers of boxes sold.

a. Complete the table for the remaining values of m.

b. Write an equation for the amount of money, m, that will be collected if b boxes of pennants are sold. Which is the independent variable and which is the dependent variable?

c. Graph the relationship by using ordered pairs from the table.

d. Calculate how much money will be collected if 100 boxes of school pennants are sold.

Boxes Sold (b)	Money Collected (m)
1	$30
2	
3	
4	
5	$150
6	
7	
8	

Solutions:

a.

Boxes Sold (b)	Money Collected (m)
1	$30
2	
3	
4	
5	$150
6	
7	
8	

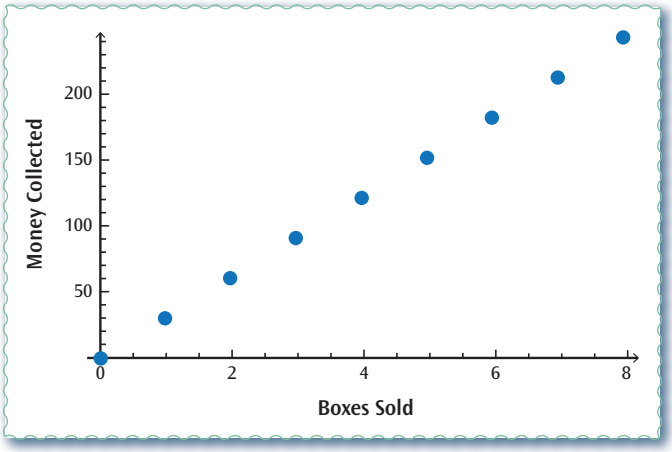
Boxes Sold (b)	Money Collected (m)
1	$30
2	**$60**
3	**$90**
4	**$120**
5	$150
6	**$180**
7	**$210**
8	**$240**

Example: 6.EE.9 *(continued)*

b. Students may derive the equation $m = 30b$, representing the fact that when b boxes are sold at \$30 per box, then the total amount of money collected is $30b$ dollars. In this case, the independent variable is the number of boxes sold, b, and the money collected is the dependent variable. This equation certainly is a valid way to make sense of the problem, in that the amount of money collected depends on the number of boxes sold.

However, if one has fund-raising goals, then it would be natural to think of the relationship as $b = \frac{m}{30}$, in the sense that the number of boxes needed to be sold depends on the fund-raising target.

c. If we graph the relationship as (b, m), then we obtain the graph shown, which illustrates the relationship $m = 30b$. (In grade seven, students will more fully explore graphs of proportional relationships such as this one.)



d. Using the equation derived in solution b, $m = 30b$, we use 100 for the value of b and find the amount of money colleted will be \$3000.

Adapted from Illustrative Mathematics 2013c.

Domain: Geometry

In grade six, students extend their understanding of length, area, and volume as they solve problems by applying formulas for the area of triangles and parallelograms and volume of rectangular prisms.

Geometry 6.G

Solve real-world and mathematical problems involving area, surface area, and volume.

1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
2. Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.
3. Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.
4. Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

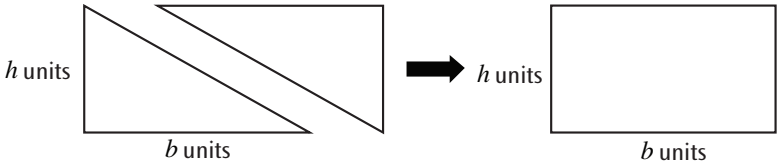
Sixth-grade students build on their work with area from previous grade levels by reasoning about relationships among shapes to determine area, surface area, and volume. Students in grade six continue to understand area as the number of squares needed to cover a plane figure. They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. As students compose and decompose shapes to determine areas, they learn that area is conserved when composing or decomposing shapes. For example, students will decompose trapezoids into triangles and/or rectangles and use this reasoning to find formulas for the area of a trapezoid. Students know area formulas for triangles and some special quadrilaterals, in the sense of having an understanding of why the formula works and how the formula relates to the measure (area) and the figure (**6.G.1**).

Prior to being exposed to the formulas for areas of different shapes, students can find areas of shapes on centimeter grid paper by duplicating, composing, and decomposing shapes. These experiences will familiarize students with the processes that result in the derivations of the following area formulas.

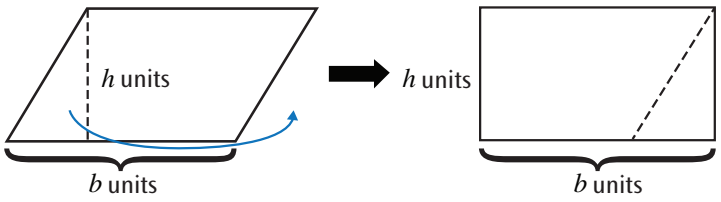
Deriving Area Formulas **6.G.1 (MP.3, MP.7)**

Starting with a basic understanding that the area of a rectangle of base b units and height h units is bh square units, along with the relationship between rectangles and triangles and the law of conservation of area, students can justify area formulas for various shapes.

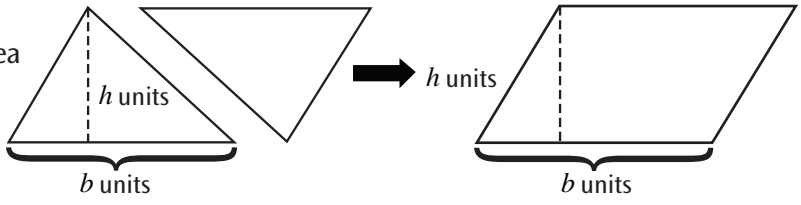
Right Triangles: Since two right triangles of base b and height h can be composed to form a rectangle of the same base and height, the triangle must have an area half that of the rectangle. Thus, the area of a right triangle of base b and height h is $\frac{1}{2}bh$ square units.



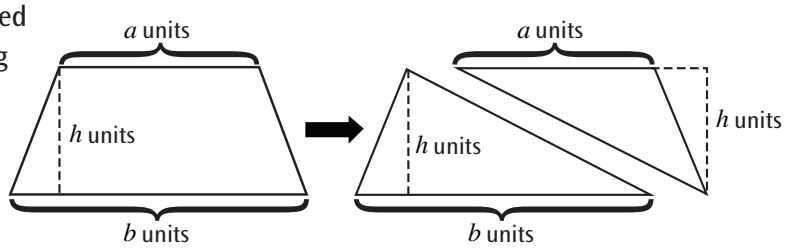
Parallelograms: If we define the height of the parallelogram to be the length of a perpendicular segment from base to base, then a parallelogram of base b and height h has the same area (bh square units) as a rectangle of the same dimensions. We cut off a right triangle as shown and move it to complete the rectangle.



Non-Right Triangles: Non-right triangles of base b units and height h units can now be duplicated to make parallelograms. By similar reasoning used with right triangles and rectangles, the area of such a triangle is $\frac{1}{2}bh$ square units. (One can show the same holds true for obtuse triangles.)



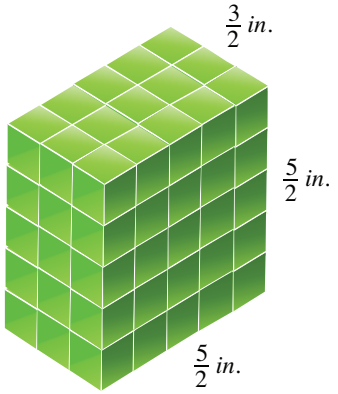
Trapezoids: Trapezoids can be deconstructed into two triangles of bases a and b, showing that the area of a trapezoid can be found by $\frac{1}{2}ah+\frac{1}{2}bh=\frac{1}{2}(a+b)h$.



Previously, students calculated the volume of right rectangular prisms using whole-number edges and understood doing so as finding the number of unit cubes (i.e., $1 \times 1 \times 1$ cubic unit) within a solid shape. In grade six, students extend this work to unit cubes with fractional edge lengths. For example, they determine volumes by finding the number of unit cubes of dimensions $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ within a figure with fractional side lengths. Students draw diagrams to represent fractional side lengths, and in doing so they connect finding these volumes with multiplication of fractions (**6.G.2**).

Example: Counting Fractional Cubic Units **6.G.2**

The model at right shows a rectangular prism with dimensions $\frac{3}{2}$ inches, $\frac{5}{2}$ inches, and $\frac{5}{2}$ inches. Each of the cubic units shown in the model has a volume of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ cubic inches. Students should reason why each of these units has this volume (i.e., by discovering that 8 of them fit in a $1 \times 1 \times 1$ cube). Furthermore, students explain why the volume of the rectangular prism is given by $\frac{3}{2} \times \frac{5}{2} \times \frac{5}{2}$ cubic inches and why the volume can also be determined by finding $(3 \times 5 \times 5) \times \left(\frac{1}{8} \text{ cubic inch}\right)$.



Adapted from ADE 2010.

When students find areas, surface areas, and volumes, modeling opportunities are presented (**MP.4**), and students must attend to precision with the types of units involved (**MP.6**).

Standard **6.G.3** calls for students to represent shapes in the coordinate plane. Students find lengths of sides that contain vertices with a common x- or y-coordinate, representing an important foundational step toward grade-eight understanding of how to use the distance formula to find the distance between any two points in the plane. In addition, grade-six students construct three-dimensional shapes using nets and build on their work with area (**6.G.4**) by finding surface areas with nets.

Example: Polygons in the Coordinate Plane **6.G.3**

On a grid map, the library is located at (–2, 2), the city hall building is located at (0, 2), and the high school is located at (0, 0).

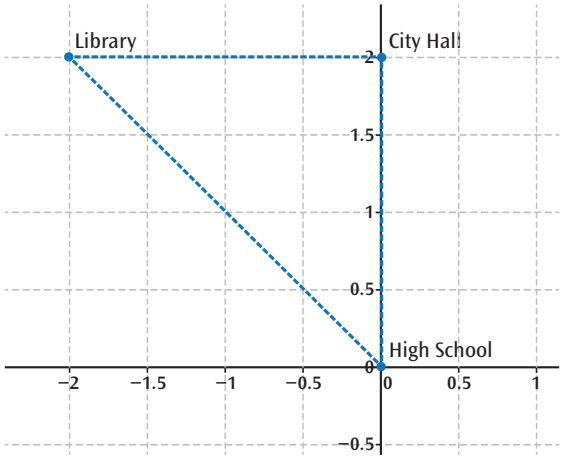
a. Represent the locations as points on a coordinate grid with a unit of 1 mile.

b. What is the distance from the library to the city hall building?

c. What is the distance from the city hall building to the high school? How do you know?

d. What is the shape that results from connecting the three locations with line segments?

e. The city council is planning to place a city park in this area. What is the area of the planned park?



Adapted from ADE 2010.

Focus, Coherence, and Rigor

The standards in the cluster "Solve real-world and mathematical problems involving area, surface area, and volume" regarding areas of triangles and volumes of right rectangular prisms (**6.G.1–2**) connect to major work in the Expressions and Equations domain (**6.EE.1–9▲**). In addition, standard **6.G.3** asks students to draw polygons in the coordinate plane, which supports major work with the coordinate plane in the Number System domain (**6.NS.8▲**).

Domain: Statistics and Probability

A critical area of instruction in grade six is developing understanding of statistical thinking. Students build on their knowledge and experiences in data analysis as they work with statistical variability and represent and analyze data distributions. They continue to think statistically, viewing statistical reasoning as a four-step investigative process (UA Progressions Documents 2011e):

- Formulate questions that can be answered with data.
- Design and use a plan to collect relevant data.
- Analyze the data with appropriate methods.
- Interpret results and draw valid conclusions from the data that relate to the questions posed.

Statistics and Probability — 6.SP

Develop understanding of statistical variability.

1. Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. *For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.*
2. Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape.
3. Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

Statistical investigations start with questions, which can result in a narrow or wide range of numerical values and ultimately result in some degree of variability (**6.SP.1**). For example, asking classmates "How old are the students in my class in years?" will result in less variability than asking "How old are the students in my class in months?" Students understand that questions need to specifically demand measurable answers. For example, if a student wants to know about the exercise habits of fellow students at her school, a statistical question for her study could be "On average, how many hours per week do students at my school exercise?" This is much more specifc than asking "Do you exercise?" Grade-six students design survey questions that anticipate variability in the responses (ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook).

One focus of grade six is the characterization of data distributions by measures of center and spread. To be useful, center and spread must have well-defined numerical descriptions that are commonly understood by those using the results of a statistical investigation (UA Progressions Documents 2011e). Grade-six students analyze the center, spread, and overall shape of a set of data (**6.SP.2**). As students analyze and/or compare data sets, they consider the context in which the data are collected and identify clusters, peaks, gaps, and symmetry in the data. Students learn that data sets contain many numerical values that can be summarized by one number, such as a measure of center (mean and median) and range.

Describing Data	**6.SP.3**
A *measure of center* gives a numerical value to represent the central tendency of the data (e.g., midpoint of an ordered list [median] or the balancing point). The *range* provides a single number that describes how widely the values vary across the data set. Another characteristic of a data set is the measure of *variability* (or spread from center) of the values.	

Measures of Center

Given a set of data values, students summarize the measure of center with the median or mean (**6.SP.3**). The *median* is the value in the middle of an ordered list of data. This value means that 50 percent of the data is greater than or equal to it and that 50 percent of the data is less than or equal to it. When there is an even number of data values, the median is the arithmetic average of the two values in the middle.

The *mean* is the arithmetic average: the sum of the values in a data set divided by the number of data values in the set. The mean measures center in the sense that it is the hypothetical value that each data point would equal if the total of the data values were redistributed equally. Students can develop an understanding of what the mean represents by redistributing data sets to be level or fair (creating an equal distribution) and by observing that the total distance of the data values above the mean is equal to the total distance of the data values below the mean (reflecting the idea of a *balance point*).

Example: Representing Data and Finding Measures of Center **6.SP.3**

Consider the data shown in the following line plot of the scores for organization skills for a group of students.

a. How many students are represented in the data set?

b. What are the mean and median of the data set? Compare the mean and median.

c. What is the range of the data? What does this value tell you?

Example: 6.SP.3 ***(continued)***

Solution:

a. Since there are 19 data points (represented by Xs) in the set, there are 19 students represented.

b. The mean of the data set can be found by adding all of the data values (scores) and dividing by 19. The calculation below is recorded as (score) × (number of students with that score):

$$\frac{0(1)+1(1)+2(2)+3(6)+4(4)+5(3)+6(2)}{19}=\frac{66}{19}\approx 3.47$$

From the line plot, the median of the data set appears to be 3. To check this, we can line up the data values in ascending order and look for the center:

0, 1, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6, 6

The median is indeed 3, since there are 9 data values to the left of the circled 3 and 9 values to the right of it. The mean is greater than the median, which makes sense because the data are slightly skewed to the right.

c. The range of the data is 6, which coincides with the range of possible scores.

Adapted from ADE 2010; KATM 2012, 6th Grade Flipbook; and NCDPI 2013.

Measures of Variability

In grade six, variability is measured by using the interquartile range or the mean absolute deviation. The *interquartile range* (IQR) describes the variability within the middle 50% of a data set. It is found by subtracting the lower quartile from the upper quartile. In a box plot, it represents the length of the box and is not affected by outliers. Students find the IQR of a data set by finding the upper and lower quartiles and taking the difference or by reading a box plot.

Mean absolute deviation (MAD) describes the variability of the data set by determining the absolute deviation (the distance) of each data piece from the mean, and then finding the average of these deviations. Both the IQR and the MAD are represented by a single numerical value. Higher values represent a greater variability in the data.

Example: Finding the IQR and MAD **6.SP.3**

In the previous example, the data set was 0, 1, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6, 6. The median (3) separated the data set into the upper 50% and the lower 50%. By further separating these two subsets, we obtain the four *quartiles* (i.e., 25%-sized parts of the data set).

0, 1, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6, 6

In this case, the IQR is $5-3=2$, indicating that the middle 50% of values differ by no more than 2 units. This is reflected in the dot plot, as most of the data appear to be clustered around 3 and 4.

To find the MAD of the data set above, the mean is rounded to 3.5 to simplify the calculations and find that there are 6 possible deviations from the mean:

$$|0-3.5|,\ |1-3.5|,\ |2-3.5|,\ |3-3.5|,\ |4-3.5|,\ |5-3.5|,\ |6-3.5|$$

Example: 6.SP.3 *(continued)*

This results in the set of deviations 3.5, 2.5, 0.5, 0.5, 1.5, and 2.5. When the average of all deviations in the data set is found, we obtain the following:

$$\frac{1(3.5)+1(2.5)+2(1.5)+6(0.5)+4(0.5)+3(1.5)+2(2.5)}{19} \approx 1.24$$

This is interpreted as saying that, on average, a student's score was 1.24 points away from the approximate mean of 3.5.

Adapted from ADE 2010; KATM 2012, 6th Grade Flipbook; and NCDPI 2013.

Statistics and Probability 6.SP

Summarize and describe distributions.

4. Display numerical data in plots on a number line, including dot plots, histograms, and box plots.
5. Summarize numerical data sets in relation to their context, such as by:
   a. Reporting the number of observations.
   b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.
   c. Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.
   d. Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.

Students in grade six use number lines, dot plots, histograms, and box plot graphs (**6.SP.4**) to display data graphically. Students learn to determine the appropriate graph for displaying data and how to read data from graphs generated by others.

Graphical Displays of Data in Grade Six **6.SP.4**

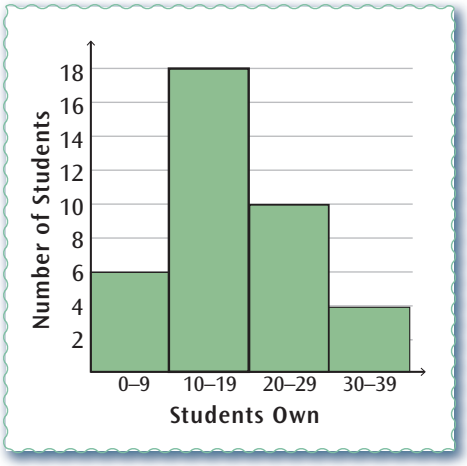
- *Dot plots* are simple plots on a number line where each dot represents a piece of data in a data set. Dot plots are suitable for small to moderately sized data sets and are useful for highlighting the distribution of the data, including clusters, gaps, and outliers.
- A *histogram* shows the distribution of data using intervals on the number line. The height of each bar represents the number of data values in that interval. In most real data sets, there is a large amount of data, and many numbers will be unique.
- A *box plot* shows the distribution of values in a data set by dividing the set into quartiles. It can be graphed either vertically or horizontally. The box plot is constructed from the five-number summary (*minimum, lower quartile, median, upper quartile,* and *maximum*). These values give a summary of the shape of a distribution. Box plots display the degree of spread of the data and the skewness of the data and can help students compare two data sets.

Students in grade six interpret data displays and determine measures of center and variability from them. They summarize numerical data sets in relation to the context of the data (**6.SP.5**).

Examples: Interpreting Data Displays **6.SP.5**

1. Students in grade six were collecting data for a project in math class. They decided to survey the other two sixth-grade classes to determine how many video games each student owns. A total of 38 students were surveyed. The data are shown in the table below, in no particular order. Create a data display. What are some observations that can be made from the data display?

11	21	5	12	10	31	19	13	23	33
10	11	25	14	34	15	14	29	8	5
22	26	23	12	27	4	25	15	7	
2	19	12	39	17	16	15	28	16	



Solution: Students might make a histogram with 4 ranges (0–9, 10–19, 20–29, 30–39) to display the data. It appears from the histogram that the mean and median are somewhere between 10 and 19, since the data of so many students lie in this range. Relatively few students own more than 30 video games; in fact, further analysis may prove the data point 39 to be an outlier.

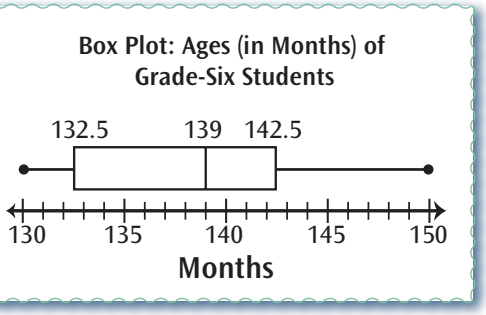
2. Ms. Wheeler asked each student in her class to write his or her age, in months, on a sticky note. The 28 students in the class brought their sticky notes to the front of the room and posted them in order on the whiteboard. The data set is listed below, in order from least to greatest. Create a data display. What are some observations that can be made from the data display?

130	130	131	131	132	132	132	133	134	136
137	137	138	139	139	139	140	141	142	142
142	143	143	144	145	147	149	150		

Solution: By finding the **five-number summary** of the data, we can create a box plot. The minimum data value is 130 months, the maximum is 150 months, and the median is 139 months. To find the first quartile (Q_1) and third quartile (Q_3), we find the middle of the upper and lower 50%. Since there is an even number of data points in each of these parts, we must find the average, so that $Q_1 = \frac{132+133}{2} = 132.5$ and $Q_3 = \frac{142+143}{2} = 142.5$. Thus, the five-number summary is as follows:

Minimum	First Quartile (Q_1)	Median	Third Quartile (Q_3)	Maximum
130	132.5	139	142.5	150

Now a box plot is easy to construct. The box plot helps to show that the middle 50% of values lie between 132.5 months and 142.5 months. Additionally, only 25% of the values are between 130 months and 132.5 months, and only 25% of the values are between 142.5 and 150.



Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Focus, Coherence, and Rigor

As students display and summarize numerical data (**6.SP.4–5**), they strengthen mathematical practices such as making sense of given data (**MP.1**), using appropriate statistical models and measures (**MP.4, MP.5**), and attending to precision in calculating and applying statistical measures (**MP.6**).

Students can use applets such as the following to create data displays (National Council of Teachers of Mathematics Illuminations 2013a):

- Box Plotter (http://illuminations.nctm.org/ActivityDetail.aspx?ID=77 [accessed November 10, 2014])
- Histogram Tool (http://illuminations.nctm.org/ActivityDetail.aspx?ID=78 [accessed November 10, 2014])

Essential Learning for the Next Grade

In grades six through eight, multiplication and division develop into powerful forms of ratio and proportional reasoning. The properties of operations take on prominence as students move from arithmetic to algebra. The theme of quantitative relationships also becomes explicit in grades six through eight, developing into the formal notion of a function by grade eight. In addition, the foundations of deductive geometry are laid. The gradual development of data representations in kindergarten through grade five leads to the study of statistics in grades six through eight: evaluation of shape, center, and spread of data distributions; possible associations between two variables; and the use of sampling in making statistical decisions (adapted from PARCC 2012).

To be prepared for grade-seven mathematics, students should be able to demonstrate mastery of particular mathematical concepts and procedural skills by the end of grade six and that they have met the fluency expectations for grade six. The expected fluencies for sixth-grade students are multi-digit whole-number division (**6.NS.2**) and multi-digit decimal operations (**6.NS.3**). These fluencies and the conceptual understandings that support them are foundational for work with fractions and decimals in grade seven.

Of particular importance at grade six are skills and understandings of division of fractions by fractions (**6.NS.1▲**); an understanding of the system of rational numbers (**6.NS.5–8▲**); the ability to use ratio concepts and reasoning to solve problems (**6.RP.1–3▲**); the extension of arithmetic to algebraic expressions (**6.EE.1–4▲**), including how to reason about and solve one-variable equations and inequalities (**6.EE.5–8▲**); and the ability to represent and analyze quantitative relationships between dependent and independent variables (**6.EE.1–9▲**).

Guidance on Course Placement and Sequences

The California Common Core State Standards for Mathematics for grades six through eight are comprehensive, rigorous, and non-redundant. Instruction in an accelerated sequence of courses will require compaction—not the former strategy of deletion. Therefore, careful consideration needs to be made before placing a student in higher-mathematics course work in grades six through eight. Acceleration may get students to advanced course work, but it may create gaps in students' mathematical background. Careful consideration and systematic collection of multiple measures of individual student performance on both the content and practice standards are required. For additional information and guidance on course placement, see appendix D (Course Placement and Sequences).

Grade 6 Overview

Ratios and Proportional Relationships

- Understand ratio concepts and use ratio reasoning to solve problems.

The Number System

- Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
- Compute fluently with multi-digit numbers and find common factors and multiples.
- Apply and extend previous understandings of numbers to the system of rational numbers.

Expressions and Equations

- Apply and extend previous understandings of arithmetic to algebraic expressions.
- Reason about and solve one-variable equations and inequalities.
- Represent and analyze quantitative relationships between dependent and independent variables.

Geometry

- Solve real-world and mathematical problems involving area, surface area, and volume.

Statistics and Probability

- Develop understanding of statistical variability.
- Summarize and describe distributions.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

6 Grade 6

Ratios and Proportional Relationships 6.RP

Understand ratio concepts and use ratio reasoning to solve problems.

1. Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*

2. Understand the concept of a unit rate a/b associated with a ratio $a:b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is $3/4$ cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."*[2]

3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

   a. Make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

   b. Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?*

   c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means $30/100$ times the quantity); solve problems involving finding the whole, given a part and the percent.

   d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

The Number System 6.NS

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share $1/2$ lb of chocolate equally? How many $3/4$-cup servings are in $2/3$ of a cup of yogurt? How wide is a rectangular strip of land with length $3/4$ mi and area $1/2$ square mi?*

Compute fluently with multi-digit numbers and find common factors and multiples.

2. Fluently divide multi-digit numbers using the standard algorithm.

3. Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

4. Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. *For example, express $36 + 8$ as $4(9 + 2)$.*

2. Expectations for unit rates in this grade are limited to non-complex fractions.

Apply and extend previous understandings of numbers to the system of rational numbers.

5. Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

6. Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

   a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite.

   b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

   c. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

7. Understand ordering and absolute value of rational numbers.

   a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. *For example, interpret –3 > –7 as a statement that –3 is located to the right of –7 on a number line oriented from left to right.*

   b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. *For example, write –3°C > –7°C to express the fact that –3°C is warmer than –7°C.*

   c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. *For example, for an account balance of –30 dollars, write* $|-30| = 30$ *to describe the size of the debt in dollars.*

   d. Distinguish comparisons of absolute value from statements about order. *For example, recognize that an account balance less than –30 dollars represents a debt greater than 30 dollars.*

8. Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

Expressions and Equations 6.EE

Apply and extend previous understandings of arithmetic to algebraic expressions.

1. Write and evaluate numerical expressions involving whole-number exponents.

2. Write, read, and evaluate expressions in which letters stand for numbers.

   a. Write expressions that record operations with numbers and with letters standing for numbers. *For example, express the calculation "Subtract y from 5" as* $5 - y$.

b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. *For example, describe the expression $2(8+7)$ as a product of two factors; view $(8+7)$ as both a single entity and a sum of two terms.*

c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). *For example, use the formulas $V = s^3$ and $A = 6s^2$ to find the volume and surface area of a cube with sides of length $s = 1/2$.*

3. Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression $3(2+x)$ to produce the equivalent expression $6+3x$; apply the distributive property to the expression $24x+18y$ to produce the equivalent expression $6(4x+3y)$; apply properties of operations to $y+y+y$ to produce the equivalent expression $3y$.*

4. Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions $y+y+y$ and $3y$ are equivalent because they name the same number regardless of which number y stands for.*

Reason about and solve one-variable equations and inequalities.

5. Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

6. Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

7. Solve real-world and mathematical problems by writing and solving equations of the form $x+p=q$ and $px=q$ for cases in which p, q and x are all non-negative rational numbers.

8. Write an inequality of the form $x>c$ or $x<c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x>c$ or $x<c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

Represent and analyze quantitative relationships between dependent and independent variables.

9. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d=65t$ to represent the relationship between distance and time.*

Geometry 6.G

Solve real-world and mathematical problems involving area, surface area, and volume.

1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
2. Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.
3. Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.
4. Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

Statistics and Probability 6.SP

Develop understanding of statistical variability.

1. Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. *For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.*
2. Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape.
3. Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

Summarize and describe distributions.

4. Display numerical data in plots on a number line, including dot plots, histograms, and box plots.
5. Summarize numerical data sets in relation to their context, such as by:
   a. Reporting the number of observations.
   b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.
   c. Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.
   d. Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.

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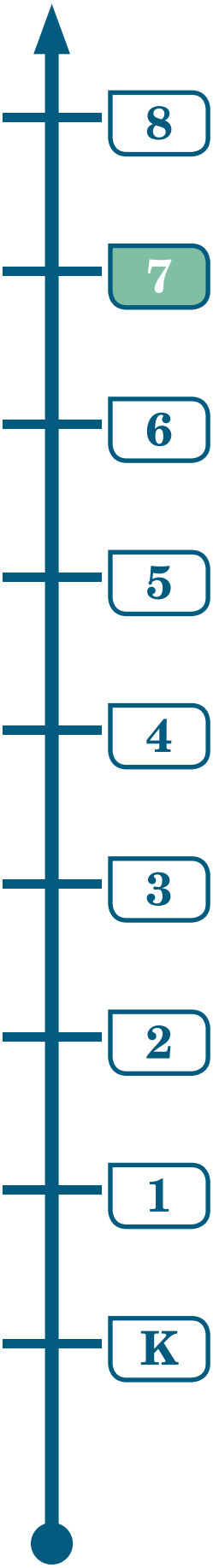


Grade Seven

As students enter grade seven, they have an understanding of variables and how to apply properties of operations to write and solve simple one-step equations. They are fluent in all positive rational number operations. Students who are entering grade seven have been introduced to ratio concepts and applications, concepts of negative rational numbers, absolute value, and all four quadrants of the coordinate plane. They have a solid foundation for understanding area, surface area, and volume of geometric figures and have been introduced to statistical variability and distributions (adapted from Charles A. Dana Center 2012).

Critical Areas of Instruction

In grade seven, instructional time should focus on four critical areas: (1) developing understanding of and applying proportional relationships, including percentages; (2) developing understanding of operations with rational numbers and working with expressions and linear equations; (3) solving problems that involve scale drawings and informal geometric constructions and working with two- and three-dimensional shapes to solve problems involving area, surface area, and volume; and (4) drawing inferences about populations based on samples (National Governors Association Center for Best Practices, Council of Chief State School Officers 2010n). Students also work toward fluently solving equations of the form $px+q=r$ and $p(x+q)=r$.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade-level examples of focus, coherence, and rigor are indicated throughout the chapter.

The standards do not give equal emphasis to all content for a particular grade level. Cluster headings can be viewed as the most effective way to communicate the **focus** and **coherence** of the standards. Some clusters of standards require a greater instructional emphasis than others based on the depth of the ideas, the time needed to master those clusters, and their importance to future mathematics or the later demands of preparing for college and careers.

Table 7-1 highlights the content emphases at the cluster level for the grade-seven standards. The bulk of instructional time should be given to "Major" clusters and the standards within them, which are indicated throughout the text by a triangle symbol (▲). However, standards in the "Additional/Supporting" clusters should not be neglected; to do so would result in gaps in students' learning, including skills and understandings they may need in later grades. Instruction should reinforce topics in major clusters by using topics in the additional/supporting clusters and including problems and activities that support natural connections between clusters.

Teachers and administrators alike should note that the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences presented in a coherent manner (adapted from Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

Table 7-1. Grade Seven Cluster-Level Emphases

Ratios and Proportional Relationships — 7.RP

Major Clusters

- Analyze proportional relationships and use them to solve real-world and mathematical problems. (**7.RP.1–3▲**)

The Number System — 7.NS

Major Clusters

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers. (**7.NS.1–3▲**)

Expressions and Equations — 7.EE

Major Clusters

- Use properties of operations to generate equivalent expressions. (**7.EE.1–2▲**)
- Solve real-life and mathematical problems using numerical and algebraic expressions and equations. (**7.EE.3–4▲**)

Geometry — 7.G

Additional/Supporting Clusters

- Draw, construct, and describe geometrical figures and describe the relationships between them. (**7.G.1–3**)
- Solve real-life and mathematical problems involving angle measure, area, surface area, and volume. (**7.G.4–6**)

Statistics and Probability — 7.SP

Additional/Supporting Clusters

- Use random sampling to draw inferences about a population.[1] (**7.SP.1–2**)
- Draw informal comparative inferences about two populations.[2] (**7.SP.3–4**)
- Investigate chance processes and develop, use, and evaluate probability models. (**7.SP.5–8**)

Explanations of Major and Additional/Supporting Cluster-Level Emphases

Major Clusters (▲) — Areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than others based on the depth of the ideas, the time needed to master them, and their importance to future mathematics or the demands of college and career readiness.

Additional Clusters — Expose students to other subjects; may not connect tightly or explicitly to the major work of the grade.

Supporting Clusters — Designed to support and strengthen areas of major emphasis.

Note of caution: Neglecting material, whether it is found in the major or additional/supporting clusters, will leave gaps in students' skills and understanding and will leave students unprepared for the challenges they face in later grades.

Adapted from Smarter Balanced Assessment Consortium 2012b, 87.

1. The standards in this cluster represent opportunities to apply percentages and proportional reasoning. In order to make inferences about a population, one needs to apply such reasoning to the sample and the entire population.

2. Probability models draw on proportional reasoning and should be connected to the major work in those standards.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The MP standards represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students.

Although the description of the MP standards remains the same at all grade levels, the way these standards look as students engage with and master new and more advanced mathematical ideas does change. Table 7-2 presents examples of how the MP standards may be integrated into tasks appropriate for students in grade seven. (Refer to the Overview of the Standards Chapters for a complete description of the MP standards.)

Table 7-2. Standards for Mathematical Practice—Explanation and Examples for Grade Seven

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	In grade seven, students solve problems involving ratios and rates and discuss how they solved them. Students solve real-world problems through the application of algebraic and geometric concepts. They seek the meaning of a problem and look for efficient ways to represent and solve it. They may check their thinking by asking themselves "Does this make sense?" or "Can I solve the problem in a different way?" When students compare arithmetic and algebraic solutions to the same problem (**7.EE.4a▲**), they identify correspondences between different approaches.
MP.2 Reason abstractly and quantitatively.	Students represent a wide variety of real-world contexts through the use of real numbers and variables in mathematical expressions, equations, and inequalities. Students contextualize to understand the meaning of the number or variable as related to the problem and decontextualize to manipulate symbolic representations by applying properties of operations.
MP.3 Construct viable arguments and critique the reasoning of others.	Students construct arguments with verbal or written explanations accompanied by expressions, equations, inequalities, models, graphs, and tables. They further refine their mathematical communication skills through mathematical discussions in which they critically evaluate their own thinking and the thinking of other students. For example, as students notice when geometric conditions determine a unique triangle, more than one triangle, or no triangle (**7.G.2**), they have an opportunity to construct viable arguments and critique the reasoning of others. Students should be encouraged to answer questions such as these: "How did you get that?" "Why is that true?" "Does that always work?"
MP.4 Model with mathematics.	Seventh-grade students model real-world situations symbolically, graphically, in tables, and contextually. Students form expressions, equations, or inequalities from real-world contexts and connect symbolic and graphical representations. Students use experiments or simulations to generate data sets and create probability models. Proportional relationships present opportunities for modeling. For example, for modeling purposes, the number of people who live in an apartment building might be taken as proportional to the number of stories in the building. Students should be encouraged to answer questions such as "What are some ways to represent the quantities?" or "How might it help to create a table, chart, or graph?"

Table 7-2 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
MP.5 Use appropriate tools strategically.	Students consider available tools (including estimation and technology) when solving a mathematical problem and decide if particular tools might be helpful. For instance, students in grade seven may decide to represent similar data sets using dot plots with the same scale to visually compare the center and variability of the data. Students might use physical objects, spreadsheets, or applets to generate probability data and use graphing calculators or spreadsheets to manage and represent data in different forms. Teachers might ask, "What approach are you considering?" or "Why was it helpful to use _____?"
MP.6 Attend to precision.	Students continue to refine their mathematical communication skills by using clear and precise language in their discussions with others and in their own reasoning. Students define variables, specify units of measure, and label axes accurately. Students use appropriate terminology when referring to rates, ratios, probability models, geometric figures, data displays, and components of expressions, equations, or inequalities. Teachers might ask, "What mathematical language, definitions, or properties can you use to explain _____?"
MP.7 Look for and make use of structure.	Students routinely seek patterns or structures to model and solve problems. For instance, students recognize patterns that exist in ratio tables, making connections between the constant of proportionality in a table with the slope of a graph. Students apply properties to generate equivalent expressions and solve equations. Students compose and decompose two- and three-dimensional figures to solve real-world problems involving scale drawings, surface area, and volume. Students examine tree diagrams or systematic lists to determine the sample space for compound events and verify that they have listed all possibilities. Solving an equation such as $8 = 4\left(x - \frac{1}{2}\right)$ is easier if students can see and make use of structure, temporarily viewing $\left(x - \frac{1}{2}\right)$ as a single entity.
MP.8 Look for and express regularity in repeated reasoning.	In grade seven, students use repeated reasoning to understand algorithms and make generalizations about patterns. After multiple opportunities to solve and model problems, they may notice that $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$ and construct other examples and models that confirm their generalization. Students should be encouraged to answer questions such as "How would we prove that _____?" or "How is this situation both similar to and different from other situations using these operations?"

Adapted from Arizona Department of Education (ADE) 2010, Georgia Department of Education 2011, and North Carolina Department of Public Instruction (NCDPI) 2013b.

Standards-Based Learning at Grade Seven

The following narrative is organized by the domains in the Standards for Mathematical Content and highlights some necessary foundational skills from previous grade levels. It also provides exemplars to explain the content standards, highlight connections to Standards for Mathematical Practice (**MP**), and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. A triangle symbol (▲) indicates standards in the major clusters (see table 7-1).

Domain: Ratio and Proportional Relationships

A critical area of instruction in grade seven is developing an understanding and application of proportional relationships, including percentages. In grade seven, students extend their reasoning about ratios and proportional relationships in several ways. Students use ratios in cases that involve pairs of rational number entries and compute associated rates. They identify unit rates in representations of proportional relationships and work with equations in two variables to represent and analyze proportional relationships. They also solve multi-step ratio and percent problems, such as problems involving percent increase and decrease (University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2011c).

Ratios and Proportional Relationships 7.RP

Analyze proportional relationships and use them to solve real-world and mathematical problems.

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. *For example, if a person walks $1/2$ mile in each $1/4$ hour, compute the unit rate as the complex fraction $\frac{1/2}{1/4}$ miles per hour, equivalently 2 miles per hour.*
2. Recognize and represent proportional relationships between quantities.
   a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
   b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
   c. Represent proportional relationships by equations. *For example, if total cost **t** is proportional to the number **n** of items purchased at a constant price **p**, the relationship between the total cost and the number of items can be expressed as $t = pn$.*
   d. Explain what a point (x,y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0,0) and $(1,r)$ where r is the unit rate.

The concept of the *unit rate* associated with a ratio is important in grade seven. For a ratio $a:b$ with a and $b \neq 0$,[3] the unit rate is the *number* $\frac{a}{b}$. In grade six, students worked primarily with ratios involving whole-number quantities and discovered what it meant to have *equivalent ratios*. In grade seven, students find unit rates in ratios involving fractional quantities (**7.RP.1▲**). For example, when a recipe calls for $1\frac{1}{2}$ cups of sugar to 3 cups of flour, this results in a unit rate of $\frac{1\frac{1}{2}}{3} = \frac{3}{6}$. The fact that any pair of quantities in a proportional relationship can be divided to find the unit rate is useful when students solve problems involving proportional relationships, as this allows students to set up an equation with equivalent fractions and use reasoning about equivalent fractions. For a simple example, if a recipe with the same ratio as given above calls for 12 cups of flour and a student wants to know how much sugar to use, he could set up an equation that sets *unit rates* equal to each other—such as $\frac{1\frac{1}{2}}{3} = \frac{3}{6} = \frac{S}{12}$, where S represents the number of cups needed in the recipe.

3. Although it is possible to define *ratio* so that a can be zero, this will rarely happen in context, so the discussion proceeds with the assumption that both a and b are non-zero.

In grade six, students worked with many examples of proportional relationships and represented them numerically, pictorially, graphically, and with equations in simple cases. In grade seven, students continue this work, but they examine more closely the characteristics of proportional relationships. In particular, they begin to identify these facts:

- When proportional quantities are represented in a table, pairs of entries represent equivalent ratios.
- The graph of a proportional relationship lies on a straight line that passes through the point (0,0), indicating that when one quantity is 0, so is the other.[4]
- Equations of proportional relationships in a ratio of $a:b$ always take the form $y = k \cdot x$, where k is the constant $\frac{b}{a}$ if the variables x and y are defined so that the ratio $x:y$ is equivalent to $a:b$. (The number k is also known as the *constant of proportionality* [**7.RP.2▲**]).

Thus, a first step for students—one that is often overlooked—is to decide when and why two quantities are actually in a proportional relationship (**7.RP.2a▲**). Students can do this by checking the characteristics listed above or by using reasoning; for example, a runner's heart rate might increase steadily as she runs faster, but her heart rate when she is standing still is not 0 beats per minute, and therefore running speed and heart rate are not proportional.

The study of proportional relationships is a foundation for the study of functions, which is introduced in grade eight and continues through higher mathematics. In grade eight, students will understand that the proportional relationships they studied in grade seven are part of a broader group of linear functions. Linear functions are characterized by having a constant rate of change (the change in the outputs is a constant multiple of the change in the corresponding inputs). The following examples show students determining whether a relationship is proportional; notice the different methods used.

Ratios, Unit Rates, and Proportional Relationships

A ***ratio*** is a pair of non-negative numbers, $A:B$, which are not both 0. When there are A units of one quantity for every B units of another quantity, a *rate* associated with the ratio $A:B$ is $\frac{A}{B}$ units of the first quantity per 1 unit of the second quantity. (Note that the two quantities may have different units.)

The associated ***unit rate*** is the number $\frac{A}{B}$. The term *unit rate* refers to the numerical part of the rate; the "unit" is used to highlight the 1 in "per 1 unit of the second quantity." Unit rates should not be confused with *unit fractions* (which have a 1 in the numerator).

A ***proportional relationship*** is a collection of pairs of numbers that are in equivalent ratios. A ratio $A:B$ with $B \neq 0$ determines a proportional relationship, namely the collection of pairs (cA,cB), where c is positive. A proportional relationship is described by an equation of the form $y = kx$, where k is a positive constant, often called a *constant of proportionality*. The constant of proportionality, k, is equal to the value $\frac{B}{A}$. The graph of a proportional relationship lies on a ray with the endpoint at the origin.

Adapted from UA Progressions Documents 2011c.

4. The formal reasoning behind this principle and the next one is not expected until grade eight (see **8.EE.5▲** and **8.EE.6▲**). However, students will notice and informally use both principles in grade seven.

Examples: Determining Proportional Relationships **7.RP.2a▲**

1. If Josh is 20 and his niece Reina is 10, how old will Reina be when Josh is 40?

Solution: If students erroneously think that this is a proportional relationship, they may decide that Reina will be 20 when Josh is 40. However, it is not true that their ages change in a ratio of 20:10 (or 2:1). As Josh ages 20 years, so does Reina, so she will be 30 when Josh is 40. Students might further investigate this situation by graphing ordered pairs (a,b), where a is Josh's age and b is Reina's age at the same time. How does the graph differ from a graph of a proportional relationship?

2. Jaime is studying proportional relationships in class. He says that if it took two people 5 hours to paint a fence, then it must take four people 10 hours to paint a fence of the same size. Is he correct? Why or why not? Is this situation a proportional relationship? Why or why not?

Solution: No, Jaime is not correct—at least not if it is assumed that each person works at the same rate. If more people contribute to the work, then it should take less time to paint the fence. This situation is not a proportional relationship because the graph would not be a straight line emanating from the origin.

3. If 2 pounds of melon cost \$4.50 at the grocery store, would 7 pounds cost \$15.75?

Solution: Since a price per pound is typically constant at a grocery store, it stands to reason that there is a proportional relationship here:

$$\frac{\$4.50}{2 \text{ pounds}} = \frac{7\times(\$4.50)}{7\times(2 \text{ pounds})} = \frac{\$31.50}{14 \text{ pounds}} = \frac{\$31.50\div 2}{(14 \text{ pounds})\div 2} = \frac{\$15.75}{7 \text{ pounds}}$$

It makes sense that 7 pounds would cost \$15.75. (Alternatively, the unit rate is $\frac{\$4.50}{2} = \$2.25$, for a rate of \$2.25 per pound. At that rate, 7 pounds costs $7\times\$2.25 = 7\times\$2 + 7\times\$0.25$. This equals $\$14 + [4 \text{ quarters}] + [3 \text{ quarters}] = \$14 + \$1 + \$0.75$, or \$15.75.)

4. The table at right gives the price for different numbers of books. Do the numbers in the table represent a proportional relationship?

Price (\$)	No. of Books
4	1
7	2
10	3
13	4

Solution: If there were a proportional relationship, it should be possible to make equivalent ratios using entries from the table. However, since the ratios 4:1 and 7:2 are not equivalent, the table does not represent a proportional relationship. (Also, the unit rate [price per book] of the first ratio is $\frac{4}{1}$, or \$4.00, and the unit rate of the second ratio is $\frac{7}{2}$, or \$3.50.)

Adapted from ADE 2010 and NCDPI 2013b.

Focus, Coherence, and Rigor

Problems involving proportional relationships support mathematical practices as students make sense of problems (**MP.1**), reason abstractly and quantitatively (**MP.2**), and model proportional relationships (**MP.4**). For example, for modeling purposes, the number of people who live in an apartment building might be taken as proportional to the number of stories in the building.

Adapted from PARCC 2012.

As students work with proportional relationships, they write equations of the form $y = kx$, where k is a constant of proportionality (i.e., a unit rate). They see this unit rate as the amount of increase in y as x increases by 1 unit in a ratio table, and they recognize the unit rate as the vertical increase in a *unit rate triangle* (or *slope triangle*) with a horizontal side of length 1 for a graph of a proportional relationship.

Example **7.RP.2▲**

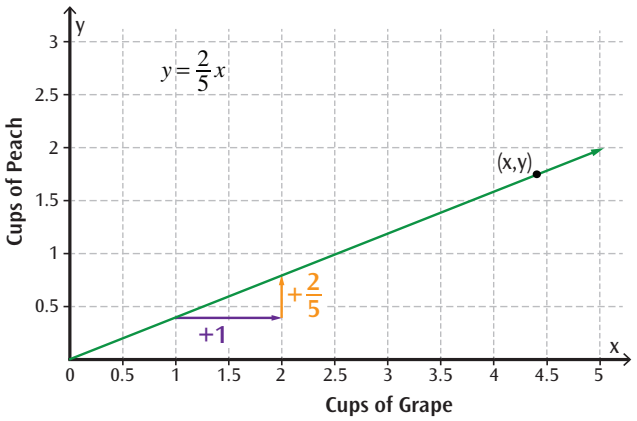
Representing Proportional Relationships. The following example from grade six is presented from a grade-seven perspective to show the progression from ratio reasoning to proportional reasoning.

A juice mixture calls for 5 cups of grape juice for every 2 cups of peach juice. Use a table to represent several different batches of juice that could be made by following this recipe. Graph the data in your table on a co-ordinate plane. Finally, write an equation to represent the relationship between cups of grape juice and cups of peach juice in any batch of juice made according to the recipe. Identify the unit rate in each of the three representations of the proportional relationship.

Using a Table. In grade seven, students identify pairs of values that include fractions as well as whole numbers. Thus, students include fractional amounts between 5 cups of grape juice and 2 cups of peach juice in their tables. They see that as amounts of cups of grape juice increase by 1 unit, the corresponding amounts of cups of peach juice increase by $\frac{2}{5}$ unit, so that if we add x cups of grape juice, then we would add $x \cdot \frac{2}{5}$ cups of peach juice. Seeing this relationship helps students to see the resulting equation, $y = \frac{2}{5}x$. Another way to derive the equation is by seeing $\frac{y}{x} = \frac{2}{5}$, and so multiplying each side by x would yield $x \cdot \left(\frac{y}{x}\right) = \left(\frac{2}{5}\right) \cdot x$, which results in $y = \frac{2}{5}x$.

	x Cups of Grape	y Cups of Peach
Batch A	0	0
Batch B	5	2
Batch C	1	$\frac{2}{5}$
Batch D	2	$2 \cdot \frac{2}{5}$
Batch E	3	$3 \cdot \frac{2}{5}$
Batch F	4	$4 \cdot \frac{2}{5}$
Any batch made according to the recipe	x	$x \cdot \frac{2}{5}$

Using a Graph. Students create a graph, realizing that even non-whole-number points represent possible combinations of grape and peach juice mixtures. They are learning to identify key features of the graph—in particular, that the resulting graph is a ray (i.e., contained in a straight line) emanating from the origin and that the point (0,0) is part of the data. They see the point $(1, \frac{2}{5})$ as the point corresponding to the unit rate, and they see that every positive horizontal movement of 1 unit (e.g., adding 1 cup of grape juice) results in a positive vertical movement of $\frac{2}{5}$ of a unit (e.g., adding $\frac{2}{5}$ cup of peach juice). The connection between this rate of change seen in the graph and the equation $y = \frac{2}{5}x$ should be made explicit for students, and they should test that every point on the graph is of the form $(x, \frac{2}{5}x)$.



Example: 7.RP.2 ***(continued)***

Deriving an Equation. Both the table and the graph show that for every 1 cup of grape juice added, $\frac{2}{5}$ cup of peach juice is added. Thus, starting with an empty bowl, when x cups of grape juice are added, $\frac{2}{5}x$ cup of peach juice must be added. On the graph, this corresponds to the fact that, when starting from (0,0), every movement horizontally of x units results in a vertical movement of $\frac{2}{5}x$ units. In either case, the equation becomes $y = \frac{2}{5}x$.

Adapted from UA Progressions Documents 2011c.

Students use a variety of methods to solve problems involving proportional relationships. They should have opportunities to solve these problems with strategies such as making tape diagrams and double number lines, using tables, using rates, and by relating proportional relationships to equivalent fractions as described above.

Examples: Proportional Reasoning in Grade Seven **7.RP.2▲**

Janet can sew 35 scarves in 2 hours. At this rate, how many scarves can she sew in 5 hours?

Solution Strategies

(a) **Using the Rate.** Since Janet can sew 35 scarves in 2 hours, this means she can sew at a rate of $35 \div 2 = 17.5$ scarves per hour. If she works for 5 hours, then she can sew

$$17.5 \text{ scarves per hour} \times 5 \text{ hours} = 87.5 \text{ scarves},$$

which means she can sew 87 scarves in 5 hours.

(b) **Setting Unit Rates Equal.** The unit rate in this case is $\frac{35}{2} = 17.5$. If C represents the number of scarves Janet can sew in 5 hours, then the following equation can be set up:

$$\frac{\text{Number of Scarves}}{\text{Number of Hours}} = \frac{35}{2} = \frac{C}{5}$$

$\frac{C}{5}$ also represents the unit rate. To solve this, we can reason that since $2 \times 2.5 = 5$, it must be true that $35 \times 2.5 = C$, yielding $C = 87.5$, which is interpreted to mean that Janet can sew 87 scarves in 5 hours.

Alternatively, one can see that the equation above is of the form $b = ax$, where a and b are rational numbers. In that case, $C = \frac{35}{2} \div \frac{1}{5}$.

(c) **Recognizing an Equation.** Students can reason that an equation relating the number of scarves, C, and the number of hours, h, takes the form $C = 17.5h$, so that C can be found by $C = 17.5(5) = 87.5$. Again, the answer is interpreted to mean that Janet can sew 87 scarves in 5 hours.

Adapted from ADE 2010.

A typical strategy for solving proportional relationship problems has been to "set up a proportion and cross-multiply." The Common Core State Standards move away from this strategy, instead prompting students to reason about solution strategies and why they work. Setting up an equation to solve a proportional relationship problem makes perfect sense if students understand that they are setting unit rates equal to each other. However, introducing the term *proportion* (or *proportion equation*) may

needlessly clutter up the curriculum; rather, students should see this as setting up an equation in a single variable. On the other hand, if after solving multiple problems by reasoning with equivalent fractions (as in strategy [b] above) students begin to see the pattern that $\frac{a}{b} = \frac{c}{d}$ precisely when $ad = bc$, then this is something to be examined, not avoided, and used as a general strategy if students are able to justify why they use it. Following are additional examples of proportional relationship problems.

Further Examples of Proportional Reasoning for Grade Seven **7.RP.2▲**

1. A truck driver averaged about 300 miles in 5.5 hours of driving. At the same rate, approximately how much more driving time will it take him to cover the remaining 1000 miles on his route?

Solution: Students might see the unit rate as $\frac{300}{5.5}$ and set up the following equation: $\frac{300}{5.5} = \frac{1000}{h}$

In this equation, h represents the number of driving hours needed to cover the remaining 1000 miles. Students might see that $1000 \div 300 = \frac{10}{3}$; so it must also be true that $h \div 5.5 = \frac{10}{3}$. This means that

$$h = \frac{10}{3} \times 5.5 = \frac{10}{3} \times \frac{11}{2} = \frac{110}{6} = 18\frac{1}{3}$$

Therefore, the truck driver has around 18 hours and 20 minutes of driving time remaining.

2. If $\frac{1}{2}$ gallon of paint covers $\frac{1}{6}$ of a wall, then how much paint is needed to cover the entire wall?

Solution: Students may see this as asking for the rate—that is, how much paint is needed per 1 wall. In that case, students would divide: $\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \bullet \frac{6}{1} = 3$, so that 3 gallons of paint will cover the entire wall.

Or, a student might see that *one* full wall could be represented by $\frac{1}{6} \times 6$, so to get the amount of paint needed to cover the entire wall, he would need to multiply the amount of paint by 6 also: $\frac{1}{2}$ gallon $\times 6 = 3$ gallons.

Adapted from Kansas Association of Teachers of Mathematics (KATM) 2012, 7th Grade Flipbook.

3. The recipe for Perfect Purple Paint calls for mixing $\frac{1}{2}$ cup blue paint with $\frac{1}{3}$ cup red paint. If a person wanted to mix blue and red paint in the same ratio to make 20 cups of Perfect Purple Paint, how many cups of blue paint and how many cups of red paint would be needed?

Solution (Strategy 1): "If I make 6 batches of purple, then that means I use 6 times more blue and red paint. This means I use $6 \bullet \frac{1}{2} = 3$ cups of blue and $6 \bullet \frac{1}{3} = 2$ cups of red, which yields a total of 5 cups of purple paint (i.e., 6 batches yields 5 cups). So to make 20 cups, I can multiply these amounts of blue and red by 4 to get 12 cups of blue and 8 cups of red."

Solution (Strategy 2): "One batch is $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ cup in volume. The fraction of one batch that is blue is then $\frac{1/2}{5/6} = \frac{1}{2} \div \frac{5}{6} = \frac{1}{2} \bullet \frac{6}{5} = \frac{6}{10}$. The fraction of one batch that is red is $\frac{1/3}{5/6} = \frac{1}{3} \div \frac{5}{6} = \frac{1}{3} \bullet \frac{6}{5} = \frac{6}{15}$. If I find these fractions of 20, that gives me how much blue and red to use: $\frac{6}{10} \bullet 20 = 12$ and $\frac{6}{15} \bullet 20 = 8$

This means I need 12 cups of blue and 8 cups of red."

Adapted from UA Progressions Documents 2011c.

Ratios and Proportional Relationships 7.RP

Analyze proportional relationships and use them to solve real-world and mathematical problems.

3. Use proportional relationships to solve multi-step ratio and percent problems. *Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.*

In grade six, students used ratio tables and unit rates to solve percent problems. In grade seven, students extend their work to solve multi-step ratio and percent problems (**7.RP.3▲**). They explain or show their work by using a representation (e.g., numbers, words, pictures, physical objects, or equations) and verify that their answers are reasonable. Models help students identify parts of the problem and how values are related (**MP.1, MP.3, MP.4**). For percentage increase and decrease, students identify the original value, determine the difference, and compare the difference in the two values to the starting value.

Examples: Multi-Step Percent Problems **7.RP.3▲**

1. A sweater is marked down 30%. The original price was $37.50. What is the price of the sweater after it is marked down?

Solution: A simple diagram like the one shown can help students see the relationship between the original price, the amount taken off, and the sale price of the sweater. In this case, students can solve the problem either by finding 70% of $37.50, or by finding 30% of $37.50 and subtracting it.

$37.50 Original price of sweater	
30% of 37.50	70% of 37.50 Sale price of sweater

Seeing many examples of problems such as this one helps students to see that discount problems take the form $(100\% - r\%) \cdot p = d$, where r is the amount of reduction, p is the original price, and d is the discounted price.

2. A shirt is on sale for 40% off. The sale price is $12. What was the original price?

Solution: Again, a simple diagram can show the relationship between the sale price and the original price. In this case, what is known is the sale price, $12, which represents 60% of the original price. A simple equation, $0.6p = 12$, can be set up and solved for p: $p = 12 \div 0.6 = 20$ The original price was $20.

Discount: 40% of original price	Sale Price: $12 60% of original price
Original Price (p)	

3. Your bill at a restaurant before tax is $52.60. The sales tax is 8%. You decide to leave a tip of 20% on the pre-tax amount. How much is the tip you'll leave? What is the total cost of dinner, including tax and tip?

Solution: To calculate the tip, students find $52.60 \cdot 0.20 = 10.52$, so the tip is $10.52. The tax is found similarly: $52.60 \cdot 0.08 \approx 4.21$. This means the total bill is $\$52.60 + \$10.52 + \$4.21 = \$67.33$. Alternatively, students may realize that they are finding 128% of the pre-tax bill, and compute $\$52.60 \cdot 1.28 \approx \$67.33$.

Adapted from ADE 2010 and NCDPI 2013b.

Problems involving percentage increase or percentage decrease require careful attention to the referent whole. For example, consider the difference between these two problems:

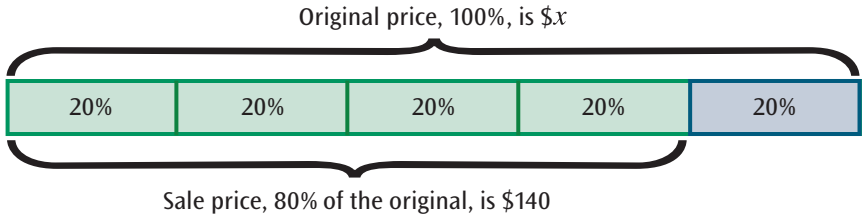
- **Skateboard Problem 1.** After a 20% discount, Eduardo paid $140 for a SuperSick skateboard. What was the price before the discount?
- **Skateboard Problem 2.** A SuperSick skateboard costs $140 today, but the price will increase by 20% tomorrow. What will the new price be after the increase?

The solutions to these two problems are presented below and are different because the 20% refers to different wholes (or 100% amounts). In the first problem, the 20% represents 20% of the larger pre-discount amount, whereas in the second problem, the 20% is 20% of the smaller pre-increase amount.

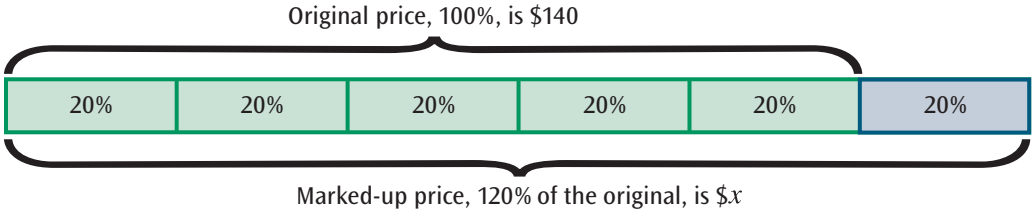
Solutions to Skateboard Problems **7.RP.3▲**

Skateboard Problem 1. The problem can be represented with a tape diagram. Students reason that since 80% is $140, 20% is $\$140 \div 4 = \$35$, so 100% is then $5 \times \$35 = \$175$.

Equivalently, $0.80 \bullet x = 140$, so that $x = 140 \div 0.08$, or $x = 140 \div \frac{4}{5} = 140 \times \frac{5}{4} = 35 \times 5 = 175$.



Skateboard Problem 2. This problem can be represented with a tape diagram as well. Students can reason that since 100% is $140, 20% is $\$140 \div 5 = 28$, so 120% is then $6 \times \$28 = \$168$. Equivalently, $x = (1.20)(140)$, so $x = 168$.



Adapted from UA Progressions Documents 2011c.

A detailed discussion of ratios and proportional relationships is provided online at http://commoncoretools.files.wordpress.com/2012/02/ccss_progression_rp_67_2011_11_12_corrected.pdf (accessed January 8, 2015) [UA Progressions Documents 2011c].

Domain: The Number System

In grade six, students completed their understanding of division of fractions and achieved fluency with multi-digit division and multi-digit decimal operations. They also worked with concepts of positive and negative rational numbers. They learned about signed numbers and the types of quantities that can be represented with these numbers. Students located signed numbers on a number line and, as a result

of this study, should have concluded that the negative side of the number line is a mirrorlike reflection of the positive side. For example, by reasoning that the reflection of a reflection is the thing itself, students will have learned that $-(-a)=a$. (Here a may be positive, negative, or zero.) Grade-six students also learned about absolute value and ordering of rational numbers, including in real-world contexts. In grade seven, a critical area of instruction is developing an understanding of operations with rational numbers. Grade-seven students extend addition, subtraction, multiplication, and division to all rational numbers by applying these operations to both positive and negative numbers.

> The rational numbers are an arithmetic system that includes 0, positive and negative whole numbers, and fractions. Wherever the term *rational numbers* is used, numbers of all types are implied, including fractions in decimal notation.

Adding, subtracting, multiplying, and dividing rational numbers is the culmination of numerical work with the four basic operations. The number system continues to develop in grade eight, expanding to become the real numbers by the introduction of irrational numbers. Because there are no specific standards for arithmetic with rational numbers in later grades—and because so much other work in grade seven depends on that arithmetic—fluency in arithmetic with rational numbers should be a primary goal of grade-seven instruction (adapted from PARCC 2012).

The Number System — 7.NS

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

1. Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.
   a. Describe situations in which opposite quantities combine to make 0. *For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged.*
   b. Understand $p+q$ as the number located a distance $|q|$ from p in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.
   c. Understand subtraction of rational numbers as adding the additive inverse, $p-q=p+(-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.
   d. Apply properties of operations as strategies to add and subtract rational numbers.

In grade six, students learned that the absolute value of a rational number is its distance from zero on the number line. In grade seven, students represent addition and subtraction with positive and negative rational numbers on a horizontal or vertical number line diagram (**7.NS.1a–c▲**). Students add and subtract, understanding $p+q$ as the number located a distance $|q|$ from p on a number line, in the positive or negative direction, depending on whether q is positive or negative. They demonstrate that a number and its opposite have a sum of 0 (i.e., they are additive inverses) and understand subtraction of rational numbers as adding the additive inverse (**MP.2, MP.4, MP.7**).

Students' work with signed numbers began in grade six, where they experienced situations in which positive and negative numbers represented (for example) credits or debits to an account, positive or negative charges, or increases or decreases, all relative to 0. Now, students realize that in each of these situations, a positive quantity and negative quantity of the same absolute value add to make 0 (**7.NS.1a▲**). For instance, the positive charge of 5 protons would neutralize the negative charge of 5 electrons, and we represent this in the following way:[5]

$$(+5)+(-5)=0$$

Students recognize that +5 and –5 are "opposites" as described in grade six, located the same distance from 0 on a number line. But they reason further that a number, a, and its opposite, $-a$, always combine to make 0:

$$a+(-a)=0$$

This crucial fact lays the foundation for understanding addition and subtraction of signed numbers.

For the sake of simplicity, many of the examples that follow involve integers, but students' work with rational numbers should include rational numbers in different forms—positive and negative fractions, decimals, and whole numbers (including combinations). Integers might be used to introduce the ideas of signed-number operations, but student work and practice should not be limited to integer operations. If students learn to compute $4+(-8)$ but not $4+\left(-\frac{1}{3}\right)$, then they are not learning the rational number system.

Addition of Rational Numbers

Through experiences starting with whole numbers and their opposites (i.e., starting with integers only), students can develop the understanding that like quantities can be combined. That is, two positive quantities combine to become a "more positive" quantity, as in $(+5)+(+7)=+12$, and two negative quantities combine to become a "more negative" quantity, as in $(-2)+(-10)=-12$. When addition problems have mixed signs, students see that positive and negative quantities combine as necessary to partially make zeros (i.e., they "cancel" each other), and the appropriate amount of positive or negative charge remains.

Examples: Adding Signed Rational Numbers **7.NS.1b▲**

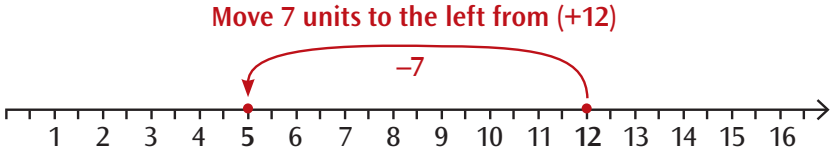
Note: The "neutral pair" approach in these examples is meant to show where the answer comes from; it is not meant to be an efficient algorithm for adding rational numbers.

1. $(+12)+(-7)=(+5)+(+7)+(-7)=(+5)+(0)=+5$
2. $(-12.55)+(+10.50)=(-2.05)+(-10.50)+(+10.50)=(-2.05)+(0)=-2.05$
3. $\left(+\frac{17}{2}\right)+\left(-\frac{9}{2}\right)=\left(+\frac{8}{2}\right)+\left(+\frac{9}{2}\right)+\left(-\frac{9}{2}\right)=\left(+\frac{8}{2}\right)+(0)=+\frac{8}{2}=+4$

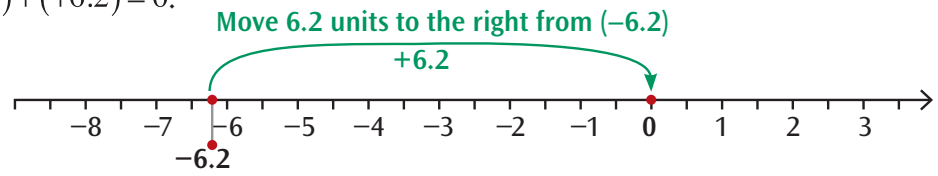
5. Teachers may wish to temporarily include the plus sign (+) to indicate positive numbers and distinguish them clearly in problems. These signs should eventually be dropped, as they are not commonly used.

Eventually, students come to realize that when adding two numbers with different signs, the sum is equal to the difference of the absolute values of the two numbers and has the same sign as the number with the larger absolute value. This understanding eventually replaces the kinds of calculations shown above, which are meant to illustrate concepts rather than serving as practical computation methods.

When students use a number line to represent the addition of integers, they can develop a general understanding that the sum $p+q$ is the number found when moving a total of $|q|$ units from p to the right if q is positive, and to the left if q is negative (**7.NS.1b▲**). The number line below represents $(+12)+(-7)$:



The concept is particularly transparent for quantities that combine to become 0, as illustrated in the example $(-6.2)+(+6.2)=0$:



Subtraction of Rational Numbers

When subtracting rational numbers, the most important concept for students to grasp is that $p-q$ gives the same result as $p+(-q)$; that is, subtracting q is equivalent to adding the opposite of q. Students have most likely already noticed that with sums such as $10+(-2)$, the result was the same as finding the difference, $10-2$. For subtraction of quantities with the same sign, teachers may find it helpful to employ typical understandings of subtraction as "taking away" or comparing to an equivalent addition problem, as in $(-12)-(-7)$ meaning to "take away –7 from –12," and compare this with $(-12)+7$. With an understanding that these numbers represent negative charges, the answer of –5 is arrived at fairly easily. However, by comparing this subtraction expression with the addition expression $(-12)+7$, students see that both result in –5. Through many examples, students can generalize these results to understand that $p-q=p+(-q)$ [**7.NS.1c▲**].

Examples: Subtracting Signed Rational Numbers **7.NS.1c▲**

1. Students interpret $15-9$ as taking away 9 positive units from 15 positive units. Students should compare this with $15+(-9)$ to see that both result in 6.
2. Students interpret $(20.5+(-17.5))$ as a credit and debit example. They compare this with $20.5-17.5$ and see that they arrive at the same result.
3. Students can use the relationship between addition and subtraction that they learned in previous grades: namely, that $a-b=c$ if and only if $c+b=a$. For example, they can use this to reason that since $10+(-1)=9$, it must be true that $1-(-9)=10$. They compare this with $1+9$ and realize that both yield the same result.
4. Students can see subtraction as a form of comparison, particularly visible on a horizontal or vertical number line. For example, they interpret $9-(-13)$ in this way: "How many degrees warmer is a temperature of 9°C compared to a temperature of –13°C?

Common Concrete Models for Addition and Subtraction of Rational Numbers

Several different concrete models may be used to represent rational numbers and operations with rational numbers. It is important for teachers to understand that all such concrete models have advantages and disadvantages, and therefore care should be taken when introducing these models to students. Not every model will lend itself well to representing every aspect of operations with rational numbers. Brief descriptions of some common concrete models are provided below.

Common Concrete Models for Representing Signed Rational Numbers **7.NS.1d▲ (MP.5)**

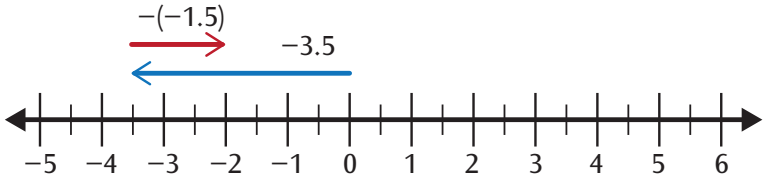
1. **Number Line Models (Vector Models).** A number line is used to represent the set of all rational numbers, and directed line segments (i.e., *vectors,* which look like arrows) are used to represent numbers. The length of the arrow is the absolute value of the number, and the direction of the arrow tells the sign of the number. Thus, the arrow emanating from 0 to –3.5 on the number line represents the number –3.5.

   Addition is then represented by placing arrows head to tail and looking at the number to which the final arrow points.

   Number Line Model for $(-3.5)+(-1.5)=-5$

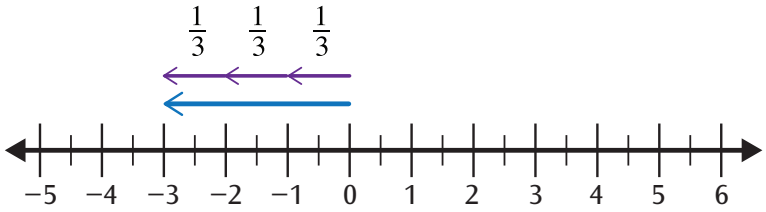
   Subtraction is equivalent to adding the opposite, so we can represent $a-b$ by reversing the arrow for b and then adding it to a.

   Number Line Model for $(-3.5)-(-1.5)=-2$



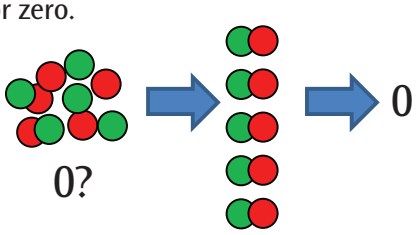
   Multiplication is interpreted as scaling. For example, the product $\frac{1}{3}(-3)$ can be interpreted as a vector one-third the length of the vector (-3) in the same direction. That is, $\frac{1}{3}(-3)=-1$.

   Number Line Model for $\frac{1}{3}(-3)=-1$



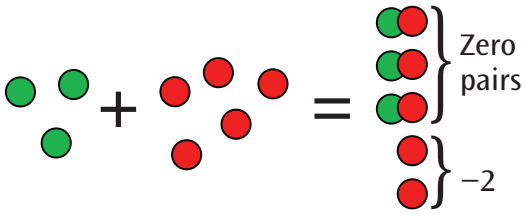
Common Concrete Models for Representing Signed Rational Numbers—7.NS.1d *(continued)*

2. **Colored-Chip Models.** Chips of one color are used to represent positive units, and chips of another color are used to represent negative units (note that plus and minus signs are sometimes written on the chips). These models make it easy to represent units that are combined, and they are especially illustrative when positive and negative units are combined to create "zero pairs" (sometimes referred to as *neutral pairs*), representing that $a+(-a)=0$. A disadvantage of these models is that multiplication and division are more difficult to represent, and chip models are typically used only to represent integer quantities (i.e., it is difficult to extend them to fractional quantities). Also, some imagination is required to view a pile of colored chips as representing "nothing" or zero.



An equal number of positive and negative chips form zero pairs, representing zero.

Colored-Chip Model for $3+(-5)=-2$



3. Money Account Models. These models are used to represent addition and subtraction of rational numbers, although such numbers typically take the form of decimal dollar amounts. Positive amounts contribute to the balance, while negative amounts subtract from it. Subtracting negatives must be interpreted delicately here, as in thinking of $-(-\$35.00)$ as "The bank forgave the negative balance of $35.00," which one would interpret as receiving a credit of $35.00.

Focus, Coherence, and Rigor

Teachers are encouraged to logically build up the rules for operations with rational numbers (**7.NS.1▲**), as modeled in the narratives on addition and subtraction, making use of the structure of the number system (**MP.7**). Students should engage in class or small-group discussions about the meaning of operations until a conceptual understanding is reached (**MP.3**). Building a foundation in using the structure of numbers with addition and subtraction will also help students understand the operations of multiplication and division of signed numbers (**7.NS.2▲**). Sufficient practice is required so that students can compute sums and products of rational numbers in all cases and apply these concepts to real-world situations.

Grade seven marks the culmination of the arithmetic learning progression for rational numbers. By the end of seventh grade, students' arithmetic repertoire includes adding, subtracting, multiplying, and dividing with rational numbers including whole numbers, fractions, decimals, and signed numbers.

The Number System **7.NS**

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

2. Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
   a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.
   b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-\left(p/q\right) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real-world contexts.
   c. Apply properties of operations as strategies to multiply and divide rational numbers.
   d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.
3. Solve real-world and mathematical problems involving the four operations with rational numbers.[6]

Students continue to develop their understanding of operations with rational numbers by seeing that multiplication and division can be extended to signed rational numbers (**7.NS.2▲**). For instance, in an account balance model, (–3)($40.00) may be thought of as a record of 3 groups of debits (indicated by the negative sign) of $40.00 each, resulting in a total contribution to the balance of –$120.00. In a vector model, students can interpret the expression (2.5)(–7.5) as the vector that points in the same direction as the vector representing –7.5, but is 2.5 times as long. Interpreting multiplication of two negatives in everyday terms may be troublesome, since negative money cannot be withdrawn from a bank. In a vector model, multiplying by a negative number reverses the direction of the vector (in addition to any stretching or compressing of the vector). Division is often difficult to interpret in everyday terms as well, but can always be understood mathematically in terms of multiplication—specifically, as multiplying by the reciprocal.

Multiplication of Signed Rational Numbers

In general, multiplication of signed rational numbers is performed as with fractions and whole numbers, but according to the following rules for determining the sign of the product:

1. Different signs: $(-a) \times b = -ab$
2. Same signs: $(-a) \times (-b) = ab$

In these equations, both a and b can be positive, negative, or zero. Of particular importance is that $-1 \cdot a = -a$. That is, multiplying a number by a negative 1 yields the opposite of the number. The first of these rules can be understood in terms of models. The second can be understood as being a result of properties of operations (refer to "A Derivation of the Fact That $(-1)(-1) = 1$," below). Students may also become more comfortable with rule 2 by examining patterns in products of signed numbers, such as in the following example, although this does not constitute a valid mathematical proof.

6. Computations with rational numbers extend the rules for manipulating fractions to complex fractions.

Example: Using Patterns to Investigate Products of Signed Rational Numbers 7.NS.2a▲

Students can look for patterns in a table like the one below. Reading from left to right, it is natural to conjecture that the missing numbers in the table should be 5, 10, 15, and 20.

5×4	5×3	5×2	5×1	5×0	$5\times(-1)$	$5\times(-2)$	$5\times(-3)$	$5\times(-4)$
20	15	10	5	0	-5	-10	-15	-20
$(-5)\times4$	$(-5)\times3$	$(-5)\times2$	$(-5)\times1$	$(-5)\times0$	$(-5)\times(-1)$	$(-5)\times(-2)$	$(-5)\times(-3)$	$(-5)\times(-4)$
-20	-15	-10	-5	0	?	?	?	?

Ultimately, if students come to understand that $(-1)(-1)=1$, then the fact that $(-a)(-b)=ab$ follows immediately using the associative and commutative properties of multiplication:

$$(-a)(-b)=(-1\cdot a)(-1\cdot b)=(-1)a(-1)b=(-1)(-1)ab=1\cdot ab=ab$$

After arriving at a general understanding of these two rules for multiplying signed numbers, students can multiply any rational numbers by finding the product of the absolute values of the numbers and then determining the sign according to the rules.

A Derivation of the Fact That (–1)(–1) = 1

Students are reminded that addition and multiplication are related by an important algebraic property, the *distributive property of multiplication over addition:*

$$a(b+c)=ab+ac$$

This property is valid for all numbers a, b, and c, and it plays an important role in the derivation here and throughout mathematics. The basis of this derivation is that the *additive inverse* of the number –1 (that is, the number you add to –1 to obtain 0) is equal to 1. We observe that if we add (–1)(–1) and (–1), the distributive property reveals something interesting:

$$\begin{aligned}(-1)(-1)+(-1) &= (-1)(-1)+(-1)(1) && \text{[Because } (-1)=(-1)(1)\text{]}\\ &= (-1)\big[(-1)+1\big] && \text{[By the distributive property]}\\ &= (-1)(0)=0 && \text{[Because } (-1)+1=0\text{]}\end{aligned}$$

Thus, when adding the quantity (–1)(–1) to –1, the result is 0. This implies that (–1)(–1) is the additive inverse of –1, which is 1. This completes the derivation.

Division of Rational Numbers

The relationship between multiplication and division allows students to infer the sign of the quotient of two rational numbers. Otherwise, division is performed as usual with whole numbers and fractions, with the sign to be determined.

Examples: Determining the Sign of a Quotient 7.NS.2b▲

If $x = (-16) \div (-5)$, then $x \cdot (-5) = -16$. It follows that whatever the value of x is, it must be a positive number. In this case, $x = \frac{-16}{-5} = \frac{16}{5}$. This line of reasoning can be used to justify the general fact that for rational numbers p and q (with $q \neq 0$), $\frac{-p}{-q} = \frac{p}{q}$.

If $y = \frac{-0.2}{50}$, then $y \cdot 50 = -0.2$. This implies that y must be negative, and therefore $y = \frac{-0.2}{50} = -\frac{2}{500} = -\frac{4}{1000} = -0.004$.

If $z = \frac{0.2}{-50}$, then $z \cdot (-50) = 0.2$. This implies that z must be negative, and thus $z = \frac{0.2}{-50} = -\frac{2}{500} = -\frac{4}{1000} = -0.004$.

The latter two examples above show that $\frac{-0.2}{50} = \frac{0.2}{-50}$. In general, it is true that $\frac{-p}{q} = \frac{p}{-q}$ for rational numbers (with $q \neq 0$). Students often have trouble interpreting the expression $-\left(\frac{p}{q}\right)$. To begin with, this should be interpreted as meaning "the opposite of the number $\frac{p}{q}$." Considering a specific example, it should be noted that because $-\left(\frac{5}{2}\right) = -(2.5)$ is a negative number, the product of 4 and $-\left(\frac{5}{2}\right)$ must also be a negative number. We determine that $4 \cdot \left(-\left(\frac{5}{2}\right)\right) = -10$. On the other hand, this equation implies that $-\left(\frac{5}{2}\right) = -10 \div 4$. In other words, $-\left(\frac{5}{2}\right) = \frac{-10}{4} = \frac{-5}{2}$. A similar line of reasoning shows that $-\left(\frac{5}{2}\right) = \frac{5}{-2}$. Examples such as these help justify that $-\left(\frac{p}{q}\right) = \frac{-p}{q} = \frac{p}{-q}$ [**7.NS.2b▲**].[7]

Students solve real-world and mathematical problems involving positive and negative rational numbers while learning to compute sums, differences, products, and quotients of rational numbers. They also come to understand that every rational number can be written as a decimal with an expansion that eventually repeats or terminates (i.e., eventually repeating with zeros [**7.NS.2c–d▲, 7.NS.3▲**] [**MP.1, MP.2, MP.5, MP.6, MP.7, MP.8**]).

Examples of Rational-Number Problems 7.NS.3▲

1. During a phone call, Melanie was told of the most recent transactions in her company's business account. There were deposits of $1,250 and $3,040.57, three withdrawals of $400 each, and the bank removed two separate $35 penalties to the account that resulted from the bank's errors. Based on this information, how much did the balance of the account change?

Solution: The deposits are considered positive changes to the account, the three withdrawals are considered negative changes, and the removal of two penalties of $35 each may be thought of as subtracting debits to the account. The total change to the balance could be represented in this way:

$$\$1{,}250.00 + \$3{,}040.57 - 3(\$400.00) - 2(-\$35.00) = \$3{,}160.57.$$

Thus, the balance of the account increased by $3,160.57.

7. This also shows why it is unambiguous to write $-\frac{p}{q}$ and drop the parentheses.

Examples: 7.NS.3 *(continued)*

2. Find the product $(-373) \cdot 8$.

Solution: "I know that the first number has a factor of (–1) in it, so the product will be negative. Then I just need to find $373 \cdot 8 = 2400 + 560 + 24 = 2984$. So $(-373) \cdot 8 = -2984$."

3. Find the quotient $\left(-\frac{25}{28}\right) \div \left(-\frac{5}{4}\right)$.

Solution: "I know that the result is a positive number. This looks like a problem where I can divide the numerator and denominator: $\frac{25}{28} \div \frac{5}{4} = \frac{25 \div 5}{28 \div 4} = \frac{5}{7}$. The quotient is $\frac{5}{7}$."

4. Represent each of the following problems with a diagram, a number line, and an equation, and solve each problem.

   (a) A weather balloon is 100,000 feet above sea level, and a submarine is 3 miles below sea level, directly under the weather balloon. How far apart are the submarine and the weather balloon?

   (b) John was $3.75 in debt, and Mary had $0.50. John found some money in his old jacket and paid his debt. Afterward, he and Mary had the same amount of money. How much money was in John's jacket?

Domain: Expressions and Equations

In grade six, students began the study of equations and inequalities and methods for solving them. In grade seven, students build on this understanding and use the arithmetic of rational numbers as they formulate expressions and equations in one variable and use these equations to solve problems. Students also work toward fluently solving equations of the form $px + q = r$ and $p(x + q) = r$.

Expressions and Equations — 7.EE

Use properties of operations to generate equivalent expressions.

1. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.
2. Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, $a + 0.05a = 1.05a$ means that "increase by 5%" is the same as "multiply by 1.05."*

This cluster of standards calls for students to work with linear expressions where the distributive property plays a prominent role (**7.EE.1▲**). A fundamental understanding is that the distributive property works "on the right" as well as "on the left," in addition to "forward" and "backward." That is, students should have opportunities to see that for numbers a, b, and c and x, y, and z:

$$a(b + c) = ab + ac \text{ and } ab + ac = a(b + c)$$
$$(x + y)z = xz + yz \text{ and } xz + yz = (x + y)z$$

Students combine their understanding of parentheses as denoting single quantities with the standard order of operations, operations with rational numbers, and the properties above to rewrite expressions in different ways (**7.EE.2▲**).

Common Misconceptions: Working with the Distributive Property 7.EE.2▲

Students see expressions like $7-2(8-1.5x)$ and realize that the expression $(8-1.5x)$ is treated as a separate quantity in its own right, being multiplied by 2 and the result being subtracted from 7 (**MP.7**). Students may mistakenly come up with the expressions below, and each case offers a chance for class discussion about why it is not equivalent to the original (**MP.3**):

- $5(8-1.5x)$, subtracting $7-2$ without realizing the multiplication must be done first
- $7-2(6.5x)$, erroneously combining 8 and $-1.5x$ by neglecting to realize that these are not like terms
- $7-16-3x$, by misapplying the distributive property or not being attentive to the rules for multiplying negative numbers

Students should have the opportunity to see this expression as equivalent to both $7+(-2)(8-1.5x)$ and $7-(-2(8-1.5x))$, which can aid in seeing the correct way to handle the –2 part of the expression.

Note that the standards do not expressly refer to "simplifying" expressions. Simplifying an expression is a special case of generating equivalent expressions. This is not to say that simplifying is never important, but whether one expression is "simpler" than another to work with often depends on the context. For example, the expression $50+(x-500)\cdot 0.20$ represents the cost of a phone plan wherein the base cost is \$50 and any minutes over 500 cost \$0.20 per minute (valid for $x \geq 500$). However, it is more difficult to see how the equivalent expression $0.20x-50$ also represents the cost of this phone plan.

Focus, Coherence, and Rigor

The work in standards **7.EE.1–2▲** is closely connected to standards **7.EE.3–4▲**, as well as the multi-step proportional reasoning problems in the domain Ratios and Proportional Relationships (**7.RP.3▲**). Students' work with rational-number arithmetic (**7.NS▲**) is particularly relevant when they write and solve equations (**7.EE▲**). Procedural fluency in solving these types of equations is an explicit goal of standard **7.EE.4a.**

As students become familiar with multiple ways of writing an expression, they also learn that different ways of writing expressions can serve varied purposes and provide different ways of seeing a problem. In example 3 below, the connection between the expressions and the figure emphasizes that both represent the same number, and the connection between the structure of each expression and a method of calculation emphasizes the fact that expressions are built from operations on numbers (adapted from UA Progressions Documents 2011d).

Examples: Working with Expressions 7.EE.2▲

1. A rectangle is twice as long as it is wide. Find as many different ways as you can to write an expression for the perimeter of such a rectangle.

Solution: If W represents the width of the rectangle and L represents the length, then the perimeter could be expressed as $L+W+L+W$. This could be rewritten as $2L+2W$. If it is known that $L=2W$, the perimeter could be represented by $W+W+2W+2W$, which could be rewritten as $6W$. Alternatively, if $W=\frac{L}{2}$, the perimeter could be given in terms of the length as $L+L+\frac{L}{2}+\frac{L}{2}$, which could be rewritten as $3L$.

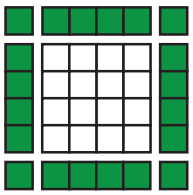
Adapted from ADE 2010.

Examples: 7.EE.2 ***(continued)***

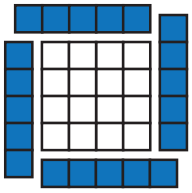
2. While Chris was driving a Canadian car, he figured out a way to mentally convert the outside temperature that the car displayed in degrees Celsius to degrees Fahrenheit. This was his method: "I took the temperature it showed and doubled it. Then I subtracted one-tenth of that doubled amount. Finally, I added 32 to get the Fahrenheit temperature." The standard expression for finding the temperature in degrees Fahrenheit when the Celsius reading is known is $\frac{9}{5}C+32$, where C is the temperature in degrees Celsius. Was Chris's method correct?

Solution: If C is the temperature in degrees Celsius, then the first step in Chris's calculation was to find $2C$. Then, he subtracted one-tenth of that quantity, which yielded $\frac{1}{10}(2C)$. Finally, he added 32. The resulting expression was $2C-\frac{1}{10}(2C)+32$. This could be rewritten as $2C-\frac{1}{5}C+32$. Combining the first two terms, we got $2C-\frac{1}{5}C+32=\left(2-\frac{1}{5}\right)C+32=\left(\frac{10}{5}-\frac{1}{5}\right)C+32=\frac{9}{5}C+32$. Chris's calculation was correct.

3. In the well-known "Pool Border Problem," students are asked to determine the number of tiles needed to construct a border for a pool (or grid) of size $n\times n$, represented by the white tiles in the figure. Students may first examine several examples and organize their counting of the border tiles, after which they can be asked to develop an expression for the number of border tiles, B (**MP.8**). Many different expressions are correct, all of which are equivalent to $4n+4$. However, different expressions arise from different ways of seeing the construction of the border. A student who sees the border as four sides of length n plus four corners might develop the expression $4n+4$, while a student who sees the border as four sides of length $n+1$ may find the expression $4(n+1)$. It is important for students to see many different representations and understand that these representations express the same quantity in different ways (**MP.7**).



$B=4n+4$



$B=4(n+1)$

Adapted from NCDPI 2013b.

Expressions and Equations — 7.EE

Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

3. Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. *For example: If a woman making $25 an hour gets a 10% raise, she will make an additional $1/10$ of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar $9\,3/4$ inches long in the center of a door that is $27\,1/2$ inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.*

4. Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

   a. Solve word problems leading to equations of the form $px+q=r$ and $p(x+q)=r$ where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. *For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?*

   b. Solve word problems leading to inequalities of the form $px+q>r$ or $px+q<r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. *For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.*

By grade seven, students begin to see whole numbers and their opposites, as well as positive and negative fractions, as belonging to a single system of rational numbers. Students solve multi-step problems involving rational numbers presented in various forms (whole numbers, fractions, and decimals), assessing the reasonableness of their answers (**MP.1**), and they solve problems that result in basic linear equations and inequalities (**7.EE.3–4▲**). This work is the culmination of many progressions of learning in arithmetic, problem solving, and mathematical practices.

Examples: Solving Equations and Inequalities **7.EE.3–4▲ (MP.2, MP.4, MP. 7)**

1. The youth group is going on a trip to the state fair. The trip costs $52.50 per student. Included in that price is $11.25 for a concert ticket and the cost of 3 passes, 2 for rides and 1 for game booths. Each of the passes costs the same price. Write an equation representing the cost of the trip, and determine the price of 1 pass.

Solution: Students can represent the situation with a tape diagram, showing that $3p+11.25$ represents the total cost of the trip if p represents the price of each pass. Students find the equation $3p+11.25=52.50$. They see the expression on the left side of the equation as some quantity $(3p)$ plus 11.25 equaling 52.50.

$52.50			
p	p	p	$11.25

In that case, the equation $52.50-11.25=41.25$ represents that quantity, by the relationship between addition and subtraction. So $3p=41.25$, which means that $p=41.25\div 3=13.75$. Thus, each pass costs $13.75.

2. The student-body government initiates a campaign to change the school mascot. The school principal has agreed to change the mascot if two-thirds of the student body plus 1 additional student vote for the change. The required number of votes is 255. How many students attend the school?

Solution: If S represents the number of students who attend the school, then $\frac{2}{3}S$ represents two-thirds of the vote, and $\frac{2}{3}S+1$ is one more than this. Since the required number of votes is 255, we can write $\frac{2}{3}S+1=255$. The quantity $\frac{2}{3}S$ plus one gives 255, so it follows that $\frac{2}{3}S=254$. To solve this, we can find

$$\frac{2}{3}S=254$$
$$S=254\div\frac{2}{3}$$
$$S=\frac{254}{1}\bullet\frac{3}{2}$$
$$S=\frac{762}{2}$$
$$S=381$$

or

$$S=254\div\frac{2}{3}=\frac{254}{1}\bullet\frac{3}{2}=\frac{762}{2}=381$$

Thus, there are 381 total students.

Alternatively, students may solve the equation $\frac{2}{3}S=254$ by multiplying each side by $\frac{3}{2}$, giving $\frac{3}{2}\bullet\frac{2}{3}S=254\bullet\frac{3}{2}$, and since $\frac{3}{2}\bullet\frac{2}{3}\bullet S=1\bullet S$, we get $S=254\bullet\frac{3}{2}=\frac{762}{2}=381$.

Examples: 7.EE.3–4 ***(continued)***

3. Florencia can spend at most \$60 on clothes. She wants to buy a pair of jeans for \$22 and spend the rest on T-shirts. Each shirt costs \$8. Write an inequality for the number of T-shirts she can purchase.

Solution: If t represents the number of T-shirts Florencia buys, then an expression for the total amount she spends on clothes is $8t + 22$, since each T-shirt costs \$8. The term *at most* might be new to students, but it indicates that the amount Florencia spends must be less than or equal to \$60. The inequality that results is $8t + 22 \leq 60$. Note that the symbol "≤" is used here to denote that the amount Florencia spends can be less than *or* equal to \$60. This symbol should be introduced in grade seven.

Adapted from NCDPI 2013b.

Estimation Strategies for Assessing Reasonableness of Answers (MP.1, MP.5)

Below are a few examples of estimation strategies that students may use to evaluate the reasonableness of their answers:

- **Front-end estimation with adjusting** — using the highest place value and estimating from the front end, making adjustments to the estimate by taking into account the remaining amounts
- **Clustering around an average** — when the values are close together, an average value is selected and multiplied by the number of values to determine an estimate
- **Rounding and adjusting** — rounding down or rounding up and then adjusting the estimate based on how much the rounding affected the original values
- **Using friendly or compatible numbers such as factors** — fitting numbers together (e.g., rounding to factors and grouping numbers together that have round sums like 100 or 1000)
- **Using benchmark numbers that are easy to compute** — selecting close whole numbers for fractions or decimals to determine an estimate

Adapted from KATM 2012, 7th Grade Flipbook.

Table 7-3 presents a sample classroom activity that connects the Standards for Mathematical Content and Standards for Mathematical Practice.

Table 7-3. Connecting to the Standards for Mathematical Practice—Grade Seven

Standards Addressed	Explanation and Examples
Connections to Standards for Mathematical Practice **MP.2.** Students must reason quantitatively with regard to percentages and should be able to flexibly compute with the given numbers in various forms. **MP.3.** A class discussion may be held for students to debate why four reductions of 25% do not constitute a total reduction of the original price by 100%. Moreover, students can explain to each other how to solve the problem correctly, and the teacher can discuss student misconceptions about percentages. **MP.5.** Students apply percentages correctly and use percentage reductions correctly. **Standards for Mathematical Content** **7.EE.3.** Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. *For example: If a woman making $25 an hour gets a 10% raise, she will make an additional $\frac{1}{10}$ of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar $9\frac{3}{4}$ inches long in the center of a door that is $27\frac{1}{2}$ inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.*	**Sample Problem.** Julie sees a jacket that costs $32 before a sale. During the sale, prices on all items are reduced by 25%. 1. What is the cost of the jacket during the sale? In the second week of the sale, all prices are reduced by 25% of the previous week's price. In the third week of the sale, prices are again reduced by 25% of the previous week's price. Likewise, in the fourth week of the sale, prices are again reduced by 25% of the previous week's price. 2. Julie thinks that this means her jacket will be reduced to $0 at the end of the fourth week. Why might she think this, and why is she wrong? 3. If Julie decides to buy the jacket at the end of the fourth week, then how much will she have to pay for it? **Classroom Connections.** Teachers can assess students' basic understanding of percentages and percent-off concepts with question 1 above. However, when students are asked to reason why Julie is incorrect in thinking that the jacket will cost $0, since $4 \times 25\% = 100\%$, they are required to understand that the number from which 25% is taken off changes each week. This is where the concept of ***the whole*** comes into play. In each situation involving percentages, ratios, or fractions, what constitutes the whole (unit, 1, 100%) is important. Finally, the third question challenges students to compute the correct cost of the jacket, by either successively subtracting 0.25 times the new price, or even by multiplying successively by 0.75. The equivalence of these two methods may be explained in this problem situation. **Jacket Price: $32** / **Discount** / **Sale Price** Sale (week 1) / 25% off / $32 - 8 = \$24$ Sale (week 2) / 25% off / $24 - 6 = \$18$ Sale (week 3) / 25% off / $18 - 4.50 = \$13.50$ Sale (week 4) / 25% off / $13.50 - 3.375 = \$10.125 \approx \$10.13$

Domain: Geometry

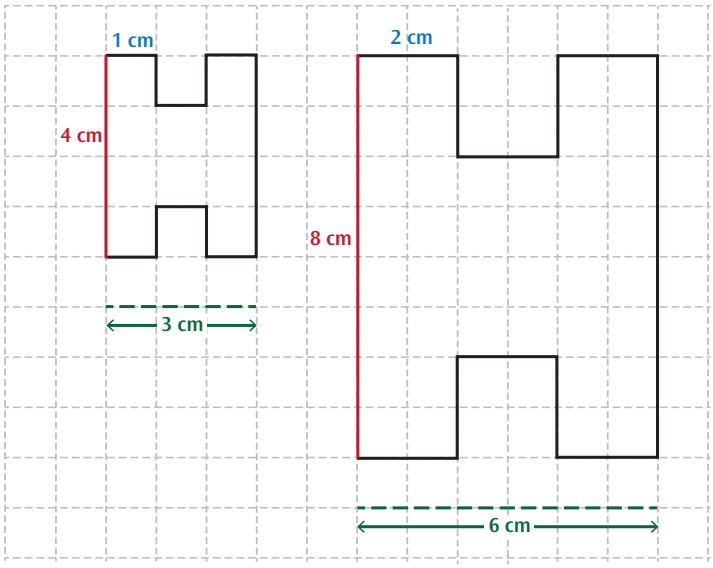
In grade seven, a critical area of instruction is for students to extend their study of geometry as they solve problems involving scale drawings and informal geometric constructions. Students also work with two- and three-dimensional shapes to solve problems involving area, surface area, and volume.

Geometry	7.G
Draw, construct, and describe geometrical figures and describe the relationships between them.	
1. Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.	

Standard **7.G.1** lays the foundation for students to understand dilations as geometric transformations. This will lead to a definition of the concept of *similar shapes* in eighth grade: shapes that can be obtained from one another through dilation. It is critical for students to grasp these ideas, as students need this comprehension to understand the derivation of the equations $y = mx$ and $y = mx + b$ by using similar triangles and the relationships between them. Thus standard **7.G.1** should be given significant attention in grade seven. Students solve problems involving scale drawings by applying their understanding of ratios and proportions, which started in grade six and continues in the grade-seven domain Ratios and Proportional Relationships (**7.RP.1–3▲**).

Teachers should note that the notion of *similarity* has not yet been addressed. Attempts to define similar shapes as those that have the "same shape but not necessarily the same size" should be avoided. Similarity will be defined precisely in grade eight, and imprecise notions of similarity may detract from student understanding of this important concept. Shapes drawn to scale are indeed similar to each other, but could safely be referred to as "scale drawings of each other" at this grade level.

The concept of a scale drawing may be effectively introduced by allowing students to blow up or shrink pictures on grid paper. For example, students may be asked to re-create the image on the left (below) on the same sheet of grid paper, but using 2 units of length for every 1 unit in the original picture:



By recording measurements in many examples, students come to see there are two important ratios with scale drawings: the ratios between two figures and the ratios within a single figure. For instance, in the illustrations above, students notice that the ratio of the topmost shorter segments and the ratio of the leftmost longer segments are equal (ratios "between" figures are equal):

$$\frac{1 \text{ cm}}{2 \text{ cm}} = \frac{4 \text{ cm}}{8 \text{ cm}}$$

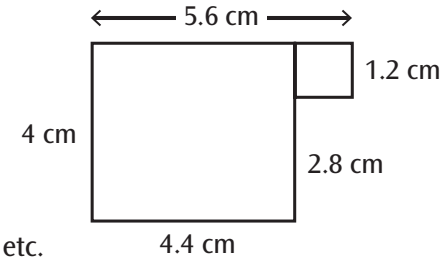
Moreover, students see that the ratios of the topmost to leftmost or topmost to total width in each shape separately are equal (ratios "within" figures are equal):

$$\frac{1 \text{ cm}}{4 \text{ cm}} = \frac{2 \text{ cm}}{8 \text{ cm}} \text{ and } \frac{1 \text{ cm}}{3 \text{ cm}} = \frac{2 \text{ cm}}{6 \text{ cm}}$$

Students should exploit these relationships when solving problems involving scale drawings, including problems that require mathematical justifications when drawings are *not* to scale.

Examples: Problems Involving Scale Drawings **7.G.1**

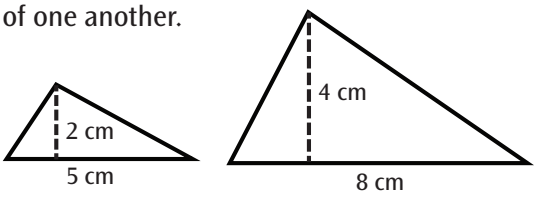
1. Julie shows you a scale drawing of her room. If 2 centimeters on the scale drawing equal 5 feet, what are the actual dimensions of Julie's room?



Solution: Since 2 centimeters in the drawing represent 5 feet, the conversion rate is $\frac{5}{2}$ ft/cm. So each measurement given in centimeters is multiplied to obtain the true measurement of the room in feet. Thus:

$5.6 \text{ cm} \rightarrow 5.6 \cdot \frac{5}{2} = 14 \text{ ft}, 1.2 \text{ cm} \rightarrow 1.2 \cdot \frac{5}{2} = 3 \text{ ft}, 2.8 \text{ cm} \rightarrow 2.8 \cdot \frac{5}{2} = 7 \text{ ft}$, etc.

2. Explain why the two triangles shown are *not* scale drawings of one another.



Solution: Since the ratios of the heights to bases of the triangles are different, one drawing cannot be a scale drawing of the other: $\frac{2}{5} \neq \frac{4}{8}$.

Adapted from ADE 2010 and NCDPI 2012.

Geometry 7.G

Draw, construct, and describe geometrical figures and describe the relationships between them.

2. Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.
3. Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

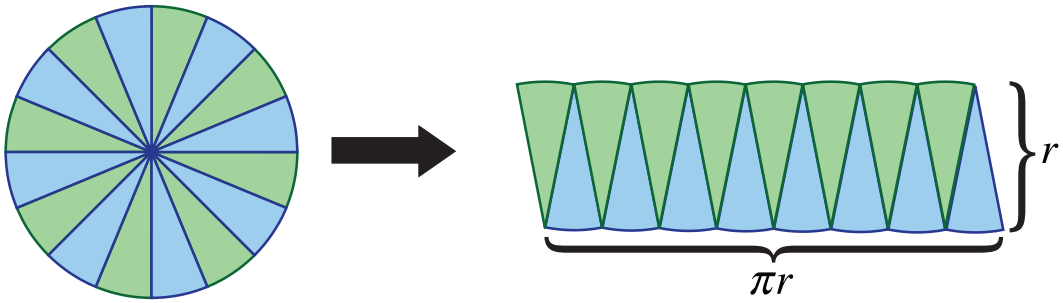
Students draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions, focusing on triangles (**7.G.2**). They work with three-dimensional figures and relate them to two-dimensional figures by examining cross-sections that result when three-dimensional figures are split (**7.G.3**). Students also describe how two or more objects are related in space (e.g., skewed lines and the possible ways in which three planes might intersect).

Geometry	7.G

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

4. Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.
5. Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.
6. Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

In grade seven, students know the formulas for the area and circumference of a circle and use them to solve problems (**7.G.4**). To "know the formula" means to have an understanding of why the formula works and how the formula relates to the measure (area and circumference) and the figure. For instance, students can cut circles into finer and finer pie pieces (sectors) and arrange them into a shape that begins to approximate a parallelogram. Because of the way the shape was created, it has a length of approximately πr and a height of approximately r. Therefore, the approximate area of this shape is πr^2, which informally justifies the formula for the area of a circle.



Adapted from KATM 2012, 7th Grade Flipbook.

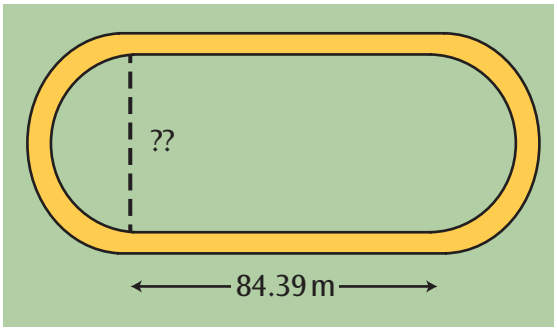
Another detailed discussion of the second part of standard **7.G.4** is available at http://www.illustrativemathematics.org/illustrations/1553 (accessed January 8, 2015) [Illustrative Mathematics 2013d].

Examples: Working with the Circumference and Area of a Circle **7.G.4**

1. Students can explore the relationship between the circumference of a circle and its diameter (or radius). For example, by tracing the circumference of a cylindrical can of beans or some other cylinder on patty paper or tracing paper and finding the diameter by folding the patty paper appropriately, students can find the approximate diameter of the base of the cylinder. If they measure a piece of string the same length as the diameter, they will find that the string can wrap around the can approximately three and one-sixth times. That is, they find that $C \approx 3\frac{1}{6} \cdot d \approx 3.16$. When students do this for a variety of objects, they start to see that the ratio of the circumference of a circle to its diameter is always approximately the same number (π).

Examples: 7.G.4 ***(continued)***

2. The total length of a standard track is 400 meters. The straight sides of the track each measure 84.39 meters. Assuming the rounded sides of the track are half-circles, find the distance from one side of the track to the other.



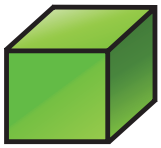
Solution: Together, the two rounded portions of the track make one circle, the circumference of which is $400 - 2(84.39) = 231.22$ meters. The length across the track is represented by the diameter of this circle. If the diameter is labeled d, then the resulting equation is $231.22 = \pi d$. Using a calculator and an approximation for π as 3.14, students arrive at $d = 231.22 \div \pi \approx 231.22 \div 3.14 \approx 73.64$ meters.

Adapted from ADE 2010.

Students continue work from grades five and six to solve problems involving area, surface area, and volume of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms (**7.G.6**).

Example: Surface Area and Volume **7.G.6**

The surface area of a cube is 96 square inches. What is the volume of the cube?



Solution: Students understand from working with nets in grade six that the cube has six faces, all with equal area. Thus, the area of one face of the cube is $96 \div 6 = 16$ square inches. Since each face is a square, the length of one side of the cube is 4 inches. This makes the volume $V = 4^3 = 64$ cubic inches.

Domain: Statistics and Probability

Students were introduced to statistics in grade six. In grade seven, they extend their work with single-data distributions to compare two different data distributions and address questions about differences between populations. They also begin informal work with random sampling.

Statistics and Probability — 7.SP

Use random sampling to draw inferences about a population.

1. Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

2. Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. *For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.*

Seventh-grade students use data from a random sample to draw inferences about a population with an unknown characteristic (**7.SP.1–2**). For example, students could predict the mean height of seventh-graders by collecting data in several random samples.

Students recognize that it is difficult to gather statistics on an entire population. They also learn that a random sample can be representative of the total population and will generate valid predictions. Students use this information to draw inferences from data (**MP.1, MP.2, MP.3, MP.4, MP.5, MP.6, MP.7**). The standards in the **7.SP.1–2** cluster represent opportunities to apply percentages and proportional reasoning. In order to make inferences about a population, one applies such reasoning to the sample and the entire population.

Example: Random Sampling **7.SP.1**

The table below shows data collected from two random samples of 100 students regarding their school lunch preferences. Make at least two inferences based on the results.

	Hamburgers	Tacos	Pizza	Total
Student Sample 1	12	14	74	100
Student Sample 2	12	11	77	100

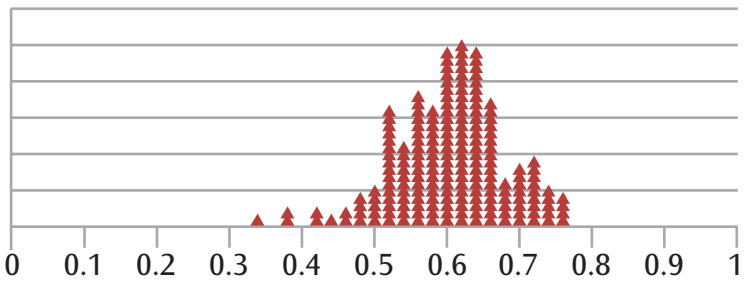
Possible solutions: Since the sample sizes are relatively large, and a vast majority in both samples prefer pizza, it would be safe to draw these two conclusions:

1. Most students prefer pizza.
2. More students prefer pizza than hamburgers and tacos combined.

Adapted from ADE 2010.

Variability in samples can be studied by using simulation (**7.SP.2**). Web-based software and spreadsheet programs may be used to run samples. For example, suppose students are using random sampling to determine the proportion of students who prefer football as their favorite sport, and suppose that 60% is the true proportion of the population. Students may simulate the sampling by conducting a simple experiment: place a collection of red and blue chips in a container in a ratio of 60:40, randomly select a chip 50 separate times with replacement, and record the proportion that came out red. If this experiment is repeated 200 times, students might obtain a distribution of the sample proportions similar to the one in figure 7-1.

Figure 7-1. Results of Simulations



Proportions of red chips in 200 random samples of size 50 from a population in which 60% of the chips are red.

This is a way for students to understand that the sample proportion can vary quite a bit, from as low as 45% to as high as 75%. Students can conjecture whether this variability will increase or decrease when the sample size increases, or if this variability depends on the true population proportion (**MP.3**) [adapted from UA Progressions Documents 2011e].

Statistics and Probability	**7.SP**

Draw informal comparative inferences about two populations.

3. Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. *For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.*
4. Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. *For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.*

Comparing two data sets is a new concept for students (**7.SP.3–4**). Students build on their understanding of graphs, mean, median, mean absolute deviation (MAD), and interquartile range from sixth grade. They know that:

- understanding data requires consideration of the measures of variability as well as the mean or median;
- variability is responsible for the overlap of two data sets, and an increase in variability can increase the overlap;
- the median is paired with the interquartile range and the mean is paired with the mean absolute deviation (adapted from NCDPI 2013b).

Example: Comparing Two Populations **7.SP.3–4**

College football teams are grouped with similar teams into divisions based on many factors. In terms of enrollment and revenue, schools from the Football Bowl Subdivision (FBS) are typically much larger than schools of other divisions. By contrast, Division III schools typically have smaller student populations and limited financial resources.

It is generally believed that, on average, the offensive linemen of FBS schools are heavier than those of Division III schools.

For the 2012 season, the University of Mount Union Purple Raiders football team won the Division III National Championship, and the University of Alabama Crimson Tide football team won the FBS National Championship. Following are the weights of the offensive linemen for both teams from that season.[8] A combined dot plot for both teams is also shown.

8. Data for Mount Union's linemen were obtained from http://athletics.mountunion.edu/sports/fball/2012-13/roster (accessed January 22, 2015). Data for the University of Alabama's linemen were obtained from http://www.rolltide.com/sports/m-footbl/mtt/alab-m-footbl-mtt.html (accessed October 31, 2014).

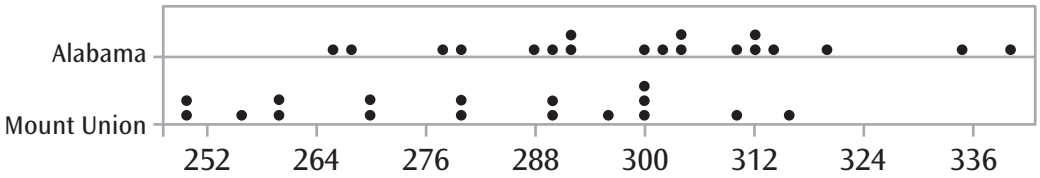
Example: 7.SP.3–4 *(continued)*

University of Alabama

277	265	292	303	303	320	300	313	267	288
311	280	302	335	310	290	312	340	292	

University of Mount Union

250	250	290	260	270	270	310	290	280	315
280	295	300	300	260	255	300			



Offensive Linemen — Weight (in pounds)

Here are some examples of conclusions that may be drawn from the data and the dot plot:

a. Based on a visual inspection of the dot plot, the mean of the Alabama group seems to be higher than the mean of the Mount Union group. However, the overall spread of each distribution appears to be similar, so we might expect the *variability* to be similar as well.

b. The Alabama mean is 300 pounds, with a MAD of 15.68 pounds. The Mount Union mean is 280.88 pounds, with a MAD of 17.99 pounds.

c. On average, it appears that an Alabama lineman's weight is about 20 pounds heavier than that of a Mount Union lineman. We also notice that the difference in the average weights of each team is greater than 1 MAD for either team. This could be interpreted as saying that for Mount Union, on average, a lineman's weight is not greater than 1 MAD above 280.88 pounds, while the average Alabama lineman's weight is already above this amount.

d. If we assume that the players from Alabama represent a random sample of players from their division (the FBS) and that Mount Union's players represent a random sample from Division III, then it is plausible that, on average, offensive linemen from FBS schools are heavier than offensive linemen from Division III schools.

Adapted from Illustrative Mathematics 2013e.

Focus, Coherence, and Rigor

Probability models draw on proportional reasoning and should be connected to major grade-seven work in the cluster "Analyze proportional relationships and use them to solve real-world and mathematical problems" (**7.RP.1–3▲**).

Seventh grade marks the first time students are formally introduced to probability. There are numerous modeling opportunities within this topic, and hands-on activities should predominate in the classroom. Technology can enhance the study of probability—for example, with online simulations of spinners, number cubes, and random number generators. The Internet is also a source of real data (e.g., on population, area, survey results, demographic information, and so forth) that can be used for writing and solving problems.

Statistics and Probability 7.SP

Investigate chance processes and develop, use, and evaluate probability models.

5. Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around $\frac{1}{2}$ indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.
6. Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. *For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.*
7. Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.
   a. Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. *For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.*
   b. Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. *For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?*

Grade-seven students interpret probability as indicating the *long-run relative frequency* of the occurrence of an event. Students may use online simulations such as the following to support their understanding:

- Marble Mania! (http://www.sciencenetlinks.com/interactives/marble/marblemania.html [Science NetLinks 2013])
- Random Drawing Tool (http://illuminations.nctm.org/activitydetail.aspx?id=67 [National Council of Teachers of Mathematics Illuminations 2013a])

Students develop and use *probability models* to find the probabilities of events and investigate both *empirical probabilities* (i.e., probabilities based on observing outcomes of a simulated random process) and *theoretical probabilities* (i.e., probabilities based on the structure of the *sample space* of an event) [**7.SP.7**].

Example: A Simple Probability Model 7.SP.7

A box contains 10 red chips and 10 black chips. Without looking, each student reaches into the box and pulls out a chip. If each of the first 5 students pulls out (and keeps) a red chip, what is the probability that the sixth student will pull a red chip?

Solution: The events in question, pulling out a red or black chip, should be considered equally likely. Furthermore, though students new to probability may believe in the "gambler's fallacy"—that since 5 red chips have already been chosen, there is a very large chance that a black chip will be chosen next—students must still compute the probabilities of events as equally likely. There are 15 chips left in the box (5 red and 10 black), so the probability that the sixth student will select a red chip is $\frac{5}{15} = \frac{1}{3}$.

Statistics and Probability 7.SP

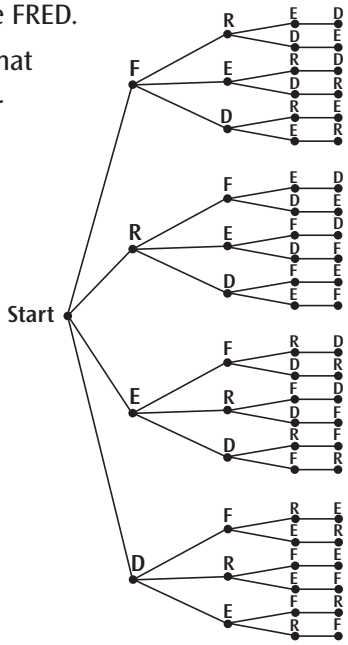
Investigate chance processes and develop, use, and evaluate probability models.

8. Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.
   a. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.
   b. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., "rolling double sixes"), identify the outcomes in the sample space which compose the event.
   c. Design and use a simulation to generate frequencies for compound events. *For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?*

Students in grade seven also examine *compound events* (such as tossing a coin and rolling a standard number cube) and use basic counting ideas for finding the total number of equally likely outcomes for such an event. For example, 2 outcomes for the coin and 6 outcomes for the number cube result in 12 total outcomes. At this grade level, there is no need to introduce formal methods of finding permutations and combinations. Students also use various means of organizing the outcomes of an event, such as two-way tables or tree diagrams (**7.SP.8a–b**).

Example: Tree Diagrams **7.SP.8a–b**

Using a tree diagram, show all possible arrangements of the letters in the name FRED. If each of the letters is on a tile and drawn at random, what is the probability that you will draw the letters F-R-E-D in that order? What is the probability that your "word" will have an F as the first letter?



Solution: A tree diagram reveals that, out of 24 total outcomes, there is only one outcome where the letters F-R-E-D appear in that order, so the probability of the event occurring is $\frac{1}{24}$. Regarding the second question, the entire top branch (6 outcomes) represents the outcomes where the first letter is F, so the probability of that occurring is $\frac{6}{24} = \frac{1}{4}$.

Adapted from ADE 2010.

Finally, students in grade seven use simulations to determine probabilities (frequencies) for compound events (**7.SP.8c**). For a more complete discussion of the Statistics and Probability domain, see "Progressions Documents for the Common Core Math Standards: Draft 6–8 Progression on Statistics and Probability" (http://ime.math.arizona.edu/progressions/ [UA Progressions Documents 2011e]).

Example **7.SP.8c**

If 40% of donors have type O blood, what is the probability that it will take at least 4 donors to find 1 with type O blood?

This problem offers a perfect opportunity for students to construct a simulation model. The proportion of donors with type O blood being 40% may be modeled by conducting blind drawings from a box containing markers labeled "O" and "Not O." One option would be to have a box with 40 "O" markers and 60 "Not O" markers. The size of the donor pool would determine the best way to model the situation. If the donor pool consisted of 25 people, one could model the situation by randomly drawing craft sticks (without replacement) from a box containing 10 "O" craft sticks and 15 "Not O" craft sticks until a type O craft stick is drawn. Reasonable estimates could be achieved in 20 trials. However, if the class were evaluating a donor pool as large as 1000, and circumstances dictated use of a box with only 10 sticks, then each stick drawn would represent a population of 100 potential donors. This situation could be modeled by having successive draws with replacement until a type O stick is drawn. In order to speed up the experiment, students might note that once 3 "Not O" sticks have been drawn, the stated conditions have been met. An advanced class could be led to the observation that if the donor pool were very large, the probability of the first three donors having blood types A, B, or AB is approximated by $(1-0.4)^3 = (0.6)^3 = 0.216$. This exercise also represents a good opportunity to collaborate with science faculty.

Essential Learning for the Next Grade

In middle grades, multiplication and division develop into powerful forms of ratio and proportional reasoning. The properties of operations take on prominence as arithmetic matures into algebra. The theme of quantitative relationships also becomes explicit in grades six through eight, developing into the formal notion of a function by grade eight. Meanwhile, the foundations of deductive geometry are laid in the middle grades. The gradual development of data representations in kindergarten through grade five leads to statistics in middle school: the study of shape, center, and spread of data distributions; possible associations between two variables; and the use of sampling in making statistical decisions (adapted from PARCC 2012).

To be prepared for grade-eight mathematics, students should be able to demonstrate mastery of particular mathematical concepts and procedural skills by the end of grade seven and that they have met the fluency expectations for grade seven. The expected fluencies for students in grade seven are to solve equations of the form $px+q=r$ and $p(x+q)=r$ (**7.EE.4▲**), which also requires fluency with rational-number arithmetic (**7.NS.1–3▲**), and to apply (to some extent) properties of operations to rewrite linear expressions with rational coefficients (**7.EE.1▲**). Also, adding, subtracting, multiplying, and dividing rational numbers (**7.NS.1–2▲**) is the culmination of numerical work with the four basic operations. The number system continues to develop in grade eight, expanding to become the real numbers with the introduction of irrational numbers, and develops further in high school, expanding again to become the complex numbers with the introduction of imaginary numbers. These fluencies and the conceptual understandings that support them are foundational for work in grade eight.

It is particularly important for students in grade seven to develop skills and understandings to analyze proportional relationships and use them to solve real-world and mathematical problems (**7.RP.1–3▲**); apply and extend previous understanding of operations with fractions to add, subtract, multiply, and divide rational numbers (**7.NS.1–3▲**); use properties of operations to generate equivalent expressions (**7.EE.1–2▲**); and solve real-life and mathematical problems using numerical and algebraic expressions and equations (**7.EE.3–4▲**).

Guidance on Course Placement and Sequences

The California Common Core State Standards for Mathematics for grades six through eight are comprehensive, rigorous, and non-redundant. Instruction in an accelerated sequence of courses will require compaction—not the former strategy of deletion. Therefore, careful consideration needs to be made before placing a student in higher-mathematics course work in grades six through eight. Acceleration may get students to advanced course work, but it may create gaps in students' mathematical background. Careful consideration and systematic collection of multiple measures of individual student performance on both the content and practice standards are required. For additional information and guidance on course placement, see appendix D (Course Placement and Sequences).

Grade 7 Overview

Ratios and Proportional Relationships

- Analyze proportional relationships and use them to solve real-world and mathematical problems.

The Number System

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Expressions and Equations

- Use properties of operations to generate equivalent expressions.
- Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

Geometry

- Draw, construct, and describe geometrical figures and describe the relationships between them.
- Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

Statistics and Probability

- Use random sampling to draw inferences about a population.
- Draw informal comparative inferences about two populations.
- Investigate chance processes and develop, use, and evaluate probability models.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

7 Grade 7

Ratios and Proportional Relationships 7.RP

Analyze proportional relationships and use them to solve real-world and mathematical problems.

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. *For example, if a person walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, compute the unit rate as the complex fraction $\frac{\frac{1}{2}}{\frac{1}{4}}$ miles per hour, equivalently 2 miles per hour.*
2. Recognize and represent proportional relationships between quantities.
   a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
   b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
   c. Represent proportional relationships by equations. *For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as $t = pn$.*
   d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0,0) and (1,r) where r is the unit rate.
3. Use proportional relationships to solve multi-step ratio and percent problems. *Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.*

The Number System 7.NS

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

1. Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.
   a. Describe situations in which opposite quantities combine to make 0. *For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged.*
   b. Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.
   c. Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.
   d. Apply properties of operations as strategies to add and subtract rational numbers.
2. Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
   a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such

as $(-1)(-1)=1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-\left(p/q\right)=(-p)/q=p/(-q)$. Interpret quotients of rational numbers by describing real-world contexts.

c. Apply properties of operations as strategies to multiply and divide rational numbers.

d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

3. Solve real-world and mathematical problems involving the four operations with rational numbers.[9]

Expressions and Equations 7.EE

Use properties of operations to generate equivalent expressions.

1. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

2. Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, $a+0.05a=1.05a$ means that "increase by 5%" is the same as "multiply by 1.05."*

Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

3. Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. *For example: If a woman making $25 an hour gets a 10% raise, she will make an additional $1/10$ of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar $9\frac{3}{4}$ inches long in the center of a door that is $27\frac{1}{2}$ inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.*

4. Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

a. Solve word problems leading to equations of the form $px+q=r$ and $p(x+q)=r$ where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. *For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?*

b. Solve word problems leading to inequalities of the form $px+q>r$ or $px+q<r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. *For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.*

9. Computations with rational numbers extend the rules for manipulating fractions to complex fractions.

7 Grade 7

Geometry 7.G

Draw, construct, and describe geometrical figures and describe the relationships between them.

1. Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.
2. Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.
3. Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

4. Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.
5. Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.
6. Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Statistics and Probability 7.SP

Use random sampling to draw inferences about a population.

1. Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.
2. Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. *For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.*

Draw informal comparative inferences about two populations.

3. Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. *For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.*
4. Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. *For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.*

Investigate chance processes and develop, use, and evaluate probability models.

5. Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around $1/2$ indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

6. Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. *For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.*

7. Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.

    a. Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. *For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.*

    b. Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. *For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?*

8. Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

    a. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.

    b. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., “rolling double sixes”), identify the outcomes in the sample space which compose the event.

    c. Design and use a simulation to generate frequencies for compound events. *For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?*

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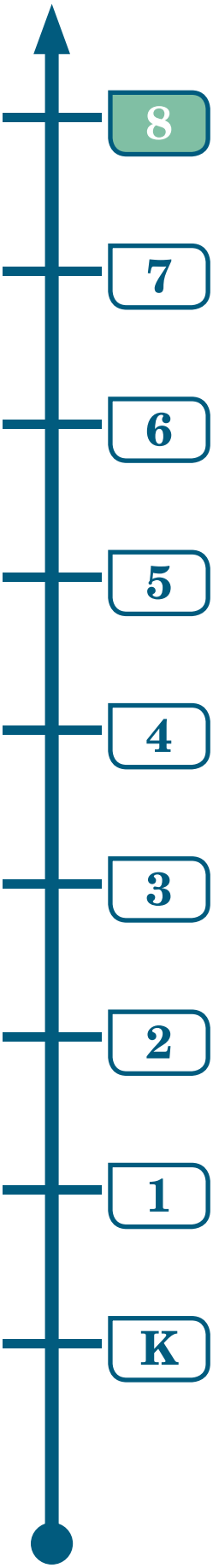


Grade Eight

Prior to entering grade eight, students wrote and interpreted expressions, solved equations and inequalities, explored quantitative relationships between dependent and independent variables, and solved problems involving area, surface area, and volume. Students who are entering grade eight have also begun to develop an understanding of statistical thinking (adapted from Charles A. Dana Center 2012).

Critical Areas of Instruction

In grade eight, instructional time should focus on three critical areas: (1) formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation, as well as solving linear equations and systems of linear equations; (2) grasping the concept of a function and using functions to describe quantitative relationships; and (3) analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010o). Students also work toward fluency in solving sets of two simple equations with two unknowns by inspection.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade-level examples of focus, coherence, and rigor are indicated throughout the chapter.

The standards do not give equal emphasis to all content for a particular grade level. Cluster headings can be viewed as the most effective way to communicate the **focus** and **coherence** of the standards. Some clusters of standards require a greater instructional emphasis than others based on the depth of the ideas, the time needed to master those clusters, and their importance to future mathematics or the later demands of preparing for college and careers.

Table 8-1 highlights the content emphases at the cluster level for the grade-eight standards. The bulk of instructional time should be given to "Major" clusters and the standards within them, which are indicated throughout the text by a triangle symbol (▲). However, standards in the "Additional/Supporting" clusters should not be neglected; to do so would result in gaps in students' learning, including skills and understandings they may need in later grades. Instruction should reinforce topics in major clusters by using topics in the additional/supporting clusters and including problems and activities that support natural connections between clusters.

Teachers and administrators alike should note that the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences presented in a coherent manner (adapted from Partnership for Assessment of Readiness for College and Careers [PARCC] 2012).

Table 8-1. Grade Eight Cluster-Level Emphases

The Number System — 8.NS

Additional/Supporting Clusters

- Know that there are numbers that are not rational, and approximate them by rational numbers.[1] (**8.NS.1–2**)

Expressions and Equations — 8.EE

Major Clusters

- Work with radicals and integer exponents. (**8.EE.1–4▲**)
- Understand the connections between proportional relationships, lines, and linear equations. (**8.EE.5–6▲**)
- Analyze and solve linear equations and pairs of simultaneous linear equations. (**8.EE.7–8▲**)

Functions — 8.F

Major Clusters

- Define, evaluate, and compare functions. (**8.F.1–3▲**)

Additional/Supporting Clusters

- Use functions to model relationships between quantities.[2] (**8.F.4–5**)

Geometry — 8.G

Major Clusters

- Understand congruence and similarity using physical models, transparencies, or geometry software. (**8.G.1–5▲**)
- Understand and apply the Pythagorean Theorem. (**8.G.6–8▲**)

Additional/Supporting Clusters

- Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres. (**8.G.9**)

Statistics and Probability — 8.SP

Additional/Supporting Clusters

- Investigate patterns of association in bivariate data.[3] (**8.SP.1–4**)

Explanations of Major and Additional/Supporting Cluster-Level Emphases

Major Clusters (▲) — Areas of intensive focus where students need fluent understanding and application of the core concepts. These clusters require greater emphasis than others based on the depth of the ideas, the time needed to master them, and their importance to future mathematics or the demands of college and career readiness.

Additional Clusters — Expose students to other subjects; may not connect tightly or explicitly to the major work of the grade.

Supporting Clusters — Designed to support and strengthen areas of major emphasis.

Note of caution: Neglecting material, whether it is found in the major or additional/supporting clusters, will leave gaps in students' skills and understanding and will leave students unprepared for the challenges they face in later grades.

Adapted from Smarter Balanced Assessment Consortium 2011, 88.

1. Work with the number system in this grade is intimately related to work with radicals, and both of these may be connected to the Pythagorean Theorem as well as to volume problems (e.g., in which a cube has known volume but unknown edge lengths).

2. The work in this cluster involves functions for modeling linear relationships and a rate of change/initial value, which supports work with proportional relationships and setting up linear equations.

3. Looking for patterns in scatter plots and using linear models to describe data are directly connected to the work in the Expressions and Equations clusters. Together, these represent a connection to the fourth Standard for Mathematical Practice, **MP.4** (Model with mathematics).

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The MP standards represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students.

Although the description of the MP standards remains the same at all grades, the way these standards look as students engage with and master new and more advanced mathematical ideas does change. Table 8-2 presents examples of how the MP standards may be integrated into tasks appropriate for students in grade eight. (Refer to the Overview of the Standards Chapters for a description of the MP standards.)

Table 8-2. Standards for Mathematical Practice—Explanation and Examples for Grade Eight

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	In grade eight, students solve real-world problems through the application of algebraic and geometric concepts. Students seek the meaning of a problem and look for efficient ways to represent and solve it. They may check their thinking by asking questions such as these: "What is the most efficient way to solve the problem?" "Does this make sense?" "Can I solve the problem in a different way?"
MP.2 Reason abstractly and quantitatively.	Students represent a wide variety of real-world contexts through the use of real numbers and variables in mathematical expressions, equations, and inequalities. They examine patterns in data and assess the degree of linearity of functions. Students *contextualize* to understand the meaning of the number(s) or variable(s) related to the problem and *decontextualize* to manipulate symbolic representations by applying properties of operations.
MP.3 Construct viable arguments and critique the reasoning of others.	Students construct arguments with verbal or written explanations accompanied by expressions, equations, inequalities, models, graphs, tables, and other data displays (e.g., box plots, dot plots, histograms). They further refine their mathematical communication skills through mathematical discussions in which they critically evaluate their own thinking and the thinking of other students. They pose questions such as these: "How did you get that?" "Why is that true?" "Does that always work?" They explain their thinking to others and respond to others' thinking.
MP.4 Model with mathematics.	Students in grade eight model real-world problem situations symbolically, graphically, in tables, and contextually. Working with the new concept of a *function*, students learn that relationships between variable quantities in the real world often satisfy a dependent relationship, in that one quantity determines the value of another. Students form expressions, equations, or inequalities from real-world contexts and connect symbolic and graphical representations. Students use scatter plots to represent data and describe associations between variables. They should be able to use any of these representations as appropriate to a particular problem context. Students should be encouraged to answer questions such as "What are some ways to represent the quantities?" or "How might it help to create a table, chart, graph, or _____?"

Table 8-2 *(continued)*

Standards for Mathematical Practice	Explanation and Examples
MP.5 Use appropriate tools strategically.	Students consider available tools (including estimation and technology) when solving a mathematical problem and decide when particular tools might be helpful. For instance, students in grade eight may translate a set of data given in tabular form into a graphical representation to compare it with another data set. Students might draw pictures, use applets, or write equations to show the relationships between the angles created by a transversal that intersects parallel lines. Teachers might ask, "What approach are you considering?" or "Why was it helpful to use _____?"
MP.6 Attend to precision.	In grade eight, students continue to refine their mathematical communication skills by using clear and precise language in their discussions with others and in their own reasoning. Students use appropriate terminology when referring to the number system, functions, geometric figures, and data displays. Teachers might ask, "What mathematical language, definitions, or properties can you use to explain _____?"
MP.7 Look for and make use of structure.	Students routinely seek patterns or structures to model and solve problems. In grade eight, students apply properties to generate equivalent expressions and solve equations. Students examine patterns in tables and graphs to generate equations and describe relationships. Additionally, students experimentally verify the effects of transformations and describe them in terms of congruence and similarity.
MP.8 Look for and express regularity in repeated reasoning.	In grade eight, students use repeated reasoning to understand the slope formula and to make sense of rational and irrational numbers. Through multiple opportunities to model linear relationships, they notice that the slope of the graph of the linear relationship and the rate of change of the associated function are the same. For example, as students repeatedly check whether points are on the line with a slope of 3 that goes through the point (1, 2), they might abstract the equation of the line in the form $\frac{y-2}{x-1}=3$. Students divide to find decimal equivalents of rational numbers (e.g., $\frac{2}{3}=0.\overline{6}$) and generalize their observations. They use iterative processes to determine more precise rational approximations for irrational numbers. Students should be encouraged to answer questions such as "How would we prove that _____?" or "How is this situation like and different from other situations using these operations?"

Adapted from Arizona Department of Education (ADE) 2010 and North Carolina Department of Public Instruction (NCDPI) 2013b.

Standards-Based Learning at Grade Eight

The following narrative is organized by the domains in the Standards for Mathematical Content and highlights some necessary foundational skills from previous grade levels. It also provides exemplars to explain the content standards, highlight connections to Standards for Mathematical Practice (**MP**), and demonstrate the importance of developing conceptual understanding, procedural skill and fluency, and application. A triangle symbol (▲) indicates standards in the major clusters (see table 8-1).

Domain: The Number System

In grade seven, adding, subtracting, multiplying, and dividing rational numbers was the culmination of numerical work with the four basic operations. The number system continues to develop in grade eight, expanding to the real numbers with the introduction of irrational numbers, and develops further in higher mathematics, expanding to become the complex numbers with the introduction of imaginary numbers (adapted from PARCC 2012).

The Number System	8.NS

Know that there are numbers that are not rational, and approximate them by rational numbers.

1. Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.
2. Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.*

In grade eight, students learn that not all numbers can be expressed in the form $\frac{a}{b}$, where a and b are positive or negative whole numbers with $b \neq 0$. Such numbers are called *irrational*, and students explore cases of both rational and irrational numbers and their decimal expansions to begin to understand the distinction. The fact that rational numbers eventually result in repeating decimal expansions is a direct result of the way in which long division is used to produce a decimal expansion.

Why Rational Numbers Have Terminating or Repeating Decimal Expansions	8.NS.1

In each step of the standard algorithm to divide a by b, a partial quotient and a remainder are determined; the requirement is that each remainder is smaller than the divisor (b). In simpler examples, students will notice (or be led to notice) that once a remainder is repeated, the decimal repeats from that point onward, as in $\frac{1}{6} = 0.16666\ldots = 0.1\overline{6}$ or $\frac{3}{11} = 0.272727\ldots = 0.\overline{27}$. If a student imagines using long division to convert the fraction $\frac{3}{13}$ to a decimal without going through the tedium of actually producing the decimal, it can be reasoned that the possible remainders are 1 through 12. Consequently, a remainder that has already occurred will present itself by the thirteenth remainder, and therefore a repeating decimal results.

The full reasoning for why the converse is true—that eventually repeating decimals represent numbers that are rational—is beyond the scope of grade eight. However, students can use algebraic reasoning to show that repeating decimals eventually represent rational numbers in some simple cases (**8.NS.1**).

Example: Converting the Repeating Decimal $0.\overline{18}$ into a Fraction of the Form $\frac{a}{b}$ **8.NS.1**

Solution: One method for converting such a decimal into a fraction is to set $N=0.\overline{18}=0.18181818\ldots$. If this is the case, then $100N=18.\overline{18}$. Subtracting $100N$ and N yields $99N$. This means that $99N=18.\overline{18}-.\overline{18}=18$. Solving for N, students see that $N=\frac{18}{99}=\frac{2}{11}$.

Since every decimal is either repeating or non-repeating, this leaves irrational numbers as those numbers whose decimal expansions do not have a repeating pattern. Students understand this informally in grade eight, and they approximate irrational numbers by rational numbers in simple cases. For example, π is irrational, so it is approximated by $\frac{22}{7}$ or 3.14.

Example: Finding Better and Better Approximations of $\sqrt{2}$ **8.NS.2**

The following reasoning may be used to approximate simple irrational square roots.

- Since $1^2<2<2^2$, then $\sqrt{1^2}<\sqrt{2}<\sqrt{2^2}$, which leads to $1<\sqrt{2}<2$. This means that $\sqrt{2}$ must be between 1 and 2.
- Since $1.4^2=1.96$ and $1.5^2=2.25$, students know by guessing and checking that $\sqrt{2}$ is between 1.4 and 1.5.
- Through additional guessing and checking, and by using a calculator, students see that since $1.41^2=1.9881$ and $1.42^2=2.0164$, $\sqrt{2}$ is between 1.41 and 1.42.

Continuing in this manner yields better and better approximations of $\sqrt{2}$. When students investigate this process with calculators, they gain some familiarity with the idea that the decimal expansion of $\sqrt{2}$ never repeats. Students should graph successive approximations on number lines to reinforce their understanding of the number line as a tool for representing real numbers.

Ultimately, students should come to an informal understanding that the set of real numbers consists of rational numbers and irrational numbers. They continue to work with irrational numbers and rational approximations when solving equations such as $x^2=18$ and in problems involving the Pythagorean Theorem. In the Expressions and Equations domain that follows, students learn to use radicals to represent such numbers (adapted from California Department of Education [CDE] 2012d, ADE 2010, and NCDPI 2013b).

Focus, Coherence, and Rigor

In grade eight, the standards in The Number System domain support major work with the Pythagorean Theorem (**8.G.6–8▲**) and connect to volume problems (**8.G.9**)—for example, a problem in which a cube has known volume but unknown edge lengths.

Domain: Expressions and Equations

In grade seven, students formulated expressions and equations in one variable, using these equations to solve problems and fluently solving equations of the form $px+q=r$ and $p(x+q)=r$. In grade eight, students apply their previous understandings of ratio and proportional reasoning to the study of linear equations and pairs of simultaneous linear equations, which is a critical area of instruction for this grade level.

Expressions and Equations — 8.EE

Work with radicals and integer exponents.

1. Know and apply the properties of integer exponents to generate equivalent numerical expressions. *For example,* $3^2\times 3^{-5}=3^{-3}=1/3^3=1/27$.
2. Use square root and cube root symbols to represent solutions to equations of the form $x^2=p$ and $x^3=p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.
3. Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. *For example, estimate the population of the United States as* 3×10^8 *and the population of the world as* 7×10^9*, and determine that the world population is more than 20 times larger.*
4. Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

Students in grade eight add the following properties of integer exponents to their repertoire of rules for transforming expressions, and they use these properties to generate equivalent expressions (**8.EE.1▲**).

Properties of Integer Exponents — 8.EE.1▲

For any non-zero numbers a and b and integers n and m:

1. $a^n a^m = a^{n+m}$
2. $(a^n)^m = a^{nm}$
3. $a^n b^n = (ab)^n$
4. $a^0 = 1$
5. $a^{-n} = \frac{1}{a^n}$

Source: University of Arizona (UA) Progressions Documents for the Common Core Math Standards 2011d.

Students in grade eight have focused on place-value relationships in the base-ten number system since elementary school, and therefore working with powers of 10 is a natural place for students to begin investigating the patterns that give rise to these properties. However, powers of numbers other than 10 should also be explored, as these foreshadow the study of exponential functions in higher mathematics courses.

Example: Reasoning About Patterns to Explore the Properties of Exponents **8.EE.1▲**

Students fill in the blanks in the table below and discuss with a neighbor any patterns they find.

	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}
Expanded	$2\times2\times2$	2×2	2	?	?	?	?
Evaluate	8	4	2	?	?	?	?

Students can reason about why the value of 2^0 should be 1, based on patterns they may see—for example, in the bottom row of the table, each value is $\frac{1}{2}$ of the value to the left of it. Students should explore similar examples with other bases to arrive at the general understanding that:

$$a^n = a\times a\times\cdots\times a \text{ (}n\text{ factors)}, a^0 = 1, \text{ and } a^{-n} = \frac{1}{a^n}.$$

Generally, Standard for Mathematical Practice **MP.3** calls for students to construct mathematical arguments; therefore, reasoning should be emphasized when it comes to learning the properties of exponents. For example, students can reason that $5^3\times5^2 = (5\times5\times5)\times(5\times5) = 5^5$. Through numerous experiences of working with exponents, students generalize the properties of exponents before using them fluently.

Students do not learn the properties of rational exponents until they reach the higher mathematics courses. However, in grade eight they start to work systematically with the symbols for square root and cube root—for example, writing $\sqrt{64} = 8$ and $\sqrt[3]{5^3} = 5$. Since $\sqrt{p}$ is defined to mean only the positive solution to the equation $x^2 = p$ (when the square root exists), it is not correct to say that $\sqrt{64} = \pm8$. However, a correct solution to $x^2 = 64$ would be $x = \pm\sqrt{64} = \pm8$. Most students in grade eight are not yet able to prove that these are the only solutions; rather, they use informal methods such as "guess and check" to verify the solutions (UA Progressions Documents 2011d).

Students recognize perfect squares and cubes, understanding that square roots of non-perfect squares and cube roots of non-perfect cubes are irrational (**8.EE.2▲**). Students should generalize from many experiences that the following statements are true (**MP.2, MP.5, MP.6, MP.7**):

- Squaring a square root of a number returns the number back (e.g., $\left(\sqrt{5}\right)^2 = 5$).
- Taking the square root of the square of a number *sometimes* returns the number back (e.g., $\sqrt{7^2} = \sqrt{49} = 7$, while $\sqrt{(-3)^2} = \sqrt{9} = 3 \neq -3$).
- Cubing a number and taking the cube root can be considered inverse operations.

Students expand their exponent work as they perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Students use scientific notation to express very large or very small numbers. Students compare and interpret scientific notation quantities in the context of the situation, recognizing that the powers of 10 indicated in quantities expressed in scientific notation follow the rules of exponents shown previously (**8.EE.3–4▲**) [adapted from CDE 2012d, ADE 2010, and NCDPI 2013b].

Example: Ants and Elephants **8.EE.4▲**

An ant has a mass of approximately 4×10^{-3} grams, and an elephant has a mass of approximately 8 metric tons. How many ants does it take to have the same mass as an elephant?

(*Note:* 1 kg = 1000 grams, 1 metric ton = 1000 kg.)

Solution: To compare the masses of an ant and an elephant, we convert the mass of an elephant into grams:

$$8 \text{ metric tons} \times \frac{1000\,\text{kg}}{1\,\text{metric ton}} \times \frac{1000\,\text{g}}{1\,\text{kg}} = 8 \times 10^3 \times 10^3 \text{ grams} = 8\times10^6 \text{ grams}$$

If N represents the number of ants that have the same mass as an elephant, then $(4\times10^{-3})N$ is their total mass in grams. This should equal 8×10^6 grams, which yields a simple equation:

$$(4\times10^{-3})N = 8\times10^6, \text{ which means that}$$

$$N = \frac{8\times10^6}{4\times10^{-3}} = 2\times10^{6-(-3)} = 2\times10^9$$

Therefore, 2×10^9 ants would have the same mass as an elephant.

Adapted from Illustrative Mathematics 2013f.

Focus, Coherence, and Rigor

As students work with scientific notation, they learn to choose units of appropriate size for measurement of very large or very small quantities (**MP.2, MP.5, MP.6**).

Students build on their work with unit rates from grade six and proportional relationships from grade seven to compare graphs, tables, and equations of proportional relationships (**8.EE.5▲**). In grade eight, students connect these concepts to the concept of the slope of a line.

Expressions and Equations 8.EE

Understand the connections between proportional relationships, lines, and linear equations.

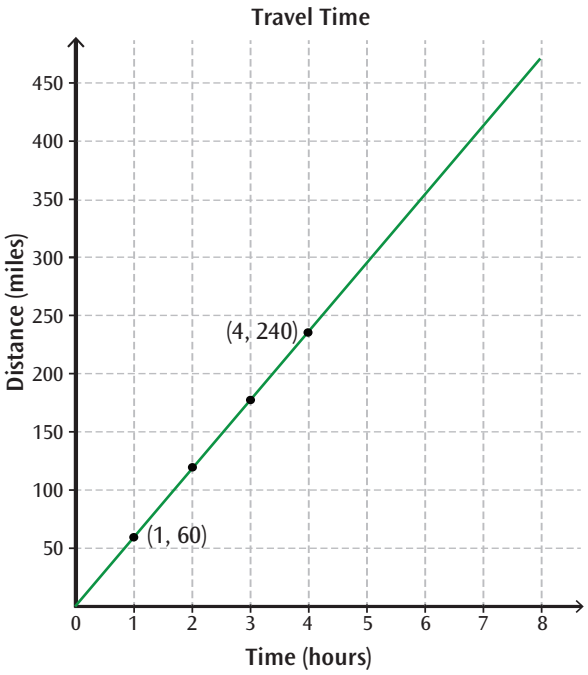
5. Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. *For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.*
6. Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

Students identify the unit rate (or slope) to compare two proportional relationships represented in different ways (e.g., as a graph of the line through the origin, a table exhibiting a constant rate of change, or an equation of the form $y=kx$). Students interpret the unit rate in a proportional relationship (e.g., r miles per hour) as the slope of the graph. They understand that the slope of a line represents a constant rate of change.

Example **8.EE.5▲**

Compare the scenarios below to determine which represents a greater speed. Include in your explanation a description of each scenario that discusses unit rates.

Scenario 1:



Scenario 2: The equation for the distance y in miles as a function of the time x in hours is:

$$y = 55x$$

Solution: "The unit rate in Scenario 1 can be read from the graph; it is 60 miles per hour. In Scenario 2, I can see that this looks like an equation $y = kx$, and in that type of equation the unit rate is the constant k. Therefore, the speed in Scenario 2 is 55 miles per hour. So the person traveling in Scenario 1 is moving at a faster rate."

Adapted from CDE 2012d, ADE 2010, and NCDPI 2013b.

Table 8-3 presents a sample classroom activity that connects the Standards for Mathematical Content and Standards for Mathematical Practice.

Table 8-3. Connecting to the Standards for Mathematical Practice—Grade Eight

Standards Addressed	Explanation and Examples
Connections to Standards for Mathematical Practice **MP.1.** Students are encouraged to persevere in solving the entire problem and make sense of each step. **MP.4.** Students model a very simple real-life cost situation. **Standards for Mathematical Content** **8.EE.5▲.** Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in two different ways.	**Task:** Below is a table that shows the cost of various amounts of almonds. Almonds (pounds): 3, 5, 8, 10, 15 Cost (dollars): 15.00, 25.00, 40.00, 50.00, 75.00 1. Graph the cost versus the number of pounds of almonds. Place the number of pounds of almonds on the horizontal axis and the cost of the almonds on the vertical axis. 2. Use the graph to find the cost of 1 pound of almonds. Explain how you arrived at your answer. 3. The table shows that 5 pounds of almonds cost \$25.00. Use your graph to find out how much 6 pounds of almonds cost. 4. Suppose that walnuts cost \$3.50 per pound. Draw a line on your graph to represent the cost of different numbers of pounds of walnuts. 5. Which are cheaper: almonds or walnuts? How do you know? ***Solution:*** 1. A graph is shown. 2. To find the cost of 1 pound of almonds, students would locate the point that has 1 as the first coordinate; this is the point (1, 5). Thus the *unit cost* is \$5 per pound. 3. Students can do this by locating 6 pounds on the horizontal axis and finding the point on the graph associated with this number of pounds. However, the teacher might also urge students to notice that when moving to the right by 1 unit along the graph, the next point on the graph is found 5 units up from the previous point. This idea is the genesis of the *slope* of a line and should be explored. 4. Ideally, students draw a line that passes through (0, 0) and the approximate point (1, 3.50). Proportional thinkers might notice that 2 pounds of walnuts cost \$7, so they can plot a point with whole-number coordinates. 5. Walnuts are cheaper. Students may explore several different ways to see this, including the unit cost, the steepness of the line, a comparison of common quantities of nuts, and so on. **Classroom Connections.** The concept of slope can be approached in its simplest form with directly proportional quantities. In this case, when two quantities x and y are directly proportional, they are related by an equation $y = kx$, which is equivalent to $\frac{y}{x} = k$, where k is a constant known as the *constant of proportionality*. In the example involving almonds, the k in an equation would represent the unit cost of almonds. Students should have several experiences graphing and exploring directly proportional relationships to build a foundation for understanding more general linear equations of the form $y = mx + b$.

Focus, Coherence, and Rigor

The connection between the unit rate in a proportional relationship and the slope of its graph depends on a connection with the geometry of similar triangles (see standards **8.G.4–5▲**). The fact that a line has a well-defined slope—that the ratio between the rise and run for any two points on the line is always the same—depends on similar triangles.

Adapted from UA Progressions Documents 2011d.

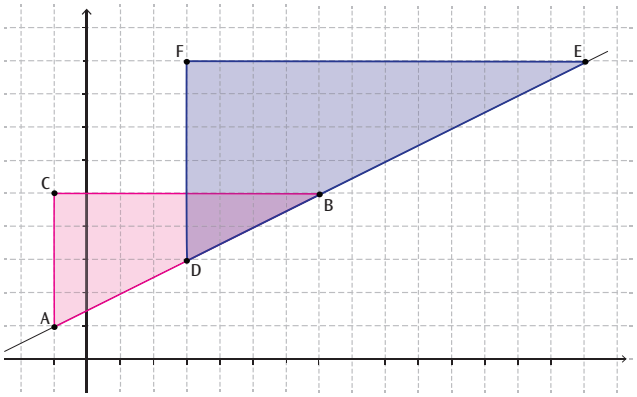
Standard **8.EE.6▲** represents a convergence of several ideas from grade eight and previous grade levels. Students have graphed proportional relationships and found the slope of the resulting line, interpreting it as the unit rate (**8.EE.5▲**). It is here that the language of "rise over run" comes into use. In the Functions domain, students see that any linear equation $y = mx + b$ determines a function whose graph is a straight line (a *linear function*), and they verify that the slope of the line is equal to m (**8.F.3▲**). Standard **8.EE.6▲** calls for students to go further and explain why the slope m is the same through any two points on a line. Students justify this fact by using similar triangles, which are studied in standards **8.G.4–5▲**.

Example of Reasoning **8.EE.6▲**

Show that the slope is the same between any two points on a line.

In grade seven, students made scale drawings of figures and observed the proportional relationships between side lengths of such figures (**7.G.1▲**). In grade eight, students generalize this idea and study *dilations* of plane figures, and they define figures as being *similar* in terms of dilations (see standard **8.G.4▲**). Students discover that similar figures share a proportional relationship between side lengths, just as scale drawings did: there is a *scale factor* $k > 0$ such that corresponding side lengths of similar figures are related by the equation $s_1 = k \bullet s_2$. Furthermore, the ratio of two sides in one shape is equal to the ratio of the corresponding two sides in the other shape. Finally, standard **8.G.5▲** calls for students to informally argue that triangles with two corresponding angles of the same measure must be similar, and this is the final piece of the puzzle for using similar triangles to show that the slope is the same between any two points on the coordinate plane (**8.EE.6▲**).

Explain why the slopes between points A and B and points D and E are the same.



Solution: "$\angle A$ and $\angle D$ are equal because they are corresponding angles formed by the transversal crossing the vertical lines through points A and D. Since $\angle C$ and $\angle F$ are both right angles, the triangles are similar.

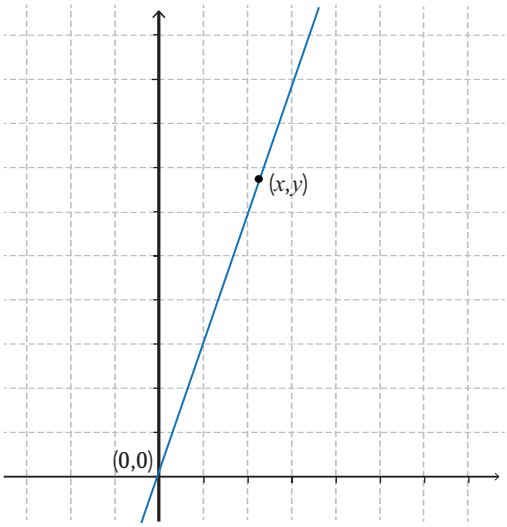
This means the ratios $\frac{AC}{BC}$ and $\frac{DF}{EF}$ are equal. But when you find the 'rise over the run,' these are the exact ratios you find, so the slope is the same between these two sets of points."

In grade eight, students build on previous work with proportional relationships, unit rates, and graphing to connect these ideas and understand that the points (x,y) on a non-vertical line are the solutions of the equation $y = mx + b$, where m is the slope of the line as well as the unit rate of a proportional relationship in the case $b = 0$.

Additional Examples of Reasoning **8.EE.6▲**

Derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

Example 1: Explain how to derive the equation $y = 3x$ for the line of slope $m = 3$ shown at right.

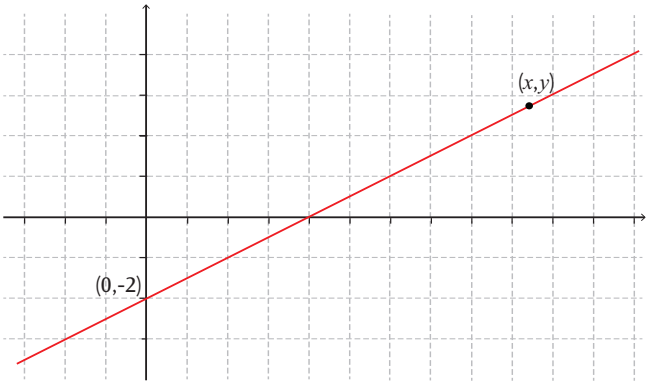


Solution: "I know that the slope is the same between any two points on a line. So I'll choose the origin $(0,0)$ and a generic point on the line, calling it (x,y). By choosing a generic point like this, I know that any point on the line will fit the equation I come up with. The slope between these two points is found by

$$3 = \frac{\text{rise}}{\text{run}} = \frac{y-0}{x-0} = \frac{y}{x}$$

This equation can be rearranged to $y = 3x$."

Example 2: Explain how to derive the equation $y = \frac{1}{2}x - 2$ for the line of slope $m = \frac{1}{2}$ with intercept $b = -2$.



Solution: "I know the slope is $\frac{1}{2}$, so I'll determine the equation of the line using the slope formula, with the point $(0,-2)$ and the generic point (x,y). The slope between these two points is found by

$$\frac{1}{2} = \frac{\text{rise}}{\text{run}} = \frac{y-(-2)}{x-0} = \frac{y+2}{x}$$

This can be rearranged to $y + 2 = \frac{1}{2}x$, which is the same as $y = \frac{1}{2}x - 2$."

Students have worked informally with one-variable linear equations as early as kindergarten. This important line of development culminates in grade eight, as much of the students' work involves analyzing and solving linear equations and pairs of simultaneous linear equations.

Expressions and Equations 8.EE

Analyze and solve linear equations and pairs of simultaneous linear equations.

7. Solve linear equations in one variable.
   a. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).
   b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.
8. Analyze and solve pairs of simultaneous linear equations.
   a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
   b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because $3x + 2y$ cannot simultaneously be 5 and 6.*
   c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

Grade-eight students solve linear equations in one variable, including cases with one solution, an infinite number of solutions, and no solutions (**8.EE.7▲**). Students show examples of each of these cases by successively transforming an equation into simpler forms ($x = a$, $a = a$, and $a = b$, where a and b represent different numbers). Some linear equations require students to expand expressions by using the distributive property and to collect like terms.

Solutions to One-Variable Equations **8.EE.7a▲**

- When an equation has only one solution, there is only one value of the variable that makes the equation true (e.g., $12 - 4y = 16$).
- When an equation has an infinite number of solutions, the equation is true for all real numbers and is sometimes referred to as an *identity*—for example, $7x + 14 = 7(x + 2)$. Solving this equation by using familiar steps might yield $14 = 14$, a statement that is true regardless of the value of x. Students should be encouraged to think about why this means that any real number solves the equation and relate it back to the original equation (e.g., the equation is showing the distributive property).
- When an equation has no solutions, it is sometimes described as *inconsistent*—for example, $5x - 2 = 5(x + 1)$. Attempting to solve this equation might yield $-2 = 5$, which is a false statement regardless of the value of x. Students should be encouraged to reason why there are no solutions to the equation; for example, they observe that the original equation is equivalent to $5x - 2 = 5x + 5$ and reason that it is never the case that $N - 2 = N + 5$, no matter what N is.

Adapted from ADE 2010.

Grade-eight students also analyze and solve pairs of simultaneous linear equations (**8.EE.8 a–c▲**). Solving pairs of simultaneous linear equations builds on the skills and understandings students used to solve linear equations with one variable, and systems of linear equations may also have one solution, an infinite number of solutions, or no solutions. Students will discover these cases as they graph systems of linear equations and solve them algebraically. For a system of linear equations, students in grade eight learn the following:

- If the graphs of the lines meet at one point (the lines intersect), then there is one solution, the ordered pair of the point of intersection representing the solution of the system.
- If the graphs of the lines do not meet (the lines are parallel), the system has no solutions, and the slopes of these lines are the same.
- If the graphs of the lines are *coincident* (the graphs are exactly the same line), then the system has an infinite number of solutions, the solutions being the set of all ordered pairs on the line (adapted from ADE 2010).

Example: Introducing Systems of Linear Equations **8.EE.8a▲**

To introduce the concept of a system of linear equations, a teacher might ask students to assemble in small groups and think about how they would start a business selling smoothies at school during lunch. Then each group would create a budget that details the cost of the items that would have to be purchased each month (students could use the Internet to acquire pricing or use their best estimate), as well as a monthly total. Each group would also establish a price for its smoothies. Students can also discuss a model (equation) for the profit their business will make in a month. The teacher might ask the students questions such as these:

1. What are some variable quantities in our situation? (The number of smoothies sold and monthly profit are important.)
2. What is the profit at the beginning of the month? (This would be a negative number corresponding to the monthly total of items purchased.)
3. How many smoothies will you need to sell to make a profit? (The teacher instructs students to make a table that shows profit versus the number of smoothies sold, for multiples of 10 smoothies to 200. Students are also asked to create a graph from the data in their table. The teacher can demonstrate the graphs of the lines $y = 0$ and $y = 50$, and then ask students to draw the same lines on their graph. Students should also be asked to explain the meaning of those lines.)

Solution: The line $y = 0$ represents the point when the business is no longer losing money; the line $y = 50$ represents the point at which the company makes a $50 profit. The teacher can demonstrate the points of intersection and discuss the importance of these two coordinates. Finally, the teacher asks two students from different groups (groups whose graphs will intersect should be selected) to graph their data on the same axis for the whole class to see. Students discuss the significance of the point of intersection of the two lines, including the concept that the number of smoothies sold and the profit will be the same at that point. As a class, students write equations for both lines and demonstrate by substitution that the coordinates of the intersection point are solutions to both equations.

By making connections between algebraic and graphical solutions and the context of the system of linear equations, students are able to make sense of their solutions.

Students in grade eight also solve real-world and mathematical problems leading to two linear equations in two variables (**8.EE.8c▲**). Below is an example of how reasoning about real-world situations can be used to introduce and make sense out of solving systems of equations by elimination. The technique of elimination may be used in general cases to solve systems of equations.

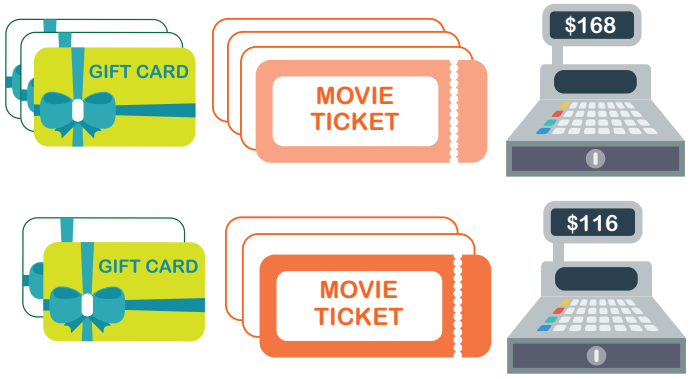
Example: Solving a System of Equations by Elimination **8.EE.8c▲**

Suppose you know that the total cost of 3 gift cards and 4 movie tickets is \$168, while 2 gift cards and 3 movie tickets cost \$116.

1. Explain how to use this information to find the cost of 1 gift card and 1 movie ticket.
2. Next, explain how you could find the cost of 1 movie ticket.
3. Explain how you would find the cost of 1 gift card.

Solution:

1. If g represents the cost of a gift card and t represents the cost of a movie ticket, then I know that $3g+4t=168$ and $2g+3t=116$. I can represent this in a diagram:



If I subtract the 2 gift cards and 3 movie tickets from the 3 gift cards and 4 movie tickets, I get \$168 – \$116 = \$52. This means the cost of 1 of each item together is \$52. I can represent this by

$$\begin{array}{r} 3g+4t=168 \\ 2g+3t=116 \\ \hline g+t=52 \end{array}$$

2. Now I can see that 2 of each item would cost \$104. If I subtract this result from the second equation above, I am left with 1 movie ticket, and it costs \$12.

$$\begin{array}{r} 2g+3t=116 \\ 2g+2t=104 \\ \hline 1t=12 \end{array}$$

3. Now it is easy to see that if 1 gift card and 1 ticket together cost \$52, then 1 gift card alone would cost \$52 – \$12 = \$40.

Domain: Functions

In grade seven, students learned to determine if two quantities represented a proportional relationship. Proportional reasoning is a transitional topic that falls between arithmetic and algebra. Underlying the progression from proportional reasoning through algebra and beyond is the idea of a *function*—a rule that assigns to each input exactly one output. The concept of a function is a critical area of instruction in grade eight. Students are introduced to functions and learn that proportional relationships are part of a broader group of linear functions.

Functions	**8.F**
Define, evaluate, and compare functions.	
1. Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.[4]	
2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*	
3. Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function $A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4), and (3,9), which are not on a straight line.*	

In grade eight, students understand two main points regarding functions (**8.F.1▲**):

- A *function* is a rule that assigns to each input exactly one output.
- The *graph* of a function is the set of ordered pairs consisting of an input and the corresponding output.

In general, students understand that functions describe situations in which one quantity determines another. The main work in grade eight concerns linear functions, though students are exposed to non-linear functions to contrast them with linear functions. Thus, students may view a linear equation such as $y = -0.75x + 12$ as a rule that defines a quantity y whenever the quantity x is given. In this case, the function may describe the amount of money remaining after a number of turns, x, when a student who starts with \$12 plays a game that costs \$0.75 per turn. Alternatively, students may view the formula for the area of a circle, $A = \pi r^2$, as a (non-linear) function in the sense that the area of a circle is dependent on its radius. Student work with functions at grade eight remains informal but sets the stage for more formal work in higher mathematics courses.

4. Function notation is not required in grade eight.

Example: Introduction to Functions **8.F.1▲**

To introduce the concept of a function, a teacher might have students contrast two workers' wages at two different jobs, one with an hourly wage and the other based on a combination of an hourly wage and tips. Students read through scenarios and make a table for each of the two workers, listing hours worked and money earned during 20 different shifts varying from 3 to 8 hours in length. Students answer questions about the data, including the level of predictability of the wage of each worker, based on the number of hours worked. Students graph the data and observe the patterns of the graph. Next, the teacher could introduce the concept of a function and relate the tables and graphs from the activity to the idea of a function, emphasizing that an input value completely determines an output value. Students could then be challenged to find other quantities that are functions and to create and discuss corresponding tables and/or graphs.

Students are able to connect foundational understandings about functions to their work with proportional relationships. The same kinds of tables and graphs that students used in grade seven to recognize and represent proportional relationships between quantities are used in grade eight when students compare the properties of two functions that are represented in different ways (e.g., numerically in tables or visually in graphs). Students also compare the properties of two functions that are represented algebraically or verbally (**8.F.2▲**).

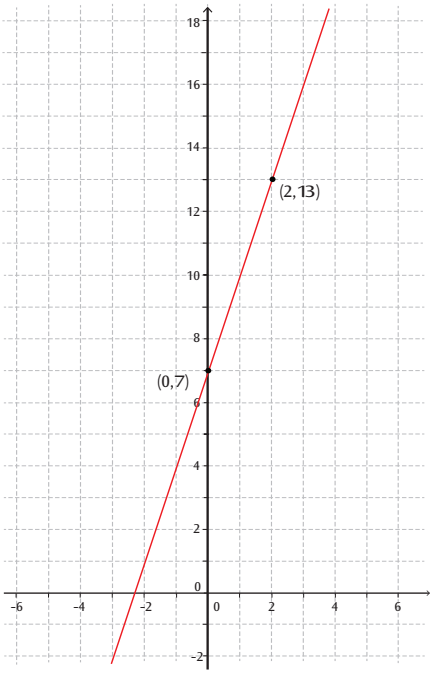
Example: Functions Represented Differently **8.F.2▲**

Which function has a greater rate of change? (**MP.1, MP.2, MP.4, MP.8**)

Function 1: The function represented by the graph shown.

Function 2: The function whose input x and output y are related by the equation $y = 4x + 7$.

Solution: The graph of the function shows that when $x = 0$, the value of the function is $y = 7$, and when $x = 2$ the value of the function is $y = 13$. This means that function 1 increases by 6 units when x increases by 2 units. Function 2 also has an output of $y = 7$ when $x = 0$, but when $x = 2$, the value of function 2 is $y = 15$. This means that function 2 increases by 8 units when x increases by 2 units. Therefore, function 2 has a greater rate of change.



Adapted from ADE 2010.

Students' understanding of the equation $y = mx + b$ deepens as they learn that the equation defines a linear function whose graph is a straight line (**8.F.3▲**), a concept closely related to standard **8.EE.6▲**. To avoid the mistaken impression that all functional relationships are linear, students also work with non-linear functions and provide examples of non-linear functions, recognizing that the graph of a non-linear function is not a straight line.

Functions 8.F

Use functions to model relationships between quantities.

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x,y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
5. Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or non-linear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

In grade eight, students learn to use functions to represent relationships between quantities. This work is also closely tied to **MP.4** (Model with mathematics). There are many real-world problems that can be modeled with linear functions, including instances of constant payment plans (e.g., phone plans), costs associated with running a business, and relationships between associated bivariate data (see standard **8.SP.3**). Students also recognize that linear functions in which $b = 0$ are proportional relationships, something they have studied since grade six.

Standard **8.F.4** refers to students finding the *initial value* of a linear function. Thus, if f represents a linear function with a domain $[a,b]$—that is, the input values for f are between the values a and b—then the initial value for f would be $f(a)$. The term *initial value* takes its name from an interpretation of the independent variable as representing time, although the term can apply to any function. Note that formal introduction of the term *domain* does not occur until the higher mathematics courses, but teachers may wish to include this language if it clarifies these ideas for students. The following example illustrates the definition of *initial value*.

Example: Modeling With a Linear Function **8.F.4**

A car rental company charges \$45 per day to rent a car as well as a one-time \$25 fee for the car's GPS navigation system. Write an equation for the cost in dollars, c, as a function of the number of days the car is rented, d. What is the initial value for this function?

Solution: There are several aids that may help students determine an equation for the cost:

- A verbal description: "Each day adds \$45 to the cost, but there is a one-time \$25 GPS fee. This means that the cost should be \$25 plus \$45 times the number of days you rent the car, or $c = 25 + 45d$. Since a customer must rent the car for a minimum of 1 day, the initial value is $25 + 45 = 70$, which means it costs \$70 to rent the car for 1 day."

Continued on next page

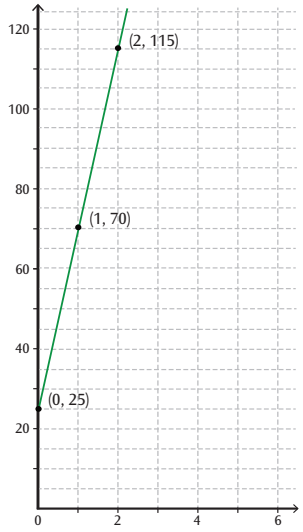
Example: 8.F.4 ***(continued)***

- A table: "I made a table to give me a feel for how much the car rental might cost after d days.

d (days)	c (dollars)
1	$70 = 25 + (1)45$
2	$115 = 25 + (2)45$
3	$160 = 25 + (3)45$
4	$205 = 25 + (4)45$
d	$c = 25 + (d)45$

The table helped me see that the cost in dollars is represented by $c = 25 + (d)45$. Since the car must be rented for 1 day or more, the initial value is when $d = 1$, which is $c = 70$, or $70."

- A graph: "I made a rough graph and saw that the relationship between the cost and the days rented appeared to be linear. I found the slope of the line to be 45 (which is the cost per day) and the y intercept to be 25. This means the equation is $c = 45d + 25$. It is important to see that even though the y intercept of the graph is 25, that is not the initial value—because the initial value is when someone rents the car for 1 day. The point on the graph is $(1, 70)$ so the initial value is $70."



Students analyze graphs and then describe qualitatively the functional relationship between two quantities (e.g., where the function is increasing or decreasing, linear or non-linear). They are able to sketch graphs that illustrate the qualitative features of functions that are described verbally (**8.F.5**).

Focus, Coherence, and Rigor

Work in the cluster "Use functions to model relationships between quantities" involves functions for modeling linear relationships and computing a rate of change or initial value, which supports major work at grade eight with proportional relationships and setting up linear equations (**8.EE.5–8▲**).

Domain: Geometry

In grade seven, students solved problems involving scale drawings and informal geometric constructions, and they worked with two- and three-dimensional shapes to solve problems involving area, surface area, and volume. Students in grade eight complete their work on volume by solving problems involving cones, cylinders, and spheres. They also analyze two- and three-dimensional space and figures using distance, angle, similarity, and congruence and by understanding and applying the Pythagorean Theorem, which is a critical area of instruction at this grade level.

Geometry	**8.G**
Understand congruence and similarity using physical models, transparencies, or geometry software.	
1. Verify experimentally the properties of rotations, reflections, and translations:	
a. Lines are taken to lines, and line segments to line segments of the same length.	
b. Angles are taken to angles of the same measure.	
c. Parallel lines are taken to parallel lines.	
2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.	
3. Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.	

In this grade-eight Geometry domain, a major shift in the traditional curriculum occurs with the introduction of basic transformational geometry. In particular, the notion of *congruence* is defined differently than it has been in the past. Previously, two shapes were understood to be congruent if they had the "same size and same shape." This imprecise notion is exchanged for a more precise one: that a two-dimensional figure is congruent to another if the second figure can be obtained from the first by a sequence of rotations, reflections, and translations. Students need ample opportunities to explore these three geometric transformations and their properties. The work in the Geometry domain is designed to provide a seamless transition to the Geometry conceptual category in higher mathematics courses, which begins by approaching transformational geometry from a more advanced perspective.

With the aid of physical models, transparencies, and geometry software, students in grade eight gain an understanding of transformations and their relationship to congruence of shapes. Through experimentation, students verify the properties of rotations, reflections, and translations, including discovering that these transformations change the position of a geometric figure but not its shape or size (**8.G.1a–c▲**). Finally, students come to understand that congruent shapes are precisely those that can be "mapped" one onto the other by using rotations, reflections, or translations (**8.G.2▲**).

Characteristics of Rotations, Reflections, and Translations **8.G.1a–c▲**

Students come to understand that the following transformations result in shapes that are *congruent* to one another.

- Students understand a *rotation* as the spinning of a figure around a fixed point known as the *center of rotation*. Unless specified otherwise, rotations are usually performed counterclockwise according to a particular angle of rotation.
- Students understand a *reflection* as the flipping of an object over a line known as the *line of reflection*.
- Students understand a *translation* as the shifting of an object in one direction for a fixed distance, so that any point lying on the shape moves the same distance in the same direction.

An example of an interactive online tool that shows transformation is Shodor Education's "Interactivate Transmographer" (http://www.shodor.org/interactivate/activities/Transmographer/ [Shodor Education Foundation, Inc. 2015]), which allows students to work with rotation, reflection, and translation.

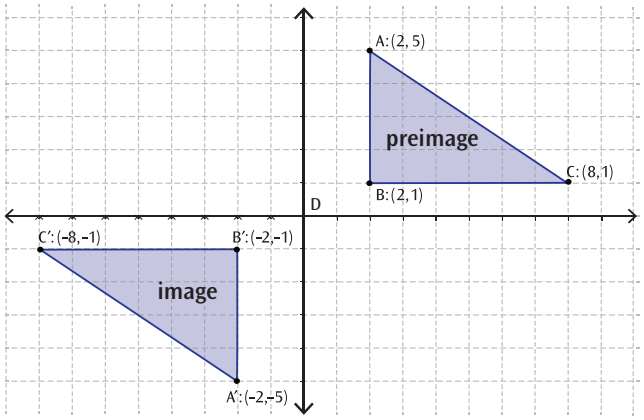
Standard **8.G.3▲** also calls for students to study dilations. A *dilation* with scale factor $k > 0$ can be thought of as a stretching (if $k > 1$) or shrinking (if $k < 1$) of an object. In a dilation, a point is specified from which the distance to the points of a figure is multiplied to obtain new points, and hence a new figure.

Examples of Four Geometric Transformations **8.G.2▲**

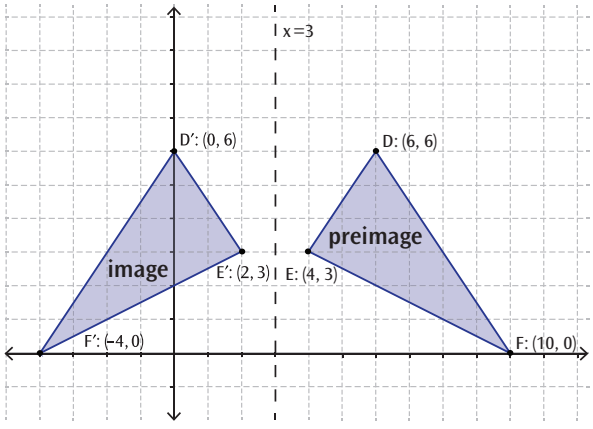
(Note that the original figure is called the *preimage*, and the new figure is called the *image*.)

Rotation: A figure can be rotated up to 360° about the center of rotation.

Consider when ΔABC is rotated 180° clockwise about the origin. The coordinates of ΔABC are $A(2,5)$, $B(2,1)$, and $C(8,1)$. When rotated 180°, the image triangle $\Delta A'B'C'$ has coordinates $A'(-2,-5)$, $B'(-2,-1)$, $C'(-8,-1)$. Each coordinate of the image is the opposite of its preimage point's coordinate.



Reflection: In the picture shown, ΔDEF has been reflected across the line $x = 3$. Notice the change in the orientation of the points, in the sense that the counterclockwise order of the preimage D–E–F is reversed in the image to D'–F'–E'. Notice also that each point on the image is at the same distance from the line of reflection as its corresponding point on the preimage.



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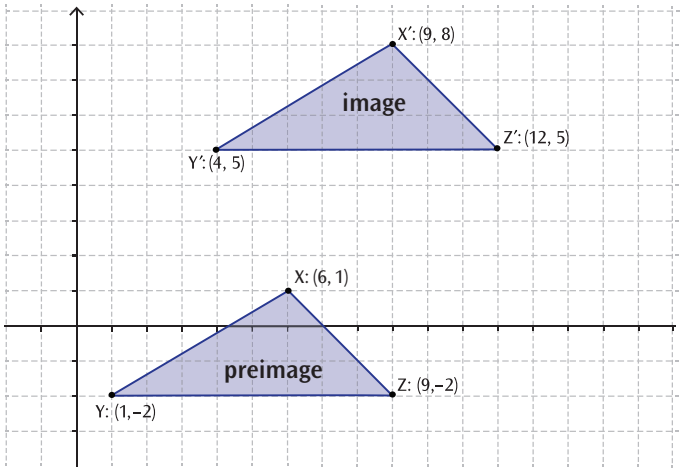
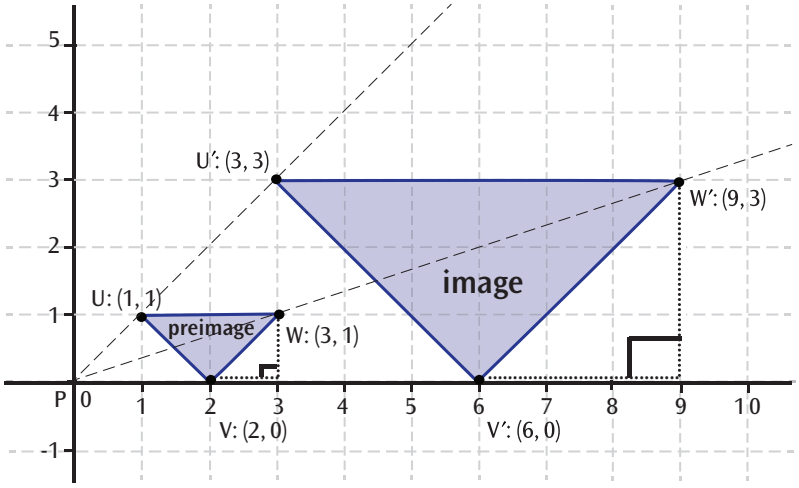
Example: 8.G.2▲ ***(continued)***

Translation: Here, ΔXYZ has been translated 3 units to the right and 7 units up. Orientation is preserved. It is not too difficult to see that under this transformation, a preimage point (x, y) yields the image point $(x+3, y+7)$.



Dilation: In the picture, ΔUVW has been dilated from the origin $P: (0,0)$ by a factor of $k = 3$. The picture shows that the segments PU, PV, and PW have all been multiplied by the factor $k = 3$, which results in a new triangle, $\Delta U'V'W'$. By definition, ΔUVW and $\Delta U'V'W'$ are similar triangles. Students should experiment and find that the ratios of corresponding side lengths satisfy $\frac{U'V'}{UV} = \frac{V'W'}{VW} = \frac{U'W'}{UW} = 3$,



which corresponds to k. Students can apply the Pythagorean Theorem (**8.G.7–8▲**) to find the side lengths and justify this result. For example, they may find the lengths of VW and $V'W'$:

$$\overline{VW} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\text{and } \overline{V'W'} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

Students can check informally that $\sqrt{18} = 3\sqrt{2}$, as formal work with radicals has not yet begun in grade eight.

Geometry 8.G

Understand congruence and similarity using physical models, transparencies, or geometry software.

4. Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.
5. Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle–angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.*

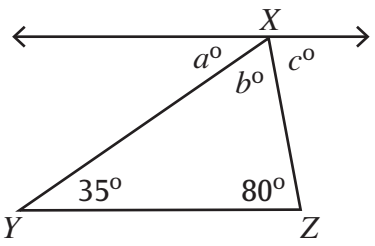
The definition of *similar shapes* is analogous to the new definition of *congruence*, but it has been refined to be more precise. Previously, shapes were said to be similar if they had the "same shape but not necessarily the same size." Now, two shapes are said to be similar if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations (**8.G.4▲**). By investigating dilations and using reasoning such as in the previous example, students learn that the following statements are true:

1. When two shapes are similar, the length of a segment AB in the first shape is multiplied by the scale factor k to give the length of the corresponding segment $A'B'$ in the second shape: $A'B' = k \bullet AB$.
2. Because the previous fact is true for all sides of a dilated shape, the ratio of the lengths of any two corresponding sides of the first and second shape is equal to k.
3. It is also true that the ratio of any two side lengths from the first shape is the same as the ratio of the corresponding side lengths from the second shape—for example, $\frac{AB}{BC} = \frac{A'B'}{B'C'}$. (Students can justify this algebraically, because fact 2 yields that $\frac{AB}{A'B'} = \frac{BC}{B'C'}$.)

Students use informal arguments to establish facts about the angle sum and exterior angles of triangles (e.g., consecutive exterior angles are supplementary), the angles created when parallel lines are cut by a transversal (e.g., corresponding angles are congruent), and the angle–angle criterion for similarity of triangles (if two angles of a triangle are congruent to two angles of another triangle, the two triangles are similar) [**8.G.5▲**]. When coupled with the previous three properties of similar shapes, the angle–angle criterion for triangle similarity allows students to justify the fact that the slope of a line is the same between any two points on the line (see discussion of standard **8.EE.6▲**).

Example: The sum of the measures of the angles of a triangle is 180°. **8.G.5▲**

In the figure shown, the line through point X is parallel to segment YZ.



We know that $a = 35$ because it is the measure of an angle that is alternating with $\angle Y$. For a similar reason, $c = 80$. Because all lines have an angle measure of 180°, we know that $a + b + c = 180$, which leads to $b = 180 - (35 + 80) = 65$. So the sum of the measures of the angles in this triangle is 180°.

Adapted from ADE 2010.

Geometry 8.G

Understand and apply the Pythagorean Theorem.

6. Explain a proof of the Pythagorean Theorem and its converse.
7. Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

The Pythagorean Theorem is useful in practical problems, relates to grade-level work with irrational numbers, and plays an important role mathematically in coordinate geometry in higher mathematics. Students in grade eight explain a proof of the Pythagorean Theorem (**8.G.6▲**)—and there are many different and interesting proofs of this theorem.[5] In grade eight, students apply the Pythagorean Theorem to determine unknown side lengths in right triangles (**8.G.7▲**) and to find the distance between two points in a coordinate system (**8.G.8▲**). Application of the Pythagorean Theorem supports students' work in higher-level coordinate geometry.

Focus, Coherence, and Rigor

Understanding, modeling, and applying (**MP.4**) the Pythagorean Theorem and its converse require that students look for and make use of structure (**MP.7**) and express repeated reasoning (**MP.8**). Students also construct and critique arguments as they explain a proof of the Pythagorean Theorem and its converse to others (**MP.3**).

Adapted from Charles A. Dana Center 2012.

Geometry 8.G

Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

9. Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

In grade seven, students learned about the area of a circle. Students in grade eight learn the formulas for calculating the volumes of cones, cylinders, and spheres and use the formulas to solve real-world and mathematical problems (**8.G.9**). When students learn to solve problems involving volumes of cones, cylinders, and spheres—together with their previous grade-seven work in angle measure, area, surface area, and volume—they acquire a well-developed set of geometric measurement skills. These skills, along with proportional reasoning and multi-step numerical problem solving, can be combined and used in flexible ways as part of mathematical modeling during high school and in college and careers (adapted from PARCC 2012).

Domain: Statistics and Probability

Building on work in earlier grades with univariate measurement data and analyzing data on line plots and histograms, grade-eight students begin to work with bivariate measurement data and use scatter plots to represent and analyze the data.

Bivariate measurement data represent two separate (but usually related) measurements. Scatter plots can show the relationship between the two measured variables. Collecting and analyzing bivariate measurement data help students to answer questions such as "How does more time spent on homework affect test grades?" and "What is the relationship between annual income and the number of years of formal education a person has?"

5. One example is the geometric "Proof without Words" of the Pythagorean Theorem available at http://illuminations.nctm.org/activitydetail.aspx?id=30 (NCTM Illuminations 2013a).

Statistics and Probablity 8.SP

Investigate patterns of association in bivariate data.

1. Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and non-linear association.
2. Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.
3. Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. *For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.*
4. Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables. *For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?*

Students in grade eight construct and interpret scatter plots to investigate patterns of association between two quantities (**8.SP.1**). They also build on their previous knowledge of scatter plots to examine relationships between variables. Grade-eight students analyze scatter plots to determine positive and negative associations, the degree of association, and type of association. Additionally, they examine outliers to determine if data points are valid or represent a recording or measurement error.

Example: Creating Scatter Plots **8.SP.1**

Customer satisfaction is vital to the success of fast-food restaurants, and speed of service is a key component of that satisfaction. In order to determine the best staffing level, the owners of a local fast-food restaurant have collected the data below showing the number of staff members and the average time for filling an order. Describe the association between the number of staff and the average time for filling an order, and make a recommendation as to how many staff should be hired.

Number of staff members	3	4	5	6	7	8
Average time to fill order (seconds)	180	138	120	108	96	84

Students can use tools such as those offered by the National Center for Education Statistics (http://nces.ed.gov/nceskids/createagraph/default.aspx [National Center for Education Statistics 2013]) to create a graph or generate data sets.

Grade-eight students know that straight lines are widely used to model relationships between two quantitative variables (**8.SP.2**). For scatter plots that appear to show a linear association, students informally fit a line (e.g., by drawing a line on the coordinate plane between data points) and informally assess the fit by judging the closeness of the data points to the straight line.

Example: Informally Determining a Line of Best Fit **8.SP.2**

The capacity of the fuel tank in a car is 13.5 gallons. The table below shows the number of miles traveled and the amount of gasoline used (in gallons). Describe the relationship between the variables. If the data are linear, determine a line of best fit. Do you think the line represents a good fit for the data set? Why or why not? What is the average fuel efficiency of the car in miles per gallon?

Miles traveled	0	75	120	160	250	300
Amount of gasoline used (gallons)	0	2.3	4.5	5.7	9.7	10.7

Students in grade eight solve problems in the context of bivariate measurement data by using the equation of a linear model (**8.SP.3**). They interpret the slope and the y-intercept in the context of the problem. For example, in a linear model for a biology experiment, students interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 centimeters in the height of the plant.

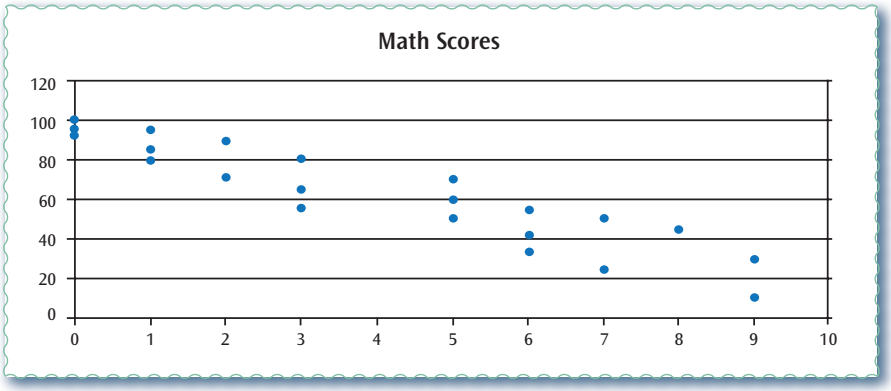
Example: Finding a Linear Model for a Data Set **8.SP.3**

Make a scatter plot by using data from students' math scores and absences. Informally fit a line to the graph and determine an approximate linear function that models the data. What would you expect to be the score of a student with 4 absences?

Solution:

Absences	3	5	1	1	3	6	5	3	0	7	8
Math scores	65	50	95	85	80	34	70	56	100	24	45
Absences	2	9	0	6	6	2	0	5	7	9	1
Math scores	71	30	95	55	42	90	92	60	50	10	80

Students would most likely use simple data software to make a scatter plot, finding a graph that looks like the following:



Students can use graphing software to find a line of best fit. Such a line might be $y = -8x + 95$. They interpret this equation as defining a function that gives the approximate score of a student based on the number of his or her absences. Thus, a student with 4 absences should have a score of approximately $y = -8(4) + 95 = 63$.

Adapted from CDE 2012d, ADE 2010, and NCDPI 2013b.

Focus, Coherence, and Rigor

Students in grade eight apply their experience with coordinate geometry and linear functions to plot bivariate data as points on a plane and to make use of the equation of a line in analyzing the relationship between two paired variables. Students develop mathematical practices as they build statistical models to explore the relationship between two variables (**MP.4**) and look for and make use of structure to describe possible associations in bivariate data (**MP.7**).

Adapted from UA Progressions Documents 2011e.

Students learn to see patterns of association in bivariate categorical data in a two-way table (**8.SP.4**). They construct and interpret a two-way table that summarizes data on two categorical variables collected from the same subjects. The two-way table displays frequencies and relative frequencies. Students use relative frequencies calculated from rows or columns to describe a possible association between the two variables. For example, students collect data from their classmates about whether they have a curfew and whether they do chores at home. The two-way table allows students to easily see if students who have a curfew also tend to do chores at home.

Example: Two-Way Tables for Categorical Data **8.SP.4**

The table at right illustrates the results when 100 students were asked these survey questions: (1) Do you have a curfew? (2) Do you have assigned chores? Students can examine the survey results to determine if there is evidence that those who have a curfew also tend to have chores.

		Curfew	
		Yes	No
Chores	Yes	40	10
	No	10	40

Solution: Of the students who answered that they had a curfew, 40 had chores and 10 did not. Of the students who answered they did not have a curfew, 10 had chores and 40 did not. From this sample, there seems to be a positive correlation between having a curfew and having chores: it appears that most students with chores have a curfew and most students without chores do not have a curfew.

Adapted from CDE 2012d, ADE 2010, and NCDPI 2013b.

Focus, Coherence, and Rigor

Work in the Statistics and Probability cluster "Investigate patterns of association in bivariate data" involves looking for patterns in scatter plots and using linear models to describe data. This is directly connected to major work in the Expressions and Equations clusters (**8.EE.1–8▲**) and provides opportunities for students to model with mathematics (**MP.4**).

A detailed discussion of statistics and probability is provided online at https://commoncoretools.files.wordpress.com/2011/12/ccss_progression_sp_68_2011_12_26_bis.pdf (UA Progressions Documents 2011e).

Essential Learning for the Next Grade

In grades six through eight, multiplication and division develop into powerful forms of ratio and proportional reasoning. The properties of operations take on prominence as arithmetic matures into algebra. The theme of quantitative relationships also becomes explicit in grades six through eight, developing into the formal notion of a function by grade eight. Meanwhile, the foundations for later courses in deductive geometry are laid in grades six through eight. The gradual development of data representations in kindergarten through grade five leads to statistics in middle school: the study of shape, center, and spread of data distributions; possible associations between two variables; and the use of sampling in making statistical decisions.

In higher mathematics courses, algebra, functions, geometry, and statistics develop with an emphasis on modeling. Students continue to take a thinking approach to algebra, learning to see and make use of structure in algebraic expressions of growing complexity (adapted from PARCC 2012). To be prepared for courses in higher mathematics, students should be able to demonstrate that they have acquired particular mathematical concepts and procedural skills by the end of grade eight. Prior to grade eight, some standards identify fluency for the grade level, but there are no explicit grade-level fluency expectations for grades seven and beyond. In grade eight, linear algebra is an instructional focus, and although the grade-eight standards do not specifically identify fluency expectations, students in grade eight who can fluently solve linear equations (**8.EE.7▲**) and pairs of simultaneous linear equations (**8.EE.8▲**) will be better prepared to complete courses in higher mathematics. These fluencies and the conceptual understandings that support them are foundational for work in higher mathematics. Many students have worked informally with one-variable linear equations since kindergarten. This important line of development culminates in grade eight with the solution of general one-variable linear equations, including cases with an infinite number of solutions or no solutions, as well as cases requiring algebraic manipulation using properties of operations.

It is particularly important for students in grade eight to obtain skills and understandings to work with radical and integer exponents (**8.EE.1–4▲**); understand connections between proportional relationships, lines, and linear equations (**8.EE.5–6▲**); analyze and solve linear equations and pairs of simultaneous linear equations (**8.EE.7–8▲**); and define, evaluate, and compare functions (**8.F.1–3▲**). In addition, the skills and understandings to use functions to model relationships between quantities (**8.F.4–5**) will better prepare students to use mathematics to model real-world problems in higher mathematics.

Guidance on Course Placement and Sequences

The California Common Core State Standards for Mathematics (CA CCSSM) support a progression of learning. Many culminating standards that remain important far beyond the particular grade level appear in grades six through eight. As stated in the national Common Core State Standards Initiative documents, "some of the highest priority content for college and career readiness comes from [g]rades 6–8. This body of material includes powerfully useful proficiencies such as applying ratio reasoning in real-world and mathematical problems, computing fluently with positive and negative fractions and decimals, and solving real-world and mathematical problems involving angle measure, area, surface area, and volume" (NGA/CCSSO 2010g).

The CA CCSSM for grades six through eight are comprehensive, rigorous, and non-redundant. Instruction in an accelerated sequence of courses will require compaction—not the former strategy of deletion. Therefore, careful consideration needs to be made before placing a student in higher-mathematics course work in grades six through eight. Acceleration may get students to advanced course work, but it may create gaps in students' mathematical background. Careful consideration and systematic collection of multiple measures of individual student performance on both the content and practice standards are required. For additional information and guidance on course placement, see appendix D (Course Placement and Sequences).

Grade 8 Overview

The Number System

- Know that there are numbers that are not rational, and approximate them by rational numbers.

Expressions and Equations

- Work with radicals and integer exponents.
- Understand the connection between proportional relationships, lines, and linear equations.
- Analyze and solve linear equations and pairs of simultaneous linear equations.

Functions

- Define, evaluate, and compare functions.
- Use functions to model relationships between quantities.

Geometry

- Understand congruence and similarity using physical models, transparencies, or geometry software.
- Understand and apply the Pythagorean Theorem.
- Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

Statistics and Probability

- Investigate patterns of association in bivariate data.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

8 Grade 8

The Number System 8.NS

Know that there are numbers that are not rational, and approximate them by rational numbers.

1. Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

2. Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.*

Expressions and Equations 8.EE

Work with radicals and integer exponents.

1. Know and apply the properties of integer exponents to generate equivalent numerical expressions. *For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.*

2. Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

3. Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. *For example, estimate the population of the United States as 3×10^8 and the population of the world as 7×10^9, and determine that the world population is more than 20 times larger.*

4. Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

Understand the connections between proportional relationships, lines, and linear equations.

5. Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. *For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.*

6. Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

Analyze and solve linear equations and pairs of simultaneous linear equations.

7. Solve linear equations in one variable.

   a. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).

   b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.

8. Analyze and solve pairs of simultaneous linear equations.
   a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
   b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, $3x+2y=5$ and $3x+2y=6$ have no solution because $3x+2y$ cannot simultaneously be 5 and 6.*
   c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

Functions 8.F

Define, evaluate, and compare functions.

1. Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.[6]
2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*
3. Interpret the equation $y=mx+b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function $A=s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4), and (3,9), which are not on a straight line.*

Use functions to model relationships between quantities.

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
5. Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or non-linear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

Geometry 8.G

Understand congruence and similarity using physical models, transparencies, or geometry software.

1. Verify experimentally the properties of rotations, reflections, and translations:
   a. Lines are taken to lines, and line segments to line segments of the same length.
   b. Angles are taken to angles of the same measure.
   c. Parallel lines are taken to parallel lines.

6. Function notation is not required in grade eight.

2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
3. Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
4. Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.
5. Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.*

Understand and apply the Pythagorean Theorem.

6. Explain a proof of the Pythagorean Theorem and its converse.
7. Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

9. Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

Statistics and Probability 8.SP

Investigate patterns of association in bivariate data.

1. Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and non-linear association.
2. Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.
3. Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. *For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.*
4. Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables. *For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?*

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Introduction to Higher Mathematics Courses

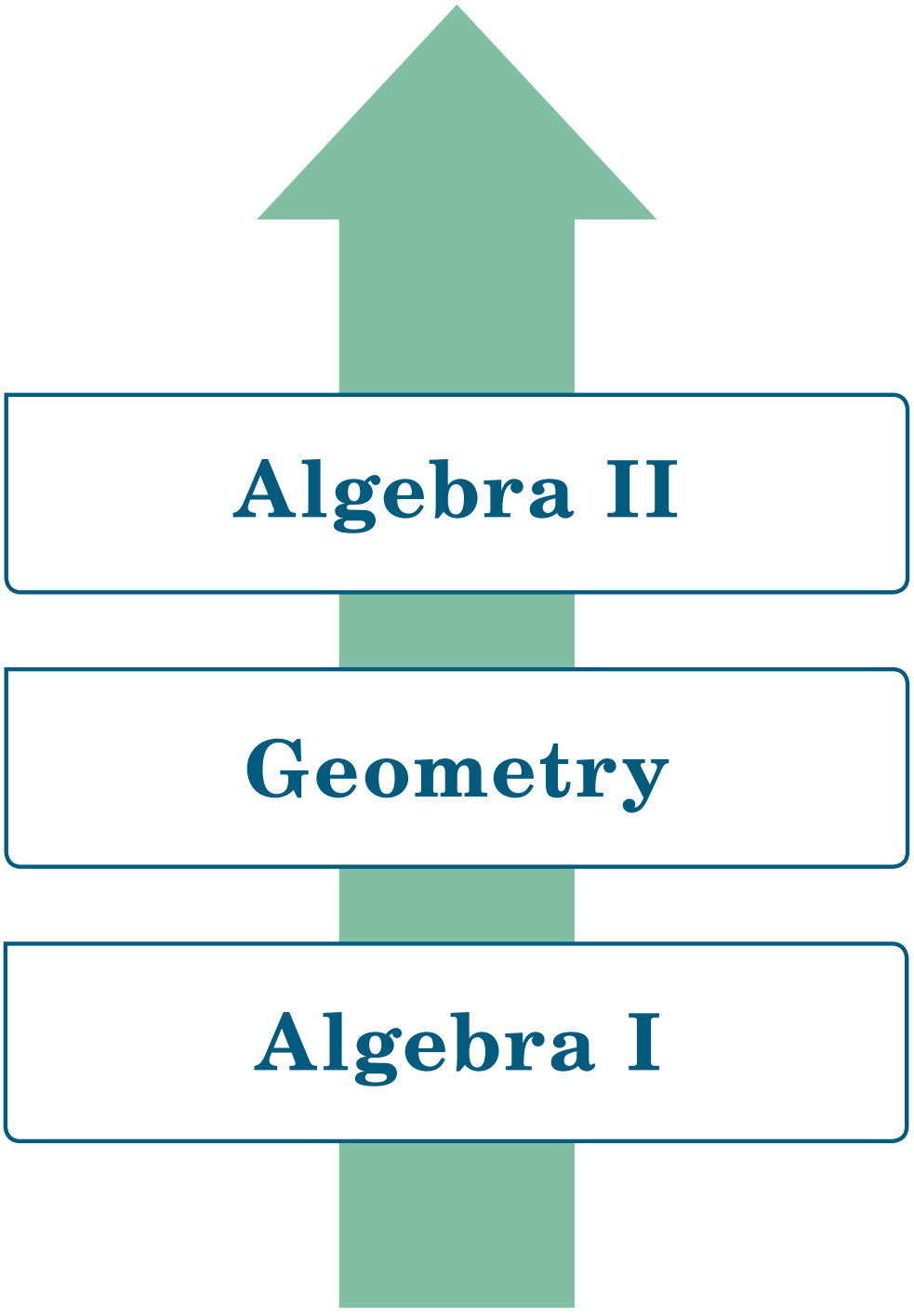
The standards for higher mathematics are organized in two ways—as model courses and in conceptual categories. The model courses consist of three courses in the Traditional Pathway (Algebra I, Geometry, and Algebra II); three courses in the Integrated Pathway (Mathematics I, II, and III); and four advanced courses (Precalculus, Statistics and Probability, Calculus, and Advanced Placement Probability and Statistics). The model courses provide guidance for developing curriculum and instruction.

The traditional and integrated pathways lay out two sequences of courses that are made up of higher mathematics standards of the CA CCSSM. The higher mathematics standards specify the mathematics that all students should study in order to be ready for college and careers. Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics are indicated by a "(+)" symbol. All standards *without* a (+) symbol should be included in the mathematics curriculum for all college- and career-ready students. Standards *with* a (+) symbol may also appear in courses intended for all students in order to increase coherence, but they are intended for students who wish to pursue higher-level mathematics courses or college majors in a science, technology, engineering, and mathematics (STEM) field. For example, the Precalculus course mainly consists of the (+) standards that are not taught in either the integrated or traditional pathway; it is designed to be an appropriate preparation course for Calculus.

Local educational agencies are not limited to offering the higher mathematics courses described in this framework. The higher mathematics standards included in the CA CCSSM are designed, in part, as a menu of standards from which educators can create customized courses. Districts must offer the mathematics courses that all students need to complete in order to fulfill state high school graduation requirements; see California *Education Code* sections 51224.5 and 51225.3. Beyond requirements set forth in state law, local districts determine which course offerings and sequences best meet the needs of students. Thus, for example, a school or district may opt to create a mathematics course based on certain CA CCSSM and the *California Career Technical Education Model Curriculum Standards* (http://www.cde.ca.gov/ci/ct/sf/ctemcstandards.asp [accessed June 2, 2014]), or it may opt to create a Mathematical Modeling course using starred (★) standards; see appendix B and the Overview of the Standards Chapters for more information. Schools and districts may want to review the *Statement on Competencies in Mathematics Expected of Entering College Students* (ICAS 2013) to help determine which courses to offer to students.

Higher Mathematics Courses

Traditional Pathway



Algebra I

The main purpose of Algebra I is to develop students' fluency with linear, quadratic, and exponential functions. The critical areas of instruction involve deepening and extending students' understanding of linear and exponential relationships by comparing and contrasting those relationships and by applying linear models to data that exhibit a linear trend. In addition, students engage in methods for analyzing, solving, and using exponential and quadratic functions. Some of the overarching elements of the Algebra I course include the notion of *function,* solving equations, rates of change and growth patterns, graphs as representations of functions, and modeling.

For the Traditional Pathway, the standards in the Algebra I course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not simply topics to be checked off from a list during isolated units of instruction; rather, they represent content that should be present throughout the school year in rich instructional experiences.

What Students Learn in Algebra I

In Algebra I, students use reasoning about structure to define and make sense of rational exponents and explore the algebraic structure of the rational and real number systems. They understand that numbers in real-world applications often have units attached to them—that is, the numbers are considered *quantities*. Students' work with numbers and operations throughout elementary and middle school has led them to an understanding of the structure of the number system; in Algebra I, students explore the structure of algebraic expressions and polynomials. They see that certain properties must persist when they work with expressions that are meant to represent numbers—which they now write in an abstract form involving variables. When two expressions with overlapping domains are set as equal to each other, resulting in an equation, there is an implied solution set (be it empty or non-empty), and students not only refine their techniques for solving equations and finding the solution set, but they can clearly explain the algebraic steps they used to do so.

Students began their exploration of linear equations in middle school, first by connecting proportional equations ($y = kx$, $k \neq 0$) to graphs, tables, and real-world contexts, and then moving toward an understanding of general linear equations ($y = mx + b$, $m \neq 0$) and their graphs. In Algebra I, students extend this knowledge to work with absolute value equations, linear inequalities, and systems of linear equations. After learning a more precise definition of *function* in this course, students examine this new idea in the familiar context of linear equations—for example, by seeing the solution of a linear equation as solving $f(x) = g(x)$ for two linear functions f and g.

Students continue to build their understanding of functions beyond linear ones by investigating tables, graphs, and equations that build on previous understandings of numbers and expressions. They make connections between different representations of the same function. They also learn to build functions in a modeling context and solve problems related to the resulting functions. Note that in Algebra I the focus is on linear, simple exponential, and quadratic equations.

Finally, students extend their prior experiences with data, using more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models. With linear models, students look at residuals to analyze the goodness of fit.

Examples of Key Advances from Kindergarten Through Grade Eight

- Having already extended arithmetic from whole numbers to fractions (grades four through six) and from fractions to rational numbers (grade seven), students in grade eight encountered specific irrational numbers such as $\sqrt{5}$ and π. In Algebra I, students begin to understand the real number *system*. (For more on the extension of number systems, refer to NGA/CCSSO 2010c.)

- Students in middle grades worked with measurement units, including units obtained by multiplying and dividing quantities. In Algebra I (conceptual category N-Q), students apply these skills in a more sophisticated fashion to solve problems in which reasoning about units adds insight.

- Algebraic themes beginning in middle school continue and deepen during high school. As early as grades six and seven, students began to use the properties of operations to generate equivalent expressions (standards **6.EE.3** and **7.EE.1**). By grade seven, they began to recognize that rewriting expressions in different forms could be useful in problem solving (standard **7.EE.2**). In Algebra I, these aspects of algebra carry forward as students continue to use properties of operations to rewrite expressions, gaining fluency and engaging in what has been called "mindful manipulation."

- Students in grade eight extended their prior understanding of proportional relationships to begin working with functions, with an emphasis on linear functions. In Algebra I, students master linear and quadratic functions. Students encounter other kinds of functions to ensure that general principles of working with functions are perceived as applying to all functions, as well as to enrich the range of quantitative relationships considered in problems.

- Students in grade eight connected their knowledge about proportional relationships, lines, and linear equations (standards **8.EE.5–6**). In Algebra I, students solidify their understanding of the analytic geometry of lines. They understand that in the Cartesian coordinate plane:

  - the graph of any linear equation in two variables is a line;
  - any line is the graph of a linear equation in two variables.

- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (e.g., standard **S-ID.6**). In a modeling context, they might informally fit a quadratic function to a set of data, graphing the data and the model function on the same coordinate axes. They also draw on skills first learned in middle school to apply basic statistics and simple probability in a modeling context. For example, they might estimate a measure of center or variation and use it as an input for a rough calculation.

- Algebra I techniques open an extensive variety of solvable word problems that were previously inaccessible or very complex for students in kindergarten through grade eight. This expands problem solving dramatically.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The Standards for Mathematical Practice represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. There are ample opportunities for students to engage in each mathematical practice in Algebra I; table A1-1 offers some general examples.

Table A1-1. Standards for Mathematical Practice—Explanation and Examples for Algebra I

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students learn that patience is often required to fully understand what a problem is asking. They discern between useful and extraneous information. They expand their repertoire of expressions and functions that can be used to solve problems.
MP.2 Reason abstractly and quantitatively.	Students extend their understanding of slope as the rate of change of a linear function to comprehend that the average rate of change of any function can be computed over an appropriate interval.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students reason through the solving of equations, recognizing that solving an equation involves more than simply following rote rules and steps. They use language such as "If _____, then _____" when explaining their solution methods and provide justification for their reasoning.
MP.4 Model with mathematics.	Students also discover mathematics through experimentation and by examining data patterns from real-world contexts. Students apply their new mathematical understanding of exponential, linear, and quadratic functions to real-world problems.
MP.5 Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret results. They construct diagrams to solve problems.
MP.6 Attend to precision.	Students begin to understand that a *rational number* has a specific definition and that *irrational numbers* exist. They make use of the definition of *function* when deciding if an equation can describe a function by asking, "Does every input value have exactly one output value?"
MP.7 Look for and make use of structure	Students develop formulas such as $(a \pm b)^2 = a^2 \pm 2ab + b^2$ by applying the distributive property. Students see that the expression $5 + (x-2)^2$ takes the form of 5 plus "something squared," and because "something squared" must be positive or zero, the expression can be no smaller than 5.
MP.8 Look for and express regularity in repeated reasoning.	Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, the equation $m = \frac{y - y_1}{x - x_1}$ will give a general equation of that line.

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star (★) symbol to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards. In the description of the Algebra I content standards that follow, Modeling is covered first to emphasize its importance in the higher mathematics curriculum.

Examples of places where specific Mathematical Practice standards can be implemented in the Algebra I standards are noted in parentheses, with the standard(s) also listed.

Algebra I Content Standards, by Conceptual Category

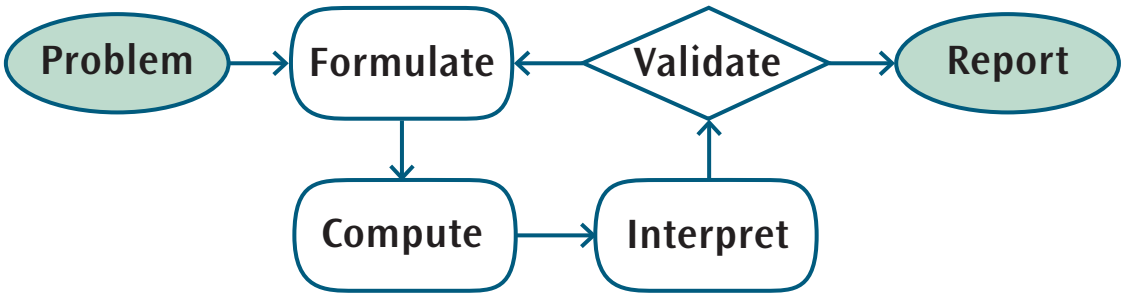
The Algebra I course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in Algebra I are described below, according to each conceptual category. Standards that are considered new for secondary-grades teachers are discussed more thoroughly than other standards.

Conceptual Category: Modeling

Throughout the California Common Core State Standards for Mathematics (CA CCSSM), specific standards for higher mathematics are marked with a ★ symbol to indicate they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: Which of the quantities present in this situation are known, and which are unknown? Can a table of data be made? Is there a functional relationship in this situation? Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new equation or function will apply. In addition, students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, etc.), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure A1-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Figure A1-1. The Modeling Cycle



The examples in this chapter are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding linear and exponential functions, graphing, solving equations, and rates of change are explored through this lens. Readers are encouraged to consult appendix B (Mathematical Modeling) for a further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

Functions describe situations where one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually form theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models. In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v; the rule $T(v) = \frac{100}{v}$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its *domain*. We often assume the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context. When describing relationships between quantities, the defining characteristic of a function is that the input value determines the output value, or equivalently, that the output value depends upon the input value (University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2013c, 2).

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city"; by an assignment, such as the fact that each individual is given a unique Social Security Number; by an algebraic expression, such as $f(x) = a + bx$; or by a recursive rule, such as $f(n+1) = f(n) + b, f(0) = a$. The graph of a function is often a useful way of visualizing the relationship that the function models, and manipulating a mathematical expression for a function can shed light on the function's properties.

Interpreting Functions F-IF

Understand the concept of a function and use function notation. [Learn as general principle; focus on linear and exponential and on arithmetic and geometric sequences.]

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$.

Interpret functions that arise in applications in terms of the context. [Linear, exponential, and quadratic]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

While the grade-eight standards call for students to work informally with functions, students in Algebra I begin to refine their understanding and use the formal mathematical language of functions. Standards **F-IF.1–9** deal with understanding the concept of a function, interpreting characteristics of functions in context, and representing functions in different ways (**MP.6**). In **F-IF.1–3**, students learn the language of functions and that a function has a domain that must be specified as well as a corresponding range. For instance, the function f where $f(n) = 4(n-2)^2$, defined for n, an integer, is a different function than the function g where $g(x) = 4(x-2)^2$ and g is defined for all real numbers x. Students make the connection between the graph of the equation $y = f(x)$ and the function itself—namely, that the coordinates of any point on the graph represent an input and output, expressed as $(x, f(x))$, and understand that the graph is a *representation* of a function. They connect the domain and range of a function to its graph (**F-IF.5**). Note that there is neither an exploration of the notion of *relation vs. function* nor the *vertical line test* in the CA CCSSM. This is by design. The core question when investigating functions is, "Does each element of the domain correspond to exactly one element of the range?" (UA Progressions Documents 2013c, 8).

Standard **F-IF.3** represents a topic that is new to the traditional Algebra I course: *sequences*. Sequences are functions with a domain consisting of a subset of the integers. In grades four and five, students began to explore number patterns, and this work led to a full progression of ratios and proportional relationships in grades six and seven. Patterns are examples of sequences, and the work here is intended to formalize and extend students' earlier understandings. For a simple example, consider the sequence 4, 7, 10, 13, 16 . . . , which might be described as a "plus 3 pattern" because terms are computed by adding 3 to the previous term. If we decided that 4 is the first term of the sequence, then we can make a table, a graph, and eventually a recursive rule for this sequence: $f(1) = 4, f(n+1) = f(n) + 3$ for $n \geq 2$. Of course, this sequence can also be described with the explicit formula $f(n) = 3n + 1$ for $n \geq 1$. Notice that the domain is included in the description of the rule (adapted from UA Progressions Documents 2013c, 8). In Algebra I, students should have opportunities to work with linear, quadratic, and exponential sequences and to interpret the parameters of the expressions defining the terms of the sequence when they arise in context.

Interpreting Functions F-IF

Analyze functions using different representations. [Linear, exponential, quadratic, absolute value, step, piecewise-defined]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, *and* $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

In standards **F-IF.7–9**, students represent functions with graphs and identify key features in the graph. In Algebra I, linear, exponential, and quadratic functions are given extensive treatment because they have their own group of standards (the **F-LE** standards) dedicated to them. Students are expected to develop fluency only with linear, exponential, and quadratic functions in Algebra I, which includes the ability to graph them by hand.

In this set of three standards, students represent the same function algebraically in different forms and interpret these differences in terms of the graph or context. For instance, students may easily see that the graph of the equation $f(x) = 3x^2 + 9x + 6$ crosses the y-axis at $(0,6)$, since the terms containing x are simply 0 when $x = 0$—but then they factor the expression defining f to obtain $f(x) = 3(x+2)(x+1)$, easily revealing that the function crosses the x-axis at $(-2,0)$ and $(-1,0)$, since this is where $f(x) = 0$ (**MP.7**).

Building Functions — F-BF

Build a function that models a relationship between two quantities. [For **F-BF.1–2**, linear, exponential, and quadratic]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

Build new functions from existing functions. [Linear, exponential, quadratic, and absolute value; for **F-BF.4a**, linear only]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

Knowledge of functions and expressions is only part of the complete picture. One must be able to understand a given situation and apply function reasoning to model how quantities change together. Often, the function created sheds light on the situation at hand; one can make predictions of future changes, for example. This is the content of standards **F-BF.1** and **F-BF.2** (starred to indicate they are modeling standards). A strong connection exists between standard **F-BF.1** and standard **A-CED.2**, which discusses creating equations. The following example shows that students can create functions based on prototypical ones.

Example: Exponential Growth **F-BF.1–2**

When a quantity grows with time by a multiplicative factor greater than 1, it is said the quantity grows exponentially. Hence, if an initial population of bacteria, P_0, doubles each day, then after t days, the new population is given by $P(t) = P_0 2^t$. This expression can be generalized to include different growth rates, r, as in $P(t) = P_0 r^t$. A more specific example illustrates the type of problem that students may face after they have worked with basic exponential functions:

On June 1, a fast-growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If the algae continue to grow unabated, the lake will be totally covered, and the fish in the lake will suffocate. Based on the current rate at which the algae are growing, this will happen on June 30.

Possible Questions to Ask:

a. When will the lake be covered halfway?

b. Write an equation that represents the percentage of the surface area of the lake that is covered in algae, as a function of time (in days) that passes since the algae were introduced into the lake.

Solution and Comments:

a. Since the population doubles each day, and since the entire lake will be covered by June 30, this implies that half the lake was covered on June 29.

b. If $P(t)$ represents the *percentage* of the lake covered by the algae, then we know that $P(29) = P_0 2^{29} = 100$ (note that June 30 corresponds to $t = 29$). Therefore, we can solve for the initial percentage of the lake covered, $P_0 = \frac{100}{2^{29}} \approx 1.86 \times 10^{-7}$. The equation for the percentage of the lake covered by algae at time t is therefore $P(t) = \left(1.86 \times 10^{-7}\right) 2^t$.

Adapted from Illustrative Mathematics 2013i.

As mentioned earlier, the study of arithmetic and geometric sequences, written both explicitly and recursively (**F-BF.2**), is new to the Algebra I course in California. When presented with a sequence, students can often manage to find the recursive pattern of the sequence (i.e., how the sequence changes from term to term). For instance, a simple doubling pattern can lead to an exponential expression of the form $a2^n$, for $n \geq 0$. Ample experience with linear and exponential functions—which show equal differences over equal intervals and equal ratios over equal intervals, respectively—can provide students with tools for finding explicit rules for sequences. Investigating the simple sequence of squares, $f(x) = n^2$ (where $n \geq 1$), provides a prototype for other basic quadratic sequences. Diagrams, tables, and graphs can help students make sense of the different rates of growth all three sequences exhibit.

Example **F-BF.2**

Cellular Growth

Populations of cyanobacteria can double every 6 hours under ideal conditions, at least until the nutrients in its supporting culture are depleted. This means a population of 500 such bacteria would grow to 1000 in the first 6-hour period, 2000 in the second 6-hour period, 4000 in the third 6-hour period, and so on. Evidently, if n represents the number of 6-hour periods from the start, the population at that time $P(n)$ satisfies $P(n) = 2 \cdot P(n-1)$. This is a *recursive* formula for the sequence $P(n)$, which gives the population at a given time period n in terms of the population at time period $n-1$. To find a closed, *explicit*, formula for $P(n)$, students can reason that

$$P(0) = 500,\ P(1) = 2 \cdot 500,\ P(2) = 2 \cdot 2 \cdot 500,\ P(3) = 2 \cdot 2 \cdot 2 \cdot 500, \ldots$$

A pattern emerges: that $P(n) = 2^n \cdot 500$. In general, if an initial population P_0 grows by a factor $r > 1$ over a fixed time period, then the population after n time periods is given by $P(n) = P_0 r^n$.

The content of standard **F-BF.3** has typically been left to later courses. In Algebra I, the focus is on linear, exponential, and quadratic functions. Even and odd functions are addressed in later courses. In keeping with the theme of the input–output interpretation of a function, students should work toward developing an understanding of the effect on the output of a function under certain transformations, such as in the table below:

Expression	Interpretation
$f(a+2)$	The output when the input is 2 greater than a
$f(a)+3$	3 more than the output when the input is a
$2f(x)+5$	5 more than twice the output of f when the input is x

Such understandings can help students to see the effect of transformations on the graph of a function, and in particular, can aid in understanding why it appears that the effect on the graph is the opposite to the transformation on the variable. For example, the graph of $y = f(x+2)$ is the graph of f shifted 2 units to the left, not to the right (UA Progressions Documents 2013c, 7).

Also new to the Algebra I course is standard **F-BF.4**, which calls for students to find inverse functions in simple cases. For example, an Algebra I student might solve the equation $F = \frac{9}{5}C + 32$ for C. The student starts with this formula, showing how Fahrenheit temperature is a function of Celsius temperature, and by solving for C finds the formula for the inverse function. This is a contextually appropriate way to find the expression for an inverse function, in contrast with the practice of simply swapping x and y in an equation and solving for y.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★
   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★
   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Interpret expressions for functions in terms of the situation they model.

5. Interpret the parameters in a linear or exponential function in terms of a context ★ [Linear and exponential of form $f(x) = b^x + k$]
6. **Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity. CA** ★

Modeling the world often involves investigating rates of change and patterns of growth. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. In standards **F-LE.1a–c**, students recognize and understand the defining characteristics of linear, quadratic, and exponential functions. Students have already worked extensively with linear equations. They have developed an understanding that an equation in two variables of the form $y = mx + b$ exhibits a special relationship between the variables x and y—namely, that a change of Δx in the variable x, the independent variable, results in a change of $\Delta y = m \bullet \Delta x$ in the dependent variable y. They have seen this informally, in graphs and tables of linear relationships, starting in the grade-eight standards (**8.EE.5, 8.EE.6, 8.F.3**). For example, students recognize that for successive whole-number input values, x and $x+1$, a linear function f, defined by $f(x) = mx + b$, exhibits a constant rate of change:

$$f(x+1) - f(x) = \left[m(x+1) + b\right] - (mx + b) = m(x + 1 - x) = m$$

Standard **F-LE.1a** requires students to prove that linear functions exhibit such growth patterns.

In contrast, an exponential function exhibits a constant percent change in the sense that such functions exhibit a constant *ratio* between output values for successive input values.[1] For instance, a t-table for the equation $y = 3^x$ illustrates the constant ratio of successive y-values for this equation:

1. In Algebra I of the California Common Core State Standards for Mathematics, only integer values for x are considered in exponential equations such as $y = b^x$.

x	$y = 3^x$	Ratio of successive y-values
1	3	
2	9	$\frac{9}{3} = 3$
3	27	$\frac{27}{9} = 3$
4	81	$\frac{81}{27} = 3$

In general, a function g, defined by $g(x) = ab^x$, can be shown to exhibit this constant ratio growth pattern:

$$\frac{g(x+1)}{g(x)} = \frac{ab^{x+1}}{ab^x} = \frac{b^{x+1}}{b^x} = b^{(x+1)-x} = b$$

In Algebra I, students are not required to prove that exponential functions exhibit this growth rate; however, they must be able to recognize situations that represent both linear and exponential functions and construct functions to describe the situations (**F-LE.2**). Finally, students interpret the *parameters* in linear, exponential, and quadratic expressions and model physical problems with such functions. The meaning of parameters often becomes much clearer when they are presented in a modeling situation rather than in an abstract way.

A graphing utility, spreadsheet, or computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions (**MP.5**). Real-world examples where this can be explored involve half-lives of pharmaceuticals, investments, mortgages, and other financial instruments. For example, students can develop formulas for annual compound interest based on a general formula, such as $P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$, where r is the interest rate, n is the number of times the interest is compounded per year, and t is the number of years the money is invested. They can explore values after different time periods and compare different rates and plans using computer algebra software or simple spreadsheets (**MP.5**). This hands-on experimentation with such functions helps students develop an understanding of the functions' behavior.

Conceptual Category: Number and Quantity

In the grade-eight standards, students encountered some examples of irrational numbers, such as π and $\sqrt{2}$ (or $\sqrt{n}$ where n is a non-square number). In Algebra I, students extend this understanding beyond the fact that there are numbers that are not rational; they begin to understand that the rational numbers form a closed system. Students have witnessed that with each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—whole numbers, rational numbers, and real numbers—the distributive law continues to hold, and the commutative and associative laws are still valid for both addition and multiplication. However, in Algebra I students go further along this path.

The Real Number System — N-RN

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a non-zero rational number and an irrational number is irrational.

With **N-RN.1**, students make meaning of the representation of radicals with rational exponents. Students are first introduced to exponents in grade six; by the time they reach Algebra I, they should have an understanding of the basic properties of exponents (e.g., that $x^n \cdot x^m = x^{n+m}$, $\left(x^n\right)^m = x^{nm}$, $\frac{x^n}{x^m} = x^{n-m}$, $x^0 = 1$ for $x \neq 0$,). In fact, they may have justified certain properties of exponents by reasoning with other properties (**MP.3, MP.7**), for example, justifying why any non-zero number to the power 0 is equal to 1:

$$x^0 = x^{n-n} = \frac{x^n}{x^n} = 1, \text{ for } x \neq 0$$

They further their understanding of exponents in Algebra I by using these properties to explain the meaning of rational exponents. For example, properties of whole-number exponents suggest that $\left(5^{1/3}\right)^3$ should be the same as $5^{[(1/3)\cdot 3]} = 5^1 = 5$, so that $5^{1/3}$ should represent the cube root of 5. In addition, $(ab)^n = a^n \cdot b^n$ reveals that $\sqrt{20} = (4 \cdot 5)^{1/2} = 4^{1/2} \cdot 5^{1/2} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$, which shows that $\sqrt{20} = 2\sqrt{5}$. The intermediate steps of writing the square root as a rational exponent are necessary at first, but eventually students can work more quickly, understanding the reasoning underpinning this process. Students extend such work with radicals and rational exponents to variable expressions as well—for example, rewriting an expression like $\left(a^2 b^5\right)^{3/2}$ using radicals (**N-RN.2**).

In standard **N-RN.3**, students explain that the sum or product of two rational numbers is rational, arguing that the sum of two fractions with integer numerator and denominator is also a fraction of the same type, which shows that the rational numbers are *closed* under the operations of addition and multiplication (**MP.3**). The notion that this set of numbers is closed under these operations will be extended to the sets of polynomials and rational functions in later courses. Moreover, students argue that the sum of a rational and an irrational is irrational, and the product of a non-zero rational and an irrational is still irrational, showing that the irrational numbers are truly another unique set of numbers that, along with the rational numbers, forms a larger *system*, the system of real numbers (**MP.3, MP.7**).

Quantities N-Q

Reason quantitatively and use units to solve problems. [Foundation for work with expressions, equations and functions]

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★

In real-world problems, the answers are usually not pure numbers, but *quantities:* numbers with units, which involve measurement. In their work in measurement up through grade eight, students primarily measure commonly used attributes such as length, area, and volume. In higher mathematics, students encounter a wider variety of units in modeling—for example, when considering acceleration, currency conversions, derived quantities such as person-hours and heating degree-days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages.

In Algebra I, students use units to understand problems and make sense of the answers they deduce. The following example illustrates the facility with units that students are expected to attain in this domain.

Example N-Q.1–3

As Felicia gets on the freeway to drive to her cousin's house, she notices that she is a little low on fuel. There is a gas station at the exit she normally takes, and she wonders if she will need gas before reaching that exit. She normally sets her cruise control at the speed limit of 70 mph, and the freeway portion of the drive takes about an hour and 15 minutes. Her car gets about 30 miles per gallon on the freeway. Gas costs $3.50 per gallon.

a. Describe an estimate that Felicia might form in her head while driving to decide how many gallons of gas she needs to make it to the gas station at her usual exit.

b. Assuming she makes it, how much does Felicia spend per mile on the freeway?

Solution:

a. To estimate the amount of gas she needs, Felicia calculates the distance traveled at 70 mph for 1.25 hours. She might calculate as follows:

$$70 \cdot 1.25 = 70 + (0.25 \cdot 70) = 70 + 17.5 = 87.5 \text{ miles}$$

Since 1 gallon of gas will take her 30 miles, 3 gallons of gas will take her 90 miles—a little more than she needs. So she might figure that 3 gallons is enough.

b. Since Felicia pays $3.50 for one gallon of gas, and one gallon of gas takes her 30 miles, it costs her $3.50 to travel 30 miles. Therefore:

$$\frac{\$3.50}{30 \text{ miles}} \approx \frac{\$0.12}{1 \text{ mile}}$$

Which means it costs Felicia 12 cents to travel each mile on the freeway.

Adapted from Illustrative Mathematics 2013o.

Conceptual Category: Algebra

In the Algebra conceptual category, students extend the work with expressions that they started in the middle-grades standards. They create and solve equations in context, utilizing the power of variable expressions to model real-world problems and solve them with attention to units and the meaning of the answers they obtain. They continue to graph equations, understanding the resulting picture as a representation of the points satisfying the equation. This conceptual category accounts for a large portion of the Algebra I course and, along with the Functions category, represents the main body of content.

The Algebra conceptual category in higher mathematics is very closely related to the Functions conceptual category (UA Progressions Documents 2013b, 2):

- An expression in one variable can be viewed as defining a function: the act of evaluating the expression is an act of producing the function's output given the input.
- An equation in two variables can sometimes be viewed as defining a function, if one of the variables is designated as the input variable and the other as the output variable, and if there is just one output for each input. This is the case if the expression is of the form $y = (\text{expression in } x)$ or if it can be put into that form by solving for y.
- The notion of equivalent expressions can be understood in terms of functions: if two expressions are equivalent, they define the same function.
- The solutions to an equation in one variable can be understood as the input values that yield the same output in the two functions defined by the expressions on each side of the equation. This insight allows for the method of finding approximate solutions by graphing functions defined by each side and finding the points where the graphs intersect.

Thus, in light of understanding functions, the main content of the Algebra category (solving equations, working with expressions, and so forth) has a very important purpose.

Seeing Structure in Expressions	**A-SSE**

Interpret the structure of expressions. [Linear, exponential, and quadratic]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* ★
2. Use the structure of an expression to identify ways to rewrite it.

An expression can be viewed as a recipe for a calculation, with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price, p. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor. Students began this work in grades six and seven and continue this work with more complex expressions in Algebra I.

Seeing Structure in Expressions	**A-SSE**

Write expressions in equivalent forms to solve problems. [Quadratic and exponential]

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★
   a. Factor a quadratic expression to reveal the zeros of the function it defines. ★
   b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. ★
   c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $\left(1.15^{1/12}\right)^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* ★

In Algebra I, students work with examples of more complicated expressions, such as those that involve multiple variables and exponents. Students use the distributive property to investigate equivalent forms of quadratic expressions—for example, by writing

$$\begin{aligned}(x+y)(x-y) &= x(x-y)+y(x-y)\\ &= x^2 - xy + xy - y^2\\ &= x^2 - y^2.\end{aligned}$$

This yields a special case of a factorable quadratic, the difference of squares.

Students factor second-degree polynomials by making use of such special forms and by using factoring techniques based on properties of operations (**A-SSE.2**). Note that the standards avoid talking about "simplification," because the simplest form of an expression is often unclear, and even in cases where it is clear, it is not obvious that the simplest form is desirable for a given purpose. The standards emphasize purposeful transformation of expressions into equivalent forms that are suitable for the purpose at hand, as the following example shows.

Example **A-SSE.2**

Which is the simpler form? A particularly rich mathematical investigation involves finding a general expression for the sum of the first n consecutive natural numbers:

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n.$$

A famous tale speaks of a young C. F. Gauss being able to add the first 100 natural numbers quickly in his head, wowing his classmates and teachers alike. One way to find this sum is to consider the "reverse" of the sum:

$$S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$

Then the two expressions for S are added together:

$$2S = (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1),$$

where there are n terms of the form $(n+1)$. Thus, $2S = n(n+1)$, so that $S = \frac{n(n+1)}{2}$. While students may be tempted to transform this expression into $\frac{1}{2}n^2 + \frac{1}{2}n$, they are obscuring the ease with which they can evaluate the first expression. Indeed, since n is a natural number, one of either n or $n+1$ is even, so evaluating $\frac{n(n+1)}{2}$, especially mentally, is often easier. In Gauss's case, $\frac{100(101)}{2} = 50(101) = 5050$.

Students also use different forms of the same expression to reveal important characteristics of the expression. For instance, when working with quadratics, they complete the square in the expression $x^2 - 3x + 4$ to obtain the equivalent expression $\left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$. Students can then reason with the new expression that the term being squared is always greater than or equal to 0; hence, the value of the expression will always be greater than or equal to $\frac{7}{4}$ (**A-SSE.3, MP.3**). A spreadsheet or a computer algebra system may be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave, further contributing to students' understanding of work with expressions (**MP.5**).

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials. [Linear and quadratic]

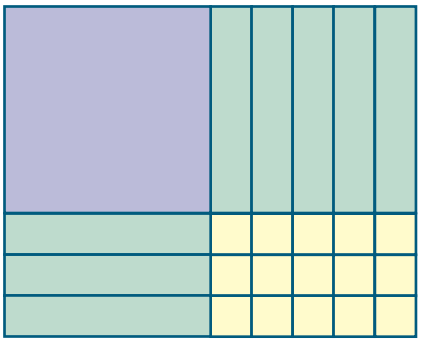
1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

In Algebra I, students begin to explore the set of polynomials in x as a system in its own right, subject to certain operations and properties. To perform operations with polynomials meaningfully, students are encouraged to draw parallels between the set of integers—wherein integers can be added, subtracted, and multiplied according to certain properties—and the set of all polynomials with real coefficients (**A-APR.1, MP.7**). If the function concept is developed before or concurrently with the study of

polynomials, then a polynomial can be identified with the function it defines. In this way, x^2-2x-3, $(x+1)(x-3)$, and $(x-1)^2-4$ are all the same polynomial because they all define the same function.

In Algebra I, students are required only to add linear or quadratic polynomials and to multiply linear polynomials to obtain quadratic polynomials, since in later courses they will explore polynomials of higher degree. Students fluently add, subtract, and multiply linear expressions of the form $ax+b$, and add and subtract expressions of the form ax^2+bx+c, with a, b, and c real numbers, understanding that the result is yet another expression of one of these forms. The explicit notion of *closure* of the set of polynomials need not be explored in Algebra I.

Figure A1-2. Algebra Tiles



The rectangle above has height $(x+3)$ and base $(x+5)$. The total area represented, the product of these binomials, is seen to be $x^2+5x+3x+15=x^2+8x+15$.

Manipulatives such as "algebra tiles" may be used to support understanding of addition and subtraction of polynomials and the multiplication of monomials and binomials. Algebra tiles may be used to offer a concrete representation of the terms in a polynomial (**MP.5**). The tile representation relies on the area interpretation of multiplication: the notion that the product ab can be thought of as the area of a rectangle of dimensions a units and b units. With this understanding, tiles can be used to represent 1 square unit (a 1 by 1 tile), x square units (a 1 by x tile), and x^2 square units (an x by x tile). Finding the product $(x+5)(x+3)$ amounts to finding the area of an abstract rectangle of dimensions $(x+5)$ and $(x+3)$, as illustrated in figure A1-2 (**MP.2**).

Care must be taken in the way negative numbers are handled with this representation. The tile representation of polynomials is also very useful for understanding the notion of completing the square, as described later in this chapter.

Creating Equations — A-CED

Create equations that describe numbers or relationships. [Linear, quadratic, and exponential (integer inputs only); for A-CED.3 linear only]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.* ★

An equation is a statement of equality between two expressions. The values that make the equation true are the solutions to the equation. An identity, in contrast, is true for all values of the variables; rewriting an expression in an equivalent form often creates identities. The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers which can be plotted in the coordinate plane. In this set of standards, students create equations to solve problems, they correctly graph the equations on coordinate axes, and they interpret solutions in a modeling context. The following example requires students to understand the multiple variables that appear in a given equation and to reason with them.

Example **A-CED.4**

The height h of a ball at time t seconds thrown vertically upward at a speed of v feet/second is given by the equation $h = 6 + vt - 16t^2$.

Write an equation whose solution is:

a. the time it takes a ball thrown at a speed of 88 ft/sec to rise 20 feet;

b. the speed with which the ball must be thrown to rise 20 feet in 2 seconds.

Solution and Comments:

a. We want $h = 20$, and we are told that $v = 88$, so the equation is $20 = 6 + 88t - 16t^2$.

b. We want $h = 20$, and we are told that $t = 2$, so the equation is $20 = 6 + 2v - 16 \bullet 2^2$.

Although this is a straightforward example, students must be able to flexibly see some of the variables in the equation as constants when others are given values. In addition, the example does not explicitly state that $v = 88$ or $t = 2$, so students must understand the meaning of the variables in order to proceed with the problem (**MP.1**).

Adapted from Illustrative Mathematics 2013h.

One change in the CA CCSSM is the creation of equations involving absolute values (**A-CED.1**). The basic absolute value function has at least two very useful definitions: (1) a descriptive, verbal definition and (2) a formula definition. A common definition of the absolute value of x is:

$|x|$ = the distance from the number x to 0 (on a number line).

An understanding of the number line easily yields that, for example, $|0| = 0$, $|7| = 7$, and $|-3.9| = 3.9$. However, an equally valid "formula" definition of absolute value reads:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

In other words, $|x|$ is simply x whenever x is 0 or positive, but $|x|$ is the opposite of x whenever x is negative. Either definition can be extended to an understanding of the expression $|x - a|$ as the distance between x and a on a number line, an interpretation that has many uses. For a simple application of this idea, suppose a type of bolt is to be mass-produced in a factory with the specification that its width be 5mm with an error no larger than 0.01mm. If w represents the width of a given bolt produced on the production line, then we want w to satisfy the inequality $|w - 5| \leq 0.01$; that is, the difference between the actual width w and the target width should be less than or equal to 0.01 (**MP.4, MP.6**). Students should become comfortable with the basic properties of absolute values (e.g., $|x| + a \neq |x + a|$) and with solving absolute value equations and interpreting the solution.

In higher mathematics courses, intervals on the number line are often denoted by an inequality of the form $|x-a| \le d$ for a positive number d. For example, $|x-2| \le \frac{1}{2}$ represents the closed interval $1\frac{1}{2} \le x \le 2\frac{1}{2}$. This can be seen by interpreting $|x-2| \le \frac{1}{2}$ as "the distance from x to 2 is less than or equal to $\frac{1}{2}$" and deciding which numbers fit this description.

On the other hand, in the case where $x-2<0$, we would have $|x-2| = -(x-2) \le \frac{1}{2}$, so that $x \ge 1\frac{1}{2}$. In the case where $x-2 \ge 0$, we have $|x-2| = x-2 \le \frac{1}{2}$, which means that $x \le 2\frac{1}{2}$. Since we are looking for all x that satisfy both inequalities, the interval is $1\frac{1}{2} \le x \le 2\frac{1}{2}$. This shows how the formula definition can be used to find this interval.

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Master linear; learn as general principle.]

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable. [Linear inequalities; literal equations that are linear in the variables being solved for; quadratics with real solutions]

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

3.1. Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA

4. Solve quadratic equations in one variable.

   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions. In Algebra I, students solve linear equations and inequalities in one variable, including equations and inequalities with absolute values and equations with coefficients represented by letters (**A-REI.3, A-REI.3.1**). In addition, this is the students' first exposure to quadratic equations, and they learn various techniques for solving them and the relationships between those techniques (**A-REI.4.a–b**). When solving equations, students

make use of the symmetric and transitive properties, as well as properties of equality with regard to operations (e.g., "Equals added to equals are equal"). Standard **A-REI.1** requires that in any situation, students can solve an equation *and explain the steps* as resulting from previous true equations and using the aforementioned properties (**MP.3**). In this way, the idea of *proof,* while not explicitly named, is given a prominent role in the solving of equations, and the reasoning and justification process is not simply relegated to a future mathematics course.

On Solving Equations: A written sequence of steps is code for a narrative line of reasoning that would use words such as *if, then, for all,* and *there exists.* In the process of learning to solve equations, students should learn certain "if–then" moves—for example, "If $x = y$, then $x + c = y + c$ for any c." The danger in learning algebra is that students emerge with nothing but the moves, which may make it difficult to detect incorrect or made-up moves later on. Thus the first requirement in this domain (**REI**) is that students understand that solving equations is a process of reasoning (**A-REI.1**).

Fragments of Reasoning

$$x^2 = 4$$
$$x^2 - 4 = 0$$
$$(x-2)(x+2) = 0$$
$$x = 2, -2$$

This sequence of equations is shorthand for a line of reasoning: *"If x is a number whose square is 4, then $x^2 - 4 = 0$, by properties of equality. Since $x^2 - 4 = (x-2)(x+2)$ for all numbers, it follows that $(x-2)(x+2) = 0$. So either $(x-2) = 0$, in which case $x = 2$, or $(x+2) = 0$, in which case $x = -2$."*

Adapted from UA Progressions Documents 2013b, 13.

Students in Algebra I extend their work with exponents to working with quadratic functions and equations that have real roots. To extend their understanding of these quadratic expressions and the functions they define, students investigate properties of quadratics and their graphs in the Functions domain.

Example **A-REI.4.a**

Standard **A-REI.4.a** calls for students to solve quadratic equations of the form $(x-p)^2 = q$. In doing so, students rely on the understanding that they can take square roots of both sides of the equation to obtain the following:

$$\sqrt{(x-p)^2} = \sqrt{q} \qquad (1)$$

In the case where $\sqrt{q}$ is a real number, we can solve this equation for x. A common mistake is to quickly introduce the symbol $\pm$ here, without understanding where it comes from. Doing so without care often leads students to think that $\sqrt{9} = \pm 3$, for example.

Note that the quantity $\sqrt{a^2}$ is simply a when $a \geq 0$ (as in $\sqrt{5^2} = \sqrt{25} = 5$), while $\sqrt{a^2}$ is equal to $-a$ (the opposite of a) when $a < 0$ (as in $\sqrt{(-4)^2} = \sqrt{16} = 4$). But this means that $\sqrt{a^2} = |a|$. Applying this to equation (1), shown above, yields $|x-p| = \sqrt{q}$. Solving this simple absolute value equation yields that $(x-p) = \sqrt{q}$ or $-(x-p) = \sqrt{q}$. This results in the two solutions $p + \sqrt{q}$, $p - \sqrt{q}$.

Students also transform quadratic equations into the form $ax^2+bx+c=0$ for $a\neq 0$, which is the *standard form* of a quadratic equation. In some cases, the quadratic expression factors nicely and students can apply the zero product property of the real numbers to solve the resulting equation. The *zero product property* states that for two real numbers m and n, $m\cdot n=0$ if and only if either $m=0$ or $n=0$. Hence, when a quadratic polynomial can be rewritten as $a(x-r)(x-s)=0$, the solutions can be found by setting each of the linear factors equal to 0 separately, and obtaining the solution set $\{r,s\}$. In other cases, a means for solving a quadratic equation arises by *completing the square*. Assuming for simplicity that $a=1$ in the standard equation above, and that the equation has been rewritten as $x^2+bx=-c$, we can "complete the square" by adding the square of half the coefficient of the x-term to each side of the equation:

$$x^2+bx+\left(\frac{b}{2}\right)^2=-c+\left(\frac{b}{2}\right)^2 \qquad (2)$$

The result of this simple step is that the quadratic on the left side of the equation is a perfect square, as shown here:

$$x^2+bx+\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{b}\right)^2$$

Thus, we have now converted equation (2) into an equation of the form $(x-p)^2=q$:

$$\left(x+\frac{b}{2}\right)^2=-c+\frac{b^2}{4}$$

This equation can be solved by the method described above, as long as the term on the right is non-negative. When $a\neq 1$, the case can be handled similarly and ultimately results in the familiar quadratic formula. Tile representations of quadratics illustrate that the process of completing the square has a geometric interpretation that explains the origin of the name. Students should be encouraged to explore these representations in order to make sense out of the process of completing the square (**MP.1, MP.5**). Completing the square is an example of a recurring theme in algebra: finding ways of transforming equations into certain standard forms that have the same solutions.

Example: Completing the Square **A-REI.4.a**

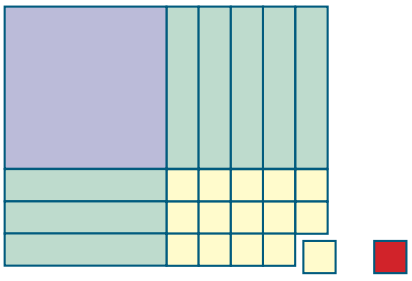
The method of completing the square is a useful skill in algebra. It is generally used to change a quadratic in standard form, ax^2+bx+c, into one in vertex form, $a(x-h)^2+k$. The vertex form can help determine several properties of quadratic functions. Completing the square also has applications in geometry (**G-GPE.1**) and later higher mathematics courses.

To complete the square for the quadratic $y=x^2+8x+15$, we take half the coefficient of the x-term and square it to yield 16. We realize that we need only to add 1 and subtract 1 to the quadratic expression:

$$\begin{aligned} y &= x^2+8x+15+1-1 \\ &= x^2+8x+16-1. \end{aligned}$$

Factoring gives us $y=(x+4)^2-1$.

In the picture at right, note that the tiles used to represent $x^2+8x+15$ have been rearranged to try to form a square, and that a positive unit tile and a "negative" unit tile are added into the picture to "complete the square."



The same solution techniques used to solve equations can be used to rearrange formulas to highlight specific quantities and explore relationships between the variables involved. For example, the formula for the area of a trapezoid, $A = \left(\frac{b_1 + b_2}{2}\right)h$, can be solved for h using the same deductive process (**MP.7, MP.8**). As will be discussed later, functional relationships can often be explored more deeply by rearranging equations that define such relationships; thus, the ability to work with equations that have letters as coefficients is an important skill.

Reasoning with Equations and Inequalities	**A-REI**

Solve systems of equations. [Linear-linear and linear-quadratic]

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system. The process of adding one equation to another is understood in this way: if the two sides of one equation are equal, and the two sides of another equation are equal, then the sum (or difference) of the left sides of the two equations is equal to the sum (or difference) of the right sides. The reversibility of these steps justifies that we achieve an equivalent system of equations by doing this. This crucial point should be consistently noted when reasoning about solving systems of equations (UA Progressions Documents 2013b, 11).

Example **A-REI.6**

Solving simple systems of equations. To get started with understanding how to solve systems of equations by linear combinations, students can be encouraged to interpret a system in terms of real-world quantities, at least in some cases. For instance, suppose one wanted to solve this system:

$$3x + y = 40$$
$$4x + 2y = 58$$

Now consider the following scenario: Suppose 3 CDs and a magazine cost \$40, while 4 CDs and 2 magazines cost \$58.

- What happens to the price when you add 1 CD and 1 magazine to your purchase?
- What is the price if you decided to buy only 2 CDs and no magazine?

Answering these questions amounts to realizing that since $(3x + y) + (x + y) = 40 + 18$, we must have that $x + y = 18$. Therefore, $(3x + y) + (-1)(x + y) = 40 + (-1)18$, which implies that $2x = 22$, or 1 CD costs \$11. The value of y can now be found using either of the original equations: $y = 7$.

When solving systems of equations, students also make frequent use of substitution—for example, when solving the system $2x-9y=5$ and $y=\frac{1}{3}x+1$, the expression $\frac{1}{3}x+1$ can be substituted for y in the first equation to obtain $2x-9\left(\frac{1}{3}x+1\right)=5$. Students also solve such systems approximately, by using graphs and tables of values (**A-REI.5–7**).

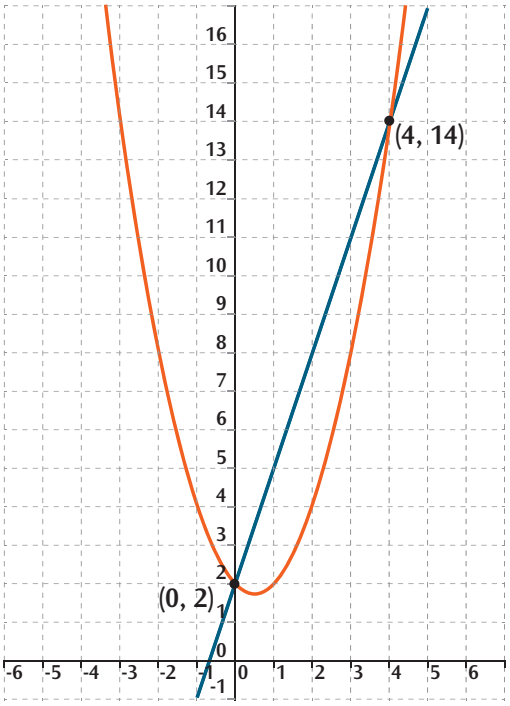
Reasoning with Equations and Inequalities	**A-REI**

Represent and solve equations and inequalities graphically. [Linear and exponential; learn as general principle.]

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
11. Explain why the x-coordinates of the points where the graphs of the equations $y=f(x)$ and $y=g(x)$ intersect are the solutions of the equation $f(x)=g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

One of the most important goals of instruction in mathematics is to illuminate connections between different mathematical concepts. In particular, in standards **A-REI.10–12**, students learn the relationship between the algebraic representation of an equation and its graph plotted in the coordinate plane and understand geometric interpretations of solutions to equations and inequalities. In Algebra I, students work only with linear, exponential, quadratic, step, piecewise, and absolute value functions. As students become more comfortable with function notation — for example, writing $f(x)=3x-2$ and $g(x)=x^2-x+2$ — they begin to see solving the equation $3x-2=x^2-x+2$ as solving the equation $f(x)=g(x)$. That is, they find those x-values where two functions take on the same output value. Moreover, they graph the two equations (see figure A1-3) and see that the x-coordinate(s) of the point(s) of intersection of the graphs of $y=f(x)$ and $y=g(x)$ are the solutions to the original equation.

Figure A1-3. Graphs of $y = 3x + 2$ and $y = x^2 - x + 2$



Students also create tables of values for functions to approximate or find exact solutions to equations such as those plotted in figure A1-3. For example, they may use spreadsheet software to construct a table (see table A1-2).

Table A1-2. Values for $f(x) = 3x + 2$ and $g(x) = x^2 - x + 2$

x	$f(x) = 3x + 2$	$g(x) = x^2 - x + 2$
-6	-16	44
-5	-13	32
-4	-10	22
-3	-7	14
-2	-4	8
-1	-1	4
0	2	2
1	5	2
2	8	4
3	11	8
4	14	14
5	17	22
6	20	32
7	23	44
8	26	58
9	29	74

Although a table like this one does not offer sufficient proof that all solutions to a given equation have been found, students can reason in certain situations why they have found all solutions (**MP.3, MP.6**). In this example, since the original equation is of degree two, we know that there are at most two solutions, so that the solution set is $\{0,4\}$.

Conceptual Category: Statistics and Probability

In Algebra I, students build on their understanding of key ideas for describing distributions—shape, center, and spread—presented in the standards for grades six through eight. This enhanced understanding allows them to give more precise answers to deeper questions, often involving comparisons of data sets.

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★

Summarize, represent, and interpret data on two categorical and quantitative variables. [Linear focus, discuss general principle.]

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★
   b. Informally assess the fit of a function by plotting and analyzing residuals. ★
   c. Fit a linear function for a scatter plot that suggests a linear association. ★

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★

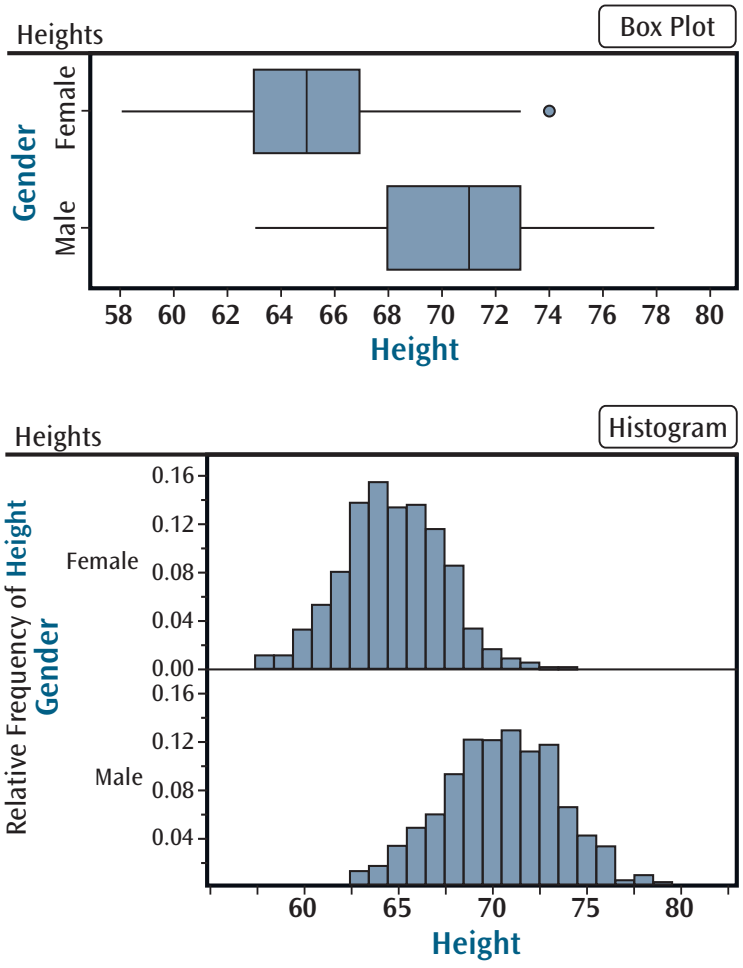
Standards **S-ID.1–6** extend concepts that students began learning in grades six through eight and, as such, may be considered supporting standards for **S-ID.7–9.** In general, students use shape and the question(s) to be answered to decide on the median or mean as the more appropriate measure of center and to justify their choice through statistical reasoning. Students may use parallel box plots or histograms to compare differences in the shape, center, and spread of comparable data sets (**S-ID.1–2**).

Example **S-ID.2**

The following graphs show two ways of comparing height data for males and females in the 20–29 age group. Both involve plotting the data or data summaries (box plots or histograms) on the same scale, resulting in what are called parallel (or side-by-side) box plots and parallel histograms (**S-ID.1**). The parallel box plots show an obvious difference in the medians and the interquartile ranges (IQRs) for the two groups; the medians for males and females are, respectively, 71 inches and 65 inches, while the IQRs are 5 inches and 4 inches. Thus, male heights center at a higher value but are slightly more variable.

The parallel histograms show the distributions of heights to be mound shaped and fairly symmetrical (approximately normal) in shape. Therefore, the data can be succinctly described using the mean and standard deviation. Heights for males and females have means of 70.4 inches and 64.7 inches, respectively, and standard deviations of 3.0 inches and 2.6 inches. Students should be able to sketch each distribution and answer questions about it solely from knowledge of these three facts (shape, center, and spread). For either group, about 68% of the data values will be within one standard deviation of the mean (**S-ID.2, S-ID.3**). Students also observe that the two measures of center—median and mean—tend to be close to each other for symmetric distributions.

Comparing heights of males and females



Heights of U.S. males and females in the 20–29 age group

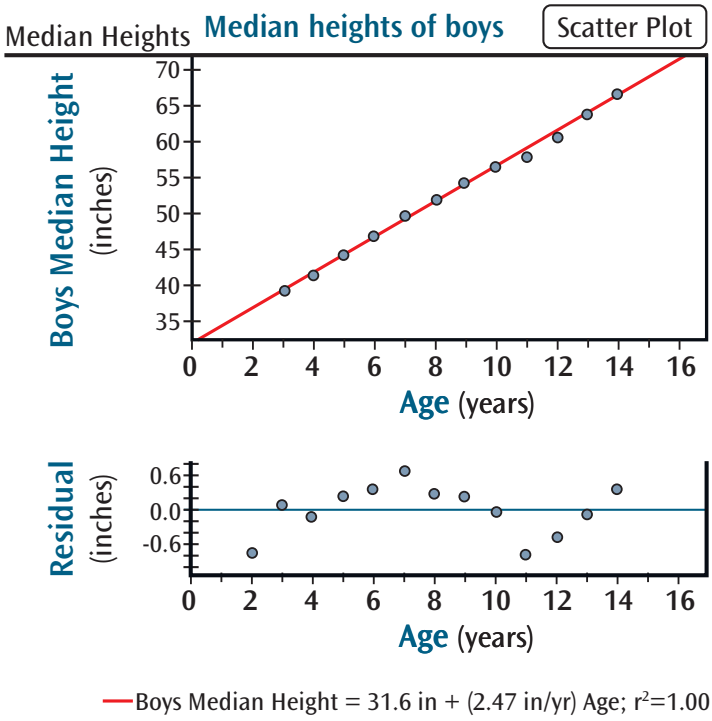
Source: United States Census Bureau 2009 (Statistical Abstract of the United States, Table 201).

Adapted from UA Progressions Documents 2012d, 3.

As with univariate data analysis, students now take a deeper look at bivariate data, using their knowledge of proportions to describe categorical associations and using their knowledge of functions to fit models to quantitative data (**S-ID.5–6**). Students have seen scatter plots in the grade-eight standards and now extend that knowledge to fit mathematical models that capture key elements of the relationship between two variables and to explain what the model tells us about the relationship. Students must learn to take a careful look at scatter plots, as sometimes the "obvious" pattern does not tell the whole story and may be misleading. A line of best fit may appear to fit data almost perfectly, while an examination of the *residuals*—the collection of differences between corresponding coordinates on a least squares line and the actual data value for a variable—may reveal more about the behavior of the data.

Example **S-ID.6b**

Students must learn to look carefully at scatter plots, as sometimes the "obvious" pattern may not tell the whole story and could even be misleading. The graphs below show the median heights of growing boys from the ages of 2 through 14. The line (least squares regression line) with slope 2.47 inches per year of growth looks to be a perfect fit (**S-ID.6c**). However, the residuals—the differences between the corresponding coordinates on the least squares line and the actual data values for each age—reveal additional information. A plot of the residuals shows that growth does not proceed at a constant rate over those years.



Source: Centers for Disease Control and Prevention (CDC) 2002.

Adapted from UA Progressions Documents 2012d, 5.

Finally, students extend their work from topics covered in the grade-eight standards and other topics in Algebra I to interpret the parameters of a linear model in the context of data that it represents. They compute *correlation coefficients* using technology and interpret the value of the coefficient (**MP.4, MP.5**). Students see situations where correlation and causation are mistakenly interchanged, and they are careful to closely examine the story that data and computed statistics are trying to tell (**S-ID.7–9**).

Algebra I Overview

Number and Quantity

The Real Number System

- Extend the properties of exponents to rational exponents.
- Use properties of rational and irrational numbers.

Quantities

- Reason quantitatively and use units to solve problems.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Solve systems of equations.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Understand the concept of a function and use function notation.
- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.
- Interpret expressions for functions in terms of the situation they model.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.
- Summarize, represent, and interpret data on two categorical and quantitative variables.
- Interpret linear models.

A1 Algebra I

Number and Quantity

The Real Number System — N-RN

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{1/3}$ *to be the cube root of* 5 *because we want* $(5^{1/3})^3 = 5^{(1/3)3}$ *to hold, so* $(5^{1/3})^3$ *must equal* 5.
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a non-zero rational number and an irrational number is irrational.

Quantities — N-Q

Reason quantitatively and use units to solve problems. [Foundation for work with expressions, equations and functions]

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★

Algebra

Seeing Structure in Expressions — A-SSE

Interpret the structure of expressions. [Linear, exponential, and quadratic]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of* P *and a factor not depending on* P. ★
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems. [Quadratic and exponential]

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★
   a. Factor a quadratic expression to reveal the zeros of the function it defines. ★

Note: ★ Indicates a modeling standard linking mathematics to everyday life, work, and decision making.
(+) Indicates additional mathematics to prepare students for advanced courses.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. ★

c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $\left(1.15^{1/12}\right)^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* ★

Arithmetic with Polynomials and Rational Expressions A-APR

Perform arithmetic operations on polynomials. [Linear and quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Creating Equations A-CED

Create equations that describe numbers or relationships. [Linear, quadratic, and exponential (integer inputs only); for **A-CED.3** linear only]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★

Reasoning with Equations and Inequalities A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Master linear; learn as general principle.]

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable. [Linear inequalities; literal equations that are linear in the variables being solved for; quadratics with real solutions]

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

3.1 Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA

4. Solve quadratic equations in one variable.

   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations. [Linear-linear and linear-quadratic]

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

Represent and solve equations and inequalities graphically. [Linear and exponential; learn as general principle.]

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Functions

Interpreting Functions F-IF

Understand the concept of a function and use function notation. [Learn as general principle; focus on linear and exponential and on arithmetic and geometric sequences.]

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$.

Interpret functions that arise in applications in terms of the context. [Linear, exponential, and quadratic]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Linear, exponential, quadratic, absolute value, step, piecewise-defined]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, *and* $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Building Functions F-BF

Build a function that models a relationship between two quantities. [For **F-BF.1–2,** linear, exponential, and quadratic]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

Build new functions from existing functions. [Linear, exponential, quadratic, and absolute value; for **F-BF.4a,** linear only]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

4. Find inverse functions.

   a. Solve an equation of the form $f(x)=c$ for a simple function f that has an inverse and write an expression for the inverse.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★

   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★

   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★

   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★

2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★

3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Interpret expressions for functions in terms of the situation they model.

5. Interpret the parameters in a linear or exponential function in terms of a context. ★ [Linear and exponential of form $f(x)=b^x+k$]

6. **Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity. CA** ★

Statistics and Probability

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★

2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★

3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★

Summarize, represent, and interpret data on two categorical and quantitative variables. [Linear focus, discuss general principle.]

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★

6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★

   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★

   b. Informally assess the fit of a function by plotting and analyzing residuals. ★

   c. Fit a linear function for a scatter plot that suggests a linear association. ★

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★

8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★

9. Distinguish between correlation and causation. ★

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Geometry

The fundamental purpose of the Geometry course is to introduce students to formal geometric proofs and the study of plane figures, culminating in the study of right-triangle trigonometry and circles. Students begin to formally prove results about the geometry of the plane by using previously defined terms and notions. Similarity is explored in greater detail, with an emphasis on discovering trigonometric relationships and solving problems with right triangles. The correspondence between the plane and the Cartesian coordinate system is explored when students connect algebra concepts with geometry concepts. Students explore probability concepts and use probability in real-world situations. The major mathematical ideas in the Geometry course include geometric transformations, proving geometric theorems, congruence and similarity, analytic geometry, right-triangle trigonometry, and probability.

The standards in the traditional Geometry course come from the following conceptual categories: Modeling, Geometry, and Statistics and Probability. The content of the course is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Geometry

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). In the higher mathematics courses, students begin to formalize their geometry experiences from elementary and middle school, using definitions that are more precise and developing careful proofs. The standards for grades seven and eight call for students to see two-dimensional shapes as part of a generic plane (i.e., the *Euclidean plane*) and to explore transformations of this plane as a way to determine whether two shapes are congruent or similar. These concepts are formalized in the Geometry course, and students use transformations to prove geometric theorems. The definition of *congruence* in terms of rigid motions provides a broad understanding of this means of proof, and students explore the consequences of this definition in terms of congruence criteria and proofs of geometric theorems.

Students investigate triangles and decide when they are similar—and with this newfound knowledge and their prior understanding of proportional relationships, they define trigonometric ratios and solve problems by using right triangles. They investigate circles and prove theorems about them. Connecting to their prior experience with the coordinate plane, they prove geometric theorems by using coordinates and describe shapes with equations. Students extend their knowledge of area and volume formulas to those for circles, cylinders, and other rounded shapes. Finally, continuing the development of statistics and probability, students investigate probability concepts in precise terms, including the independence of events and conditional probability.

Examples of Key Advances from Previous Grade Levels or Courses

- Because concepts such as *rotation*, *reflection*, and *translation* were treated in the grade-eight standards mostly in the context of hands-on activities and with an emphasis on geometric intuition, the Geometry course places equal weight on precise definitions.
- In kindergarten through grade eight, students worked with a variety of geometric measures: length, area, volume, angle, surface area, and circumference. In Geometry, students apply these component skills in tandem with others in the course of modeling tasks and other substantial applications (**MP.4**).
- The skills that students develop in Algebra I around simplifying and transforming square roots will be useful when solving problems that involve distance or area and that make use of the Pythagorean Theorem.
- Students in grade eight learned the Pythagorean Theorem and used it to determine distances in a coordinate system (**8.G.6–8**). In Geometry, students build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of circles (**G-GPE.1**).
- The algebraic techniques developed in Algebra I can be applied to study analytic geometry. Geometric objects can be analyzed by the algebraic equations that give rise to them. Algebra can be used to prove some basic geometric theorems in the Cartesian plane.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The Standards for Mathematical Practice represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. The Geometry course offers ample opportunities for students to engage with each MP standard; table G-1 offers some examples.

Table G-1. Standards for Mathematical Practice—Explanation and Examples for Geometry

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students construct accurate diagrams of geometry problems to help make sense of them. They organize their work so that others can follow their reasoning (e.g., in proofs).
MP.2 Reason abstractly and quantitatively.	Students understand that the coordinate plane can be used to represent geometric shapes and transformations, and therefore they connect their understanding of number and algebra to geometry.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students reason through the solving of equations, recognizing that solving an equation involves more than simply following rote rules and steps. They use language such as "If _____, then _____" when explaining their solution methods and provide justification for their reasoning.
MP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems. They learn how transformational geometry and trigonometry can be used to model the physical world.
MP.5 Use appropriate tools strategically.	Students make use of visual tools for representing geometry, such as simple patty paper, transparencies, or dynamic geometry software.
MP.6 Attend to precision.	Students develop and use precise definitions of geometric terms. They verify that a particular shape has specific properties and justify the categorization of the shape (e.g., a rhombus versus a quadrilateral).
MP.7 Look for and make use of structure.	Students construct triangles in quadrilaterals or other shapes and use congruence criteria of triangles to justify results about those shapes.
MP.8 Look for and express regularity in repeated reasoning.	Students explore rotations, reflections, and translations, noticing that some attributes of shapes (e.g., parallelism, congruency, orientation) remain the same. They develop properties of transformations by generalizing these observations.

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific stan-

dards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a high place in instruction. Some standards are marked with a star (★) symbol to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards.

In places where specific MP standards may be implemented with the geometry standards, the MP standards are noted in parentheses.

Geometry Content Standards, by Conceptual Category

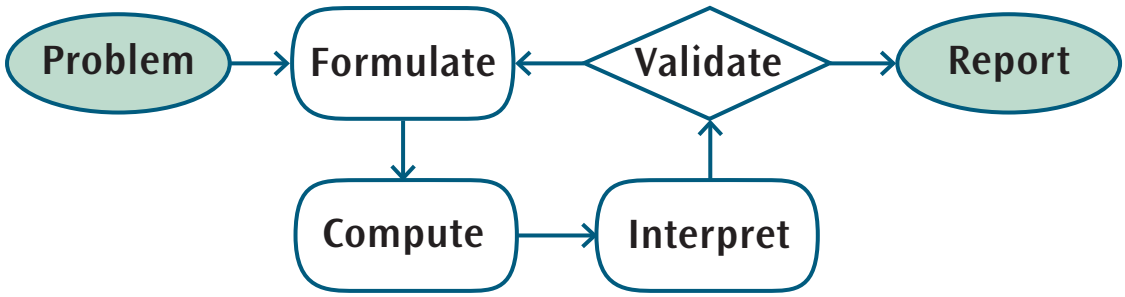
The Geometry course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in the Geometry course are described below, according to each conceptual category. Standards that are considered new for secondary-grades teachers are discussed more thoroughly than other standards.

Conceptual Category: Modeling

Throughout the California Common Core State Standards for Mathematics (CA CCSSM), specific standards for higher mathematics are marked with a ★ symbol to indicate they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: *Which of the quantities present in this situation are known and which are unknown? What can I generalize? Is there some way to introduce into this diagram a known shape that gives more information?* Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They validate their work by moving between calculations done by hand and software-assisted computations.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, and the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure G-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Figure G-1. The Modeling Cycle



The examples in this chapter are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas of proving geometric theorems, congruence and similarity, analytic geometry, right-triangle trigonometry, and probability can be explored in this way. Readers are encouraged to consult appendix B (Mathematical Modeling) for further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Geometry

A large portion of instruction in the traditional Geometry course is formed by the standards of the Geometry conceptual category. Here, students develop the ideas of congruence and similarity through transformations. They prove theorems, both with and without the use of coordinates. They explore right-triangle trigonometry, as well as circles and parabolas. Standard **MP.3**, "Construct viable arguments and critique the reasoning of others," with the California addition **MP.3.1** ("Students build proofs by induction and proofs by contradiction"), plays a predominant role throughout the Geometry course.

Congruence G-CO

Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions. [Build on rigid motions as a familiar starting point for development of concept of geometric proof.]

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

In geometry, the commonly held (but imprecise) definition that shapes are congruent when they "have the same size and shape" is replaced by a more mathematically precise one (**MP.6**): *Two shapes are congruent if there is a sequence of rigid motions in the plane that takes one shape exactly onto the other.*

This definition is explored intuitively in the grade-eight standards, but in the Geometry course it is investigated more closely. In grades seven and eight, students experimented with transformations in the plane; however, the Geometry course requires that students build more precise definitions for the *rigid motions* (rotation, reflection, and translation) based on previously defined and understood terms such as *point*, *line*, *between*, *angle*, *circle*, *perpendicular*, and so forth (**G-CO.1, 3–4**). Students base their understanding of these definitions on their experience with transforming figures using patty paper, transparencies, or geometry software (**G-CO.2–3, 5; MP.5**), something they started doing in grade eight. These transformations should be investigated both in a general plane as well as on a coordinate system—especially when transformations are explicitly described by using precise names of points, translation vectors, and lines of symmetry or reflection.

Example: Defining Rotations **G-CO.4**

Mrs. B wants to help her class understand the following definition of a *rotation*:

> A *rotation* about a point P through angle α is a transformation $A \mapsto A'$ such that (1) if point A is different from P, then $PA = PA'$ and the measure of $\angle APA' = \alpha$; and (2) if point A is the same as point P, then $A' = A$.

Mrs. B gives her students a handout with several geometric shapes on it and a point, P, indicated on the page. In pairs, students copy the shapes onto a transparency sheet and rotate them through various angles about P; then they transfer the rotated shapes back onto the original page and measure various lengths and angles as indicated in the definition.

While justifying that the properties of the definition hold for the shapes given to them by Mrs. B, the students also make some observations about the effects of a rotation on the entire plane. For example:

- Rotations preserve lengths.
- Rotations preserve angle measures.
- Rotations preserve parallelism.

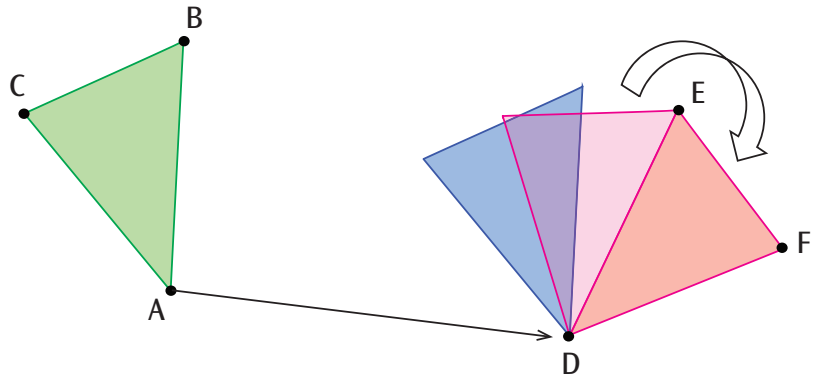
In a subsequent exercise, Mrs. B plans to allow students to explore more rotations on dynamic geometry software, asking them to create a geometric shape and rotate it by various angles about various points, both part of the object and not part of the object.

In standards **G-CO.6–8**, geometric transformations are given a more prominent role in the higher mathematics geometry curriculum than perhaps ever before. The new definition of *congruence* in terms of rigid motions applies to any shape in the plane, whereas previously, congruence seemed to depend on criteria that were specific only to particular shapes. For example, the side–side–side (SSS) congruence criterion for triangles did not extend to quadrilaterals, which seemed to suggest that congruence was a notion dependent on the shape that was considered. Although it is true that there are specific criteria for determining congruence of certain shapes, the basic notion of congruence is the same for all shapes. In the CA CCSSM, the SSS criterion for triangle congruence is a *consequence* of the definition of

congruence, just as the fact that if two polygons are congruent, then their sides and angles can be put into a correspondence such that each corresponding pair of sides and angles is congruent. This concept comprises the content of standards **G-CO.7** and **G-CO.8**, which derive congruence criteria for triangles from the new definition of congruence.

Standard **G-CO.7** explicitly states that students show that two triangles are congruent *if and only if* corresponding pairs of sides and corresponding pairs of angles are congruent (**MP.3**). The depth of reasoning here is fairly substantial, as students must be able to show, using rigid motions, that congruent triangles have congruent corresponding parts and that, conversely, if the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions that takes one triangle to the other. The second statement may be more difficult to justify than the first for most students, so a justification is presented here. Suppose there are two triangles ΔABC and ΔDEF such that the correspondence $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$ results in pairs of sides and pairs of angles being congruent. If one triangle were drawn on a fixed piece of paper and the other drawn on a separate transparency, then a student could illustrate a translation, T, that takes point A to point D. A simple rotation R about point A, if necessary, takes point B to point E, which is certain to occur because $\overline{AB} \cong \overline{DE}$ and rotations preserve lengths. A final step that may be needed is a reflection S about the side AB, to take point C to point F. It is important to note why the image of point C is actually F. Since $\angle A$ is reflected about line $\overleftrightarrow{AB}$, its measure is preserved. Therefore, the image of side $\overline{AC}$ at least lies on line $\overline{DF}$, since $\angle A \cong \angle D$. But since $\overline{AC} \cong \overline{DF}$, it must be the case that the image of point C coincides with F. The previous discussion amounts to the fact that the sequence of rigid motions, T, followed by R, followed by S, maps ΔABC exactly onto ΔDEF. Therefore, if it is known that the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions carrying one onto the other; that is, they are congruent. The informal proof presented here should be accessible to students in the Geometry course; see figure G-2.

Figure G-2. Illustration of the Reasoning That Congruent Corresponding Parts Imply Triangle Congruence



Point A is translated to D, the resulting image of ΔABC is rotated so as to place B onto E, and the image is then reflected along line segment DE to match point C to F.

Similar reasoning applies for standard **G.CO.8**, in which students justify the typical triangle congruence criteria such as ASA, SAS, and SSS. Experimentation with transformations of triangles where only two of the criteria are satisfied will result in counterexamples, and geometric constructions of triangles of prescribed side lengths (e.g., in the case of SSS) will leave little doubt that any triangle constructed with these side lengths will be congruent to another, and therefore that SSS holds (**MP.7**).

Congruence G-CO

Prove geometric theorems. [Focus on validity of underlying reasoning while using variety of ways of writing proofs.]

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

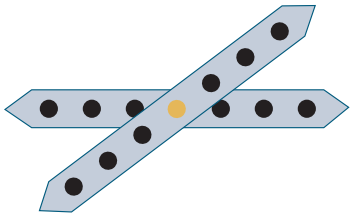
Make geometric constructions. [Formalize and explain processes.]

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

It is important to note that when the triangle criteria for congruence are established, students can begin to prove geometric theorems. Examples of such theorems are listed in standards **G.CO.9–11**. The triangle congruence criteria are established results that can be used to prove new results. Instructors are encouraged to use a variety of strategies for engaging students to understand and write proofs, such as using numerous pictures to demonstrate results and generate strategies; using patty paper, transparencies, or dynamic geometry software to explore the steps in a proof; creating flowcharts and other organizational diagrams for outlining a proof; and writing step-by-step or paragraph formats for a completed proof (**MP.5**). Above all else, instructors should emphasize the reasoning involved in connecting one step in the logical argument to the next. Students should be encouraged to make conjectures based on experimentation, to justify their conjectures, and to communicate their reasoning to their peers (**MP.3**). The following example illustrates how students can be encouraged to experiment and construct hypotheses based on their observations.

Example: The Kite Factory **G.CO.11**

Kite engineers want to know how the shape of a kite—the length of the rods, where they are attached, the angle at which the rods are attached, and so on—affects how the kite flies. In this activity, students are given pieces of cardstock of various lengths, hole-punched at regular intervals so they can be attached in different places.



These two "rods" form the frame for a kite at the kite factory. By changing the angle at which the sticks are held and the places where the sticks are attached, students discover different properties of quadrilaterals.

Students are challenged to make conjectures and use precise language to describe their findings about which diagonals result in which quadrilaterals. They can discover properties unique to certain quadrilaterals, such as the fact that diagonals that are perpendicular bisectors of each other imply the quadrilateral is a rhombus. To see videos of this lesson being implemented in a high school classroom, visit http://www.insidemathematics.org/ (accessed March 26, 2015).

Similarity, Right Triangles, and Trigonometry G-SRT

Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle–Angle (AA) criterion for two triangles to be similar.

Prove theorems involving similarity.

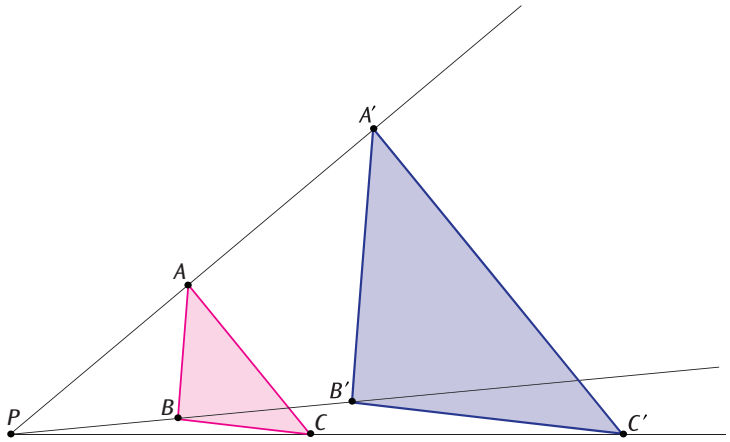
4. Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Because right triangles and triangle relationships play such an important role in applications and future mathematics learning, they are given a prominent role in the Geometry conceptual category. A discussion of *similarity* is necessary first—and again, a more precise mathematical definition of similarity is given in the higher mathematics standards. Students worked with *dilations* as a transformation in the grade-eight standards; now they explore the properties of dilations in more detail and develop an understanding of the notion of *scale factor* (**G-SRT.1**). Whereas it is common to say that objects that are similar have "the same shape," the new definition for two objects being similar is that there is a sequence of *similarity transformations*—translation, rotation, reflection, or dilation—that maps one object exactly onto the other. Standards **G-SRT.2** and **G-SRT.3** call for students to explore the consequences of two triangles being similar: that they have congruent angles and that their side lengths are in the same proportion. This new understanding gives rise to more results that are encapsulated in standards **G-SRT.4** and **G-SRT.5**.

Example: Experimenting with Dilations **G-SRT.2–3**

Students are given opportunities to experiment with dilations and determine how they affect planar objects. Students first make sense of the definition of a *dilation of scale factor* $k > 0$ *with center* P as the transformation that moves a point A along the ray $\overrightarrow{PA}$ to a new point A', so that $\left|\overline{PA'}\right| = k \bullet \left|\overline{PA}\right|$. For example, using a ruler, students apply the dilation of scale factor 2.5 with center P to the points A, B, and C illustrated below. Once this is done, the students consider the two triangles ΔABC and $\Delta A'B'C'$, and they discover that the lengths of the corresponding sides of the triangles have the same ratio dictated by the scale factor (**G-SRT.2**).

Students learn that parallel lines are taken to parallel lines by dilations; thus corresponding segments of ΔABC and $\Delta A'B'C'$ are parallel. After students have proved results about parallel lines intersected by a transversal, they can deduce that the angles of the triangles are congruent. Through experimentation, they see that the congruence of corresponding angles is a necessary and sufficient condition for the triangles to be similar, leading to the AA criterion for triangle similarity (**G.SRT.3**).



For a simple investigation, students can observe how the distance at which a projector is placed from a screen affects the size of the image on the screen (**MP.4**).

Similarity, Right Triangles, and Trigonometry G-SRT

Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★

8.1 Derive and use the trigonometric ratios for special right triangles (30°, 60°, 90° and 45°, 45°, 90°). CA

Once the angle–angle (AA) similarity criterion for triangles is established, it follows that any two *right* triangles ΔABC and ΔDEF are similar when at least one pair of angles are congruent (say $\angle A \cong \angle D$), since the right angles are obviously congruent (say $\angle B \cong \angle E$). By similarity, the corresponding sides of the triangles are in proportion:

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Notice the first and third expressions in the statement of equality above can be rearranged to yield that:

$$\frac{AB}{AC} = \frac{DE}{DF}.$$

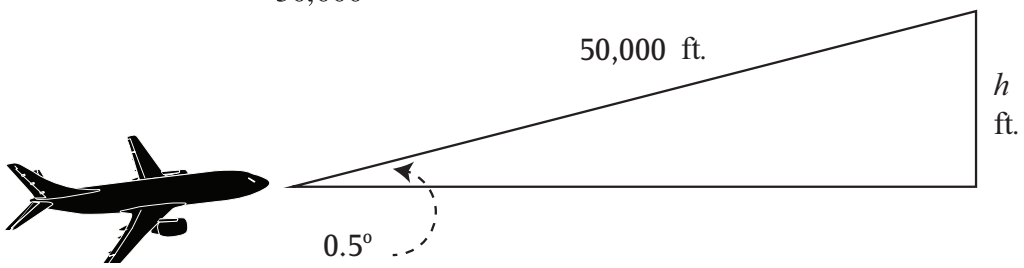
Since the triangles in question are arbitrary, this implies that for any right triangle with an angle congruent to $\angle A$, the ratio of the side opposite to $\angle A$ and the hypotenuse of the triangle is a certain constant. This allows us to define unambiguously the *sine* of $\angle A$, denoted by $\sin A$, as the value of this ratio. In this way, students come to understand the trigonometric functions as relationships completely determined by angles (**G-SRT.6**). They further their understanding of these functions by investigating relationships between sine, cosine, and tangent; by exploring the relationship between the sine and cosine of complementary angles; and by applying their knowledge of right triangles to real-world situations (**MP.4**), such as in the example below (**G-SRT.6–8**). Experience working with many different triangles, finding their measurements, and computing ratios of the measurements found will help students understand the basics of the trigonometric functions.

Example: Using Trigonometric Relationships **G-SRT.6–8**

Airplanes that travel at high speeds and low elevations often have onboard radar systems to detect possible obstacles in their path. The radar can determine the range of an obstacle and the angle of elevation to the top of the obstacle. Suppose that the radar detects a tower that is 50,000 feet away, with an angle of elevation of 0.5 degrees. By how many feet must the plane rise in order to pass above the tower?

Solution: The sketch below shows that there is a right triangle with a hypotenuse of 50,000 (ft) and smallest angle 0.5 (degrees). To find the side opposite this angle, which represents the minimum height the plane should rise, students would use

$$\sin 0.5^\circ = \frac{h}{50{,}000}, \text{ so that } h = (50{,}000)\sin 0.5^\circ \approx 436.33 \text{ ft.}$$



Similarity, Right Triangles, and Trigonometry G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Students advance their knowledge of right-triangle trigonometry by applying trigonometric ratios in non-right triangles. For instance, students see that by dropping an altitude in a given triangle, they divide the triangle into right triangles to which these relationships can be applied. By seeing that the base of the triangle is a and the height is $b \bullet \sin C$, students derive a general formula for the area of any triangle $A = \frac{1}{2} ab \sin C$ (**G SRT.9**). In addition, they use reasoning about similarity and trigonometric identities to derive the Laws of Sines and Cosines only in acute triangles, and they use these and other relationships to solve problems (**G-SRT.10–11**). Instructors will need to address the ideas of the sine and cosine of angles larger than or equal to 90 degrees to fully discuss Laws of Sine and Cosine, although full unit-circle trigonometry need not be discussed in this course.

Circles G-C

Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.
4. (+) Construct a tangent line from a point outside a given circle to the circle.

Students extend their understanding of the usefulness of similarity transformations by investigating circles (**G-C.1**). For instance, students can reason that any two circles are similar by describing precisely how to transform one into the other, as the following example illustrates with two specific circles. Students continue investigating properties of circles and relationships among angles, radii, and chords (**G-C.2–4**).

Example **G-C.1**

Students can show that the two circles C and D given by the equations below are similar.

$$C: (x-1)^2 + (y-4)^2 = 9$$

$$D: (x+2)^2 + (y-1)^2 = 25$$

Solution: Because the centers of the circles are $(1,4)$ and $(-2,1)$, respectively, the first step is to translate the center of circle C to the center of circle D using the translation $T(x,y) = (x-3, y-3)$. The final step is to dilate from the point $(-2,1)$ by a scale factor of $\frac{5}{3}$, since the radius of circle C is 3 and the radius of circle D is 5.

Circles G-C

Find arc lengths and areas of sectors of circles. [Radian introduced only as unit of measure.]

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. **Convert between degrees and radians. CA**

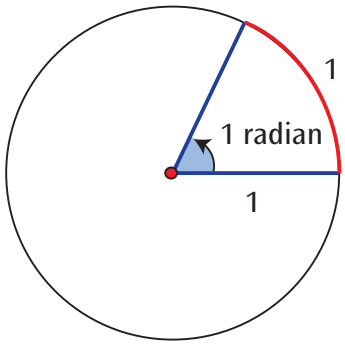
Another important application of the concept of similarity is the definition of the radian measure of an angle. Students can derive this result in the following way: given a sector of a circle C of radius r and central angle α, and a sector of a circle D of radius s and central angle also α, it stands to reason that because these sectors are similar,

$$\frac{\text{length of arc on circle } C}{r} = \frac{\text{length of arc on circle } D}{s}.$$

Therefore, much like defining the trigonometric functions, there is a constant m such that for an arc subtended by an angle α on any circle:

$$\frac{\text{length of arc subtended by angle } \alpha}{\text{radius of the circle}} = m.$$

This constant of proportionality is the *radian measure* of angle α. It follows that an angle that subtends an arc on a circle that is the same length as the radius measures 1 radian. By investigating circles of different sizes, using string to measure arcs subtended by the same angle, and finding the ratios described above, students can apply their proportional-reasoning skills to discover this constant ratio, thereby developing an understanding of the definition of *radian measure*.



Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically. [Include distance formula; relate to Pythagorean Theorem.]

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point* $(1, \sqrt{3})$ *lies on the circle centered at the origin and containing the point (0, 2).*
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

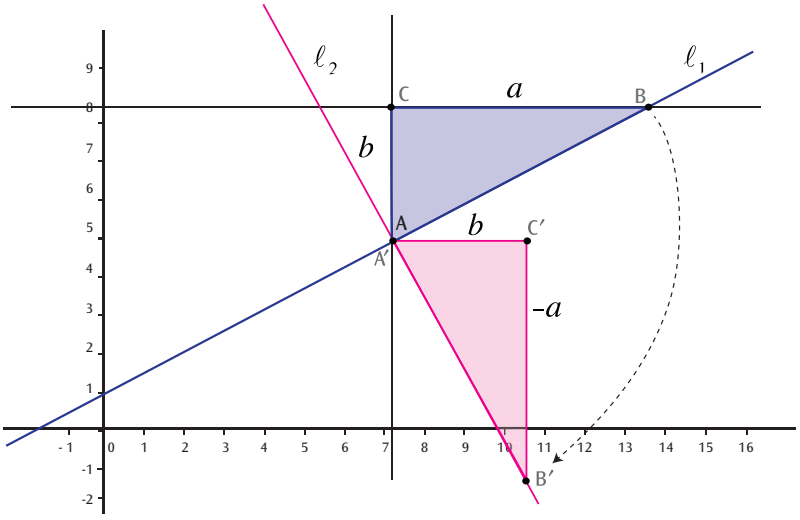
The largest intersection of algebra concepts and geometry occurs here, wherein two-dimensional shapes are represented on a coordinate system and can be described using algebraic equations and inequalities. A derivation of the equation of a circle by the Pythagorean Theorem and the definition of a circle (**G-GPE.1**) is as follows: given that a circle consists of all points (x, y) that are at a distance $r > 0$ from a fixed center (h, k), students see that $\sqrt{(x-h)^2+(y-k)^2} = r$ for any point lying on the circle, so that $(x-h)^2+(y-k)^2 = r^2$ determines this circle. Students can derive this equation and flexibly change an equation into this form by completing the square as necessary. By understanding the derivation of this equation, students develop a clear meaning of the variables h, k, and r. Standard **G-GPE.2** calls for students to do the same for the definition of a parabola in terms of a focus and directrix.

Standards **G.GPE.4** and **G.GPE.6** call for students to continue their work of using coordinates to prove geometric theorems with algebraic techniques. In standard **G.GPE.6**, given a directed line segment represented by a vector emanating from the origin to the point $(4, 6)$, students may be asked to find the point on this vector that partitions it into a ratio of 2:1. Students may construct right triangles and use triangle similarity to find this point, or they may represent the vector as $x = 4t$, $y = 6t$ for $0 \le t \le 1$ and reason that the point they seek can be found when $t = \frac{2}{3}$.

Many simple geometric theorems can be proved algebraically, but two results of high importance are the slope criteria for parallel and perpendicular lines. Students in grade seven began to study lines and linear equations; in the Geometry course, they not only use relationships between slopes of parallel and perpendicular lines to solve problems, but they also justify why these relationships are true. An intuitive argument for why parallel lines have the same slope might read, "Since the two lines never meet, each line must *keep up* with the other as we travel along the slopes of the lines. So it seems obvious that their slopes must be equal." This intuitive thought leads to an equivalent statement: If given a pair of linear equations $\ell_1 : y = m_1x + b_1$ and $\ell_2 : y = m_2x + b_2$ (for $m_1, m_2 \neq 0$) such that $m_1 \neq m_2$—that is, such that *their slopes are different*—then the lines must intersect. Solving for the intersection of the two lines yields the x-coordinate of their intersection to be $x = \frac{b_2 - b_1}{m_1 - m_2}$, which surely exists because $m_1 \neq m_2$. It is important for students to understand the steps of the argument and comprehend why proving this statement is equivalent to proving the statement "If $\ell_1 \parallel \ell_2$, then $m_1 = m_2$" (**MP.1**, **MP.2**).

In addition, students are expected to justify why the slopes of two non-vertical perpendicular lines ℓ_1 and ℓ_2 satisfy the relationship $m_1 = -\frac{1}{m_2}$, or $m_1 \bullet m_2 = -1$. Although there are numerous ways to do this, only one way is presented here, and dynamic geometry software can be used to illustrate it well (**MP.4**). Let ℓ_1 and ℓ_2 be any two non-vertical perpendicular lines. Let A be the intersection of the two lines, and let B be any other point on ℓ_1 above A. A vertical line is drawn through A, a horizontal line is drawn through B, and C is the intersection of those two lines. ΔABC is a right triangle. If a is the horizontal displacement Δx from C to B, and b is the length of $\overline{AC}$, then the slope of ℓ_1 is $m_1 = \frac{\Delta y}{\Delta x} = \frac{b}{a}$. By rotating ΔABC clockwise around A by 90 degrees, the hypotenuse $\overline{AB'}$ of the rotated triangle $\Delta AB'C'$ lies on ℓ_2. Using the legs of $\Delta AB'C'$, students see that the slope of ℓ_2 is $m_2 = \frac{\Delta y}{\Delta x} = \frac{-a}{b}$. Thus $m_1 \bullet m_2 = \frac{b}{a} \bullet \frac{-a}{b} = -1$. Figure G-3 gives a visual presentation of this proof (**MP.1**, **MP.7**).

Figure G-3. Illustration of the Proof That the Slopes of Two Perpendicular Lines Are Opposite Reciprocals of One Another



The proofs described above make use of several ideas that students learned in Geometry and prior courses—for example, the relationship between equations and their graphs in the plane (**A.REI.10**) and solving equations with variable coefficients (**A.REI.3**). An investigative approach that first uses several examples of lines that are perpendicular and their equations to find points, construct triangles, and decide if the triangles formed are right triangles will help students ramp up to the second proof (**MP.8**). Once more, the reasoning required to make sense of such a proof and to communicate the essence of the proof to a peer is an important goal of geometry instruction (**MP.3**).

Geometric Measurement and Dimension — G-GMD

Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★

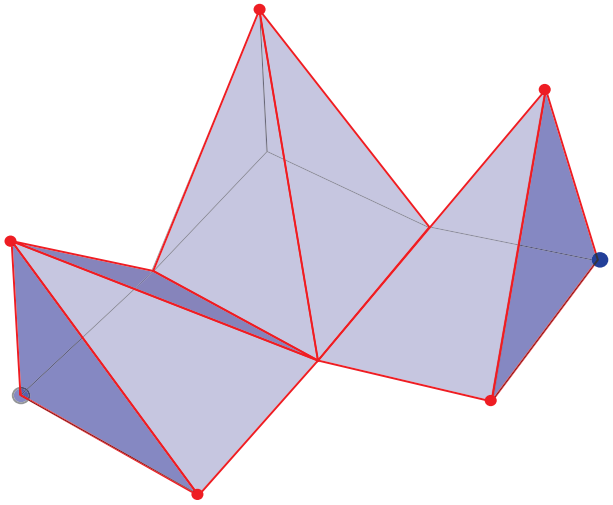
Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.
5. **Know that the effect of a scale factor k greater than zero on length, area, and volume is to multiply each by k, k^2, and k^3, respectively; determine length, area and volume measures using scale factors. CA**
6. **Verify experimentally that in a triangle, angles opposite longer sides are larger, sides opposite larger angles are longer, and the sum of any two side lengths is greater than the remaining side length; apply these relationships to solve real-world and mathematical problems. CA**

The ability to visualize two- and three-dimensional shapes is a useful skill. This group of standards addresses that skill and includes understanding and using volume and area formulas for curved objects. Students also have the opportunity to make use of the notion of a *limiting process*—an idea that plays a large role in calculus and advanced mathematics courses—when they investigate the formula for the area of a circle. By experimenting with grids of finer and finer mesh, students can repeatedly approximate the area of a unit circle and thereby get a better and better approximation for the irratio-

nal number π. They also dissect shapes and make arguments based on these dissections. For instance, as shown in figure G-4 below, a cube can be dissected into three congruent pyramids, which can lend weight to the formula that the volume of a pyramid of base area B and height h is $\frac{1}{3}Bh$ (**MP.2**).

Figure G-4. Three Congruent Pyramids That Form a Cube



Source: Park City Mathematics Institute 2013.

Modeling with Geometry — G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

This set of standards is rich with opportunities for students to apply modeling (**MP.4**) with geometric concepts. The implementation of these standards should not be limited to the end of a Geometry course simply because they are later in the sequence of standards; they should be employed throughout the geometry curriculum. In standard **G-MG.1**, students use geometric shapes, their measures, and their properties to describe objects. This standard can involve two- and three-dimensional shapes, and it is not relegated to simple applications of formulas. In standard **G-MG.3**, students solve design problems by modeling with geometry, such as the one illustrated below.

Example: Ice-Cream Cone **G-MG.3**

The owner of a local ice-cream parlor has hired you to assist with his company's new venture: the company will soon sell its ice-cream cones in the freezer section of local grocery stores. The manufacturing process requires that each ice-cream cone be wrapped in a cone-shaped paper wrapper with a flat, circular disc covering the top. The company wants to minimize the amount of paper that is wasted in the process of wrapping the cones. Use a real ice-cream cone or the dimensions of a real ice-cream cone to complete the following tasks:

a. Sketch a wrapper like the one described above, using the actual size of your cone. Ignore any overlap required for assembly.

b. Use your sketch to help develop an equation the owner can use to calculate the surface area of a wrapper (including the lid) for another cone, given that its base had a radius of length r and a slant height s.

c. Using measurements of the radius of the base and slant height of your cone, and your equation from step b, find the surface area of your cone.

d. The company has a large rectangular piece of paper that measures 100 centimeters by 150 centimeters. Estimate the maximum number of complete wrappers sized to fit your cone that could be cut from this single piece of paper, and explain your estimate. (Solutions can be found at https://www.illustrativemathematics.org/ [accessed April 1, 2015].)

Source: Illustrative Mathematics 2013l.

Conceptual Category: Statistics and Probability

In grades seven and eight, students learned some basic concepts related to probability, including chance processes, probability models, and sample spaces. In higher mathematics, the relative-frequency approach to probability is extended to conditional probability and independence, rules of probability and their use in finding probabilities of compound events, and the use of probability distributions to solve problems involving expected value (University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2012d). Building on probability concepts that began in the middle grades, students in the Geometry course use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students should make use of geometric probability models wherever possible. They use probability to make informed decisions (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010a).

Conditional Probability and the Rules of Probability — S-CP

Understand independence and conditional probability and use them to interpret data. [Link to data from simulations or experiments.]

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*. ★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. ★

To develop student understanding of conditional probability, students should experience two types of problems: those in which the uniform probabilities attached to outcomes lead to independence of the outcomes, and those in which they do not (**S-CP.1–3**). The following examples illustrate these two distinct possibilities.

Example: Guessing on a True–False Quiz — S-CP.1–3

If there are four true-or-false questions on a quiz, then the possible outcomes based on guessing on each question may be arranged as in the table below:

Possible outcomes: Guessing on four true–false questions

Number correct	Out-comes	Number correct	Out-comes	Number correct	Out-comes
4	CCCC	2	CCII	1	CIII
3	ICCC	2	CICI	1	ICII
3	CICC	2	CIIC	1	IICI
3	CCIC	2	ICCI	1	IIIC
3	CCCI	2	ICIC	0	IIII
		2	IICC		

C indicates a correct answer; I indicates an incorrect answer.

By counting outcomes, one can find various probabilities. For example:

$$P(\text{C on first question}) = \frac{1}{2}$$

and

$$P(\text{C on second question}) = \frac{1}{2}$$

Noticing that $P[(\text{C on first}) \text{ AND } (\text{C on second})] = \frac{4}{16} = \frac{1}{2} \bullet \frac{1}{2}$ shows that the two events—getting the first question correct and the second question correct—are independent.

Adapted from UA Progressions Documents 2012d.

Example: Work-Group Leaders S-CP.1–3

Suppose a five-person work group consisting of three girls (April, Briana, and Cyndi) and two boys (Daniel and Ernesto) wants to randomly choose two people to lead the group. The first person is the discussion leader and the second is the recorder, so order is important in selecting the leadership team. In the table below, "A" represents April, "B" represents Briana, "C" represents Cyndi, "D" represents Daniel, and "E" represents Ernesto. There are 20 outcomes for this situation:

Selecting two students from three girls and two boys

Number of girls	Outcomes	
2	AB	BA
2	AC	CA
2	BC	CB
1	AD	DA
1	AE	EA
1	BD	DB
1	BE	EB
1	CD	DC
1	CE	EC
0	DE	ED

Notice that the probability of selecting two girls as the leaders is as follows:

$$P(\text{two girls chosen}) = \frac{6}{20} = \frac{3}{10}$$

whereas

$$P(\text{girl selected on first draw}) = \frac{12}{20} = \frac{3}{5}$$

and

$$P(\text{girl selected on second draw}) = \frac{3}{5}$$

But since $\frac{3}{5} \bullet \frac{3}{5} \neq \frac{3}{10}$, the two events are not independent.

One can also use the conditional-probability perspective to show that these events are not independent.

$$\text{Since } P(\text{girl on second} \mid \text{girl on first}) = \frac{6}{12} = \frac{1}{2}$$

and

$$P(\text{girl selected on second}) = \frac{3}{5},$$

these events are seen to be dependent.

Adapted from UA Progressions Documents 2012d.

Conditional Probability and the Rules of Probability S-CP

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems.★

Students also explore finding probabilities of compound events (**S-CP.6–9**) by using the Addition Rule ($P(A \text{ or } B) = P(A)+P(B) - P(A \text{ and } B)$) and the general Multiplication Rule ($P(A \text{ and } B)=P(A) \bullet P(B|A)= P(B) \bullet P(A|B)$). A simple experiment in which students roll two number cubes and tabulate the possible outcomes can shed light on these formulas before they are extended to application problems.

Example **S-CP.6–9**

On April 15, 1912, the RMS *Titanic* rapidly sank in the Atlantic Ocean after hitting an iceberg. Only 710 of the ship's 2,204 passengers and crew members survived. Some believe that the rescue procedures favored the wealthier first-class passengers. Data on survival of passengers are summarized in the table at the end of this example, and these data will be used to investigate the validity of such claims. Students can use the fact that two events A and B are independent if $P(A|B)=P(A) \bullet P(B)$. A represents the event that a passenger survived, and B represents the event that the passenger was in first class. The conditional probability $P(A|B)$ is compared with the probability $P(A)$.

For a first-class passenger, the probability of surviving is the fraction of all first-class passengers who survived. That is, the sample space is restricted to include only first-class passengers to obtain:

$$P(A|B) = \frac{202}{325} \approx 0.622$$

The probability that a passenger survived is the number of all passengers who survived divided by the total number of passengers:

$$P(A) = \frac{498}{1316} \approx 0.378$$

Since $0.622 \neq 0.378$, the two given events are not independent. Moreover, it can be said that being a passenger in first class did increase the chances of surviving the accident.

Students can be challenged to further investigate where similar reasoning would apply today. For example, what are similar statistics for Hurricane Katrina, and what would a similar analysis conclude about the distribution of damages? (**MP.4**)

***Titanic* passengers**	**Survived**	**Did not survive**	**Total**
First-class	202	123	325
Second-class	118	167	285
Third-class	178	528	706
Total passengers	498	818	1,316

Adapted from Illustrative Mathematics 2013q.

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions. [Introductory; apply counting rules.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

Standards **S-MD.6** and **S-MD.7** involve students' use of probability models and probability experiments to make decisions. These standards set the stage for more advanced work in Algebra II, where the ideas of statistical inference are introduced. See the University of Arizona Progressions document titled "High School Statistics and Probability" for further explanation and examples: http://ime.math.arizona.edu/progressions/ (UA Progressions Documents 2012d [accessed April 6, 2015]).

Geometry Overview

Geometry

Congruence

- Experiment with transformations in the plane.
- Understand congruence in terms of rigid motions.
- Prove geometric theorems.
- Make geometric constructions.

Similarity, Right Triangles, and Trigonometry

- Understand similarity in terms of similarity transformations.
- Prove theorems involving similarity.
- Define trigonometric ratios and solve problems involving right triangles.
- Apply trigonometry to general triangles.

Circles

- Understand and apply theorems about circles.
- Find arc lengths and area of sectors of circles.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.
- Use coordinates to prove simple geometric theorems algebraically.

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems.
- Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- Apply geometric concepts in modeling situations.

Statistics and Probability

Conditional Probability and the Rules of Probability

- Understand independence and conditional probability and use them to interpret data.
- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

G Geometry

Congruence G-CO

Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions. [Build on rigid motions as a familiar starting point for development of concept of geometric proof.]

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems. [Focus on validity of underlying reasoning while using variety of ways of writing proofs.]

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Make geometric constructions. [Formalize and explain processes.]

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Similarity, Right Triangles, and Trigonometry G-SRT

Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle–Angle (AA) criterion for two triangles to be similar.

Prove theorems involving similarity.

4. Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★

8.1 Derive and use the trigonometric ratios for special right triangles (30°, 60°, 90° and 45°, 45°, 90°). CA

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2} ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles G-C

Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

Note: ★ Indicates a modeling standard linking mathematics to everyday life, work, and decision making.
(+) Indicates additional mathematics to prepare students for advanced courses.

3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.
4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles. [Radian introduced only as unit of measure.]

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. **Convert between degrees and radians. CA**

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically. [Include distance formula; relate to Pythagorean Theorem.]

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point* $(1, \sqrt{3})$ *lies on the circle centered at the origin and containing the point (0, 2).*
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

Geometric Measurement and Dimension G-GMD

Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.
5. **Know that the effect of a scale factor *k* greater than zero on length, area, and volume is to multiply each by *k*, k^2, and k^3, respectively; determine length, area, and volume measures using scale factors. CA**
6. **Verify experimentally that in a triangle, angles opposite longer sides are larger, sides opposite larger angles are longer, and the sum of any two side lengths is greater than the remaining side length; apply these relationships to solve real-world and mathematical problems. CA**

Modeling with Geometry G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

Statistics and Probability

Conditional Probability and the Rules of Probability S-CP

Understand independence and conditional probability and use them to interpret data. [Link to data from simulations or experiments.]

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
2. Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B. ★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. ★

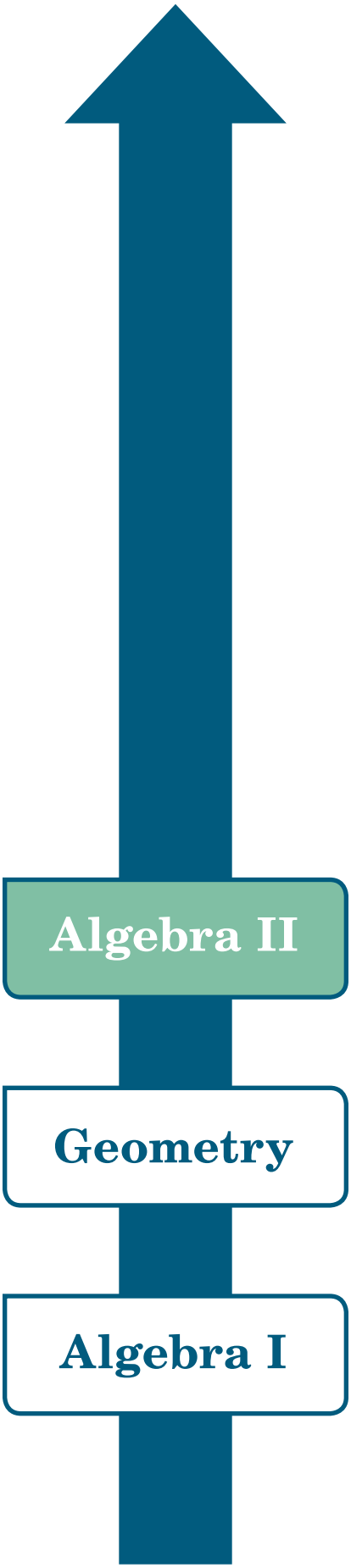
Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions. [Introductory; apply counting rules.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★



Algebra II

The Algebra II course extends students' understanding of functions and real numbers and increases the tools students have for modeling the real world. Students in Algebra II extend their notion of *number* to include complex numbers and see how the introduction of this set of numbers yields the solutions of polynomial equations and the Fundamental Theorem of Algebra. Students deepen their understanding of the concept of *function* and apply equation-solving and function concepts to many different types of functions. The system of polynomial functions, analogous to integers, is extended to the field of rational functions, which is analogous to rational numbers. Students explore the relationship between exponential functions and their inverses, the logarithmic functions. Trigonometric functions are extended to all real numbers, and their graphs and properties are studied. Finally, students' knowledge of statistics is extended to include understanding *the normal distribution*, and students are challenged to make inferences based on sampling, experiments, and observational studies.

For the Traditional Pathway, the standards in the Algebra II course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not simply topics to be checked off from a list during isolated units of instruction; rather, they represent content that should be present throughout the school year in rich instructional experiences.

What Students Learn in Algebra II

Building on their work with linear, quadratic, and exponential functions, students in Algebra II extend their repertoire of functions to include polynomial, rational, and radical functions.[1] Students work closely with the expressions that define the functions and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms. Based on their previous work with functions, and on their work with trigonometric ratios and circles in Geometry, students now use the coordinate plane to extend trigonometry to model periodic phenomena. They explore the effects of transformations on graphs of diverse functions, including functions arising in applications, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of underlying function. They identify appropriate types of functions to model a situation, adjust parameters to improve the model, and compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. Students see how the visual displays and summary statistics learned in earlier grade levels relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role of randomness and careful design in the conclusions that can be drawn.

Examples of Key Advances from Previous Grade Levels or Courses

- In Algebra I, students added, subtracted, and multiplied polynomials. Students in Algebra II divide polynomials that result in remainders, leading to the factor and remainder theorems. This is the underpinning for much of advanced algebra, including the algebra of rational expressions.
- Themes from middle-school algebra continue and deepen during high school. As early as grade six, students began thinking about solving equations as a process of reasoning (**6.EE.5**). This perspective continues throughout Algebra I and Algebra II (**A-REI**). "Reasoned solving" plays a role in Algebra II because the equations students encounter may have extraneous solutions (**A-REI.2**).
- In Algebra I, students worked with quadratic equations with no real roots. In Algebra II, they extend their knowledge of the number system to include complex numbers, and one effect is that they now have a complete theory of quadratic equations: Every quadratic equation with complex coefficients has (counting multiplicity) two roots in the complex numbers.
- In grade eight, students learned the Pythagorean Theorem and used it to determine distances in a coordinate system (**8.G.6–8**). In the Geometry course, students proved theorems using coordinates (**G-GPE.4–7**). In Algebra II, students build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of conic sections (for example, refer to standard **G-GPE.1**).
- In Geometry, students began trigonometry through a study of right triangles. In Algebra II, they extend the three basic functions to the entire unit circle.
- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (for example, refer to standard **S-ID.6**). In a modeling context, students might informally fit an exponential function to a set of data, graphing the data and the model function on the same coordinate axes (Partnership for Assessment of Readiness for College and Careers 2012).

1. In this course, rational functions are limited to those with numerators having a degree not more than 1 and denominators having a degree not more than 2; radical functions are limited to square roots or cube roots of at most quadratic polynomials (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010a).

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The Standards for Mathematical Practice represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. There are ample opportunities for students to engage in each mathematical practice in Algebra II; table A2-1 offers some general examples.

Table A2-1. Standards for Mathematical Practice—Explanation and Examples for Algebra II

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students apply their understanding of various functions to real-world problems. They approach complex mathematics problems and break them down into smaller problems, synthesizing the results when presenting solutions.
MP.2 Reason abstractly and quantitatively.	Students deepen their understanding of variables—for example, by understanding that changing the values of the parameters in the expression $A\sin(Bx+C)+D$ has consequences for the graph of the function. They interpret these parameters in a real-world context.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function when modeling a real-world situation.
MP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems, making use of their expanding repertoire of functions in modeling. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MP.5 Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MP.6 Attend to precision.	Students make note of the precise definition of *complex number*, understanding that real numbers are a subset of complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers.
MP.7 Look for and make use of structure.	Students see the operations of complex numbers as extensions of the operations for real numbers. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena.

(Table continues on next page)

Table A2-1 *(continued)*

MP.8 Look for and express regularity in repeated reasoning.	Students observe patterns in geometric sums—for example, that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written as follows: $1 = 2^1 - 1$ $1 + 2 = 2^2 - 1$ $1 + 2 + 4 = 2^3 - 1$ $1 + 2 + 4 + 8 = 2^4 - 1$ Students use this observation to make a conjecture about any such sum.

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star (★) symbol to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards. In the description of the Algebra II content standards that follow, Modeling is covered first to emphasize its importance in the higher mathematics curriculum.

Examples of places where specific Mathematical Practice standards can be implemented in the Algebra II standards are noted in parentheses, with the standard(s) also listed.

Algebra II Content Standards, by Conceptual Category

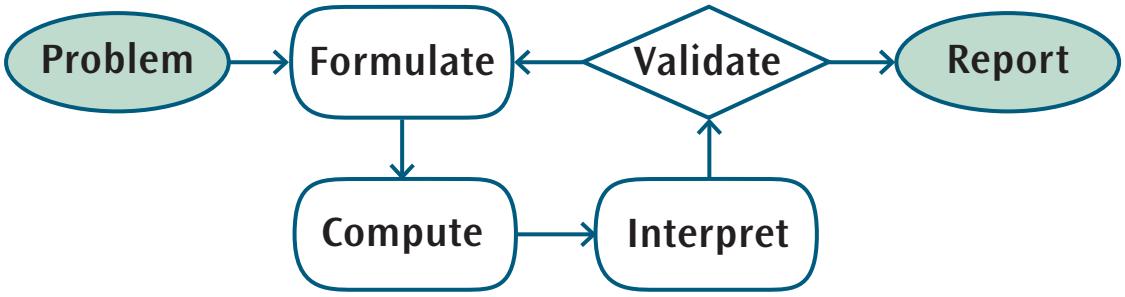
The Algebra II course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in Algebra II are described below, according to each conceptual category. Standards that are considered new for secondary-grades teachers are discussed more thoroughly than other standards.

Conceptual Category: Modeling

Throughout the California Common Core State Standards for Mathematics (CA CCSSM), specific standards for higher mathematics are marked with a ★ symbol to indicate they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: Which of the quantities present in this situation are known and unknown? Can a table of data be made? Is there a functional relationship in this situation? Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new equation or function will apply. In addition, students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, or the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure A2-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Figure A2-1. The Modeling Cycle



The examples in this chapter are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding polynomial and rational functions, graphing, trigonometric functions and their inverses, and applications of statistics are explored through this lens. Readers are encouraged to consult appendix B (Mathematical Modeling) for further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

Work on functions began in Algebra I. In Algebra II, students encounter more sophisticated functions, such as polynomial functions of degree greater than 2, exponential functions having all real numbers as the domain, logarithmic functions, and extended trigonometric functions and their inverses. Several standards of the Functions category are repeated here, illustrating that the standards attempt to reach depth of understanding of the *concept* of a function. As stated in the University of Arizona (UA) Progressions Documents for the Common Core Math Standards, "students should develop ways of thinking that are general and allow them to approach any function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary" (UA Progressions Documents 2013c, 7). For instance, students in Algebra II see quadratic, polynomial, and rational functions as belonging to the same system.

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Emphasize selection of appropriate models.]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Focus on using key features to guide selection of appropriate type of model function.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

In this domain, students work with functions that model data and choose an appropriate model function by considering the context that produced the data. Students' ability to recognize rates of change, growth and decay, end behavior, roots, and other characteristics of functions is becoming more sophisticated; they use this expanding repertoire of families of functions to inform their choices for models. This group of standards focuses on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate (**F-IF.4–9**). The following example illustrates some of these standards.

Example: The Juice Can **F-IF.4–5; F-IF.7–9**

Students are asked to find the minimal surface area of a cylindrical can of a fixed volume. The surface area is represented in units of square centimeters (cm^2), the radius in units of centimeters (cm), and the volume is fixed at 355 milliliters (ml), or 355 cm^3. Students can find the surface area of this can as a function of the radius:

$$S(r) = \frac{2(355)}{r} + 2\pi r^2$$

(See The Juice-Can Equation example that appears in the Algebra conceptual category of this chapter.)

This representation allows students to examine several things. First, a table of values will provide a hint about what the minimal surface area is. The table below lists several values for S based on r:

r(cm)	S(cm^2)
0.5	1421.6
1.0	716.3
1.5	487.5
2.0	380.1
2.5	323.3
3.0	293.2
3.5	279.8
4.0	278.0
4.5	284.9
5.0	299.0
5.5	319.1
6.0	344.4
6.5	374.6
7.0	409.1
7.5	447.9
8.0	490.7

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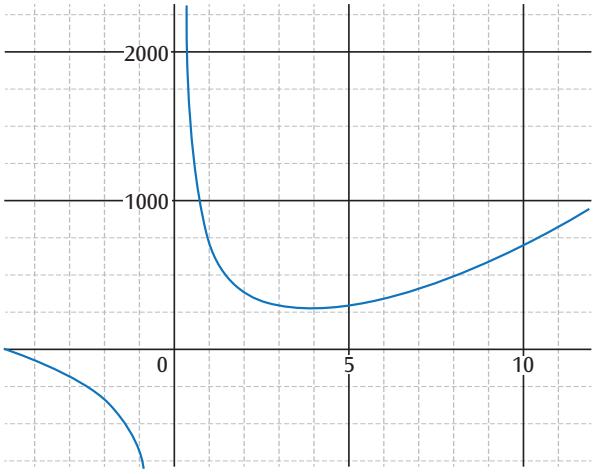
Example: The Juice Can *(continued)*

The data suggest that the minimal surface area occurs when the radius of the base of the juice can is between 3.5 and 4.5 centimeters. Successive approximation using values of r between these values will yield a better estimate. But how can students be sure that the minimum is truly located here? A graph of $S(r)$ provides a clue:



Furthermore, students can deduce that as r gets smaller, the term $\frac{2(355)}{r}$ gets larger and larger, while the term $2\pi r$ gets smaller and smaller, and that the reverse is true as r grows larger, so that there is truly a minimum somewhere in the interval $[3.5, 4.5]$.

Graphs help students reason about rates of change of functions (**F-IF.6**). In grade eight, students learned that the *rate of change* of a linear function is equal to the slope of the graph of that function. And because the slope of a line is constant, the phrase "rate of change" is clear for linear functions. For non-linear functions, however, rates of change are not constant, and thus average rates of change over an interval are used. For example, for the function g defined for all real numbers by $g(x) = x^2$, the average rate of change from $x = 2$ to $x = 5$ is

$$\frac{g(5) - g(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7.$$

This is the slope of the line containing the points $(2, 4)$ and $(5, 25)$ on the graph of g. If g is interpreted as returning the area of a square of side length x, then this calculation means that over this interval the area changes, on average, by 7 square units for each unit increase in the side length of the square (UA Progressions Documents 2013c, 9). Students could investigate similar rates of change over intervals for the Juice Can problem shown previously.

Building Functions F-BF

Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities. ★

   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

Build new functions from existing functions. [Include simple radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.

   a. Solve an equation of the form $f(x)=c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x)=2x^3$ *or* $f(x)=(x+1)/(x-1)$ *for* $x \neq 1$.

Students in Algebra II develop models for more complex situations than in previous courses, due to the expansion of the types of functions available to them (**F-BF.1**). Modeling contexts provide a natural place for students to start building functions with simpler functions as components. Situations in which cooling or heating are considered involve functions that approach a limiting value according to a decaying exponential function. Thus, if the ambient room temperature is 70 degrees Fahrenheit and a cup of tea is made with boiling water at a temperature of 212 degrees Fahrenheit, a student can express the function describing the temperature as a function of time by using the constant function $f(t)=70$ to represent the ambient room temperature and the exponentially decaying function $g(t)=142e^{-kt}$ to represent the decaying difference between the temperature of the tea and the temperature of the room, which leads to a function of this form:

$$T(t)=70+142e^{-kt}$$

Students might determine the constant k experimentally (**MP.4, MP.5**).

Example: Population Growth **F-BF.1**

The approximate population of the United States, measured each decade starting in 1790 through 1940, can be modeled with the following function:

$$P(t)=\frac{(3,900,000\times 200,000,000)e^{0.31t}}{200,000,000+3,900,000\,(e^{0.31t}-1)}$$

In this function, t represents the number of decades after 1790. Such models are important for planning infrastructure and the expansion of urban areas, and historically accurate long-term models have been difficult to derive.

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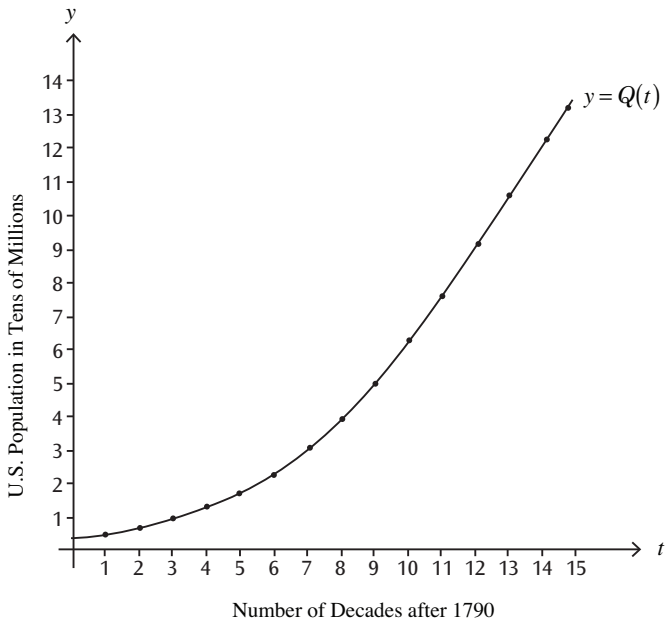
Example: Population Growth *(continued)*



Questions:

a. According to this model, what was the population of the United States in the year 1790?

b. According to this model, when did the U.S. population first reach 100,000,000? Explain your answer.

c. According to this model, what should the U.S. population be in the year 2010? Find the actual U.S. population in 2010 and compare with your result.

d. For larger values of t, such as $t = 50$, what does this model predict for the U.S. population? Explain your findings.

Solutions:

a. The population in 1790 is given by $P(0)$, which is easily found to be 3,900,000 because $e^{0.31(0)} = 1$.

b. This question asks students to find t such that $P(t) = 100{,}000{,}000$. Dividing the numerator and denominator on the left by 100,000,000 and dividing both sides of the equation by 100,000,000 simplifies this equation to

$$\frac{3.9 \times 2 \times e^{0.31t}}{200 + 3.9\left(e^{0.31t} - 1\right)} = 1.$$

Algebraic manipulation and solving for t result in $t \approx \frac{1}{0.31} \ln 50.28 \approx 12.64$. This means that after 1790, it would take approximately 126.4 years for the population to reach 100 million.

c. Twenty-two (22) decades after 1790, the population would be approximately 190,000,000, which is far less (by about 119,000,000) than the estimated U.S. population of 309,000,000 in 2010.

d. The structure of the expression reveals that for very large values of t, the denominator is dominated by $3{,}900{,}000e^{0.31t}$. Thus, for very large values of t,

$$P(t) \approx \frac{3{,}900{,}000 \times 200{,}000{,}000 \times e^{0.31t}}{3{,}900{,}000e^{0.31t}} = 200{,}000{,}000.$$

Therefore, the model predicts a population that stabilizes at 200,000,000 as t increases.

Adapted from Illustrative Mathematics 2013m.

Students can make good use of graphing software to investigate the effects of replacing a function $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for different types of functions (**MP.5**). For example, starting with the simple quadratic function $f(x)=x^2$, students see the relationship between these transformed functions and the vertex form of a general quadratic, $f(x)=a(x-h)^2+k$. They understand the notion of a *family of functions* and characterize such function families based on their properties. These ideas are explored further with trigonometric functions (**F-TF.5**).

With standard **F-BF.4a**, students learn that some functions have the property that an input can be recovered from a given output; for example, the equation $f(x)=c$ can be solved for x, given that c lies in the range of f. Students understand that this is an attempt to "undo" the function, or to "go backwards." Tables and graphs should be used to support student understanding here. This standard dovetails nicely with standard **F-LE.4** described below and should be taught in progression with it. Students will work more formally with inverse functions in advanced mathematics courses, and so standard **F-LE.4** should be treated carefully to prepare students for deeper understanding of functions and their inverses.

Linear, Quadratic, and Exponential Models — F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct}=d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★ [Logarithms as solutions for exponentials]

4.1 Prove simple laws of logarithms. CA ★

4.2 Use the definition of logarithms to translate between logarithms in any base. CA ★

4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA ★

Students worked with exponential models in Algebra I and continue this work in Algebra II. Since the exponential function $f(x)=b^x$ is always increasing or always decreasing for $b\neq 0$, 1, it can be deduced that this function has an inverse, called the *logarithm to the base b*, denoted by $g(x)=\log_b x$. The logarithm has the property that $\log_b x=y$ if and only if $b^y=x$, and this arises in contexts where one wishes to solve an exponential equation. Students find logarithms with base b equal to 2, 10, or e by hand and using technology (**MP.5**). Standards **F-LE.4.1–4.3** call for students to explore the properties of logarithms, such as $\log_b xy=\log_b x+\log_b y$, and students connect these properties to those of exponents (e.g., the previous property comes from the fact that the logarithm represents an exponent and that $b^{n+m}=b^n\bullet b^m$). Students solve problems involving exponential functions and logarithms and express their answers using logarithm notation (**F-LE.4**). In general, students understand logarithms as functions that *undo* their corresponding exponential functions; instruction should emphasize this relationship.

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

2.1 Graph all 6 basic trigonometric functions. CA

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

This set of standards calls for students to expand their understanding of the trigonometric functions first developed in Geometry. At first, the trigonometric functions apply only to angles in right triangles; $\sin\theta$, $\cos\theta$, and $\tan\theta$ make sense only for $0 < \theta < \frac{\pi}{2}$. By representing right triangles with hypotenuse 1 in the first quadrant of the plane, it can be seen that $(\cos\theta, \sin\theta)$ represents a point on the unit circle. This leads to a natural way to extend these functions to any value of θ that remains consistent with the values for acute angles: interpreting θ as the radian measure of an angle traversed from the point (1,0) counterclockwise around the unit circle, $\cos\theta$ is taken to be the x-coordinate of the point corresponding to this rotation and $\sin\theta$ to be the y-coordinate of this point. This interpretation of sine and cosine immediately yields the Pythagorean Identity: that $\cos^2\theta + \sin^2\theta = 1$. This basic identity yields others through algebraic manipulation and allows values of other trigonometric functions to be found for a given θ if one of the values is known (**F-TF.1, 2, 8**).

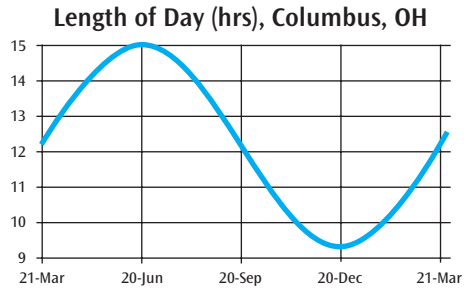
The graphs of the trigonometric functions should be explored with attention to the connection between the unit-circle representation of the trigonometric functions and their properties—for example, to illustrate the periodicity of the functions, the relationship between the maximums and minimums of the sine and cosine graphs, zeros, and so forth. Standard **F-TF.5** calls for students to use trigonometric functions to model periodic phenomena. This is connected to standard **F-BF.3** (families of functions), and students begin to understand the relationship between the parameters appearing in the general cosine function $f(x) = A \bullet \cos(Bx - C) + D$ (and sine function) and the graph and behavior of the function (e.g., amplitude, frequency, line of symmetry).

Example: Modeling Daylight Hours **F-TF.5**

By looking at data for length of days in Columbus, Ohio, students see that the number of daylight hours is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference of the maximum and minimum. Approximations of these values are set as $A = 12.17$ and $B = 2.83$. With some support, students determine that for the period to be 365 days (per cycle), or for the frequency to be $\frac{1}{365}$ cycles per day, $C = \frac{2\pi}{365}$, and if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$.

Thus, $f(t) = 12.17 + 2.83\sin\left(\frac{2\pi t}{365}\right)$ is a function that gives the approximate length of day for t, the day of the year from March 21. Considering questions such as when to plant a garden (i.e., when there are at least 7 hours of midday sunlight), students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$ and find that May 1 and August 10 mark this interval of time.

- or for the frequency to be $\frac{1}{365}$ cycles/day



Students can investigate many other trigonometric modeling situations, such as simple predator–prey models, sound waves, and noise-cancellation models.

Source: UA Progressions Documents 2013c, 19.

Conceptual Category: Number and Quantity

The Complex Number System N-CN

Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations. [Polynomials with real coefficients]

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

In Algebra I, students worked with examples of quadratic functions and solved quadratic equations, encountering situations in which a resulting equation did not have a solution that is a real number—for example, $(x-2)^2=-25$. In Algebra II, students complete their extension of the concept of *number* to include complex numbers, numbers of the form $a+bi$, where i is a number with the property that $i^2=-1$. Students begin to work with complex numbers and apply their understanding of properties of operations (the commutative, associative, and distributive properties) and exponents and radicals to solve equations like those above, by finding square roots of negative numbers—for example, $\sqrt{-25}=\sqrt{25\bullet(-1)}=\sqrt{25}\bullet\sqrt{-1}=5i$ (**MP.7**). They also apply their understanding of properties of operations and exponents and radicals to solve equations:

$$(x-2)^2=-25, \text{ which implies } |x-2|=5i, \text{ or } x=2\pm 5i.$$

Now equations like these have solutions, and the extended number system forms yet another system that behaves according to familiar rules and properties (**N-CN.1–2; N-CN.7–9**). By exploring examples of polynomials that can be factored with real and complex roots, students develop an understanding of the Fundamental Theorem of Algebra; they can show that the theorem is true for quadratic polynomials by an application of the quadratic formula and an understanding of the relationship between roots of a quadratic equation and the linear factors of the quadratic polynomial (**MP.2**).

Conceptual Category: Algebra

Along with the Number and Quantity standards in Algebra II, the Algebra conceptual category standards develop the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers and division of polynomials with long division of integers. Rational numbers extend the arithmetic of integers by allowing division by all numbers except zero; similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of this section is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Seeing Structure in Expressions — A-SSE

Interpret the structure of expressions. [Polynomial and rational]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* ★
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

In Algebra II, students continue to pay attention to the meaning of expressions in context and interpret the parts of an expression by "chunking"—that is, viewing parts of an expression as a single entity (**A-SSE.1–2**). For example, their facility in using special cases of polynomial factoring allows them to fully factor more complicated polynomials:

$$x^4 - y^4 = \left(x^2\right)^2 - \left(y^2\right)^2 = \left(x^2 + y^2\right)\left(x^2 - y^2\right) = \left(x^2 + y^2\right)(x + y)(x - y).$$

In a physics course, students may encounter an expression such as $L_0\sqrt{1 - \frac{v^2}{c^2}}$, which arises in the theory of special relativity. Students can see this expression as the product of a constant L_0 and a term that is equal to 1 when $v = 0$ and equal to 0 when $v = c$. Furthermore, they might be expected to see this mentally, without having to go through a laborious process of evaluation. This involves combining large-scale structure of the expression—a product of L_0 and another term—with the meaning of internal components such as $\frac{v^2}{c^2}$ (UA Progressions Documents 2013b, 4).

By examining the sums of examples of finite geometric series, students can look for patterns to justify why the equation for the sum holds: $\sum_{k=0}^{n} ar^k = \frac{a\left(1 - r^{n+1}\right)}{(1 - r)}$. They may derive the formula with proof by mathematical induction (**MP.3**) or by other means (**A-SSE.4**), as shown in the following example.

Example: Sum of a Geometric Series **A-SSE.4**

Students should investigate several concrete examples of finite geometric series (with $r \neq 1$) and use spreadsheet software to investigate growth in the sums and patterns that arise (**MP.5, MP.8**).

Geometric series have applications in several areas, including calculating mortgage payments, calculating totals for annual investments such as retirement accounts, finding total payout amounts for lottery winners, and more (**MP.4**). In general, a finite geometric series has this form:

$$\sum_{k=0}^{n} ar^k = a\left(1 + r + r^2 + \cdots r^{n-1} + r^n\right)$$

If the sum of this series is denoted by S, then some algebraic manipulation shows that

$$S - rS = a - ar^{n+1}.$$

Applying the distributive property to the common factors and solving for S shows that

$$S(1 - r) = a\left(1 - r^{n+1}\right),$$

so that

$$S = \frac{a\left(1 - r^{n+1}\right)}{1 - r}.$$

Students hone their ability to flexibly see expressions such as $A_n = A_0\left(1+\frac{.15}{12}\right)^n$ as describing the total value of an investment at 15% interest, compounded monthly, for a number of compoundings, n. Moreover, they can interpret the following equation as a type of geometric series that would calculate the total value in an investment account at the end of one year if \$100 is deposited at the beginning of each month (**MP.2, MP.4, MP.7**):

$$A_1 + A_2 + \cdots + A_{12} = 100\left(1+\frac{.15}{12}\right)^1 + 100\left(1+\frac{.15}{12}\right)^2 + \cdots + 100\left(1+\frac{.15}{12}\right)^{12}$$

They apply the formula for geometric series to find this sum.

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials. [Beyond quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x-a$ is $p(a)$, so $p(a)=0$ if and only if $(x-a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $\left(x^2+y^2\right)^2 = \left(x^2-y^2\right)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
5. $(+)$ Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[2]

Rewrite rational expressions. [Linear and quadratic denominators]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x)+r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. $(+)$ Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

Note: $(+)$ Indicates additional mathematics to prepare students for advanced courses.

2. The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

In Algebra II, students continue to develop their understanding of the set of polynomials as a system analogous to the set of integers that exhibits particular properties, and they explore the relationship between the factorization of polynomials and the roots of a polynomial (**A-APR.1–3**). It is shown that when a polynomial $p(x)$ is divided by $(x-a)$, $p(x)$ is written as $p(x)=q(x)\bullet(x-a)+r$, where r is a constant. This can be done by inspection or by polynomial long division (**A-APR.6**). It follows that $p(a)=q(a)\bullet(a-a)+r=q(a)\bullet 0+r=r$, so that $(x-a)$ is a factor of $p(x)$ if and only if $p(a)=0$. This result is generally known as the Remainder Theorem (**A-APR.2**), and provides an easy check to see if a polynomial has a given linear polynomial as a factor. This topic should not be simply reduced to "synthetic division," which reduces the theorem to a method of carrying numbers between registers, something easily done by a computer, while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a technique (**MP.3**) [UA Progressions Documents 2013b, 7].

Students use the zeros of a polynomial to create a rough sketch of its graph and connect the results to their understanding of polynomials as functions (**A-APR.3**). The notion that polynomials can be used to approximate other functions is important in higher mathematics courses such as Calculus, and standard **A-APR.3** is the first step in a progression that can lead students, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane.

In Algebra II, students explore rational functions as a system analogous to that of rational numbers. They see rational functions as useful for describing many real-world situations—for instance, when rearranging the equation $d=rt$ to express the rate as a function of the time for a fixed distance d_0 and obtaining $r=\frac{d_o}{t}$. Now students see that any two polynomials can be divided in much the same way as with numbers (provided the divisor is not zero). Students first understand rational expressions as similar to other expressions in algebra, except that rational expressions have the form $\frac{a(x)}{b(x)}$ for both $a(x)$ and $b(x)$ polynomials. They should have opportunities to evaluate various rational expressions for many values of x, both by hand and using software, perhaps discovering that when the degree of $b(x)$ is larger than the degree of $a(x)$, the value of the expression gets smaller in absolute value as $|x|$ gets larger. When students understand the behavior of rational expressions in this way, it helps them see rational expressions as functions and sets the stage for working with simple rational functions.

Example: The Juice-Can Equation **A-APR.6; F-BF.1**

If someone wanted to investigate the shape of a juice can of minimal surface area, the investigation could begin in the following way. If the volume V_0 is fixed, then the expression for the volume of the can is $V_0 = \pi r^2 h$, where h is the height of the can and r is the radius of the circular base. On the other hand, the surface area S is given by the following formula:

$$S = 2\pi rh + 2\pi r^2$$

This is because the two circular bases of the can contribute $2\pi r^2$ units of surface area, and the outside surface of the can contributes an area in the shape of a rectangle with length equal to the circumference of the base, $2\pi r$, and height equal to h. Since the volume is fixed, h can be found in terms of r: $h = \frac{V_0}{\pi r^2}$. Then this can be substituted into the equation for the surface area:

$$S = 2\pi r \bullet \frac{V_0}{\pi r^2} + 2\pi r^2$$

$$= \frac{2V_0}{r} + 2\pi r^2$$

This equation expresses the surface area S as a (rational) function of r, which can then be analyzed. (Also refer to standards **A-CED.4** and **F-BF.4–9**.)

In addition, students are able to rewrite rational expressions in the form $a(x) = q(x) \bullet b(x) + r(x)$, where $r(x)$ is a polynomial of degree less than $b(x)$, by inspection or by using polynomial long division. They can flexibly rewrite this expression as $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ as necessary—for example, to highlight the end behavior of the function defined by the expression $\frac{a(x)}{b(x)}$. In order to make working with rational expressions more than just an exercise in the proper manipulation of symbols, instruction should focus on the characteristics of rational functions that can be understood by rewriting them in the ways described above (e.g., rates of growth, approximation, roots, axis intersections, asymptotes, end behavior, and so on).

Creating Equations — A-CED

Create equations that describe numbers or relationships. [Equations using all available types of expressions, including simple root functions]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★

Students in Algebra II work with all available types of functions to create equations (**A-CED.1**). Although the functions referenced in standards **A-CED.2–4** will often be linear, exponential, or quadratic, the types of problems should draw from more complex situations than those addressed in Algebra I. For example, knowing how to find the equation of a line through a given point perpendicular to another line makes it possible to find the distance from a point to a line. The Juice-Can Equation example presented previously in this section is connected to standard **A-CED.4**.

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Simple radical and rational]

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

3.1 Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA

Represent and solve equations and inequalities graphically. [Combine polynomial, rational, radical, absolute value, and exponential functions.]

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Students extend their equation-solving skills to those involving rational expressions and radical equations; they make sense of extraneous solutions that may arise (**A-REI.2**). In particular, students understand that when solving equations, the flow of reasoning is generally forward, in the sense that it is assumed a number x is a solution of the equation and then a list of possibilities for x is found. However, not all steps in this process are reversible. For example, although it is true that if $x = 2$, then $x^2 = 4$, it is not true that if $x^2 = 4$, then $x = 2$, as $x = -2$ also satisfies this equation (UA Progressions Documents 2013b, 10). Thus students understand that some steps are reversible and some are not, and they anticipate extraneous solutions. In addition, students continue to develop their understanding of solving equations as solving for values of x such that $f(x) = g(x)$, now including combinations of linear, polynomial, rational, radical, absolute value, and exponential functions (**A-REI.11**). Students also understand that some equations can be solved only approximately with the tools they possess.

Conceptual Category: Geometry

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

3.1 **Given a quadratic equation of the form $ax^2+by^2+cx+dy+e=0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola and graph the equation.** [In Algebra II, this standard addresses only circles and parabolas.] **CA**

No traditional Algebra II course would be complete without an examination of planar curves represented by the general equation $ax^2+by^2+cx+dy+e=0$. In Algebra II, students use "completing the square" (a skill learned in Algebra I) to decide if the equation represents a circle or parabola. They graph the shapes and relate the graph to the equation. The study of ellipses and hyperbolas is reserved for a later course.

Conceptual Category: Statistics and Probability

Students in Algebra II move beyond analysis of data to make sound statistical decisions based on probability models. The reasoning process is as follows: develop a statistical question in the form of a hypothesis (supposition) about a population parameter, choose a probability model for collecting data relevant to that parameter, collect data, and compare the results seen in the data with what is expected under the hypothesis. If the observed results are far from what is expected and have a low probability of occurring under the hypothesis, then that hypothesis is called into question. In other words, the evidence against the hypothesis is weighed by probability (**S-IC.1**) [UA Progressions Documents 2012d]. By investigating simple examples of simulations of experiments and observing outcomes of the data, students gain an understanding of what it means for a model to fit a particular data set (**S-IC.2**). This includes comparing theoretical and empirical results to evaluate the effectiveness of a treatment.

Interpreting Categorical and Quantitative Data — S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Although students may have heard of the normal distribution, it is unlikely that they will have prior experience using the normal distribution to make specific estimates. In Algebra II, students build on their understanding of data distributions to help see how the normal distribution uses area to make estimates of frequencies (which can be expressed as probabilities). It is important for students to see that only some data are well described by a normal distribution (**S-ID.4**). In addition, they can learn through examples the *empirical rule*: that for a normally distributed data set, 68% of the data lie within one standard deviation of the mean and 95% are within two standard deviations of the mean.

Example: The Empirical Rule S-ID.4

Suppose that SAT mathematics scores for a particular year are approximately normally distributed, with a mean of 510 and a standard deviation of 100.

a. What is the probability that a randomly selected score is greater than 610?
b. What is the probability that a randomly selected score is greater than 710?
c. What is the probability that a randomly selected score is between 410 and 710?
d. If a student's score is 750, what is the student's percentile score (the proportion of scores below 750)?

Solutions:

a. The score 610 is one standard deviation above the mean, so the tail area above that is about half of 0.32, or 0.16. The calculator gives 0.1586.
b. The score 710 is two standard deviations above the mean, so the tail area above that is about half of 0.05, or 0.025. The calculator gives 0.0227.
c. The area under a normal curve from one standard deviation below the mean to two standard deviations above the mean is about 0.815. The calculator gives 0.8186.
d. Using either the normal distribution given or the standard normal (for which 750 translates to a z-score of 2.4), the calculator gives 0.9918.

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

In earlier grade levels, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data are collected determines the scope and nature of the conclusions that can be drawn from those data. The concept of *statistical significance* is developed informally through simulation as meaning a result that is unlikely to have occurred solely through random selection in sampling or

random assignment in an experiment (NGA/CCSSO 2010a). When covering standards **S-IC.4–5**, instructors should focus on the variability of results from experiments—that is, on statistics as a way of handling, not eliminating, inherent randomness. Because standards **S-IC.1–6** are all modeling standards, students should have ample opportunities to explore statistical experiments and informally arrive at statistical techniques.

Example: Estimating a Population Proportion **S-IC.1–6**

Suppose a student wishes to investigate whether 50% of homeowners in her neighborhood will support a new tax to fund local schools. If she takes a random sample of 50 homeowners in her neighborhood, and 20 support the new tax, then the *sample proportion* agreeing to pay the tax would be 0.4. But is this an accurate measure of the true proportion of homeowners who favor the tax? How can this be determined?

If this sampling situation (**MP.4**) is simulated with a graphing calculator or spreadsheet software under the assumption that the true proportion is 50%, then the student can arrive at an understanding of the *probability* that her randomly sampled proportion would be 0.4. A simulation of 200 trials might show that 0.4 arose 25 out of 200 times, or with a probability of 0.125. Thus, the chance of obtaining 40% as a sample proportion is not insignificant, meaning that a true proportion of 50% is plausible.

Adapted from UA Progressions Documents 2012d.

Using Probability to Make Decisions — S-MD

Use probability to evaluate outcomes of decisions. [Include more complex situations.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

As in Geometry, students apply probability models to make and analyze decisions. In Algebra II, this skill is extended to more complex probability models, including situations such as those involving quality control or diagnostic tests that yield both false-positive and false-negative results. See the University of Arizona Progressions document titled "High School Statistics and Probability" for further explanation and examples: http://ime.math.arizona.edu/progressions/ (UA Progressions Documents 2012d [accessed April 6, 2015]).

Algebra II is the culmination of the Traditional Pathway in mathematics. Students completing this pathway will be well prepared for higher mathematics and should be encouraged to continue their study of mathematics with Precalculus or other mathematics electives, such as Statistics and Probability or a course in modeling.

Algebra II Overview

Number and Quantity

The Complex Number System

- Perform arithmetic operations with complex numbers.
- Use complex numbers in polynomial identities and equations.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.
- Understand the relationship between zeros and factors of polynomials.
- Use polynomial identities to solve problems.
- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

Geometry

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

A2 Algebra II

Number and Quantity

The Complex Number System — N-CN

Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations. [Polynomials with real coefficients]

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

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Interpret the structure of expressions. [Polynomial and rational]

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   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of* P *and a factor not depending on* P. ★
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials. [Beyond quadratic]

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Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
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Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$ *can be used to generate Pythagorean triples.*

5. (+) Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[3]

Rewrite rational expressions. [Linear and quadratic denominators]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x)+r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations — A-CED

Create equations that describe numbers or relationships. [Equations using all available types of expressions, including simple root functions]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★

3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. ★

4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Simple radical and rational]

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

3.1 Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA

Note: ★ Indicates a modeling standard linking mathematics to everyday life, work, and decision making.
(+) Indicates additional mathematics to prepare students for advanced courses.
3. The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

Represent and solve equations and inequalities graphically. [Combine polynomial, rational, radical, absolute value, and exponential functions.]

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Emphasize selection of appropriate models.]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Focus on using key features to guide selection of appropriate type of model function.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions F-BF

Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

Build new functions from existing functions. [Include simple radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

4. Find inverse functions.

   a. Solve an equation of the form $f(x)=c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x)=2x^3$ *or* $f(x)=(x+1)/(x-1)$ *for* $x \neq 1$.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct}=d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★ [Logarithms as solutions for exponentials]

4.1 Prove simple laws of logarithms. CA ★

4.2 Use the definition of logarithms to translate between logarithms in any base. CA ★

4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA ★

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

2.1 Graph all 6 basic trigonometric functions. CA

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

A2 Algebra II

Geometry

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

3.1 **Given a quadratic equation of the form $ax^2+by^2+cx+dy+e=0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola and graph the equation.** [In Algebra II, this standard addresses only circles and parabolas.] **CA**

Statistics and Probability

Interpreting Categorical and Quantitative Data — S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Making Inferences and Justifying Conclusions — S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population. ★

2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★

4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★

5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★

6. Evaluate reports based on data. ★

Using Probability to Make Decisions — S-MD

Use probability to evaluate outcomes of decisions. [Include more complex situations.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★

7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

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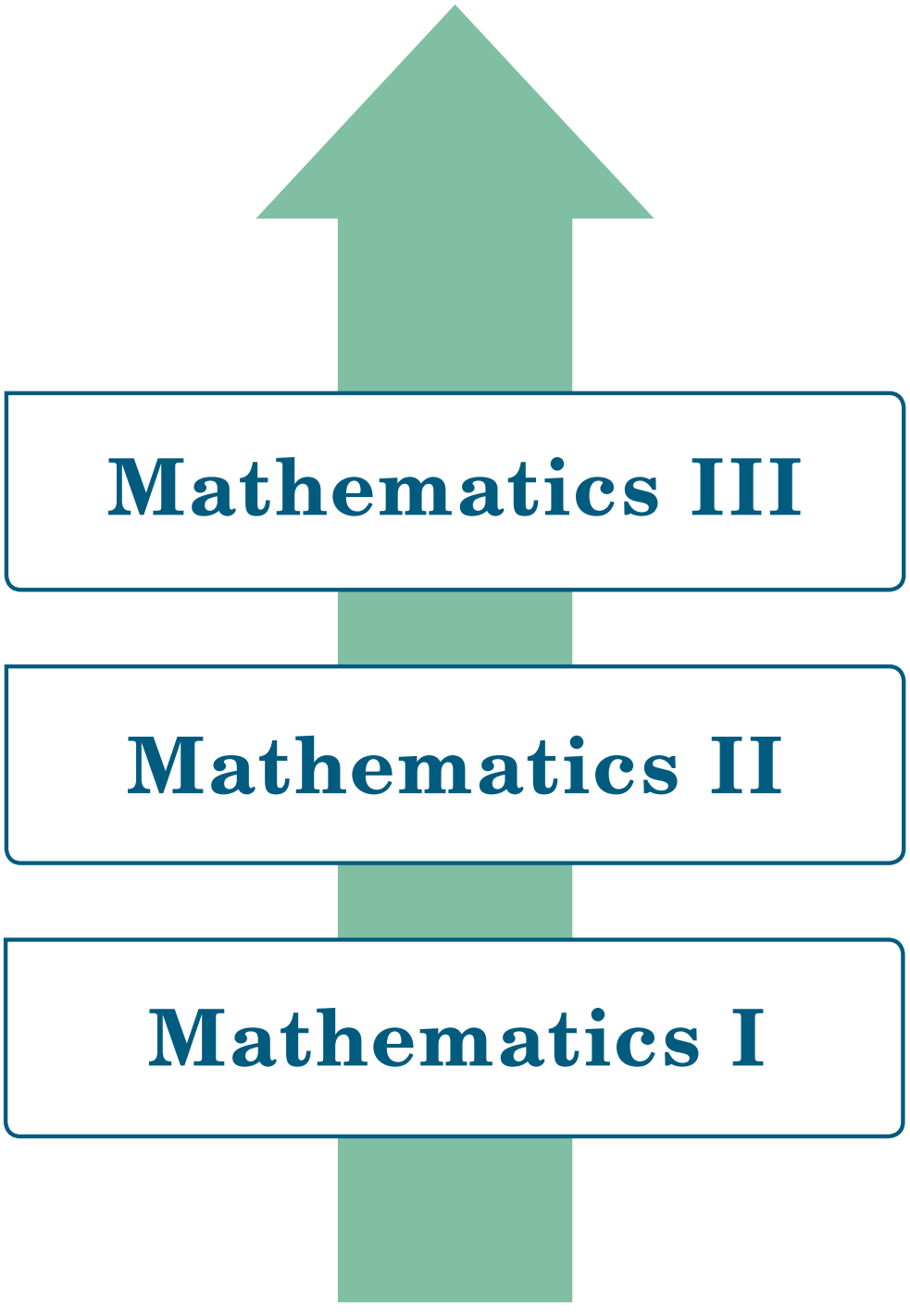
Higher Mathematics Courses

Integrated Pathway



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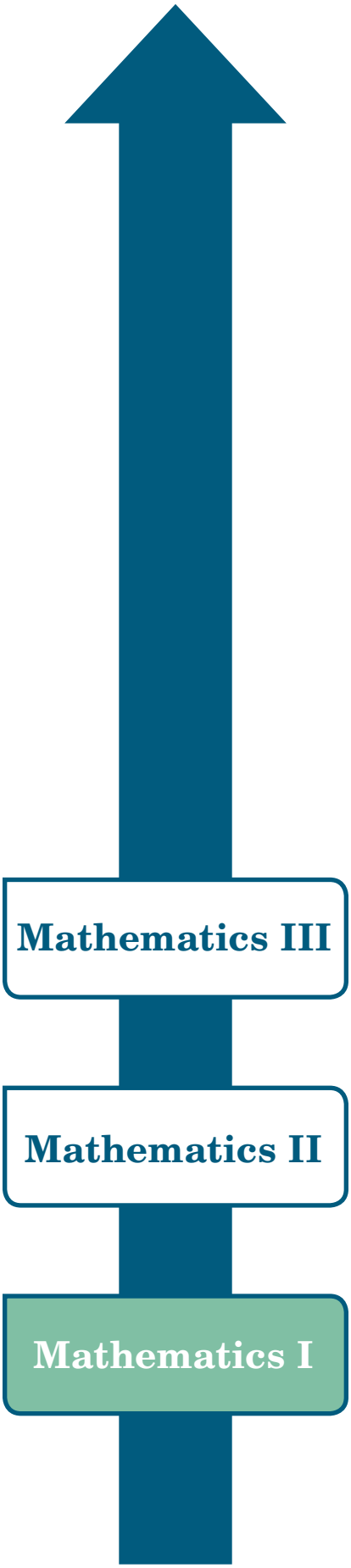


Mathematics I

The fundamental purpose of the Mathematics I course is to formalize and extend students' understanding of linear functions and their applications. The critical topics of study deepen and extend understanding of linear relationships—in part, by contrasting them with exponential phenomena and, in part, by applying linear models to data that exhibit a linear trend. Mathematics I uses properties and theorems involving congruent figures to deepen and extend geometric knowledge gained in prior grade levels. The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the California Common Core State Standards for Mathematics (CA CCSSM); they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics I course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The content of the course is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics I

Students in Mathematics I continue their work with expressions and modeling and analysis of situations. In previous grade levels, students informally defined, evaluated, and compared functions, using them to model relationships between quantities. In Mathematics I, students learn function notation and develop the concepts of *domain* and *range*. Students move beyond viewing functions as processes that take inputs and yield outputs and begin to view functions as objects that can be combined with operations (e.g., finding $(f+g)(x)=f(x)+g(x)$). They explore many examples of functions, including sequences. They interpret functions that are represented graphically, numerically, symbolically, and verbally, translating between representations and understanding the limitations of various representations. They work with functions given by graphs and tables, keeping in mind that these representations are likely to be approximate and incomplete, depending upon the context. Students' work includes functions that can be described or approximated by formulas, as well as those that cannot. When functions describe relationships between quantities arising from a context, students reason with the units in which those quantities are measured. Students build on and informally extend their understanding of integer exponents to consider exponential functions. They compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They also interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.

Students who are prepared for Mathematics I have learned to solve linear equations in one variable and have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. Mathematics I builds on these earlier experiences by asking students to analyze and explain the process of solving an equation and to justify the process used in solving a system of equations. Students develop fluency in writing, interpreting, and translating between various forms of linear equations and inequalities and using them to solve problems. They master solving linear equations and apply related solution techniques and the laws of exponents to the creation and solving of simple exponential equations. Students explore systems of equations and inequalities, finding and interpreting solutions. All of this work is based on understanding quantities and the relationships between them.

In Mathematics I, students build on their prior experiences with data, developing more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models. With linear models, they look at residuals to analyze the goodness of fit.

In previous grade levels, students were asked to draw triangles based on given measurements. They also gained experience with rigid motions (translations, reflections, and rotations) and developed notions about what it means for two objects to be congruent. In Mathematics I, students establish triangle congruence criteria based on analyses of rigid motions and formal constructions. They solve problems about triangles, quadrilaterals, and other polygons. They apply reasoning to complete geometric constructions and explain why the constructions work. Finally, building on their work with the Pythagorean Theorem in the grade-eight standards to find distances, students use a rectangular coordinate system to verify geometric relationships, including properties of special triangles and quadrilaterals and slopes of parallel and perpendicular lines.

Examples of Key Advances from Kindergarten Through Grade Eight

- Students build on previous work with solving linear equations and systems of linear equations from grades seven and eight in two ways: (a) They extend to more formal solution methods, including attending to the structure of linear expressions; and (b) they solve linear inequalities.
- Students' work with patterns and number sequences in the early grades extends to an understanding of sequences as functions.
- Students formalize their understanding of the definition of a function, particularly their understanding of linear functions, emphasizing the structure of linear expressions. Students also begin to work with exponential functions by comparing them to linear functions.
- Work with congruence and similarity transformations that started in grades six through eight progresses. Students consider sufficient conditions for the congruence of triangles.
- Work with bivariate data and scatter plots in grades six through eight is extended to working with lines of best fit (Partnership for Assessment of Readiness for College and Careers [PARCC] 2012, 26).

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The Standards for Mathematical Practice represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards.

The CA CCSSM call for an intense focus on the most critical material, allowing depth in learning, which is carried out through the MP standards. Connecting practices and content happens in the context of *working on problems;* the very first MP standard is to make sense of problems and persevere in solving them. Table M1-1 gives examples of how students can engage in the MP standards in Mathematics I.

Table M1-1. Standards for Mathematical Practice—Explanation and Examples for Mathematics I

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students persevere when attempting to understand the differences between linear and exponential functions. They make diagrams of geometric problems to help make sense of the problems.
MP.2 Reason abstractly and quantitatively.	Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students reason through the solving of equations, recognizing that solving an equation involves more than simply following rote rules and steps. They use language such as "If _____, then _____" when explaining their solution methods and provide justification for their reasoning.
MP.4 Model with mathematics.	Students apply their mathematical understanding of linear and exponential functions to many real-world problems, such as linear and exponential growth. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MP.5 Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the results.
MP.6 Attend to precision.	Students use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem.
MP.7 Look for and make use of structure.	Students recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects.
MP.8 Look for and express regularity in repeated reasoning.	Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, the equation $m = \frac{y - y_1}{x - x_1}$ will give a general equation of that line.

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star (★) symbol to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards. In the description of the Mathematics I content standards that follow, Modeling is covered first to emphasize its importance in the higher mathematics curriculum.

Examples of places where specific Mathematical Practice standards can be implemented in the Mathematics I standards are noted in parentheses, with the standard(s) also listed.

Mathematics I Content Standards, by Conceptual Category

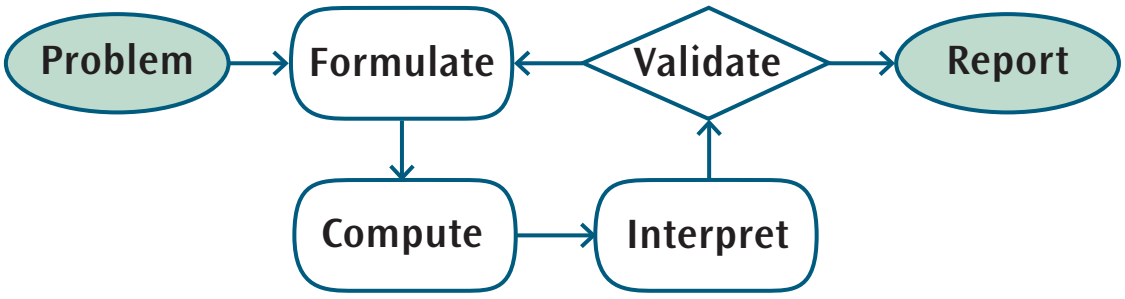
The Mathematics I course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in Mathematics I are described below, according to each conceptual category. Standards that are considered new for secondary-grades teachers are discussed more thoroughly than other standards.

Conceptual Category: Modeling

Throughout the CA CCSSM, specific standards for higher mathematics are marked with a ★ symbol to indicate they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: Which of the quantities present in this situation are known and unknown? Can a table of data be made? Is there a functional relationship in this situation? Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new equation or function will apply. In addition, students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, and the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure M1-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Figure M1-1. The Modeling Cycle



The examples in this chapter are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding linear and exponential functions, graphing, solving equations, and rates of change are explored through this lens. Readers are encouraged to consult appendix B (Mathematical Modeling) for further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

The standards in the Functions conceptual category can serve as motivation for the study of standards in the other Mathematics I conceptual categories. For instance, an equation wherein one is asked to "solve for x" can be seen as a search for the input of a function f that gives a specified output, and solving the equation amounts to undoing the work of the function. Or, the graph of an equation such as $y = \frac{1}{3}x + 5$ can be seen as a representation of a function f where $f(x) = \frac{1}{3}x + 5$. Solving a more complicated equation can be seen as asking, "For which values of x do two functions f and g agree? (i.e., when does $f(x) = g(x)$?)," and the intersection of the two graphs $y = f(x)$ and $y = g(x)$ is then connected to the solution of this equation. In general, functions describe in a precise way how two quantities are related and can be used to make predictions and generalizations, keeping true to the emphasis on modeling in higher mathematics.

Functions describe situations in which one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because theories are continually formed about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models. In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v; the rule $T(v) = \frac{100}{v}$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its *domain*. The domain is often assumed to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context. When relationships between quantities are described, the defining characteristic of a function is that the input value determines the output value, or equivalently, that the output value depends

upon the input value (University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2013c, 2).

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city"; by an assignment, such as the fact that each individual is given a unique Social Security Number; by an algebraic expression, such as $f(x) = a + bx$; or by a recursive rule, such as $f(n+1) = f(n) + b$, $f(0) = a$. The graph of a function is often a useful way of visualizing the relationship that the function models, and manipulating a mathematical expression for a function can shed light on the function's properties.

Interpreting Functions F-IF

Understand the concept of a function and use function notation. [Learn as general principle. Focus on linear and exponential (integer domains) and on arithmetic and geometric sequences.]

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$.

Interpret functions that arise in applications in terms of the context. [Linear and exponential (linear domain)]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

While the grade-eight standards called for students to work informally with functions, students in Mathematics I begin to refine their understanding and use the formal mathematical language of functions. Standards **F-IF.1–9** deal with understanding the concept of a function, interpreting characteristics of functions in context, and representing functions in different ways (**MP.6**). Standard **F-IF.3** calls for students to learn the language of functions and that a function has a domain that must be specified as well as a corresponding range. For instance, by itself, the equation $f(x) = 2^x$ does not describe a function entirely. Similarly, though the expressions in the equations $f(x) = 3x - 4$ and $g(n) = 3n - 4$

look the same, except for the variables used, f may have as its domain all real numbers, while g may have as its domain the natural numbers (i.e., g defines a sequence). Students make the connection between the graph of the equation $y = f(x)$ and the function itself—namely, that the coordinates of any point on the graph represent an input and output, expressed as $(x, f(x))$—and understand that the graph is a *representation* of a function. They connect the domain and range of a function to its graph (**F-IF.5**). Note that there is neither an exploration of the notion of *relation versus function* nor the *vertical line test* in the CA CCSSM. This is by design. The core question when students investigate functions is, "Does each element of the domain correspond to exactly one element of the range?" (UA Progressions Documents 2013c, 8).

Standard **F-IF.3** introduces sequences as functions. In general, a sequence is a function whose inputs consist of a subset of the integers, such as {0, 1, 2, 3, 4, 5, …}. Students can begin to study sequences in simple contexts, such as when calculating their total pay, P, when working for n days at \$65 per day, obtaining a general expression $P(n) = 65 \bullet n$. Students investigate geometric sequences of the form $g(n) = ar^n$, $n \geq 1$, or $g(1) = ar$, $g(n+1) = r \bullet g(n)$, for $n \geq 2$, when they study population growth or decay, as in the availability of a medical drug over time, or financial mathematics, such as when determining compound interest. Notice that the domain is included in the description of the rule (adapted from UA Progressions Documents 2013c, 8).

Interpreting Functions **F-IF**

Analyze functions using different representations. [Linear and exponential]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Standards **F-IF.7** and **F-IF.9** call for students to represent functions with graphs and identify key features in the graph. In Mathematics I, students study only linear, exponential, and absolute value functions. They represent the same function algebraically in different forms and interpret these differences in terms of the graph or context.

Building Functions F-BF

Build a function that models a relationship between two quantities. [For F.BF.1, 2, linear and exponential (integer inputs)]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

Build new functions from existing functions. [Linear and exponential; focus on vertical translations for exponential.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Knowledge of functions and expressions is only part of the complete picture. One must be able to understand a given situation and apply function reasoning to model how quantities change together. Often, the function created sheds light on the situation at hand; one can make predictions of future changes, for example. This is the content of standards **F-BF.1** and **F-BF.2** (starred to indicate they are modeling standards). Mathematics I features the introduction of arithmetic and geometric sequences, written both explicitly and recursively. Students can often see the recursive pattern of a sequence—that is, how the sequence changes from term to term—but they may have a difficult time finding an explicit formula for the sequence.

For example, a population of cyanobacteria can double every 6 hours under ideal conditions, at least until the nutrients in its supporting culture are depleted. This means a population of 500 such bacteria would grow to 1000 in the first 6-hour period, to 2000 in the second 6-hour period, to 4000 in the third 6-hour period, and so on. So if n represents the number of 6-hour periods from the start, the population at that time $P(n)$ satisfies $P(n)=2\bullet P(n-1)$. This is a *recursive* formula for the sequence $P(n)$, which gives the population at a given time period n in terms of the population at time period $n-1$. To find a closed, *explicit* formula for $P(n)$, students can reason that

$$P(0)=500,\ P(1)=2\bullet 500,\ P(2)=2\bullet 2\bullet 500,\ P(3)=2\bullet 2\bullet 2\bullet 500,\ \ldots$$

A pattern emerges: that $P(n)=2^n\bullet 500$. In general, if an initial population P_0 grows by a factor $r>1$ over a fixed time period, then the population after n time periods is given by $P(n)=P_0r^n$.

The following example shows that students can create functions based on prototypical ones.

Example: Exponential Growth **F-BF.3**

The following example illustrates the type of problem that students can solve after they have worked with basic exponential functions.

On June 1, a fast-growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If the algae continue to grow unabated, the lake will be totally covered, and the fish in the lake will suffocate. Based on the current rate at which the algae are growing, this will happen on June 30.

Possible Questions to Ask:

a. When will the lake be covered halfway?

b. Write an equation that represents the percentage of the surface area of the lake that is covered in algae, as a function of time (in days) that passes since the algae were introduced into the lake.

Solution and Comments:

a. Since the population doubles each day, and since the entire lake will be covered by June 30, this implies that half the lake was covered on June 29.

b. If $P(t)$ represents the *percentage* of the lake covered by the algae, then we know that $P(29) = P_0 2^{29} = 100$ (note that June 30 corresponds to $t = 29$). Therefore, one can solve for the initial percentage of the lake covered, $P_0 = \frac{100}{2^{29}} \approx 1.86 \times 10^{-7}$. The equation for the percentage of the lake covered by algae at time t is therefore $P(t) = \left(1.86 \times 10^{-7}\right) 2^t$.

Adapted from Illustrative Mathematics 2013i.

It should be noted that sequences often *do not* lend themselves to compact, explicit formulas such as those in the preceding example. When provided with a sufficient number of examples, students will be able to see this. The means for deciding which sequences do have explicit formulas, such as arithmetic and geometric sequences, is an important area of instruction.

The content of standard **F-BF.3** has typically been left to later courses. In Mathematics I, the focus is on linear and exponential functions. Even and odd functions are addressed in later courses. In keeping with the theme of the input–output interpretation of a function, students should work toward developing an understanding of the effect on the output of a function under certain transformations, such as in the table below:

Expression	Interpretation
$f(a+2)$	The output when the input is 2 greater than a
$f(a)+3$	3 more than the output when the input is a
$2f(x)+5$	5 more than twice the output of f when the input is x

Such understandings can help students to see the effect of transformations on the graph of a function, and in particular, can aid in understanding why it appears that the effect on the graph is the opposite to the transformation on the variable. For example, the graph $y = f(x+2)$ is the graph of f shifted 2 to the left, not to the right (UA Progressions Documents 2013c, 7).

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems. [Linear and exponential]

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★
   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★
   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Interpret expressions for functions in terms of the situation they model. [Linear and exponential of form $f(x) = b^x + k$]

5. Interpret the parameters in a linear or exponential function in terms of a context. ★

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. In standards **F-LE.1a–c,** students recognize and understand the defining characteristics of linear and exponential functions. Students have already worked extensively with linear equations. They have developed an understanding that an equation in two variables of the form $y = mx + b$ exhibits a special relationship between the variables x and y—namely, that a change of Δx in the variable x, the independent variable, results in a change of $\Delta y = m \bullet \Delta x$ in the dependent variable y. They have seen this informally, in graphs and tables of linear relationships, starting in the grade-eight standards (**8.EE.5, 8.EE.6, 8.F.3**). If one considers only integer values of x, so that the incremental change in x is simply 1 unit, then the change in y is exactly m; it is this constant rate of change, m, that defines linear relationships, both in discrete linear sequences and in general linear functions of one real variable. Stated in a different way, students recognize that for successive whole-number input values, x and $x+1$, a linear function $f(x) = mx + b$ exhibits a constant rate of change:

$$f(x+1) - f(x) = \left[m(x+1)+b\right] - (mx+b) = m(x+1-x) = m$$

In contrast, exponential equations such as $g(n) = ab^n$ exhibit a constant percent change. For instance, a t-table for the equation $y = 3^n$ illustrates the constant ratio of successive y-values for this equation:

n	$y = 3^n$	Ratio of successive y-values
1	3	
2	9	$\frac{9}{3} = 3$
3	27	$\frac{27}{9} = 3$
4	81	$\frac{81}{27} = 3$

This table shows that each value of y is 3 times the value preceding it (i.e., 300% of the value preceding it), illustrating the constant percent change of this exponential. In the general case, we have

$$\frac{g(n+1)}{g(n)} = \frac{ab^{n+1}}{ab^n} = \frac{b^{n+1}}{b^n} = b^{(n+1)-n} = b,$$

which illustrates the constant ratio of successive values of g.[1]

The standards require students to prove the result above for linear functions (**F-LE.1a**). In general, students must also be able to recognize situations that represent both linear and exponential functions and construct functions to describe the situations (**F-LE.2**). Finally, students interpret the parameters in linear and exponential functions and model physical problems with such functions.

A graphing utility, spreadsheet, or computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions (**MP.4** and **MP.5**). Real-world examples where this can be explored involve investments, mortgages, and other financial instruments. For example, students can develop formulas for annual compound interest based on a general formula, such as $P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$, where P_0 is the initial amount invested, r is the interest rate, n is the number of times the interest is compounded per year, and t is the number of years the money is invested. They can explore values after different time periods and compare different rates and plans using computer algebra software or simple spreadsheets (**MP.5**). This hands-on experimentation with such functions helps students develop an understanding of the functions' behavior.

Conceptual Category: Number and Quantity

In real-world problems, the answers are usually not numbers, but *quantities*: numbers with units, which involve measurement. In their work in measurement up through grade eight, students primarily measure commonly used attributes such as length, area, and volume. In higher mathematics, students encounter a wider variety of units in modeling—for example, when considering acceleration, currency

1. In Mathematics I of the CA CCSSM, only integer values for x are considered in exponential equations such as $y = b^x$.

conversions, derived quantities such as person-hours and heating degree-days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages.

Quantities	N-Q

Reason quantitatively and use units to solve problems. [Foundation for work with expressions, equations, and functions]

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★

In Mathematics I, students reason through problems with careful selection of units, and they use units to understand problems and make sense of the answers they deduce. Standards **N-Q.1–3** are modeling standards that refer to students' appropriate use of units and definition of quantities. For instance, students can evaluate the accuracy of the following conclusion made in a magazine:

> On average the human body is more than 50 percent water [by weight]. Runners and other endurance athletes average around 60 percent. This equals about 120 soda cans' worth of water in a 160-pound runner! (Illustrative Mathematics 2013p)

Students look for appropriate unit conversions. For example, a typical soda can holds 12 ounces of fluid, a pound is equivalent to 16 dry ounces, and an ounce of water weighs approximately 1 dry ounce (at the temperature of the human body).

Conceptual Category: Algebra

In the Algebra conceptual category, students extend the work with expressions that they started in grades six through eight. They create and solve equations in context, utilizing the power of variable expressions to model real-world problems and solve them with attention to units and the meaning of the answers they obtain. They continue to graph equations, understanding the resulting picture as a representation of the points satisfying the equation. This conceptual category accounts for a large portion of the Mathematics I course and, along with the Functions category, represents the main body of content.

The Algebra conceptual category in higher mathematics is very closely related to the Functions conceptual category (UA Progressions Documents 2013b, 2):

- An expression in one variable can be viewed as defining a function: the act of evaluating the expression is an act of producing the function's output given the input.

- An equation in two variables can sometimes be viewed as defining a function, if one of the variables is designated as the input variable and the other as the output variable, and if there is just one output for each input. This is the case if the expression is of the form $y = (\text{expression in } x)$ or if it can be put into that form by solving for y.
- The notion of equivalent expressions can be understood in terms of functions: if two expressions are equivalent, they define the same function.
- The solutions to an equation in one variable can be understood as the input values that yield the same output in the two functions defined by the expressions on each side of the equation. This insight allows for the method of finding approximate solutions by graphing functions defined by each side and finding the points where the graphs intersect.

Thus, in light of understanding functions, the main content of the Algebra category (solving equations, working with expressions, and so forth) has a very important purpose.

Seeing Structure in Expressions	**A-SSE**

Interpret the structure of expressions. [Linear expressions and exponential expressions with integer exponents]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* ★

An expression can be viewed as a recipe for a calculation, with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price, p. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor. Students began this work in grades six and seven and continue this work with more complex expressions in Mathematics I.

The following example might arise in a modeling context. It emphasizes the importance of understanding the meaning of expressions in a given problem.

Example **A-SSE.1**

A company uses two different-sized trucks to deliver sand. The first truck can transport x cubic yards, and the second truck can transport y cubic yards. The first truck makes S trips to a job site, while the second makes T trips. What do the following expressions represent in practical terms?

a. $S+T$

b. $x+y$

c. $xS+yT$

d. $\dfrac{xS+yT}{S+T}$

Solutions:

a. $S+T$ = the total number of trips both trucks make to a job site.

b. $x+y$ = the total amount of sand that both trucks can transport together.

c. $xS+yT$ = the total amount of sand (in cubic yards) being delivered to a job site by both trucks.

d. $\dfrac{xS+yT}{S+T}$ = the average amount of sand being transported per trip.

Creating Equations A-CED

Create equations that describe numbers or relationships. [Linear and exponential (integer inputs only); for A.CED.3, linear only]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.* ★

An *equation* is a statement of equality between two expressions. The values that make the equation true are the solutions to the equation. An *identity,* in contrast, is true for all values of the variables; rewriting an expression in an equivalent form often creates an identity. The solutions of an equation in one variable form a set of numbers that can be plotted on a number line; the solutions of an equation in two variables form a set of ordered pairs of numbers that can be plotted in the coordinate plane. This set of standards (**A-CED.1–4**) calls for students to create equations to solve problems, correctly graph the equations on the coordinate plane, and interpret solutions in a modeling context.

The following example is designed to have students think about the meaning of the quantities presented in the context and choose which quantities are appropriate for the two different constraints presented. In particular, note that the purpose of the task is to have students *generate* the constraint equations for each part (although the problem statements avoid using this particular terminology), not *solve* the equations (Illustrative Mathematics 2013g).

Example **A-CED.2–3**

The arabica coffee variety yields about 750 kilograms of coffee beans per hectare, and the robusta coffee variety yields approximately 1200 kilograms per hectare. Suppose that a plantation has a hectares of arabica and r hectares of robusta.

a. Write an equation relating a and r if the plantation yields 1,000,000 kilograms of coffee.

b. On August 14, 2003, the world market price of coffee was \$1.42 per kilogram of arabica and \$0.73 per kilogram of robusta. Write an equation relating a and r if the plantation produces coffee worth \$1,000,000.

Solution:

a. The quantity a hectares of arabica will yield $750a$ kilograms (kg) of beans, and r hectares of robusta will yield $1200r$ kg of beans. So the constraint equation is

$$750a + 1200r = 1{,}000{,}000.$$

b. Since a hectares of arabica yield $750a$ kg of beans worth \$1.42/kg, the total dollar value of $1.42(750a) = 1065a$. Likewise, r hectares of robusta yield $1200r$ kg of beans worth \$0.73/kg, for a total dollar value of $0.73(1200r) = 876r$. So the equation governing the possible values of a and r coming from the total market value of the coffee is

$$1065a + 876r = 1{,}000{,}000.$$

One California addition to the Common Core State Standards for Mathematics is the creation of equations involving absolute values (**A-CED.1 [CA]**). The basic absolute value function has at least two useful definitions: (1) a descriptive, verbal definition and (2) a formula definition. A common definition of the absolute value of x is

$$|x| = \text{the distance from the number } x \text{ to 0 (on a number line).}$$

An understanding of the number line easily yields that, for example, $|0| = 0$, $|7| = 7$, and $|-3.9| = 3.9$. However, an equally valid "formula" definition of absolute value reads as follows:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

In other words, $|x|$ is simply x whenever x is 0 or positive, but $|x|$ is the opposite of x whenever x is negative. Either definition can be extended to an understanding of the expression $|x - a|$ as the distance between x and a on a number line, an interpretation that has many uses. For a simple application of this idea, suppose a type of bolt is to be mass-produced in a factory with the specification that its width be 5 mm with an error no larger than 0.01 mm. If w represents the width of a given bolt produced on the production line, then w must satisfy the inequality $|w - 5| \leq 0.01$; that is, the difference between the actual width w and the target width should be less than or equal to 0.01 (**MP.4, MP.6**). Students should become comfortable with the basic properties of absolute values (e.g., $|x| + a \neq |x + a|$) and with solving absolute value equations and interpreting the solution.

In higher mathematics courses, intervals on the number line are often denoted by an inequality of the form $|x-a| \le d$ for a positive number d. For example, $|x-2| \le \frac{1}{2}$ represents the closed interval $1\frac{1}{2} \le x \le 2\frac{1}{2}$. This can be seen by interpreting $|x-2| \le \frac{1}{2}$ as "the distance from x to 2 is less than or equal to $\frac{1}{2}$" and deciding which numbers fit this description.

On the other hand, in the case where $x-2<0$, the following would hold true: $|x-2| = -(x-2) \le \frac{1}{2}$, so that $x \ge 1\frac{1}{2}$. In the case where $x-2 \ge 0$, $|x-2| = x-2 \le \frac{1}{2}$, which means that $x \le 2\frac{1}{2}$. Since students are looking for all values of x that satisfy both inequalities, the interval is $1\frac{1}{2} \le x \le 2\frac{1}{2}$. This shows how the formula definition can be used to find this interval.

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Master linear; learn as general principle.]

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. [Linear inequalities; literal equations that are linear in the variables being solved for; exponential of a form, such as $2^x = 1/16$.]

3.1 Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions. In Mathematics I, students solve linear equations and inequalities in one variable, including equations and inequalities with absolute values and equations with coefficients represented by letters (**A-REI.3, A REI.3.1**). When solving equations, students make use of the symmetric and transitive properties and particular properties of equality regarding operations (e.g., "Equals added to equals are equal"). Standard **A-REI.1** requires that in any situation, students can solve an equation *and explain the steps* as resulting from previous true equations and using the aforementioned properties (**MP.3**). In this way, the idea of *proof,* while not explicitly named, is given a prominent role in the solving of equations, and the reasoning and justification process is not simply relegated to a future mathematics course. The following example illustrates the justification process that may be expected in Mathematics I.

On Solving Equations: A written sequence of steps is code for a narrative line of reasoning that would use words such as *if, then, for all,* and *there exists.* In the process of learning to solve equations, students should learn certain "if–then" moves—for example, "If $x = y$, then $x + c = y + c$ for any c." The danger in learning algebra is that students emerge with nothing but the moves, which may make it difficult to detect incorrect or made-up moves later on. Thus the first requirement in this domain (**REI**) is that students understand that solving equations is a process of reasoning (**A-REI.1**).

Fragments of Reasoning

$$2x - 5 = 16 - x$$
$$2x - 5 + x = 16 - x + x$$
$$3x - 5 = 16$$
$$3x = 21$$
$$x = \frac{21}{3} = 7$$

This sequence of equations is shorthand for a line of reasoning: *"If twice a number minus 5 equals 16 minus that number, then three of that number minus 5 must be 16, by the properties of equality. But that means three times that number is 21, so the number is 7."*

Adapted from UA Progressions Documents 2013b, 13.

The same solution techniques used to solve equations can be used to rearrange formulas to highlight specific quantities and explore relationships between the variables involved. For example, the formula for the area of a trapezoid, $A = \left(\frac{b_1 + b_2}{2}\right)h$, can be solved for h using the same deductive process (**MP.7, MP.8**). As will be discussed later, functional relationships can often be explored more deeply by rearranging equations that define such relationships; thus, the ability to work with equations that have letters as coefficients is an important skill.

Reasoning with Equations and Inequalities — A-REI

Solve systems of equations. [Linear systems]

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system. The process of adding one equation to another is understood in this way: If the two sides of one equation are equal, and the two sides of another equation are equal, then the sum (or difference) of the left sides of the two equations is equal to the sum (or difference) of the right sides. The reversibility of these steps justifies that an equivalent system of equations has been achieved. This crucial point should be consistently noted when students reason about solving systems of equations (UA Progressions Documents 2013b, 11).

When solving systems of equations, students also make frequent use of the substitution property of equality—for example, when solving the system $2x-9y=5$ and $y=\frac{1}{3}x+1$, the expression $\frac{1}{3}x+1$ can be substituted for y in the first equation to obtain $2x-9\left(\frac{1}{3}x+1\right)=5$. Students also solve such systems approximately, by using graphs and tables of values (**A-REI.5–6**). Presented in context, the method of solving a system of equations by elimination takes on meaning, as the following example shows.

Example: Solving Simple Systems of Equations **A-REI.6**

To get started with understanding how to solve systems of equations by linear combinations, students can be encouraged to interpret a system in terms of real-world quantities, at least in some cases. For instance, suppose one wanted to solve this system:

$$3x+y=40$$
$$4x+2y=58$$

Now consider the following scenario: Suppose 3 CDs and a magazine cost $40, while 4 CDs and 2 magazines cost $58.

- What happens to the price when you add 1 CD and 1 magazine to your purchase?
- What is the price if you decided to buy only 2 CDs and no magazine?

Answering these questions amounts to realizing that since $(3x+y)+(x+y)=40+18$, we must have that $x+y=18$. Therefore, $(3x+y)+(-1)(x+y)=40+(-1)18$, which implies that $2x=22$, or 1 CD costs $11. The value of y can now be found using either of the original equations: $y=7$.

Reasoning with Equations and Inequalities — A-REI

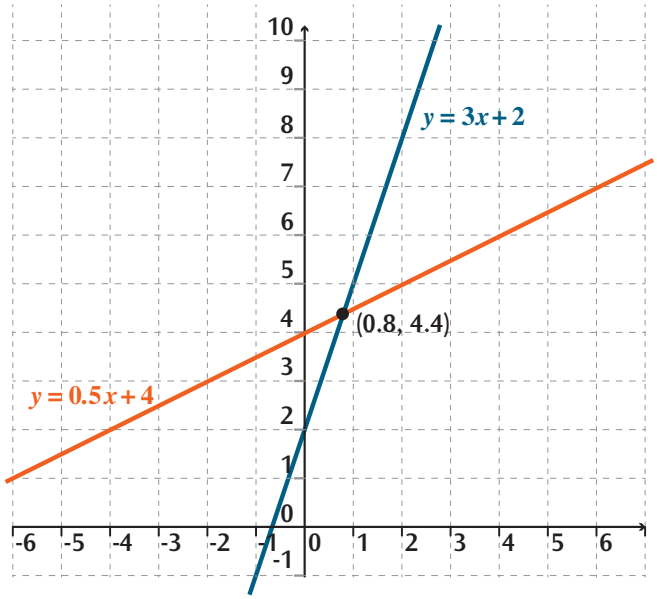
Represent and solve equations and inequalities graphically. [Linear and exponential; learn as general principle.]

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
11. Explain why the x-coordinates of the points where the graphs of the equations $y=f(x)$ and $y=g(x)$ intersect are the solutions of the equation $f(x)=g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

One of the most important goals of instruction in mathematics is to illuminate connections between different mathematical concepts. In particular, standards **A-REI.10–12** call for students to learn the relationship between the algebraic representation of an equation and its graph plotted in the

coordinate plane and understand geometric interpretations of solutions to equations and inequalities. As students become more comfortable with function notation after studying standards **F-IF.1–2** — for example, writing $f(x) = 3x + 2$ and $g(x) = -\frac{1}{2}x + 4$ — they begin to see solving the equation $3x + 2 = -\frac{1}{2}x + 4$ as solving the equation $f(x) = g(x)$. That is, they find those x-values where two functions take on the same output value. Moreover, they graph the two equations (see figure M1-2) and see that the x-coordinate(s) of the point(s) of intersection of the graphs of $y = f(x)$ and $y = g(x)$ are the solutions to the original equation.

Figure M1-2. Graph of a System of Two Linear Equations



Students also create tables of values for functions to approximate or find exact solutions to equations such as that above. For example, they may use spreadsheet software to construct a table (see table M1-2).

Table M1-2. Values for $f(x) = 3x + 2$ and $g(x) = -(0.5)x + 4$

x	$f(x) = 3x + 2$	$g(x) = -(0.5)x + 4$
-3	-7	5.5
-2.5	-5.5	5.25
-2	-4	5
-1.5	-2.5	4.75
-1	-1	4.5
-0.5	0.5	4.25
0	2	4
0.5	3.5	3.75
1	5	3.5
1.5	6.5	3.25
2	8	3
2.5	9.5	2.75
3	11	2.5
3.5	12.5	2.25
4	14	2
4.5	15.5	1.75

Using this table, students can reason that since $f(x) = 3.5$ at $x = 0.5$ and f is increasing, and $g(x) = 3.5$ at $x = 1$ and g is decreasing, the two functions must take on the same value somewhere between these values (**MP.3, MP.6**). In this example, since the original equation is of degree one, students know that there is only one solution, and using finer increments of x will approximate the solution. Examining graphs and tables and solving equations algebraically help students to make connections between these various representations of functions and equations.

Conceptual Category: Geometry

The standards for grades seven and eight introduced students to seeing two-dimensional shapes as part of a generic plane (the *Euclidean plane*) and exploring transformations of this plane as a way to determine whether two shapes are congruent or similar. These notions are formalized In Mathematics I, and students use transformations to prove geometric theorems about triangles. Students then apply these triangle congruence theorems to prove other geometric results, engaging throughout in standard **MP.3.**

Congruence G-CO

Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions. [Build on rigid motions as a familiar starting point for development of concept of geometric proof.]

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Make geometric constructions. [Formalize and explain processes.]

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

In the Geometry conceptual category, the commonly held (but imprecise) definition that shapes are congruent when they "have the same size and shape" is replaced by a more mathematically precise one (**MP.6**): *Two shapes are congruent if there is a sequence of rigid motions in the plane that takes one shape exactly onto the other.* This definition is explored intuitively in the grade-eight standards, but it is investigated more closely in Mathematics I. In grades seven and eight, students experimented with transformations in the plane, but in Mathematics I they build more precise definitions for the *rigid motions* (rotation, reflection, and translation) based on previously defined and understood terms such as *angle, circle, perpendicular line, point, line, between,* and so forth (**G-CO.1, 3–4**). Students base their understanding of these definitions on their experience with transforming figures using patty paper, transparencies, or geometry software (**G-CO.2–3, 5; MP.5**), something they started doing in grade eight. These transformations should be investigated both in a general plane as well as on a coordinate system—especially when transformations are explicitly described by using precise names of points, translation vectors, and specific lines.

Example: Defining Rotations **G-CO.4**

Mrs. B wants to help her class understand the following definition of a *rotation:*

> A *rotation* about a point P through angle α is a transformation $A \mapsto A'$ such that (1) if point A is different from P, then $PA = PA'$ and the measure of $\angle APA' = \alpha$; and (2) if point A is the same as point P, then $A' = A$.

Mrs. B gives her students a handout with several geometric shapes on it and a point, P, indicated on the page. In pairs, students copy the shapes onto a transparency sheet and rotate them through various angles about P; then they transfer the rotated shapes back onto the original page and measure various lengths and angles as indicated in the definition.

While justifying that the properties of the definition hold for the shapes given to them by Mrs. B, the students also make some observations about the effects of a rotation on the entire plane. For example:

- Rotations preserve lengths.
- Rotations preserve angle measures.
- Rotations preserve parallelism.

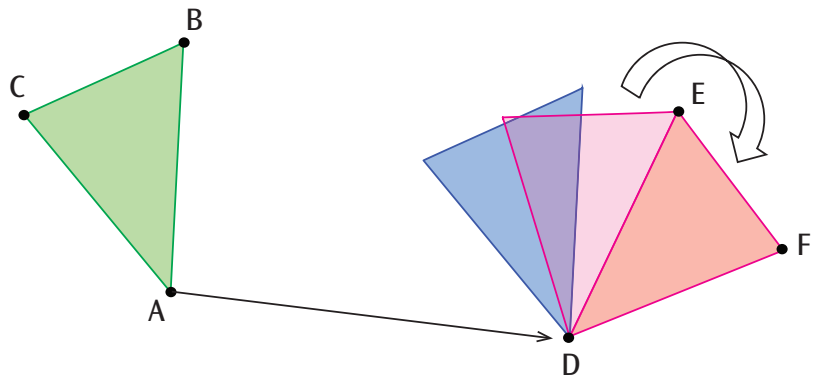
In a subsequent exercise, Mrs. B plans to allow students to explore more rotations on dynamic geometry software, asking them to create a geometric shape and rotate it by various angles about various points, both part of the object and not part of the object.

In standards **G-CO.6–8,** geometric transformations are given a more prominent role in the higher mathematics geometry curriculum than perhaps ever before. The new definition of *congruence* in terms of rigid motions applies to any shape in the plane, whereas previously, congruence seemed to depend on criteria that were specific only to particular shapes. For example, the side–side–side (SSS) congruence criterion for triangles did not extend to quadrilaterals, which seemed to suggest that congruence was a notion dependent on the shape that was considered. Although it is true that there are specific

criteria for determining congruence of certain shapes, the basic notion of congruence is the same for all shapes. In the CA CCSSM, the SSS criterion for triangle congruence is a *consequence* of the definition of congruence, just as the fact that if two polygons are congruent, then their sides and angles can be put into a correspondence such that each corresponding pair of sides and angles is congruent. This concept comprises the content of standards **G-CO.7** and **G-CO.8,** which derive congruence criteria for triangles from the new definition of congruence.

Further discussion of standards **G-CO.7** and **G-CO.8** is warranted here. Standard **G-CO.7** explicitly states that students show that two triangles are congruent *if and only if* corresponding pairs of sides and corresponding pairs of angles are congruent (**MP.3**). The depth of reasoning here is fairly substantial at this level, as students must be able to show, using rigid motions, that congruent triangles have congruent corresponding parts and that, conversely, if the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions that takes one triangle to the other. The second statement may be more difficult to justify than the first for most students, so a justification is presented here. Suppose there are two triangles ΔABC and ΔDEF such that the correspondence $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$ results in pairs of sides and pairs of angles being congruent. If one triangle were drawn on a fixed piece of paper and the other drawn on a separate transparency, then a student could illustrate a translation, T, that takes point A to point D. A simple rotation R about point A, if necessary, takes point B to point E, which is certain to occur because $\overline{AB} \cong \overline{DE}$ and rotations preserve lengths. A final step that may be needed is a reflection S about the side AB, to take point C to point F. It is important to note why the image of point C is actually F. Since $\angle A$ is reflected about line $\overleftrightarrow{AB}$, its measure is preserved. Therefore, the image of side $\overline{AC}$ at least lies on line $\overline{DF}$, since $\angle A \cong \angle D$. But since $\overline{AC} \cong \overline{DF}$, it must be the case that the image of point C coincides with F. The previous discussion amounts to the fact that the sequence of rigid motions, T, followed by R, followed by S, maps ΔABC exactly onto ΔDEF. Therefore, if it is known that the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions carrying one onto the other; that is, they are congruent. Figure M1-3 presents the steps in this reasoning.

Figure M1-3. Illustration of the Reasoning That Congruent Corresponding Parts Imply Triangle Congruence



Point A is translated to D, the resulting image of ΔABC is rotated so as to place B onto E, and the image is then reflected along line segment DE to match point C to F.

Similar reasoning applies for standard **G-CO.8**, in which students justify the typical triangle congruence criteria such as ASA, SAS, and SSS. Experimentation with transformations of triangles where only two of the criteria are satisfied will result in counterexamples, and geometric constructions of triangles of prescribed side lengths (e.g., in the case of SSS) will leave little doubt that any triangle constructed with these side lengths will be congruent to another, and therefore that SSS holds (**MP.7**). Note that in standards **G-CO.1–8,** formal proof is not required. Students are asked to use transformations to show that particular results are true.

Expressing Geometric Properties with Equations — G-GPE

Use coordinates to prove simple geometric theorems algebraically. [Include distance formula; relate to Pythagorean Theorem.]

4. Use coordinates to prove simple geometric theorems algebraically.
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

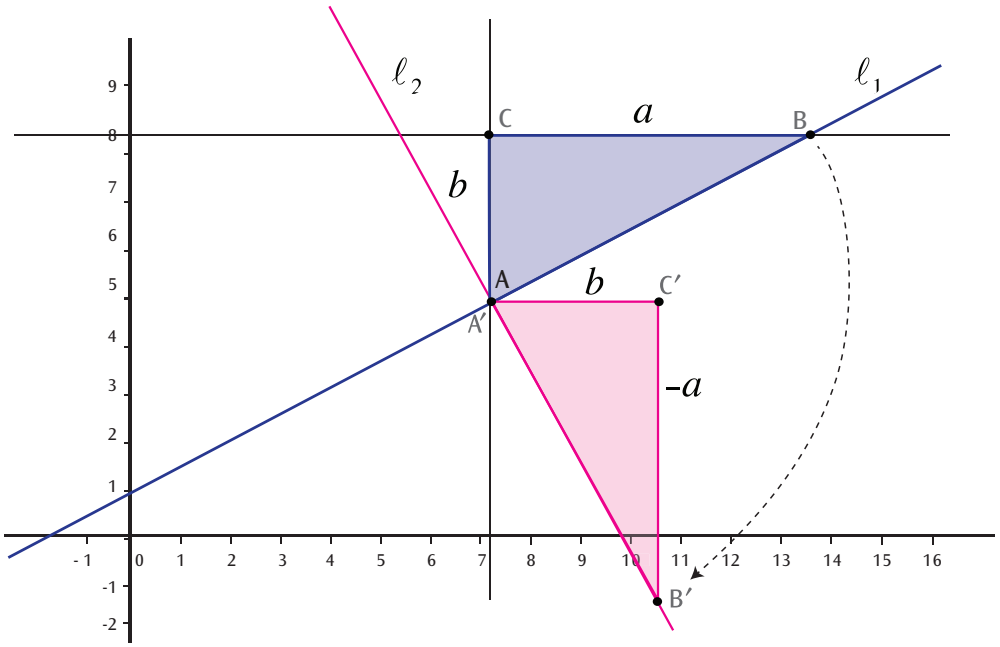
The intersection of algebra and geometry is explored in this cluster of standards. Standard **G-GPE.4** calls for students to use coordinates to prove simple geometric theorems. For instance, they prove that a figure defined by four points is a rectangle by proving that lines containing opposite sides of the figure are parallel and lines containing adjacent sides are perpendicular. Students must be fluent in finding slopes and equations of lines (where necessary) and understand the relationships between the slopes of parallel and perpendicular lines (**G-GPE.5**).

Many simple geometric results can be proved algebraically, but two results of high importance are the slope criteria for parallel and perpendicular lines. Students in grade seven began to study lines and linear equations; in Mathematics I, they not only use relationships between slopes of parallel and perpendicular lines to solve problems, but they also justify why these relationships are true. An intuitive argument for why parallel lines have the same slope might read, "Since the two lines never meet, each line must *keep up* with the other as we travel along the slopes of the lines. So it seems obvious that their slopes must be equal." This intuitive thought leads to an equivalent statement: if given a pair of linear equations $\ell_1: y = m_1x + b_1$ and $\ell_2: y = m_2x + b_2$ (for $m_1, m_2 \neq 0$) such that $m_1 \neq m_2$—that is, such that *their slopes are different*—then the lines must intersect. Solving for the intersection of the two lines yields the x-coordinate of their intersection to be $x = \frac{b_2 - b_1}{m_1 - m_2}$, which surely exists because $m_1 \neq m_2$. It is important for students to understand the steps of the argument and comprehend why proving this statement is equivalent to proving the statement "If $\ell_1 \parallel \ell_2$, then $m_1 = m_2$" (**MP.1, MP.2**).

In addition, students are expected to justify why the slopes of two non-vertical perpendicular lines ℓ_1 and ℓ_2 satisfy the relationship $m_1 = -\frac{1}{m_2}$, or $m_1 \bullet m_2 = -1$. Although there are numerous ways to do this, only one way is presented here, and dynamic geometry software can be used to illustrate it well (**MP.4**). Let ℓ_1 and ℓ_2 be any two non-vertical perpendicular lines. Let A be the intersection of the two lines, and let B be any other point on ℓ_1 above A. A vertical line is drawn through A, a horizontal line is drawn through B, and C is the intersection of those two lines. ΔABC is a right triangle. If a is the

horizontal displacement Δx from C to B, and b is the length of $\overline{AC}$, then the slope of ℓ_1 is $m_1 = \frac{\Delta y}{\Delta x} = \frac{b}{a}$. By rotating ΔABC clockwise around A by 90 degrees, the hypotenuse $\overline{AB'}$ of the rotated triangle $\Delta AB'C'$ lies on ℓ_2. Using the legs of $\Delta AB'C'$, students see that the slope of ℓ_2 is $m_2 = \frac{\Delta y}{\Delta x} = \frac{-a}{b}$. Thus $m_1 \bullet m_2 = \frac{b}{a} \bullet \frac{-a}{b} = -1$. Figure M1-4 illustrates this proof (**MP.1, MP.7**).

Figure M1-4. Illustration of the Proof That the Slopes of Two Perpendicular Lines Are Opposite Reciprocals of One Another



The proofs described above make use of several ideas that students learned in Mathematics I and prior courses—for example, the relationship between equations and their graphs in the plane (**A-REI.10**) and solving equations with variable coefficients (**A-REI.3**). An investigative approach that first uses several examples of lines that are perpendicular and their equations to find points, construct triangles, and decide if the triangles formed are right triangles will help students ramp up to the second proof (**MP.8**). Once more, the reasoning required to make sense of such a proof and to communicate the essence of the proof to a peer is an important goal of geometry instruction (**MP.3**).

Conceptual Category: Statistics and Probability

In Mathematics I, students build on their understanding of key ideas for describing distributions—shape, center, and spread—presented in the standards for grades six through eight. This enhanced understanding allows students to give more precise answers to deeper questions, often involving comparisons of data sets.

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★

Summarize, represent, and interpret data on two categorical and quantitative variables. [Linear focus; discuss general principle.]

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★
   b. Informally assess the fit of a function by plotting and analyzing residuals. ★
   c. Fit a linear function for a scatter plot that suggests a linear association. ★

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★

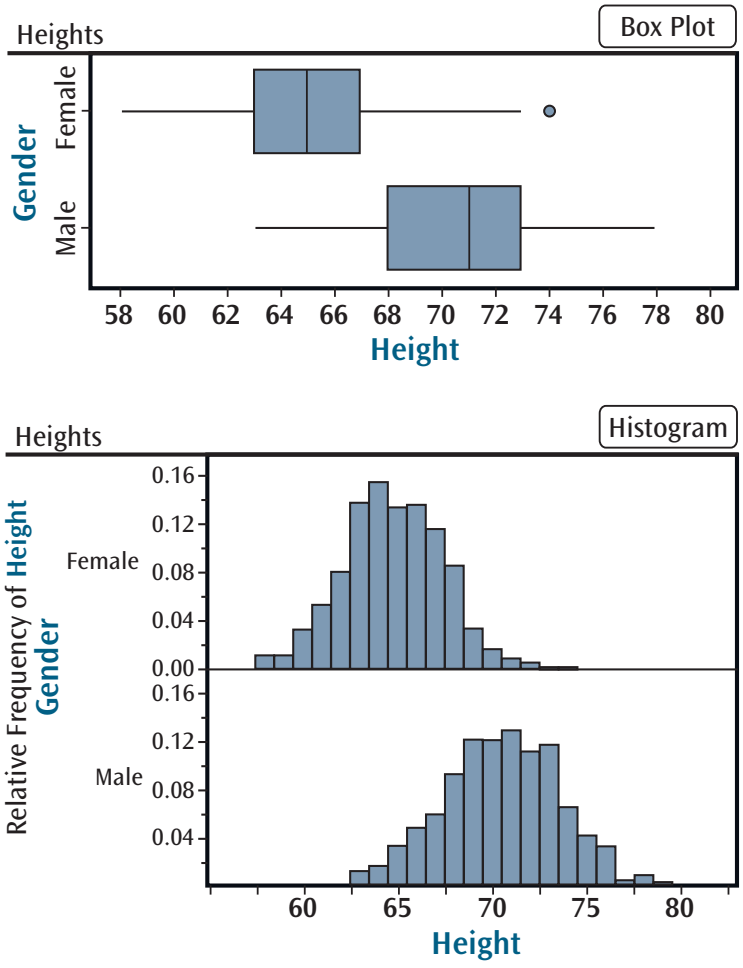
Standards **S-ID.1–6** support standards **S-ID.7–9,** in the sense that the former standards extend concepts students began to learn in grades six through eight. Students use the shape of the distribution and the question(s) to be answered to decide on the median or mean as the more appropriate measure of center and to justify their choice through statistical reasoning. Students may use parallel box plots or histograms to compare differences in the shape, center, and spread of comparable data sets (**S-ID.1–2**).

Example **S-ID.2**

The following graphs show two ways of comparing height data for males and females in the 20–29 age group. Both involve plotting the data or data summaries (box plots or histograms) on the same scale, resulting in what are called parallel (or side-by-side) box plots and parallel histograms (**S-ID.1**). The parallel box plots show an obvious difference in the medians and the interquartile ranges (IQRs) for the two groups; the medians for males and females are, respectively, 71 inches and 65 inches, while the IQRs are 5 inches and 4 inches. Thus, male heights center at a higher value but are slightly more variable.

The parallel histograms show the distributions of heights to be mound shaped and fairly symmetrical (approximately normal) in shape. Therefore, the data can be succinctly described using the mean and standard deviation. Heights for males and females have means of 70.4 inches and 64.7 inches, respectively, and standard deviations of 3.0 inches and 2.6 inches. Students should be able to sketch each distribution and answer questions about it solely from knowledge of these three facts (shape, center, and spread). For either group, about 68% of the data values will be within one standard deviation of the mean (**S-ID.2–3**). Students also observe that the two measures of center—median and mean—tend to be close to each other for symmetric distributions.

Comparing heights of males and females



Heights of U.S. males and females in the 20–29 age group

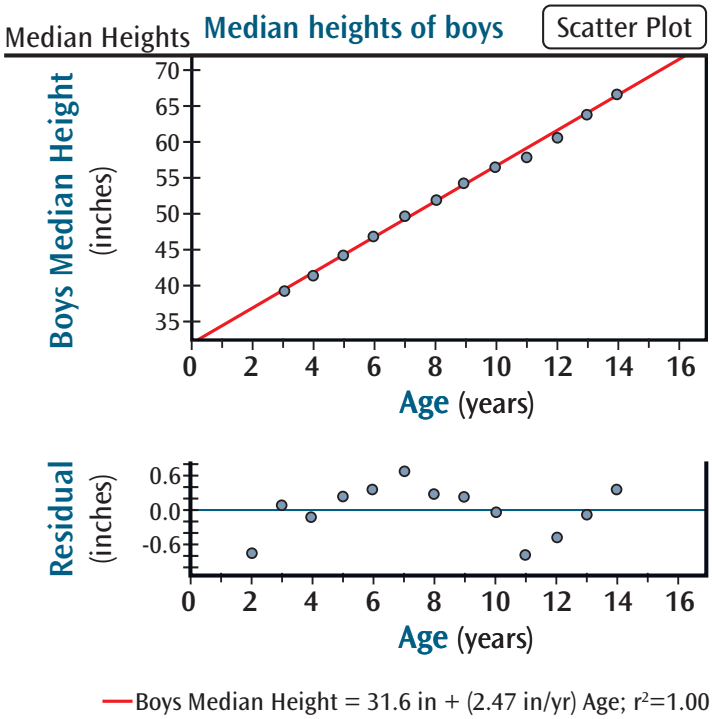
Source: United States Census Bureau 2009 (Statistical Abstract of the United States, Table 201).

Adapted from UA Progressions Documents 2012d, 3.

Students now take a deeper look at bivariate data, using their knowledge of proportions to describe categorical associations and using their knowledge of functions to fit models to quantitative data (**S-ID.5–6**). Students have seen scatter plots in the grade-eight standards and now extend that knowledge to fit mathematical models that capture key elements of the relationship between two variables and to explain what the model indicates about the relationship. Students must learn to take a careful look at scatter plots, as sometimes the "obvious" pattern does not tell the whole story and may be misleading. A line of best fit may appear to fit data almost perfectly, while an examination of the *residuals*—the collection of differences between corresponding coordinates on a least squares line and the actual data value for a variable—may reveal more about the behavior of the data.

Example **S-ID.6b**

Students must learn to look carefully at scatter plots, as sometimes the "obvious" pattern may not tell the whole story and could even be misleading. The graphs below show the median heights of growing boys from the ages of 2 through 14. The line (least squares regression line) with slope 2.47 inches per year of growth looks to be a perfect fit (**S-ID.6c**). However, the residuals—the differences between the corresponding coordinates on the least squares line and the actual data values for each age—reveal additional information. A plot of the residuals shows that growth does not proceed at a constant rate over those years.



Source: Centers for Disease Control and Prevention (CDC) 2002.

Adapted from UA Progressions Documents 2012d, 5.

Finally, students extend their work from topics covered in the grade-eight standards and other topics in Mathematics I to interpret the parameters of a linear model in the context of data that it represents. They compute *correlation coefficients* using technology and interpret the value of the coefficient (**MP.4, MP.5**). Students see situations where correlation and causation are mistakenly interchanged, and they are careful to closely examine the story that data and computed statistics try to tell (**S-ID.7–9**).

Mathematics I Overview

Number and Quantity

Quantities

- Reason quantitatively and use units to solve problems.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Solve systems of equations.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Understand the concept of a function and use function notation.
- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.
- Interpret expressions for functions in terms of the situation they model.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Geometry

Congruence

- Experiment with transformations in the plane.
- Understand congruence in terms of rigid motions.
- Make geometric constructions.

Expressing Geometric Properties with Equations

- Use coordinates to prove simple geometric theorems algebraically.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.
- Summarize, represent, and interpret data on two categorical and quantitative variables.
- Interpret linear models.

Number and Quantity

Quantities N-Q

Reason quantitatively and use units to solve problems. [Foundation for work with expressions, equations, and functions]

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★

Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions. [Linear expressions and exponential expressions with integer exponents]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of* P *and a factor not depending on* P. ★

Creating Equations A-CED

Create equations that describe numbers or relationships. [Linear and exponential (integer inputs only); for **A-CED.3**, linear only]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance* R. ★

M1 Mathematics I

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Master linear; learn as general principle.]

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. [Linear inequalities; literal equations that are linear in the variables being solved for; exponential of a form, such as $2^x = 1/16$.]

3.1 Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA

Solve systems of equations. [Linear systems]

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Represent and solve equations and inequalities graphically. [Linear and exponential; learn as general principle.]

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Functions

Interpreting Functions — F-IF

Understand the concept of a function and use function notation. [Learn as general principle. Focus on linear and exponential (integer domains) and on arithmetic and geometric sequences.]

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$.

Interpret functions that arise in applications in terms of the context. [Linear and exponential (linear domain)]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Linear and exponential]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions F-BF

Build a function that models a relationship between two quantities. [For F-BF.1, 2, linear and exponential (integer inputs)]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

Build new functions from existing functions. [Linear and exponential; focus on vertical translations for exponential.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems. [Linear and exponential]

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★
   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★
   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Interpret expressions for functions in terms of the situation they model. [Linear and exponential of form $f(x)=b^x+k$]

5. Interpret the parameters in a linear or exponential function in terms of a context. ★

Geometry

Congruence G-CO

Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions. [Build on rigid motions as a familiar starting point for development of concept of geometric proof.]

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Make geometric constructions. [Formalize and explain processes.]

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Expressing Geometric Properties with Equations — G-GPE

Use coordinates to prove simple geometric theorems algebraically. [Include distance formula; relate to Pythagorean Theorem.]

4. Use coordinates to prove simple geometric theorems algebraically.
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

Statistics and Probability

Interpreting Categorical and Quantitative Data — S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

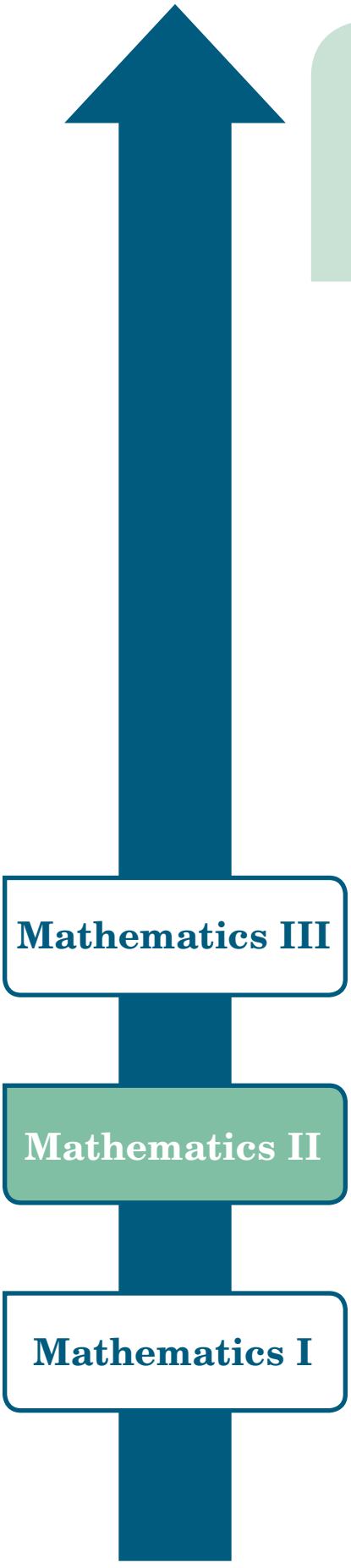
1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★

Summarize, represent, and interpret data on two categorical and quantitative variables. [Linear focus; discuss general principle.]

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
    a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★
    b. Informally assess the fit of a function by plotting and analyzing residuals. ★
    c. Fit a linear function for a scatter plot that suggests a linear association. ★

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★



Mathematics II

The Mathematics II course focuses on quadratic expressions, equations, and functions and on comparing the characteristics and behavior of these expressions, equations, and functions to those of linear and exponential relationships from Mathematics I. The need for extending the set of rational numbers arises, and students are introduced to real and complex numbers. Links between probability and data are explored through conditional probability and counting methods and involve the use of probability and data in making and evaluating decisions. The study of similarity leads to an understanding of right-triangle trigonometry and connects to quadratics through Pythagorean relationships. Circles, with their quadratic algebraic representations, finish out the course. The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the California Common Core State Standards for Mathematics (CA CCSSM); they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics II course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics II

In Mathematics II, students extend the laws of exponents to rational exponents and explore distinctions between rational and irrational numbers by considering their decimal representations. Students learn that when quadratic equations do not have real solutions, the number system can be extended so that solutions exist, analogous to the way in which extending whole numbers to negative numbers allows $x+1=0$ to have a solution. Students explore relationships between number systems: whole numbers, integers, rational numbers, real numbers, and complex numbers. The guiding principle is that equations with no solutions in one number system may have solutions in a larger number system.

Students consider quadratic functions, comparing the key characteristics of quadratic functions to those of linear and exponential functions. They select from these functions to model phenomena. Students learn to anticipate the graph of a quadratic function by interpreting various forms of quadratic expressions. In particular, they identify the real solutions of a quadratic equation as the zeros of a related quadratic function. Students also learn that when quadratic equations do not have real solutions, the graph of the related quadratic function does not cross the horizontal axis. Additionally, students expand their experience with functions to include more specialized functions—absolute value, step, and other piecewise-defined functions.

Students in Mathematics II focus on the structure of expressions, writing equivalent expressions to clarify and reveal aspects of the quantities represented. Students create and solve equations, inequalities, and systems of equations involving exponential and quadratic expressions.

Building on probability concepts introduced in the middle grades, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students use probability to make informed decisions, and they should make use of geometric probability models whenever possible.

Students apply their earlier experience with dilations and proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use similarity to solve problems, and apply similarity in right triangles to understand right-triangle trigonometry, with particular attention to special right triangles and the Pythagorean Theorem. In Mathematics II, students develop facility with geometric proof. They use what they know about congruence and similarity to prove theorems involving lines, angles, triangles, and other polygons. They also explore a variety of formats for writing proofs.

In Mathematics II, students prove basic theorems about circles, chords, secants, tangents, and angle measures. In the Cartesian coordinate system, students use the distance formula to write the equation of a circle when given the radius and the coordinates of its center, and the equation of a parabola with

a vertical axis when given an equation of its horizontal directrix and the coordinates of its focus. Given an equation of a circle, students draw the graph in the coordinate plane and apply techniques for solving quadratic equations to determine intersections between lines and circles, between lines and parabolas, and between two circles. Students develop informal arguments to justify common formulas for circumference, area, and volume of geometric objects, especially those related to circles.

Examples of Key Advances from Mathematics I

Students extend their previous work with linear and exponential expressions, equations, and systems of equations and inequalities to quadratic relationships.

- A parallel extension occurs from linear and exponential functions to quadratic functions: students begin to analyze functions in terms of transformations.
- Building on their work with transformations, students produce increasingly formal arguments about geometric relationships, particularly around notions of similarity.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The Standards for Mathematical Practice represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards.

The CA CCSSM call for an intense focus on the most critical material, allowing depth in learning, which is carried out through the Standards for Mathematical Practice. Connecting content and practices happens in the context of *working on problems,* as is evident in the first MP standard ("Make sense of problems and persevere in solving them"). Table M2-1 offers examples of how students can engage in each mathematical practice in the Mathematics II course.

Table M2-1. Standards for Mathematical Practice—Explanation and Examples for Mathematics II

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students persevere when attempting to understand the differences between quadratic functions and linear and exponential functions studied previously. They create diagrams of geometric problems to help make sense of the problems.
MP.2 Reason abstractly and quantitatively.	Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students construct proofs of geometric theorems based on congruence criteria of triangles. They understand and explain the definition of *radian measure*.
MP.4 Model with mathematics.	Students apply their mathematical understanding of quadratic functions to real-world problems. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MP.5 Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the result.
MP.6 Attend to precision.	Students begin to understand that a *rational number* has a specific definition and that *irrational numbers* exist. When deciding if an equation can describe a function, students make use of the definition of *function* by asking, "Does every input value have exactly one output value?"
MP.7 Look for and make use of structure.	Students apply the distributive property to develop formulas such as $(a \pm b)^2 = a^2 \pm 2ab + b^2$. They see that the expression $5 + (x-2)^2$ takes the form of "5 plus 'something' squared," and therefore that expression can be no smaller than 5.
MP.8 Look for and express regularity in repeated reasoning.	Students notice that consecutive numbers in the sequence of squares 1, 4, 9, 16, and 25 always differ by an odd number. They use polynomials to represent this interesting finding by expressing it as $(n+1)^2 - n^2 = 2n+1$.

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star (★) symbol to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards. Modeling in higher mathematics centers on problems that arise in everyday life, society, and the workplace. Such problems may draw upon mathematical content knowledge and skills articulated in the standards prior to or during the Mathematics II course.

Examples of places where specific Mathematical Practice standards can be implemented in the Mathematics II standards are noted in parentheses, with the standard(s) also listed.

Mathematics II Content Standards, by Conceptual Category

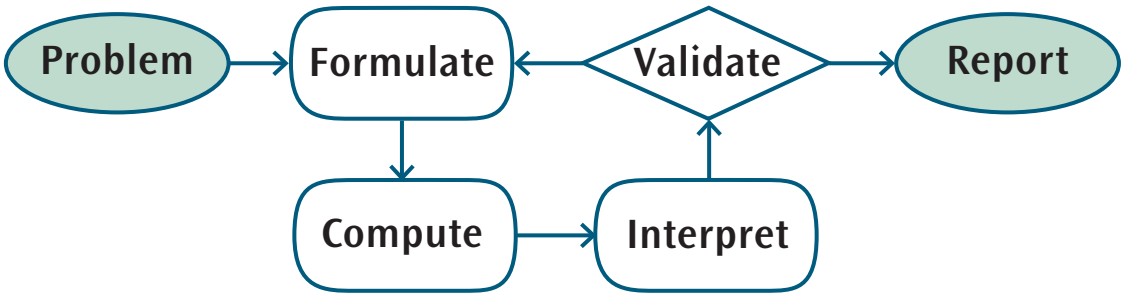
The Mathematics II course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in Mathematics II are described below, according to each conceptual category. Standards that are considered new for secondary-grades teachers are discussed more thoroughly than other standards.

Conceptual Category: Modeling

Throughout the CA CCSSM, specific standards for higher mathematics are marked with a ★ symbol to indicate they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise (e.g., Which of the quantities present in this situation are known, and which are unknown? Can a table of data be made? Is there a functional relationship in this situation?). Students need to decide on a solution path that may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new formula or function will apply. Students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, or the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure M2-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Figure M2-1. The Modeling Cycle



The examples in this chapter are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding quadratic functions, graphing, solving equations, and rates of change are explored through this lens. Readers are encouraged to consult appendix B (Mathematical Modeling) for further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

The standards of the Functions conceptual category can serve as motivation for the study of standards in the other Mathematics II conceptual categories. Students have already worked with equations in which they have to "solve for x" as a search for the input of a function f that gives a specified output; solving the equation amounts to undoing the work of the function. The types of functions that students encounter in Mathematics II have new properties. For example, while linear functions show constant additive change and exponential functions show constant multiplicative change, quadratic functions exhibit a different change and can be used to model new situations. New techniques for solving equations need to be constructed carefully, as extraneous solutions may arise or no real-number solutions may exist. In general, functions describe how two quantities are related in a precise way and can be used to make predictions and generalizations, keeping true to the emphasis on modeling that occurs in higher mathematics. The core question when students investigate functions is, "Does each element of the domain correspond to exactly one element of the range?" (University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2013c, 8).

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Quadratic]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Linear, exponential, quadratic, absolute value, step, piecewise-defined]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y=(1.02)^t$, $y=(0.97)^t$, $y=(1.01)^{12t}$, *and* $y=(1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Standards **F-IF.4–9** deal with understanding the concept of a function, interpreting characteristics of functions in context, and representing functions in different ways (**MP.6**). Standards **F-IF.7–9** call for students to represent functions with graphs and identify key features of the graph. They represent the same function algebraically in different forms and interpret these differences in terms of the graph or context. For instance, students may easily see that the function $f(x)=3x^2+9x+6$ crosses the y-axis at (0,6), since the terms involving x are simply 0 when $x=0$. But then they factor the expression defining f to obtain $f(x)=3(x+2)(x+1)$, revealing that the function crosses the x-axis at (–2,0) and (–1,0) because those points correspond to where $f(x)=0$ (**MP.7**). In Mathematics II, students work with linear, exponential, and quadratic functions and are expected to develop fluency with these types of functions, including the ability to graph them by hand.

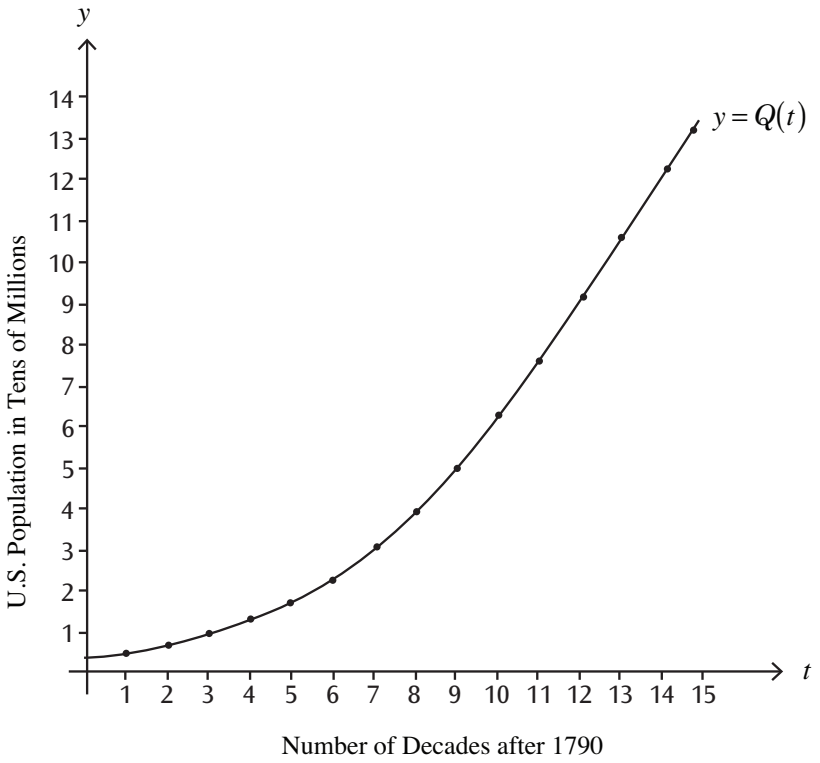
Students work with functions that model data and with choosing an appropriate model function by considering the context that produced the data. Students' ability to recognize rates of change, growth and decay, end behavior, roots, and other characteristics of functions becomes more sophisticated; they use this expanding repertoire of families of functions to inform their choices for models. Standards **F-IF.4–9** focus on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.

Example: Population Growth **F-IF.4–9**

The approximate population of the United States, measured each decade starting in 1790 through 1940, can be modeled with the following function:

$$P(t) = \frac{(3{,}900{,}000 \times 200{,}000{,}000)e^{0.31t}}{200{,}000{,}000 + 3{,}900{,}000(e^{0.31t} - 1)}$$

In this function, t represents the number of decades after 1790. Such models are important for planning infrastructure and the expansion of urban areas, and historically accurate long-term models have been difficult to derive.



Questions:

a. According to this model, what was the population of the United States in the year 1790?

b. According to this model, when did the U.S. population first reach 100,000,000? Explain your answer.

c. According to this model, what should the U.S. population be in the year 2010? Find the actual U.S. population in 2010 and compare with your result.

d. For larger values of t, such as $t = 50$, what does this model predict for the U.S. population? Explain your findings.

Example: Population Growth *(continued)* **F-IF.4–9**

Solutions:

a. The population in 1790 is given by $P(0)$, which is easily found to be 3,900,000 because $e^{0.31(0)} = 1$.

b. This question asks students to find t such that $P(t) = 100{,}000{,}000$. Dividing the numerator and denominator on the left by 100,000,000 and dividing both sides of the equation by 100,000,000 simplifies this equation to

$$\frac{3.9 \times 2 \times e^{0.31t}}{200 + 3.9\left(e^{0.31t} - 1\right)} = 1.$$

Algebraic manipulation and solving for t result in $t \approx \frac{1}{0.31} \ln 50.28 \approx 12.64$. This means that after 1790, it would take approximately 126.4 years for the population to reach 100 million.

c. Twenty-two (22) decades after 1790, the population would be approximately 190,000,000, which is far less (by about 119,000,000) than the estimated U.S. population of 309,000,000 in 2010.

d. The structure of the expression reveals that for very large values of t, the denominator is dominated by $3{,}900{,}000e^{0.31t}$. Thus, for very large values of t,

$$P(t) \approx \frac{3{,}900{,}000 \times 200{,}000{,}000 \times e^{0.31t}}{3{,}900{,}000e^{0.31t}} = 200{,}000{,}000.$$

Therefore, the model predicts a population that stabilizes at 200,000,000 as t increases.

Adapted from Illustrative Mathematics 2013m.

Building Functions F-BF

Build a function that models a relationship between two quantities. [Quadratic and exponential]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. ★

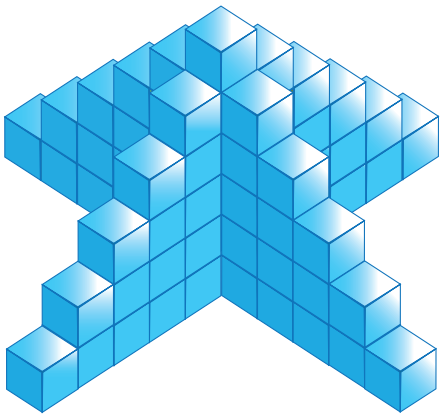
Build new functions from existing functions. [Quadratic, absolute value]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2x^3$.

Students in Mathematics II develop models for more complex or sophisticated situations than in previous courses because the types of functions available to them have expanded (**F-BF.1**). The following example illustrates the type of reasoning with functions that students are expected to develop in standard **F-BF.1.**

Example: The Skeleton Tower **F-BF.1**

The tower shown at right measures 6 cubes high.



a. How many cubes are needed to build this tower? (Organize your counting so other students can follow your reasoning.)

b. How many cubes would be needed to build a tower just like this one, but 12 cubes high? Justify your reasoning.

c. Find a way to calculate the number of cubes needed to build a similar tower that is n cubes high.

Solution:

a. The top layer has a single cube. The layer below has one cube beneath the top cube, plus 4 new ones, making a total of 5. The next layer has cubes below these 5, plus 4 new ones, to make 9. Continuing to add 4 each time gives a total of $1+5+9+13+17+21=66$ cubes in the skeleton tower with 6 layers.

b. Building upon the reasoning established in (a), the number of cubes in the bottom (12th) layer will be $1+4\times11$, since it is 11 layers below the top. So for this total, students need to add $1+5+9+\ldots+45$. One way to do this would be to add the numbers. Another method is the Gauss method: Rewrite the sum backward as $45+41+37+\ldots+1$. Now if this sum is placed below the previous sum, students can see that each pair of addends, one above the other, sums to 46. There are 12 columns, so the answer to this problem is half of $12\times46=552$, or 276.

c. Let $f(n)$ be the number of cubes in the nth layer counting down from the top. Then $f(1)=1$, $f(2)=5$, $f(3)=9$, and so on. In general, because each term is obtained from the previous one by adding 4, $f(n)=4(n-1)+1$. Therefore, the total for n layers in the tower is $1+5+9+\ldots+f(n)=1+5+9+\ldots+(4(n-1)+1)$. If the method from solution (b) is used here, twice this sum will be equal to $n\bullet(4(n-1)+2)$, so the general solution for the number of cubes in a skeleton tower with n layers is $\frac{n(4(n-1)+2)}{2}=n(2n-1)$.

Note: For an alternative solution, visit https://www.illustrativemathematics.org/.

Adapted from Illustrative Mathematics 2013j.

For standard **F-BF.3,** students can make good use of graphing software to investigate the effects of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for different types of functions. For example, starting with the simple quadratic function $f(x)=x^2$, students see the relationship between the transformed functions $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ and the vertex form of a general quadratic, $f(x)=a(x-h)^2+k$. They understand the notion of a *family of functions* and characterize such

function families based on the properties of those families. In keeping with the theme of the input–output interpretation of a function, students should work toward developing an understanding of the effect on the output of a function under certain transformations, such as in the following table.

Expression	Interpretation
$f(a+2)$	The output when the input is 2 greater than a
$f(a)+3$	3 more than the output when the input is a
$2f(x)+5$	5 more than twice the output of f when the input is x

Such understandings can help students see the effect of transformations on the graph of a function and, in particular, they can help students comprehend that the effect on the graph is the opposite to the transformation on the variable. For example, the graph of $y = f(x+2)$ is the graph of f shifted 2 units to the left, not to the right (UA Progressions Documents 2013c, 7). These ideas are explored further with trigonometric functions (**F-TF.5**) in Mathematics III.

In standard **F-BF.4a,** students learn that some functions have the property that an input can be recovered from a given output—as with the equation $f(x) = c$, which can be solved for x given that c lies in the range of f. For example, a student might solve the equation $F = \frac{9}{5}C + 32$ for C. The student starts with this formula, showing how Fahrenheit temperature is a function of Celsius temperature, and by solving for C finds the formula for the inverse function. This is a contextually appropriate way to find the expression for an inverse function, in contrast with the practice of simply swapping x and y in an equation and solving for y.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems. [Include quadratic]

3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Interpret expressions for functions in terms of the situation they model.

6. Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity. CA★

In Mathematics II, students continue their investigation of exponential functions by comparing them with linear and quadratic functions, observing that exponential functions will always grow larger than any polynomial function. Standard **F-LE.6** calls for students to experiment with quadratic functions and discover how these functions can represent real-world phenomena such as projectile motion. A simple activity that involves tossing a ball and making a video recording of its height as it rises and falls can reveal that the height, as a function of time, is approximately quadratic. Afterward, students can derive a quadratic expression that determines the height of the ball at time t using a graphing calculator or other software, and they can compare the values of the function with their data.

Trigonometric Functions — F-TF

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta)+\cos^2(\theta)=1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Standard **F-TF.8** is closely linked with standards **G-SRT.6–8,** but it is included here as a property of the trigonometric functions sine, cosine, and tangent. Students use the Pythagorean identity to find the output of a trigonometric function at given angle θ when the output of another trigonometric function is known.

Conceptual Category: Number and Quantity

The Real Number System — N-RN

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{1/3}$ *to be the cube root of 5 because we want* $\left(5^{1/3}\right)^3=5^{(1/3)^3}$ *to hold, so* $\left(5^{1/3}\right)^3$ *must equal 5.*

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a non-zero rational number and an irrational number is irrational.

In grade eight, students encountered some examples of irrational numbers, such as π and $\sqrt{2}$ (or $\sqrt{p}$ for p as a non-square number). In Mathematics II, students extend this understanding beyond the fact that there are numbers that are not rational; they begin to understand that rational numbers form a closed system. Students have witnessed that, with each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, and real numbers—the distributive law continues to hold, and the commutative and associative laws are still valid for both addition and multiplication. However, in Mathematics II, students go further along this path. For example, with standard **N-RN.3**, students may explain that the sum or product of two rational numbers is rational by arguing that the sum of two fractions with integer numerator and denominator is also a fraction of the same type, showing that the rational numbers are *closed* under the operations of addition and multiplication (**MP.3**). Moreover, they argue that the sum of a rational and an irrational is irrational, and the product of a non-zero rational and an irrational is still irrational, showing that irrational numbers are truly an additional set of numbers that, along with rational numbers, form a larger *system:* real numbers (**MP.3, MP.7**).

Standard **N-RN.1** calls for students to make meaning of the representation of radicals with rational exponents. Students were first introduced to exponents in grade six; by the time they reach Mathematics II, they should have an understanding of the basic properties of exponents—for example, that

$$x^n \cdot x^m = x^{n+m},\ \left(x^n\right)^m = x^{nm},\ \frac{x^n}{x^m} = x^{n-m},\ x^0 = 1 \text{ for } x \neq 0, \text{ and so on.}$$

In fact, students may have justified certain properties of exponents by reasoning about other properties (**MP.3, MP.7**). For example, they may have justified why any non-zero number to the power 0 is equal to 1:

$$x^0 = x^{n-n} = \frac{x^n}{x^n} = 1, \text{ for } x \neq 0$$

Students in Mathematics II further their understanding of exponents by using these properties to explain the meaning of rational exponents. For example, properties of whole-number exponents suggest that $\left(5^{1/3}\right)^3$ should be the same as $5^{[(1/3)\cdot 3]} = 5^1 = 5$, so that $5^{1/3}$ should represent the cube root of 5. In addition, the fact that $(ab)^n = a^n \cdot b^n$ reveals that

$$\sqrt{20} = (4 \cdot 5)^{1/2} = 4^{1/2} \cdot 5^{1/2} = 2\sqrt{5}.$$

This shows that $\sqrt{20} = 2\sqrt{5}$. The intermediate steps of writing the square root as a rational exponent are not entirely necessary here, but the principle of how to work with radicals based on the properties of exponents is. Students extend such work with radicals and rational exponents to variable expressions as well (**N-CN.2**); for example, they rewrite an expression such as $\left(a^2b^5\right)^{3/2}$ by using radicals (**N-RN.2**).

The Complex Number System	**N-CN**

Perform arithmetic operations with complex numbers. [i^2 as highest power of i]

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations. [Quadratics with real coefficients]

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

In Mathematics II, students work with examples of quadratic functions and solve quadratic equations, encountering situations in which a resulting equation does not have a solution that is a real number (e.g., $(x-2)^2 = -25$). Students expand their extension of the concept of *number* to include complex numbers, numbers of the form $a + bi$, where i is a number with the property that $i^2 = -1$, so that such

an equation can be solved. They begin to work with complex numbers, first by finding simple square roots of negative numbers: $\sqrt{-25} = \sqrt{25 \cdot (-1)} = \sqrt{25} \cdot (-1) = 5i$ (**MP.7**). They also apply their understanding of properties of operations (the commutative, associative, and distributive properties) and exponents and radicals to solve equations such as those mentioned above:

$$(x-2)^2 = -25, \text{ which implies } |x-2| = 5i, \text{ or } x = 2 \pm 5i$$

Now equations such as these have solutions, and the extended number system forms yet another system that behaves according to certain rules and properties (**N-CN.1–2; 7–9**). Additionally, by exploring examples of polynomials that can be factored with real and complex roots, students develop an understanding of the Fundamental Theorem of Algebra; they can show that this theorem is true for a quadratic polynomial by applying the quadratic formula and understanding the relationship between roots of a quadratic equation and the linear factors of the quadratic expression (**MP.2**).

Conceptual Category: Algebra

Students began their work with expressions and equations in grades six through eight and extended their work to more complex expressions in Mathematics I. In Mathematics II, students encounter quadratic expressions for the first time and learn a new set of strategies for working with these expressions. As in Mathematics I, the Algebra conceptual category is closely tied to the Functions conceptual category, linking the writing of equivalent expressions, solving equations, and graphing to concepts involving functions.

Seeing Structure in Expressions — A-SSE

Interpret the structure of expressions. [Quadratic and exponential]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* ★
2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

Write expressions in equivalent forms to solve problems. [Quadratic and exponential]

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★
   a. Factor a quadratic expression to reveal the zeros of the function it defines. ★
   b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. ★
   c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression 1.15^t can be rewritten as $\left(1.15^{1/12}\right)^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* ★

In Mathematics II, students extend their work with expressions to include examples of more complicated expressions, such as those that involve multiple variables and exponents. Students use the distributive property to investigate equivalent forms of quadratic expressions; for example, they write

$$\begin{aligned}(x+y)(x-y) &= x(x-y)+y(x-y)\\ &= x^2-xy+xy-y^2\\ &= x^2-y^2.\end{aligned}$$

This yields a special case of a factorable quadratic: the difference of squares. Students factor second-degree polynomials and simple third-degree polynomials by making use of such special forms and by using factoring techniques based on properties of operations—for example, factoring by grouping, which arises from the distributive property (**A-SSE.2**). Note that the standards do not mention "simplification," because it is not always clear what the simplest form of an expression is, and even in cases where it is clear, it is not obvious that the simplest form is desirable for a given purpose. The standards emphasize purposeful transformation of expressions into equivalent forms that are suitable for the purpose at hand (UA Progressions Documents 2013b, 5), as the following example shows.

Example: Which Is the Simpler Form? **A-SSE.2**

A particularly rich mathematical investigation involves finding a general expression for the sum of the first n consecutive natural numbers:

$$S=1+2+3+\ldots+(n-2)+(n-1)+n.$$

A famous tale speaks of a young C. F. Gauss being able to add the first 100 natural numbers quickly in his head, wowing his classmates and teachers alike. One way to find this sum is to consider the "reverse" of the sum:

$$S=n+(n-1)+(n-2)+\ldots+3+2+1$$

Then the two expressions for S are added together:

$$2S=(n+1)+(n+1)+\ldots+(n+1)+(n+1)+(n+1),$$

where there are n terms of the form $(n+1)$. Thus, $2S=n(n+1)$, so that $S=\frac{n(n+1)}{2}$.

While students may be tempted to transform this expression into $\frac{1}{2}n^2+\frac{1}{2}n$, they are obscuring the ease with which they can evaluate the first expression. Indeed, since n is a natural number, one of either n or $n+1$ is even, so evaluating $\frac{n(n+1)}{2}$, especially mentally, is often easier. In Gauss's case, $\frac{100(101)}{2}=50(101)=5050$.

Students also use different forms of the same expression to reveal important characteristics of the expression. For instance, when working with quadratics, they complete the square in the expression x^2-3x+4 to obtain the equivalent expression $\left(x-\frac{3}{2}\right)^2+\frac{7}{4}$. Students can then reason with the new expression that the term being squared is always greater than or equal to 0; hence, the value of the expression will always be greater than or equal to $\frac{7}{4}$ (**A-SSE.3, MP.3**). A spreadsheet or a computer algebra system may be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave, further contributing to students' understanding of work with expressions (**MP.5**).

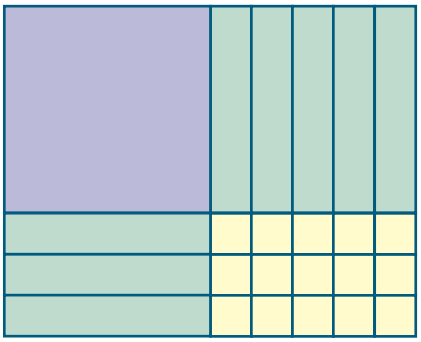
Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials. [Polynomials that simplify to quadratics]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

To perform operations with polynomials meaningfully, students are encouraged to draw parallels between the set of integers and the set of all polynomials with real coefficients (**A-APR.1, MP.7**). Manipulatives such as "algebra tiles" may be used to support understanding of addition and subtraction of polynomials and the multiplication of monomials and binomials. Algebra tiles may be used to offer a concrete representation of the terms in a polynomial (**MP.5**). The tile representation relies on the area interpretation of multiplication: the notion that the product ab can be thought of as the area of a rectangle of dimensions a units and b units. With this understanding, tiles can be used to represent 1 square unit (a 1 by 1 tile), x square units (a 1 by x tile), and x^2 square units (an x by x tile). Finding the product $(x+5)(x+3)$ amounts to finding the area of an abstract rectangle of dimensions $(x+5)$ and $(x+3)$, as illustrated in figure M2-2 (**MP.2**).

Figure M2-2. Algebra Tiles



The rectangle above has height $(x+3)$ and base $(x+5)$. The total area represented, the product of these binomials, is seen to be $x^2+5x+3x+15=x^2+8x+15$.

Care must be taken in the way negative numbers are handled with this representation, since, as with all models, there are potential limitations to connecting the mathematics to the representation. The tile representation of polynomials can support student understanding of the meaning of multiplication of variable expressions, and it is very useful for understanding the notion of completing the square (described later in this chapter).

Creating Equations — A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★ [Include formulas involving quadratic terms.]

In Mathematics II, students work with all available types of functions to create equations, including quadratic functions, absolute value functions, and simple rational and exponential functions (**A-CDE.1**). Although the functions used for standards **A-CED.1, 2,** and **4** will often be linear, exponential, or

quadratic, the types of problems should draw from more complex situations than those addressed in Mathematics I. Note that students are not required to study rational functions in Mathematics II.

Reasoning with Equations and Inequalities — A-REI

Solve equations and inequalities in one variable. [Quadratics with real coefficients]

4. Solve quadratic equations in one variable.
   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations. [Linear-quadratic systems]

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.*

Students in Mathematics II extend their work with exponents to include quadratic functions and equations. To extend their understanding of these quadratic expressions and the functions defined by such expressions, students investigate properties of quadratics and their graphs in the Functions domain. It may be best to present the solving of quadratic equations in the context of functions. For instance, if the equation $h(t) = -16t^2 + 50t + 150$ defines the height of a projectile launched with an initial velocity of 50 ft/s from a height of 150 ft, then asking at which time t the object hits the ground is asking for which t is found at $h(t) = 0$. That is, students now need to solve the equation $-16t^2 + 50t + 150 = 0$ and require new methods for doing so. Students have investigated how to "undo" linear and simple exponential functions in Mathematics I; now they do so for quadratic functions and discover that the process is more complex.

Example — **A-REI.4a**

When solving quadratic equations of the form $(x-p)^2 = q$, students rely on the understanding that they can take square roots of both sides of the equation to obtain the following:

$$\sqrt{(x-p)^2} = \sqrt{q} \qquad (1)$$

In the case that $\sqrt{q}$ is a real number, this equation can be solved for x. A common mistake is to quickly introduce the $\pm$ symbol here, without understanding where the symbol comes from. Doing so without care often leads students to think that $\sqrt{9} = \pm 3$, for example.

Note that the quantity $\sqrt{a^2}$ is simply a when $a \geq 0$ (as in $\sqrt{5^2} = \sqrt{25} = 5$), while $\sqrt{a^2}$ is equal to $-a$ (the opposite of a) when $a < 0$ (as in $\sqrt{(-4)^2} = \sqrt{16} = 4$). But this means that $\sqrt{a^2} = |a|$. Applying this to equation (1) yields $|x-p| = \sqrt{q}$. Solving this simple absolute value equation yields that $x - p = \sqrt{q}$ or $-(x-p) = \sqrt{q}$. This results in the two solutions $x = p + \sqrt{q}$, $p - \sqrt{q}$.

Students also transform quadratic equations into the form $ax^2+bx+c=0$ for $a\neq 0$, which is the *standard form* of a quadratic equation. In some cases, the quadratic expression factors nicely and students can apply the zero product property of the real numbers to solve the resulting equation. The *zero product property* states that for two real numbers m and n, $m\bullet n=0$ if and only if either $m=0$ or $n=0$. Hence, when a quadratic polynomial can be rewritten as $a(x-r)(x-s)=0$, the solutions can be found by setting each of the linear factors equal to 0 separately, and obtaining the solution set $\{r, s\}$. In other cases, a means for solving a quadratic equation arises by *completing the square*. Assuming for simplicity that $a=1$ in the standard equation above, and that the equation has been rewritten as $x^2+bx=-c$, we can "complete the square" by adding the square of half the coefficient of the x-term to each side of the equation:

$$x^2+bx+\left(\frac{b}{2}\right)^2=-c+\left(\frac{b}{2}\right)^2 \qquad (2)$$

The result of this simple step is that the quadratic on the left side of the equation is a perfect square, as shown here:

$$x^2+bx+\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)^2$$

Thus, we have now converted equation (2) into an equation of the form $(x-p)^2=q$:

$$\left(x+\frac{b}{2}\right)^2=-c+\frac{b^2}{4}$$

This equation can be solved by the method described above, as long as the term on the right is non-negative. When $a\neq 1$, the case can be handled similarly and ultimately results in the familiar quadratic formula. Tile representations of quadratics illustrate that the process of completing the square has a geometric interpretation that explains the origin of the name. Students should be encouraged to explore these representations in order to make sense out of the process of completing the square (**MP.1, MP.5**).

Example: Completing the Square **A-REI.4a**

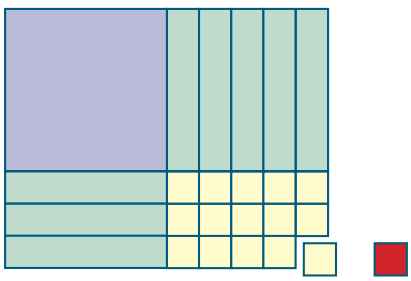
The method of completing the square is a useful skill in algebra. It is generally used to change a quadratic in standard form, ax^2+bx+c, into one in vertex form, $a(x-h)^2+k$. The vertex form can help determine several properties of quadratic functions. Completing the square also has applications in geometry (**G-GPE.1**) and later higher mathematics courses.

To complete the square for the quadratic $y=x^2+8x+15$, half the coefficient of the x-term is squared to yield 16. Then students realize that they need to add 1 and subtract 1 to the quadratic expression:

$$\begin{aligned} y&=x^2+8x+15+1-1\\ &=x^2+8x+16-1. \end{aligned}$$

Factoring yields $y=(x+4)^2-1$.

In the picture at right, note that the tiles used to represent $x^2+8x+15$ have been rearranged to try to form a square, and that a positive unit tile and a negative unit tile are added into the picture to "complete the square."



Conceptual Category: Geometry

In Mathematics I, students began to formalize their understanding of geometry by defining congruence in terms of well-defined rigid motions of the plane. They found that congruence could be deduced in certain cases by investigating other relationships (e.g., that for triangles, the ASA, SAS, and SSS congruence criteria held). In Mathematics II, students further enrich their ability to reason deductively and begin to write more formal proofs of various geometric results. They also apply to triangles their knowledge of similarity and discover powerful relationships in right triangles, leading to the discovery of trigonometric functions. Finally, students' understanding of the Pythagorean relationship and their work with quadratics leads to algebraic representations of circles and more complex proofs of results in the plane.

Congruence G-CO

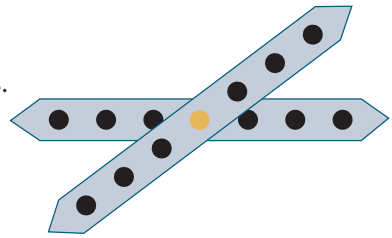
Prove geometric theorems. [Focus on validity of underlying reasoning while using variety of ways of writing proofs.]

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Students prove the congruence criteria for triangles (ASA, SAS, and SSS) with the more basic notion of congruence by rigid motions. Instructors are encouraged to use a variety of strategies to engage students in understanding and writing proofs. Strategies include using numerous pictures to demonstrate results; using patty paper, transparencies, or dynamic geometry software to explore the relationships in a proof; creating flowcharts and other organizational diagrams for outlining a proof; and writing step-by-step or paragraph formats for a completed proof (**MP.5**). Above all else, instructors should emphasize the reasoning involved in connecting one step in the logical argument to the next. Students should be encouraged to make conjectures based on experimentation, to justify their conjectures, and to communicate their reasoning to their peers (**MP.3**). Such reasoning, justification, and communication in precise language are central to geometry instruction and should be emphasized.

Example: The Kite Factory **G-CO.11**

Kite engineers want to know how the shape of a kite—the length of the rods, where they are attached, the angle at which the rods are attached, and so on—affects how the kite flies. In this activity, students are given pieces of cardstock of various lengths, hole-punched at regular intervals so they can be attached in different places.



These two "rods" form the frame for a kite at the kite factory. By changing the angle at which the sticks are held and the places where the sticks are attached, students discover different properties of quadrilaterals.

Students are challenged to make conjectures and use precise language to describe their findings about which diagonals result in which quadrilaterals. They can discover properties unique to certain quadrilaterals, such as the fact that diagonals that are perpendicular bisectors of each other imply the quadrilateral is a rhombus. To see videos of this lesson being implemented in a high school classroom, visit http://www.insidemathematics.org/ (accessed March 26, 2015).

Similarity, Right Triangles, and Trigonometry G-SRT

Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle–Angle (AA) criterion for two triangles to be similar.

Prove theorems involving similarity. [Focus on validity of underlying reasoning while using variety of formats.]

4. Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

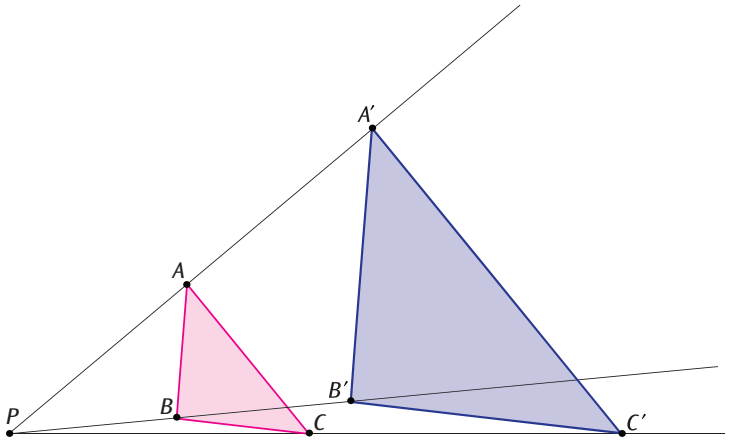
Because right triangles and triangle relationships play such an important role in applications and future mathematics learning, they are given a prominent role in the Geometry conceptual category. A discussion of *similarity* is necessary first—and again, a more precise mathematical definition of similarity is given in the higher mathematics standards. Students worked with *dilations* as a transformation in the grade-eight standards; now they explore the properties of dilations in more detail and develop an understanding of the notion of *scale factor* (**G-SRT.1**). Whereas it is common to say that objects that

are similar have "the same shape," the new definition for two objects being similar is that there is a sequence of *similarity transformations*—translation, rotation, reflection, or dilation—that maps one object exactly onto the other. Standards **G-SRT.2** and **G-SRT.3** call for students to explore the consequences of two triangles being similar: that they have congruent angles and that their side lengths are in the same proportion. This new understanding gives rise to more results that are encapsulated in standards **G-SRT.4** and **G-SRT.5.**

Example: Experimenting with Dilations **G-SRT.2–3**

Students are given opportunities to experiment with dilations and determine how they affect planar objects. Students first make sense of the definition of a *dilation of scale factor* $k > 0$ *with center* P as the transformation that moves a point A along the ray $\overrightarrow{PA}$ to a new point A', so that $\left|\overline{PA'}\right| = k \bullet \left|\overline{PA}\right|$. For example, using a ruler, students apply the dilation of scale factor 2.5 with center P to the points A, B, and C illustrated below. Once this is done, the students consider the two triangles ΔABC and $\Delta A'B'C'$, and they discover that the lengths of the corresponding sides of the triangles have the same ratio dictated by the scale factor (**G-SRT.2**).

Students learn that parallel lines are taken to parallel lines by dilations; thus corresponding segments of ΔABC and $\Delta A'B'C'$ are parallel. After students have proved results about parallel lines intersected by a transversal, they can deduce that the angles of the triangles are congruent. Through experimentation, they see that the congruence of corresponding angles is a necessary and sufficient condition for the triangles to be similar, leading to the AA criterion for triangle similarity (**G-SRT.3**).



Similarity, Right Triangles, and Trigonometry G-SRT

Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★

8.1 Derive and use the trigonometric ratios for special right triangles (30°, 60°, 90° and 45°, 45°, 90°). CA

Once the angle–angle (AA) similarity criterion for triangles is established, it follows that any two *right* triangles ΔABC and ΔDEF with at least one pair of angles congruent (say $\angle A \cong \angle D$) is similar, since the right angles are obviously congruent (say $\angle B \cong \angle E$). By similarity, the corresponding sides of the triangles are in proportion:

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Notice the first and third expressions in the statement of equality above can be rearranged to yield that

$$\frac{AB}{AC} = \frac{DE}{DF}$$

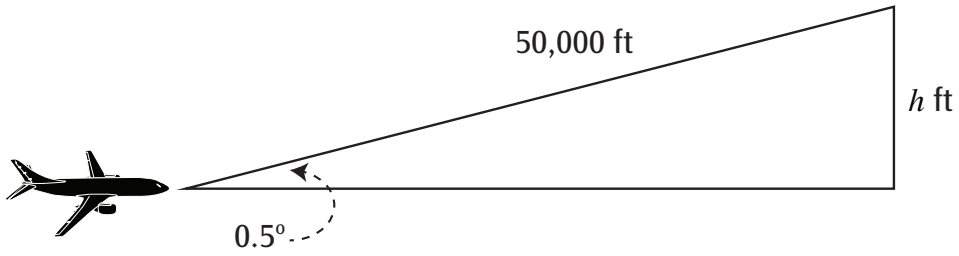
Since the triangles in question are arbitrary, this implies that for any right triangle with an angle congruent to $\angle A$, the ratio of the side adjacent to $\angle A$ and the hypotenuse of the triangle is a certain constant. This allows us to unambiguously define the *sine* of $\angle A$, denoted by $\sin A$, as this ratio. In this way, students come to understand the trigonometric functions as relationships completely determined by angles (**G-SRT.6**). They further their understanding of these functions by investigating relationships between sine, cosine, and tangent; by exploring the relationship between the sine and cosine of complementary angles; and by applying their knowledge of right triangles to real-world situations (**MP.4**), such as in the example below (**G-SRT.6–8**). Experience working with many different triangles, finding their measurements, and computing ratios of the measurements will help students understand the basics of the trigonometric functions.

Example: Using Trigonometric Relationships **G-SRT.6–8**

Airplanes that travel at high speeds and low elevations often have onboard radar systems to detect possible obstacles in their path. The radar can determine the range of an obstacle and the angle of elevation to the top of the obstacle. Suppose that the radar detects a tower that is 50,000 feet away, with an angle of elevation of 0.5 degrees. By how many feet must the plane rise in order to pass above the tower?

Solution: The sketch below shows that there is a right triangle with a hypotenuse of 50,000 (ft) and smallest angle 0.5 (degrees). To find the side opposite this angle, which represents the minimum height the plane should rise, students would use

$$\sin 0.5^\circ = \frac{h}{50{,}000}, \text{ so that } h = (50{,}000 \text{ ft}) \sin 0.5^\circ \approx 436.33 \text{ ft}.$$



Circles G-C

Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.
4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles. [Radian introduced only as unit of measure]

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. **Convert between degrees and radians. CA**

Students can extend their understanding of the usefulness of similarity transformations by investigating circles (**G-C.1**). For instance, students can reason that any two circles are similar by describing precisely how to transform one into the other.

Example **G-C.1**

Students can show that the two circles C and D given by the equations below are similar.

$$C: (x-1)^2+(y-4)^2=9$$

$$D: (x+2)^2+(y-1)^2=25$$

Solution: The centers of the circles are $(1,4)$ and $(-2,1)$, respectively, so the first step is to translate the center of circle C to the center of circle D using the translation $T(x,y)=(x-3,y-3)$. The second and final step is to dilate from the point $(-2,1)$ by a scale factor of $\frac{5}{3}$, since the radius of circle C is 3 and the radius of circle D is 5.

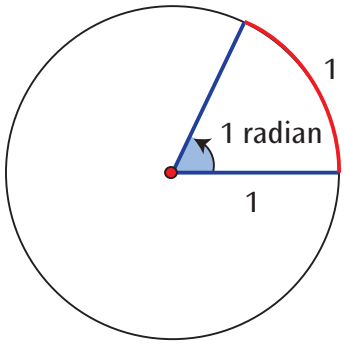
Students continue investigating properties of circles and relationships among angles, radii, and chords (**G-C.2, 3, 4**).

Another important application of the concept of similarity is the definition of the radian measure of an angle. Students can derive this result in the following way: given a sector of a circle C of radius r and central angle α, and a sector of a circle D of radius s and central angle also α, it stands to reason that because these sectors are similar,

$$\frac{\text{length of arc on circle } C}{r}=\frac{\text{length of arc on circle } D}{s}.$$

Therefore, as with the definition of trigonometric functions, there is a constant m such that for an arc subtended by an angle α on any circle:

$$\frac{\text{length of arc subtended by angle } \alpha}{\text{radius of the circle}} = m.$$



This constant of proportionality is the *radian measure* of angle α. It follows that an angle that subtends an arc on a circle that is the same length as the radius measures 1 radian. By investigating circles of different sizes, using string to measure arcs subtended by the same angle, and finding the ratios described above, students can apply their proportional-reasoning skills to discover this constant ratio, thereby developing an understanding of the definition of *radian measure.*

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).* [Include simple circle theorems.]
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

The largest intersection of algebraic and geometric concepts occurs here, wherein two-dimensional shapes are represented on a coordinate system and can be described using algebraic equations and inequalities. Readers will be familiar with the derivation of the equation of a circle by the Pythagorean Theorem and the definition of a circle (**G-GPE.1**): given that a circle consists of all points (x, y) that are at a distance $r > 0$ from a fixed center (h, k), students see that $\sqrt{(x-h)^2+(y-k)^2} = r$ for any point lying on the circle, so that $(x-h)^2+(y-k)^2 = r^2$ determines this circle. Students can derive this equation and flexibly change an equation into this form by completing the square as necessary. By understanding the derivation of this equation, students develop a clear meaning of the variables h, k, and r. Standard **G-GPE.2** calls for students to do the same for the definition of a parabola in terms of a focus and directrix. Numerous resources are available for application problems involving parabolas and should be explored to connect the geometric and algebraic aspects of these curves.

Geometric Measurement and Dimension G-GMD

Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

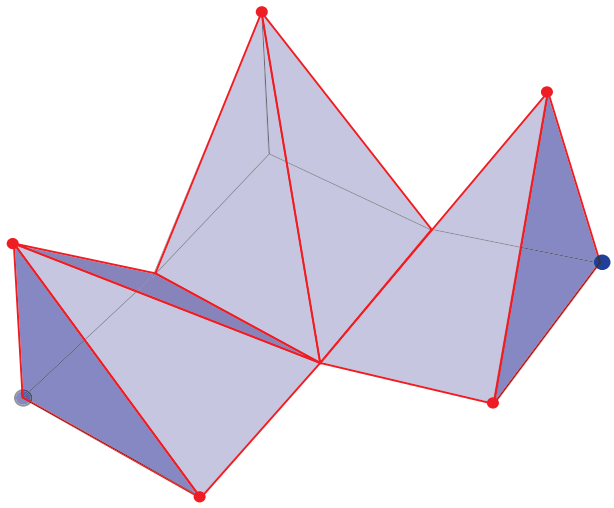
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★

5. **Know that the effect of a scale factor k greater than zero on length, area, and volume is to multiply each by k, k^2, and k^3, respectively; determine length, area, and volume measures using scale factors. CA★**

6. **Verify experimentally that in a triangle, angles opposite longer sides are larger, sides opposite larger angles are longer, and the sum of any two side lengths is greater than the remaining side length; apply these relationships to solve real-world and mathematical problems. CA**

The ability to visualize two- and three-dimensional shapes is a useful skill. This group of standards addresses that skill and includes understanding and using volume and area formulas for curved objects. Students also have the opportunity to make use of the notion of a *limiting process*—an idea that plays a large role in calculus and advanced mathematics courses—when they investigate the formula for the area of a circle. By experimenting with grids of finer and finer mesh, students can repeatedly approximate the area of a unit circle and thereby get a better and better approximation for the irrational number π. They also dissect shapes and make arguments based on these dissections. For instance, as shown in figure M2-3 below, a cube can be dissected into three congruent pyramids, which can lend weight to the formula that the volume of a pyramid of base area B and height h is $\frac{1}{3}Bh$ (**MP.2**).

Figure M2-3. Three Congruent Pyramids That Form a Cube



Source: Park City Mathematics Institute 2013.

Conceptual Category: Statistics and Probability

In grades seven and eight, students learned some basics of probability, including chance processes, probability models, and sample spaces. In higher mathematics, the relative frequency approach to probability is extended to conditional probability and independence, rules of probability and their use in finding probabilities of compound events, and the use of probability distributions to solve problems involving expected value (UA Progressions Documents 2012d, 13). Building on probability concepts that were developed in grades six through eight, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students should make use of geometric probability models wherever possible. They use probability to make informed decisions (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010a).

Conditional Probability and the Rules of Probability — S-CP

Understand independence and conditional probability and use them to interpret data. [Link to data from simulations or experiments.]

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
2. Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B. ★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. ★

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(B|A)$, and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

To develop student understanding of conditional probability, students should experience two types of problems: those in which the uniform probabilities attached to outcomes lead to independence of the outcomes, and those in which they do not (**S-CP.1–3**). The following examples illustrate these two distinct possibilities.

Example: Guessing on a True–False Quiz **S-CP.1–3**

If there are four true-or-false questions on a quiz, then the possible outcomes based on guessing on each question may be arranged as in the table below:

Possible outcomes: Guessing on four true–false questions

Number correct	Out-comes	Number correct	Out-comes	Number correct	Out-comes
4	CCCC	2	CCII	1	CIII
3	ICCC	2	CICI	1	ICII
3	CICC	2	CIIC	1	IICI
3	CCIC	2	ICCI	1	IIIC
3	CCCI	2	ICIC	0	IIII
		2	IICC		

C indicates a correct answer; I indicates an incorrect answer.

By counting outcomes, one can find various probabilities. For example:

$$P(C \text{ on first question}) = \frac{1}{2}$$

and

$$P(C \text{ on second question}) = \frac{1}{2}$$

Noticing that $P\left[(C \text{ on first}) \text{ AND } (C \text{ on second})\right] = \frac{4}{16} = \frac{1}{2} \cdot \frac{1}{2}$ shows that the two events—getting the first question correct and the second question correct—are independent.

Adapted from UA Progressions Documents 2012d.

Example: Work-Group Leaders **S-CP.1–3**

Suppose a five-person work group consisting of three girls (April, Briana, and Cyndi) and two boys (Daniel and Ernesto) wants to randomly choose two people to lead the group. The first person is the discussion leader and the second is the recorder, so order is important in selecting the leadership team. In the table below, "A" represents April, "B" represents Briana, "C" represents Cyndi, "D" represents Daniel, and "E" represents Ernesto. There are 20 outcomes for this situation:

Selecting two students from three girls and two boys

Number of girls	Outcomes	
2	AB	BA
2	AC	CA
2	BC	CB
1	AD	DA
1	AE	EA
1	BD	DB
1	BE	EB
1	CD	DC
1	CE	EC
0	DE	ED

Notice that the probability of selecting two girls as the leaders is as follows:

$$P(\text{two girls chosen}) = \frac{6}{20} = \frac{3}{10}$$

whereas

$$P(\text{girl selected on first draw}) = \frac{12}{20} = \frac{3}{5}$$

and

$$P(\text{girl selected on second draw}) = \frac{3}{5}$$

But since $\frac{3}{5} \bullet \frac{3}{5} \neq \frac{3}{10}$, the two events are not independent.

One can also use the conditional-probability perspective to show that these events are not independent.

$$\text{Since } P(\text{girl on second} \mid \text{girl on first}) = \frac{6}{12} = \frac{1}{2}$$

and

$$P(\text{girl selected on second}) = \frac{3}{5},$$

these events are seen to be dependent.

Adapted from UA Progressions Documents 2012d.

Students also explore finding probabilities of compound events (**S-CP.6–9**) by using the Addition Rule $\left(P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)\right)$ and the general Multiplication Rule $\left(P(A \text{ and } B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)\right)$. A simple experiment in which students roll two number cubes and tabulate the possible outcomes can shed light on these formulas before they are extended to application problems.

Example **S-CP.6–9**

On April 15, 1912, the RMS *Titanic* rapidly sank in the Atlantic Ocean after hitting an iceberg. Only 710 of the ship's 2,204 passengers and crew members survived. Some believe that the rescue procedures favored the wealthier first-class passengers. Data on survival of passengers are summarized in the table at the end of this example, and these data will be used to investigate the validity of such claims. Students can use the fact that two events A and B are independent if $P(A|B) = P(A) \cdot P(B)$. A represents the event that a passenger survived, and B represents the event that the passenger was in first class. The conditional probability $P(A|B)$ is compared with the probability $P(A)$.

For a first-class passenger, the probability of surviving is the fraction of all first-class passengers who survived. That is, the sample space is restricted to include only first-class passengers to obtain:

$$P(A|B) = \frac{202}{325} \approx 0.622$$

The probability that a passenger survived is the number of all passengers who survived divided by the total number of passengers:

$$P(A) = \frac{498}{1316} \approx 0.378$$

Since $0.622 \neq 0.378$, the two given events are not independent. Moreover, it can be said that being a passenger in first class did increase the chances of surviving the accident.

Students can be challenged to further investigate where similar reasoning would apply today. For example, what are similar statistics for Hurricane Katrina, and what would a similar analysis conclude about the distribution of damages? (**MP.4**)

***Titanic* passengers**	**Survived**	**Did not survive**	**Total**
First-class	202	123	325
Second-class	118	167	285
Third-class	178	528	706
Total passengers	498	818	1,316

Adapted from Illustrative Mathematics 2013q.

Using Probability to Make Decisions — S-MD

Use probability to evaluate outcomes of decisions. [Introductory; apply counting rules.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

Standards **S-MD.6** and **S-MD.7** involve students' use of probability models and probability experiments to make decisions. These standards set the stage for more advanced work in Mathematics III, where the ideas of statistical inference are introduced. See the University of Arizona Progressions document titled "High School Statistics and Probability" for further explanation and examples: http://ime.math.arizona.edu/progressions/ (UA Progressions Documents 2012d [accessed April 6, 2015]).

Mathematics II Overview

Number and Quantity

The Real Number System

- Extend the properties of exponents to rational exponents.
- Use properties of rational and irrational numbers.

The Complex Number Systems

- Perform arithmetic operations with complex numbers.
- Use complex numbers in polynomial identities and equations.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Solve equations and inequalities in one variable.
- Solve systems of equations.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.
- Interpret expressions for functions in terms of the situation they model.

Trigonometric Functions

- Prove and apply trigonometric identities.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Geometry

Congruence

- Prove geometric theorems.

Similarity, Right Triangles, and Trigonometry

- Understand similarity in terms of similarity transformations.
- Prove theorems involving similarity.
- Define trigonometric ratios and solve problems involving right triangles.

Circles

- Understand and apply theorems about circles.
- Find arc lengths and areas of sectors of circles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.
- Use coordinates to prove simple geometric theorems algebraically.

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems.

Statistics and Probability

Conditional Probability and the Rules of Probability

- Understand independence and conditional probability and use them to interpret data.
- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

Number and Quantity

The Real Number System — N-RN

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{1/3}$ *to be the cube root of 5 because we want* $\left(5^{1/3}\right)^3 = 5^{(1/3)^3}$ *to hold, so* $\left(5^{1/3}\right)^3$ *must equal 5.*
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a non-zero rational number and an irrational number is irrational.

The Complex Number System — N-CN

Perform arithmetic operations with complex numbers. [i^2 as highest power of i]

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations. [Quadratics with real coefficients]

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.*
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Algebra

Seeing Structure in Expressions — A-SSE

Interpret the structure of expressions. [Quadratic and exponential]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of* P *and a factor not depending on* P. ★
2. Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $\left(x^2\right)^2 - \left(y^2\right)^2$, *thus recognizing it as a difference of squares that can be factored as* $\left(x^2 - y^2\right)\left(x^2 + y^2\right)$.

* (+) Indicates additional mathematics to prepare students for advanced courses. ★ Indicates a modeling standard linking mathematics to everyday life, work, and decision-making.

Write expressions in equivalent forms to solve problems. [Quadratic and exponential]

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★
   a. Factor a quadratic expression to reveal the zeros of the function it defines. ★
   b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. ★
   c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $\left(1.15^{1/12}\right)^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* ★

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials. [Polynomials that simplify to quadratics]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Creating Equations — A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★ [Include formulas involving quadratic terms.]

Reasoning with Equations and Inequalities — A-REI

Solve equations and inequalities in one variable. [Quadratics with real coefficients]

4. Solve quadratic equations in one variable.
   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm b$ for real numbers a and b.

Solve systems of equations. [Linear-quadratic systems]

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ *and the circle* $x^2 + y^2 = 3$.

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Quadratic]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Linear, exponential, quadratic, absolute value, step, piecewise-defined]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y=(1.02)^t$, $y=(0.97)^t$, $y=(1.01)^{12t}$, *and* $y=(1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Building Functions F-BF

Build a function that models a relationship between two quantities. [Quadratic and exponential]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. ★

Build new functions from existing functions. [Quadratic, absolute value]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x)=c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x)=2x^3$.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems. [Include quadratic]

3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

Interpret expressions for functions in terms of the situation they model.

6. **Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity. CA** ★

Trigonometric Functions F-TF

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta)+\cos^2(\theta)=1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Geometry

Congruence G-CO

Prove geometric theorems. [Focus on validity of underlying reasoning while using variety of ways of writing proofs.]

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Similarity, Right Triangles, and Trigonometry G-SRT

Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.

Prove theorems involving similarity. [Focus on validity of underlying reasoning while using variety of formats.]

4. Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★

8.1 Derive and use the trigonometric ratios for special right triangles (30°, 60°, 90° and 45°, 45°, 90°). CA

Circles G-C

Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.
4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles. [Radian introduced only as unit of measure]

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. **Convert between degrees and radians. CA**

M2 Mathematics II

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).* [Include simple circle theorems.]
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Geometric Measurement and Dimension — G-GMD

Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★
5. **Know that the effect of a scale factor k greater than zero on length, area, and volume is to multiply each by k, k^2, and k^3, respectively; determine length, area, and volume measures using scale factors. CA★**
6. **Verify experimentally that in a triangle, angles opposite longer sides are larger, sides opposite larger angles are longer, and the sum of any two side lengths is greater than the remaining side length; apply these relationships to solve real-world and mathematical problems. CA**

Statistics and Probability

Conditional Probability and the Rules of Probability — S-CP

Understand independence and conditional probability and use them to interpret data. [Link to data from simulations or experiments.]

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
2. Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B. ★

4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. ★

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

Using Probability to Make Decisions — S-MD

Use probability to evaluate outcomes of decisions. [Introductory; apply counting rules.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

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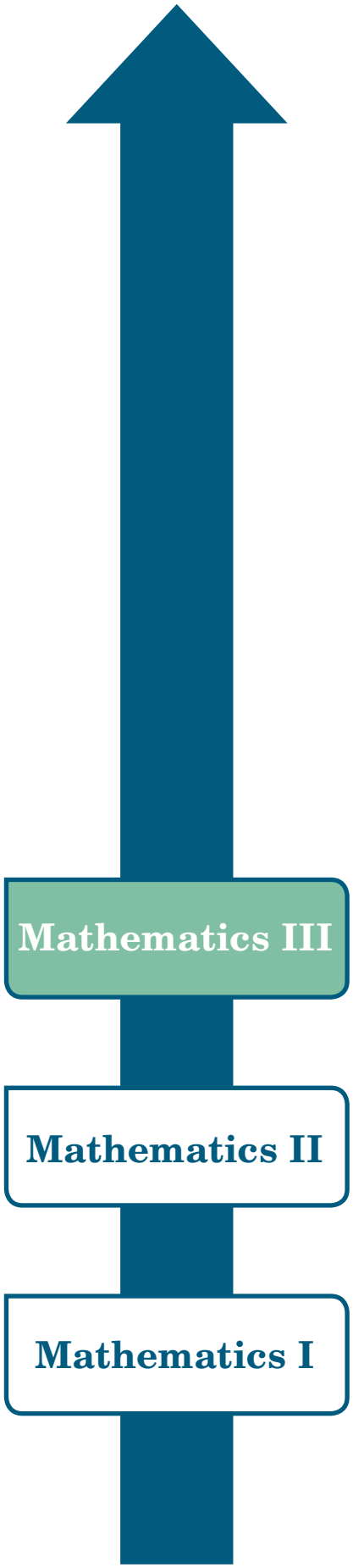


Mathematics III

In the Mathematics III course, students expand their repertoire of functions to include polynomial, rational, and radical functions. They also expand their study of right-triangle trigonometry to include general triangles. And, finally, students bring together all of their experience with functions and geometry to create models and solve contextual problems. The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the California Common Core State Standards for Mathematics (CA CCSSM); they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics III course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics III

In Mathematics III, students understand the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. They connect multiplication of polynomials with multiplication of multi-digit integers and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. Their work on polynomial expressions culminates with the Fundamental Theorem of Algebra. Rational numbers extend the arithmetic of integers by allowing division by all numbers except 0. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of working with rational expressions is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect, regardless of the type of the underlying functions.

Students develop the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles. This discussion of general triangles opens up the idea of trigonometry applied beyond the right triangle—that is, at least to obtuse angles. Students build on this idea to develop the notion of radian measure for angles and extend the domain of the trigonometric functions to all real numbers. They apply this knowledge to model simple periodic phenomena.

Students see how the visual displays and summary statistics they learned in previous grade levels or courses relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and recognize the role that randomness and careful design play in the conclusions that may be drawn.

Finally, students in Mathematics III extend their understanding of modeling: they identify appropriate types of functions to model a situation, adjust parameters to improve the model, and compare models by analyzing appropriateness of fit and by making judgments about the domain over which a model is a good fit. The description of modeling as "the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions" (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010e) is one of the main themes of this course. The discussion about modeling and the diagram of the modeling cycle that appear in this chapter should be considered when students apply knowledge of functions, statistics, and geometry in a modeling context.

Examples of Key Advances from Mathematics II

- Students begin to see polynomials as a system analogous to the integers that they can add, subtract, multiply, and so forth. Subsequently, polynomials can be extended to rational expressions, which are analogous to rational numbers.
- Students extend their knowledge of linear, exponential, and quadratic functions to include a much broader range of classes of functions.
- Students begin to examine the role of randomization in statistical design.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The Standards for Mathematical Practice represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. The Mathematics III course offers ample opportunities for students to engage with each MP standard; table M3-1 offers some examples.

Table M3-1. Standards for Mathematical Practice—Explanation and Examples for Mathematics III

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students apply their understanding of various functions to real-world problems. They approach complex mathematics problems and break them down into smaller problems, synthesizing the results when presenting solutions.
MP.2 Reason abstractly and quantitatively.	Students deepen their understanding of variables—for example, by understanding that changing the values of the parameters in the expression $A\sin(Bx+C)+D$ has consequences for the graph of the function. They interpret these parameters in a real-world context.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function when modeling a real-world situation.
MP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems, making use of their expanding repertoire of functions in modeling. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MP.5 Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MP.6 Attend to precision.	Students make note of the precise definition of *complex number,* understanding that real numbers are a subset of complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers.
MP.7 Look for and make use of structure.	Students understand polynomials and rational numbers as sets of mathematical objects that have particular operations and properties. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena.
MP.8 Look for and express regularity in repeated reasoning.	Students observe patterns in geometric sums—for example, that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written as follows: $1 = 2^1 - 1$ $1+2 = 2^2 - 1$ $1+2+4 = 2^3 - 1$ $1+2+4+8 = 2^4 - 1$ Students use this observation to make a conjecture about any such sum.

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star (★) symbol to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards.

Examples of places where specific Mathematical Practice standards can be implemented in the Mathematics III standards are noted in parentheses, with the standard(s) also listed.

Mathematics III Content Standards, by Conceptual Category

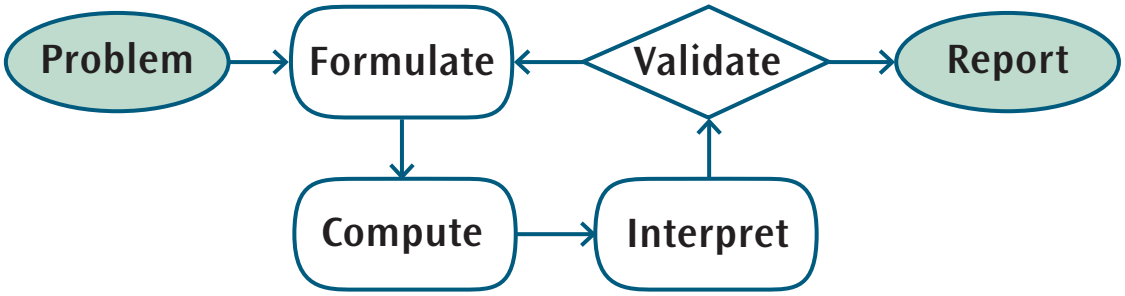
The Mathematics III course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in Mathematics III are described below, according to each conceptual category. Standards that are considered new for secondary-grades teachers are discussed more thoroughly than other standards.

Conceptual Category: Modeling

Throughout the CA CCSSM, specific standards for higher mathematics are marked with a ★ symbol to indicate that they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise (e.g., Which of the quantities present in this situation are known, and which are unknown?). Students need to decide on a solution path that may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new formula or function will apply. Students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, or the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure M3-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Figure M3-1. The Modeling Cycle



The examples in this chapter are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding rational functions, graphing, solving equations, and rates of change are explored through this lens. Readers are encouraged to consult appendix B (Mathematical Modeling) for further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

The standards in the Functions conceptual category can serve as motivation for the study of standards in the other Mathematics III conceptual categories. Students have already worked with equations in which they have to "solve for x" as a search for the input of a function f that gives a specified output; solving the equation amounts to undoing the work of the function. The types of functions that students encounter in Mathematics III have new properties. Students previously learned that quadratic functions exhibit different behavior from linear and exponential functions; now they investigate polynomial, rational, and trigonometric functions in greater generality. As in the Mathematics II course, students must discover new techniques for solving the equations they encounter. Students see how rational functions can model real-world phenomena, in particular in instances of inverse variation ($x \cdot y = k$, k a constant), and how trigonometric functions can model periodic phenomena. In general, functions describe how two quantities are related in a precise way and can be used to make predictions and generalizations, keeping true to the emphasis on modeling that occurs in higher mathematics. As stated in the University of Arizona (UA) Progressions Documents for the Common Core Math Standards, "students should develop ways of thinking that are general and allow them to approach any function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary" (UA Progressions Documents 2013c, 7).

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Include rational, square root and cube root; emphasize selection of appropriate models.]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Include rational and radical; focus on using key features to guide selection of appropriate type of model function.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

As in Mathematics II, students work with functions that model data and choose an appropriate model function by considering the context that produced the data. Students' ability to recognize rates of change, growth and decay, end behavior, roots, and other characteristics of functions becomes more sophisticated; they use this expanding repertoire of families of functions to inform their choices of models. This group of standards focuses on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate (**F-IF.4–9**). The following example illustrates some of these standards. (Note that only sine, cosine, and tangent are treated in Mathematics III.)

Example: The Juice Can **F-IF.4–9**

Students are asked to find the minimal surface area of a cylindrical can of a fixed volume. The surface area is represented in units of square centimeters (cm^2), the radius in units of centimeters (cm), and the volume is fixed at 355 milliliters (ml), or 355 cm^3. Students can find the surface area of this can as a function of the radius:

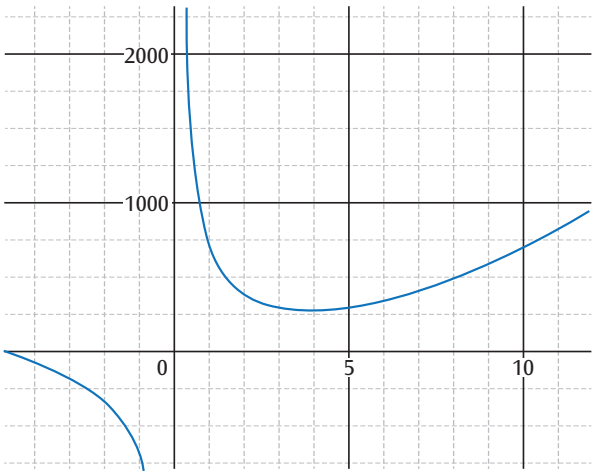
$$S(r) = \frac{2(355)}{r} + 2\pi r^2$$

(See The Juice-Can Equation example that appears in the Algebra conceptual category of this chapter.)

This representation allows students to examine several things. First, a table of values will provide a hint about what the minimal surface area is. The table below lists several values for S based on r:

r(cm)	S(cm^2)
0.5	1421.6
1.0	716.3
1.5	487.5
2.0	380.1
2.5	323.3
3.0	293.2
3.5	279.8
4.0	278.0
4.5	284.9
5.0	299.0
5.5	319.1
6.0	344.4
6.5	374.6
7.0	409.1
7.5	447.9
8.0	490.7

The data suggest that the minimal surface area occurs when the radius of the base of the juice can is between 3.5 and 4.5 centimeters. Successive approximation using values of r between these values will yield a better estimate. But how can students be sure that the minimum is truly located here? A graph of $S(r)$ provides a clue:



Furthermore, students can deduce that as r gets smaller, the term $\frac{2(355)}{r}$ gets larger and larger, while the term $2\pi r$ gets smaller and smaller, and that the reverse is true as r grows larger, so that there is truly a minimum somewhere in the interval [3.5, 4.5].

Graphs help students reason about rates of change of functions (**F-IF.6**). In grade eight, students learned that the *rate of change* of a linear function is equal to the slope of the graph of that function. And because the slope of a line is constant, the phrase "rate of change" is clear for linear functions. For non-linear functions, however, rates of change are not constant, and thus average rates of change over an interval are used. For example, for the function g defined for all real numbers by $g(x) = x^2$, the average rate of change from $x = 2$ to $x = 5$ is

$$\frac{g(5) - g(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7.$$

This is the slope of the line containing the points (2,4) and (5,25) on the graph of g. If g is interpreted as returning the area of a square of side length x, then this calculation means that over this interval the area changes, on average, by 7 square units for each unit increase in the side length of the square (UA Progressions Documents 2013c, 9). Students could investigate similar rates of change over intervals for the Juice-Can problem shown previously.

Building Functions F-BF

Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

Build new functions from existing functions. [Include simple, radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$.

Students in Mathematics III develop models for more complex or sophisticated situations than in previous courses, due to the expansion of the types of functions available to them (**F-BF.1**). Modeling contexts provide a natural place for students to start building functions with simpler functions as components. Situations in which cooling or heating are considered involve functions that approach a limiting value according to a decaying exponential function. Thus, if the ambient room temperature is 70 degrees Fahrenheit and a cup of tea is made with boiling water at a temperature of 212 degrees Fahrenheit, a student can express the function describing the temperature as a function of time by

using the constant function $f(t) = 70$ to represent the ambient room temperature and the exponentially decaying function $g(t) = 142e^{-kt}$ to represent the decaying difference between the temperature of the tea and the temperature of the room, which leads to a function of this form:

$$T(t) = 70 + 142e^{-kt}$$

Students might determine the constant k experimentally (**MP.4, MP.5**).

With standard **F-BF.4a,** students learn that some functions have the property that an input can be recovered from a given output; for example, the equation $f(x) = c$ can be solved for x, given that c lies in the range of f. Students understand that this is an attempt to "undo" the function, or to "go backwards." Tables and graphs should be used to support student understanding here. This standard dovetails nicely with standard **F-LE.4** described below and should be taught in progression with it. Students will work more formally with inverse functions in advanced mathematics courses, and so standard **F-LE.4** should be treated carefully to prepare students for deeper understanding of functions and their inverses.

Linear, Quadratic, and Exponential Models	**F-LE**

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★ [Logarithms as solutions for exponentials]

4.1. Prove simple laws of logarithms. CA ★

4.2 Use the definition of logarithms to translate between logarithms in any base. CA ★

4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA ★

Students worked with exponential models in Mathematics II. Based on the graph of the exponential function $f(x) = b^x$, students in Mathematics III can deduce that this function has an inverse—which is called *the logarithm to the base* b and denoted by $g(x) = \log_b x$. The logarithm has the property that $\log_b x = y$ if and only if $b^y = x$. Students find logarithms with base b equal to 2, 10, or e, by hand and with the assistance of technology (**MP.5**). Students may be encouraged to explore the properties of logarithms (such as $\log_b xy = \log_b x + \log_b y$) and to connect these properties to those of exponents. For example, the property just mentioned comes from the fact that the logarithm is essentially an exponent and that $b^{n+m} = b^n \cdot b^m$. Students solve problems involving exponential functions and logarithms and express their answers by using logarithm notation (**F-LE.4**). In general, students understand logarithms as functions that *undo* their corresponding exponential functions; instruction should emphasize this relationship.

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

2.1 Graph all 6 basic trigonometric functions. CA

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★

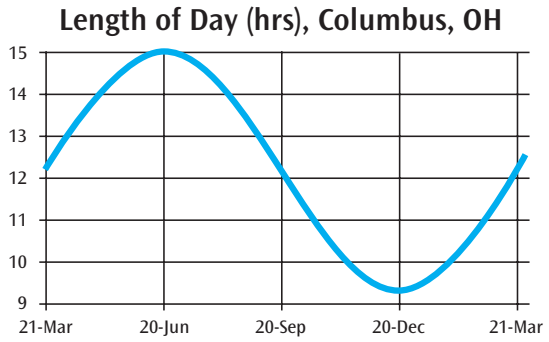
This set of standards calls for students to expand their understanding of trigonometric functions, which was first developed in Mathematics II. At first, the trigonometric functions apply only to angles in right triangles; for example, $\sin\theta$, $\cos\theta$, and $\tan\theta$ make sense only for $0<\theta<\frac{\pi}{2}$. By representing right triangles with hypotenuse 1 in the first quadrant of the plane, students see that $(\cos\theta, \sin\theta)$ represents a point on the unit circle. This leads to a natural way to extend these functions to any value of θ that remains consistent with the values for acute angles: interpreting θ as the radian measure of an angle traversed from the point (1,0) counterclockwise around the unit circle, $\cos\theta$ is taken to be the x-coordinate of the point corresponding to this rotation and $\sin\theta$ is the y-coordinate of this point. This interpretation of sine and cosine immediately yields the Pythagorean Identity: that $\cos^2\theta+\sin^2\theta=1$. This basic identity yields other identities through algebraic manipulation and allows students to find values of other trigonometric functions for a given θ if one value is known (**F-TF.1–2**).

Students should explore the graphs of trigonometric functions, with attention to the connection between the unit-circle representation of the trigonometric functions and their properties—for example, to illustrate the periodicity of the functions, the relationship between the maximums and minimums of the sine and cosine graphs, zeros, and so on. Standard **F-TF.5** calls for students to use trigonometric functions to model periodic phenomena. This is connected to standard **F-BF.3** (families of functions), and students begin to understand the relationship between the parameters appearing in the general cosine function $f(x)=A\bullet\cos(Bx-C)+D$ (and sine function), as well as the graph and behavior of the function (e.g., amplitude, frequency, line of symmetry). Additionally, students use their understanding of inverse functions to explore the inverse sine, cosine, and tangent functions at a basic level. It is important for students to understand that a function is well defined only when its domain is specified. For example, the general sine function $\sin x$, defined for all real numbers x, does not have an inverse, whereas the function $s(x)=\sin x$, defined only for values of x such that $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, *does* have an inverse function.

Example: Modeling Daylight Hours **F-TF.5**

By looking at data for length of days in Columbus, Ohio, students see that the number of daylight hours is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference of the maximum and minimum. Approximations of these values are set as $A = 12.17$ and $B = 2.83$. With some support, students determine that for the period to be 365 days (per cycle), or for the frequency to be $\frac{1}{365}$ cycles per day, $C = \frac{2\pi}{365}$, and if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$.

Thus, $f(t) = 12.17 + 2.83\sin\left(\frac{2\pi t}{365}\right)$ is a function that gives the approximate length of day for t, the day of the year from March 21. Considering questions such as when to plant a garden (i.e., when there are at least 7 hours of midday sunlight), students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$ and find that May 1 and August 10 mark this interval of time.



Students can investigate many other trigonometric modeling situations, such as simple predator–prey models, sound waves, and noise-cancellation models.

Source: UA Progressions Documents 2013c, 19.

Conceptual Category: Number and Quantity

Students continue to expand their understanding of the number system by finding complex-number roots when solving quadratic equations. Complex numbers have a practical application, and many phenomena involving real numbers become simpler when real numbers are viewed as a subsytem of complex numbers. As an example, complex solutions of differential equations can present a clear picture of the behavior of real solutions. Students are introduced to this when they study complex solutions of quadratic equations—and when complex numbers are involved, each quadratic polynomial can be expressed as a product of linear factors.

The Complex Number System	N-CN
Use complex numbers in polynomial identities and equations. [Polynomials with real coefficients; apply N-CN.9 to higher degree polynomials.]	
8. (+) Extend polynomial identities to the complex numbers.	
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.	

Standards **N-CN.8–9** call for students to continue working with complex numbers as solutions to polynomial equations. This builds on student work with quadratics that started in Mathematics II. For example, students can draw upon the Mathematics III algebra standards (e.g., **A-APR.2**) and find roots of equations such as $x^3 + 5x^2 + 8x + 6 = 0$. They experiment by using the remainder theorem and find that $x + 3$ is a root, since the polynomial expression evaluated at $x = -3$ is 0. Using polynomial long division or other factoring techniques, students find that

$$x^3 + 5x^2 + 8x + 6 = (x + 3)(x^2 + 2x + 2).$$

They use the quadratic formula to find the roots of the quadratic, $\{-1 + i, -1 - i\}$, and they write

$$x^3 + 5x^2 + 8x + 6 = (x + 3)(x^2 + 2x + 2)$$

$$x^3 + 5x^2 + 8x + 6 = (x + 3)(x - (-1 + i))(x - (-1 - i)).$$

Experimentation with examples of such polynomials and an understanding that the quadratic formula always yields solutions to a quadratic equation help students understand the Fundamental Theorem of Algebra (**N-CN.9**).

Conceptual Category: Algebra

Along with the Number and Quantity standards in Mathematics III, the standards from the Algebra domain of the Mathematics III course develop the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers and connect division of polynomials with long division of integers. Similar to the way that rational numbers extend the arithmetic of integers by allowing division by all numbers except 0, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme that arises is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Seeing Structure in Expressions	A-SSE

Interpret the structure of expressions. [Polynomial and rational]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. ★
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

In Mathematics III, students continue to pay attention to the meaning of expressions in context and interpret the parts of an expression by "chunking"—that is, by viewing parts of an expression as a single entity (**A-SSE.1–2**). For example, their facility with using special cases of polynomial factoring allows them to fully factor more complicated polynomials:

$$x^4-y^4=\left(x^2\right)^2-\left(y^2\right)^2=\left(x^2+y^2\right)\left(x^2-y^2\right)=\left(x^2+y^2\right)(x+y)(x-y)$$

Additionally, with their new understanding of complex numbers, students can factor this further into $x^4-y^4=(x+iy)(x-iy)(x+y)(x-y)$. In a physics course, students may encounter an expression such as $L_0\sqrt{1-\frac{v^2}{c^2}}$, which arises in the theory of special relativity. They can see this expression as the product of a constant L_0 and a term that is equal to 1 when $v=0$ and equal to 0 when $v=c$. Furthermore, students might be expected to see this mentally, without having to go through a laborious process of evaluation. This involves combining large-scale structure of the expression—a product of L_0 and another term—with the meaning of internal components such as $\frac{v^2}{c^2}$ (UA Progressions Documents 2013b, 4).

By examining the sum of a finite geometric series, students can look for a pattern to justify why the equation for the sum holds: $\sum_{k=0}^{n} ar^k=\frac{a\left(1-r^{n+1}\right)}{(1-r)}$. They may derive the formula, either with Proof by Mathematical Induction or by other means (**A-SSE.4**).

Example: Sum of a Geometric Series **A-SSE.4**

Students should investigate several concrete examples of finite geometric series (with $r \neq 1$) and use spreadsheet software to investigate growth in the sums and patterns that arise (**MP.5, MP.8**).

Geometric series have applications in several areas, including calculating mortgage payments, calculating totals for annual investments such as retirement accounts, finding total payout amounts for lottery winners, and more (**MP.4**). In general, a finite geometric series has this form:

$$\sum_{k=0}^{n} ar^k = a\left(1 + r + r^2 + \ldots r^{n-1} + r^n\right)$$

If the sum of this series is denoted by S, then some algebraic manipulation shows that

$$S - rS = a - ar^{n+1}.$$

Applying the distributive property to the common factors and solving for S shows that

$$S(1-r) = a\left(1 - r^{n+1}\right),$$

so that

$$S = \frac{a\left(1 - r^{n+1}\right)}{1-r}.$$

Students develop the ability to flexibly see expressions such as $A_n = A_0\left(1 + \frac{.15}{12}\right)^n$ as describing the total value of an investment at 15% interest, compounded monthly, for a number of compoundings, n. Moreover, they can interpret the following equation as a type of geometric series that would calculate the total value in an investment account at the end of one year if \$100 is deposited at the beginning of each month (**MP.2, MP.4, MP.7**):

$$A_1 + A_2 + \ldots + A_{12} = 100\left(1 + \frac{.15}{12}\right)^1 + 100\left(1 + \frac{.15}{12}\right)^2 + \ldots + 100\left(1 + \frac{.15}{12}\right)^{12}$$

They apply the formula for geometric series to find this sum.

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials. [Beyond quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x-a$ is $p(a)$, so $p(a)=0$ if and only if $(x-a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $\left(x^2+y^2\right)^2=\left(x^2-y^2\right)^2+(2xy)^2$ *can be used to generate Pythagorean triples.*
5. (+) Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

Rewrite rational expressions. [Linear and quadratic denominators]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x)+r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

Students in Mathematics III continue to develop their understanding of the set of polynomials as a system analogous to the set of integers that exhibits certain properties, and they explore the relationship between the factorization of polynomials and the roots of a polynomial (**A-APR.1–3**). When a polynomial $p(x)$ is divided by $(x-a)$, $p(x)$ is written as $p(x)=q(x)\bullet(x-a)+r$, where r is a constant. This can be done by inspection or by polynomial long division (**A-APR.7**). It follows that $p(a)=q(a)\bullet(a-a)+r=q(a)\bullet 0+r=r$, so that $(x-a)$ is a factor of $p(x)$ if and only if $p(a)=0$. This result is generally known as the Remainder Theorem (**A-APR.2**) and provides an easy check to see if a polynomial has a given linear factor. This topic should not be reduced to "synthetic division," which limits the theorem to a method of carrying numbers between registers—something easily done by a computer—while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a technique (**MP.3**) [UA Progressions Documents 2013b, 7].

1. The Binomial Theorem may be proven by mathematical induction or by a combinatorial argument.

Students use the zeros of a polynomial to create a rough sketch of its graph and connect the results to their understanding of polynomials as functions (**A-APR.3**). The notion that polynomials can be used to approximate other functions is important in advanced mathematics courses such as Calculus, and standard **A-APR.3** is the first step in a progression that can lead students, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane (UA Progressions Documents 2013b, 7).

Additionally, polynomials form a rich ground for mathematical explorations that reveal relationships in the system of integers (**A-APR.4**). For example, students can explore the sequence of squares 1, 4, 9, 16, 25, 36, … and notice the differences between them—3, 5, 7, 9 , 11—are consecutive odd integers. This mystery is explained by the polynomial identity $(n+1)^2 - n^2 = 2n+1$, which can be justified by using pictures (UA Progressions Documents 2013b, 6).

In Mathematics III, students explore rational functions as a system analogous to the rational numbers. They see rational functions as useful for describing many real-world situations—for instance, when rearranging the equation $d = rt$ to express the rate as a function of the time for a fixed distance d_0 and obtaining $r = \frac{d_o}{t}$. Now students see that any two polynomials can be divided in much the same way that numbers are (provided the divisor is not 0). Students first understand rational expressions as similar to other expressions in algebra, except that rational expressions have the special form $\frac{a(x)}{b(x)}$ for both $a(x)$ and $b(x)$ polynomials in x. Students should evaluate various rational expressions for many values of x, perhaps discovering that when the degree of $b(x)$ is larger than the degree of $a(x)$, the value of the expression gets smaller in absolute value as $|x|$ gets larger. Developing an understanding of the behavior of rational expressions in this way helps students to see these expressions as functions and sets the stage for working with simple rational functions in the Functions domain.

Example: The Juice-Can Equation **A-CED.4**

If someone wanted to investigate the shape of a juice can of minimal surface area, the investigation could begin in the following way. If the volume V_0 is fixed, then the expression for the volume of the can is $V_0 = \pi r^2 h$, where h is the height of the can and r is the radius of the circular base. On the other hand, the surface area S is given by the following formula:

$$S = 2\pi rh + 2\pi r^2$$

This is because the two circular bases of the can contribute $2\pi r^2$ units of surface area, and the outside surface of the can contributes an area in the shape of a rectangle with length equal to the circumference of the base, $2\pi r$, and height equal to h. Since the volume is fixed, h can be found in terms of r: $h = \frac{V_0}{\pi r^2}$. Then this can be substituted into the equation for the surface area:

$$S = 2\pi r \bullet \frac{V_0}{\pi r^2} + 2\pi r^2$$

$$= \frac{2V_0}{r} + 2\pi r^2$$

This equation expresses the surface area S as a (rational) function of r, which can then be analyzed. (Also refer to standard **F-IF.8**.)

Additionally, students are able to rewrite rational expressions in the form $a(x) = q(x) \bullet b(x) + r(x)$, where $r(x)$ is a polynomial of degree less than $b(x)$, by inspection or by using polynomial long division. They can flexibly rewrite this expression as $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ as necessary—for example, to highlight the end behavior of the function defined by the expression $\frac{a(x)}{b(x)}$. In order to make working with rational expressions more than just an exercise in the proper manipulation of symbols, instruction should focus on the characteristics of rational functions that can be understood by rewriting them in the ways described above (e.g., rates of growth, approximation, roots, axis intersections, asymptotes, end behavior, and so on).

Creating Equations	**A-CED**

Create equations that describe numbers or relationships. [Equations using all available types of expressions, including simple root functions]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★

Students in Mathematics III work with all available types of functions, including root functions, to create equations (**A-CED.1**). Although functions referenced in standards **A-CED.2–4** will often be linear, exponential, or quadratic, the types of problems should draw from more complex situations than those addressed in Mathematics I and Mathematics II. For example, knowing how to find the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line. The Juice-Can Equation example presented previously in this section is connected to standard **A-CED.4.**

Reasoning with Equations and Inequalities	A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Simple radical and rational]

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Represent and solve equations and inequalities graphically. [Combine polynomial, rational, radical, absolute value, and exponential functions.]

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Students extend their equation-solving skills to those involving rational expressions and radical equations, and they make sense of extraneous solutions that may arise (**A-REI.2**). In particular, students understand that when solving equations, the flow of reasoning is generally forward, in the sense that it is assumed a number x is a solution of the equation and then a list of possibilities for x is found. However, not all steps in this process are reversible. For example, although it is true that if $x = 2$, then $x^2 = 4$, it is not true that if $x^2 = 4$, then $x = 2$, as $x = -2$ also satisfies this equation (UA Progressions Documents 2013b, 10). Thus students understand that some steps are reversible and some are not, and they anticipate extraneous solutions. Additionally, students continue to develop their understanding of solving equations as solving for values of x such that $f(x) = g(x)$, now including combinations of linear, polynomial, rational, radical, absolute value, and exponential functions (**A-REI.11**). Students also understand that some equations can be solved only approximately with the tools they possess.

Conceptual Category: Geometry

In Mathematics III, students extend their understanding of the relationship between algebra and geometry as they explore the equations for circles and parabolas. They also expand their understanding of trigonometry to include finding unknown measurements in non-right triangles. The Geometry standards included in the Mathematics III course offer many rich opportunities for students to practice mathematical modeling.

Similarity, Right Triangles, and Trigonometry	G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Students advance their knowledge of right-triangle trigonometry by applying trigonometric ratios in non-right triangles. For instance, students see that by dropping an altitude in a given triangle, they divide the triangle into right triangles to which these relationships can be applied. By seeing that the base of the triangle is a and the height is $b \cdot \sin \mathrm{C}$, they derive a general formula for the area of any triangle $A = \frac{1}{2}ab\sin(\mathrm{C})$ (**G-SRT.9**). Additionally, students use reasoning about similarity and trigonometric identities to derive the Laws of Sines and Cosines only in acute triangles, and they use these and other relationships to solve problems (**G-SRT.10–11**). Instructors will need to address the ideas of the sine and cosine of angles larger than or equal to 90 degrees to fully discuss Laws of Sine and Cosine, although full unit-circle trigonometry need not be discussed in this course.

Geometric Measurement and Dimension — G-GMD

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry — G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

These standards are rich with opportunities for students to apply modeling (**MP.4**) with geometric concepts—and although these standards appear later in the sequence of the Mathematics III geometry standards, they should be incorporated throughout the geometry curriculum of the course. Standard **G-MG.1** calls for students to use geometric shapes, their measures, and their properties to describe objects. This standard can involve two- and three-dimensional shapes and is not relegated to simple applications of formulas. Standard **G-MG.3** calls for students to solve design problems by modeling with geometry.

Example: Ice-Cream Cone **G-MG.3**

The owner of a local ice-cream parlor has hired you to assist with his company's new venture: the company will soon sell its ice-cream cones in the freezer section of local grocery stores. The manufacturing process requires that each ice-cream cone be wrapped in a cone-shaped paper wrapper with a flat, circular disc covering the top. The company wants to minimize the amount of paper that is wasted in the process of wrapping the cones. Use a real ice-cream cone or the dimensions of a real ice-cream cone to complete the following tasks:

a. Sketch a wrapper like the one described above, using the actual size of your cone. Ignore any overlap required for assembly.

b. Use your sketch to help develop an equation the owner can use to calculate the surface area of a wrapper (including the lid) for another cone, given that its base had a radius of length r and a slant height s.

c. Using measurements of the radius of the base and slant height of your cone, and your equation from step b, find the surface area of your cone.

d. The company has a large rectangular piece of paper that measures 100 centimeters by 150 centimeters. Estimate the maximum number of complete wrappers sized to fit your cone that could be cut from this single piece of paper, and explain your estimate. (Solutions can be found at https://www.illustrativemathematics.org/ [accessed April 1, 2015].)

Source: Illustrative Mathematics 2013l.

Expressing Geometric Properties with Equations **G-GPE**

Translate between the geometric description and the equation for a conic section.

3.1 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola and graph the equation. [In Mathematics III, this standard addresses only circles and parabolas.] **CA**

Students further their understanding of the connection between algebra and geometry by applying the definition of circles and parabolas to derive equations and then deciding whether a given quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$ represents a circle or a parabola.

Conceptual Category: Statistics and Probability

In Mathematics III, students develop a more formal and precise understanding of statistical inference, which requires a deeper understanding of probability. They explore the conditions that meet *random sampling* of a population and that allow for generalization of results to that population. Students also learn to use significant differences to make inferences about data gathered during the course of experiments.

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Although students in Mathematics III may have heard of the normal distribution, it is unlikely that they will have had prior experience using it to make specific estimates. In Mathematics III, students build on their understanding of data distributions to see how to use the area under the normal distribution to make estimates of frequencies (which can be expressed as probabilities). It is important for students to see that only some data are well described by a normal distribution (**S-ID.4**). Additionally, they can learn through examples the *empirical rule:* that for a normally distributed data set, 68% of the data lie within one standard deviation of the mean and 95% are within two standard deviations of the mean.

Example: The Empirical Rule **S-ID.4**

Suppose that SAT mathematics scores for a particular year are approximately normally distributed, with a mean of 510 and a standard deviation of 100.

a. What is the probability that a randomly selected score is greater than 610?

b. What is the probability that a randomly selected score is greater than 710?

c. What is the probability that a randomly selected score is between 410 and 710?

d. If a student's score is 750, what is the student's percentile score (the proportion of scores below 750)?

Solutions:

a. The score 610 is one standard deviation above the mean, so the tail area above that is about half of 0.32, or 0.16. The calculator gives 0.1586.

b. The score 710 is two standard deviations above the mean, so the tail area above that is about half of 0.05, or 0.025. The calculator gives 0.0227.

c. The area under a normal curve from one standard deviation below the mean to two standard deviations above the mean is about 0.815. The calculator gives 0.8186.

d. Using either the normal distribution given or the standard normal (for which 750 translates to a z-score of 2.4), the calculator gives 0.9918.

Making Inferences and Justifying Conclusions — S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Students in Mathematics III move beyond analysis of data to make sound statistical decisions based on probability models. The reasoning process is as follows: develop a statistical question in the form of a hypothesis (supposition) about a population parameter, choose a probability model for collecting data relevant to that parameter, collect data, and compare the results seen in the data with what is expected under the hypothesis. If the observed results are far from what is expected and have a low probability of occurring under the hypothesis, then that hypothesis is called into question. In other words, the evidence against the hypothesis is weighed by probability (**S-IC.1**) [UA Progressions Documents 2012d]. By investigating simple examples of simulations of experiments and observing outcomes of the data, students gain an understanding of what it means for a model to fit a particular data set (**S-IC.2**). This includes comparing theoretical and empirical results to evaluate the effectiveness of a treatment.

In earlier grade levels, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data are collected determines the scope and nature of the conclusions that can be drawn from those data. The concept of *statistical significance* is developed informally through simulation as meaning a result that is unlikely to have occurred solely through random selection in sampling or random assignment in an experiment (NGA/CCSSO 2010a). When covering standards **S-IC.4–5,** instructors should focus on the variability of results from experiments—that is, on statistics as a way of handling, not eliminating, inherent randomness. Because standards **S-IC.1–6** are all modeling standards, students should have ample opportunities to explore statistical experiments and informally arrive at statistical techniques.

Example: Estimating a Population Proportion **S-IC.4**

Suppose a student wishes to investigate whether 50% of homeowners in her neighborhood will support a new tax to fund local schools. If she takes a random sample of 50 homeowners in her neighborhood, and 20 support the new tax, then the *sample proportion* agreeing to pay the tax would be 0.4. But is this an accurate measure of the true proportion of homeowners who favor the tax? How can this be determined?

If this sampling situation (**MP.4**) is simulated with a graphing calculator or spreadsheet software under the assumption that the true proportion is 50%, then the student can arrive at an understanding of the *probability* that her randomly sampled proportion would be 0.4. A simulation of 200 trials might show that 0.4 arose 25 out of 200 times, or with a probability of 0.125. Thus, the chance of obtaining 40% as a sample proportion is not insignificant, meaning that a true proportion of 50% is plausible.

Adapted from UA Progressions Documents 2012d.

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions. [Include more complex situations.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

As in Mathematics II, students apply probability models to make and analyze decisions. This skill is extended in Mathematics III to more complex probability models, including situations such as those involving quality control or diagnostic tests that yield both false-positive and false-negative results. See the University of Arizona Progressions document titled "High School Statistics and Probability" for further explanation and examples: http://ime.math.arizona.edu/progressions/ (UA Progressions Documents 2012d [accessed April 6, 2015]).

Mathematics III is the culmination of the Integrated Pathway. Students completing this pathway will be well prepared for advanced mathematics and should be encouraged to continue their study of mathematics with Precalculus or other mathematics electives, such as Statistics and Probability or a course in modeling.

Mathematics III Overview

Number and Quantity

The Complex Number System

- Use complex numbers in polynomial identities and equations.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.
- Understand the relationship between zeros and factors of polynomials.
- Use polynomial identities to solve problems.
- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Mathematics III Overview *(continued)*

Geometry

Similarity, Right Triangles, and Trigonometry

- Apply trigonometry to general triangles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

Geometric Measurement and Dimension

- Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- Apply geometric concepts in modeling situations.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

Number and Quantity

The Complex Number System — N-CN

Use complex numbers in polynomial identities and equations. [Polynomials with real coefficients; apply N-CN.9 to higher degree polynomials.]

8. (+) Extend polynomial identities to the complex numbers.[2]
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Algebra

Seeing Structure in Expressions — A-SSE

Interpret the structure of expressions. [Polynomial and rational]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. ★
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials. [Beyond quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x-a$ is $p(a)$, so $p(a)=0$ if and only if $(x-a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$ *can be used to generate Pythagorean triples.*
5. (+) Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[3]

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2. (+) Indicates additional mathematics to prepare students for advanced courses. ★ Indicates a modeling standard linking mathematics to everyday life, work, and decision-making.

3. The Binomial Theorem may be proven by mathematical induction or by a combinatorial argument

Rewrite rational expressions. [Linear and quadratic denominators]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x)+r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations — A-CED

Create equations that describe numbers or relationships. [Equations using all available types of expressions, including simple root functions]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★

3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★

4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.★

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Simple radical and rational]

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Represent and solve equations and inequalities graphically. [Combine polynomial, rational, radical, absolute value, and exponential functions.]

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Include rational, square root and cube root; emphasize selection of appropriate models.]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Include rational and radical; focus on using key features to guide selection of appropriate type of model function.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions F-BF

Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

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Build new functions from existing functions. [Include simple, radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

4. Find inverse functions.

   a. Solve an equation of the form $f(x)=c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x)=(x+1)/(x-1)$ *for* $x \neq 1$.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct}=d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★ [Logarithms as solutions for exponentials]

4.1. Prove simple laws of logarithms. CA ★

4.2 Use the definition of logarithms to translate between logarithms in any base. CA ★

4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA ★

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

2.1 Graph all 6 basic trigonometric functions. CA

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★

Geometry

Similarity, Right Triangles, and Trigonometry G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

3.1 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola and graph the equation. [In Mathematics III, this standard addresses only circles and parabolas.] **CA**

Geometric Measurement and Dimension G-GMD

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

M3 Mathematics III

Statistics and Probability

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

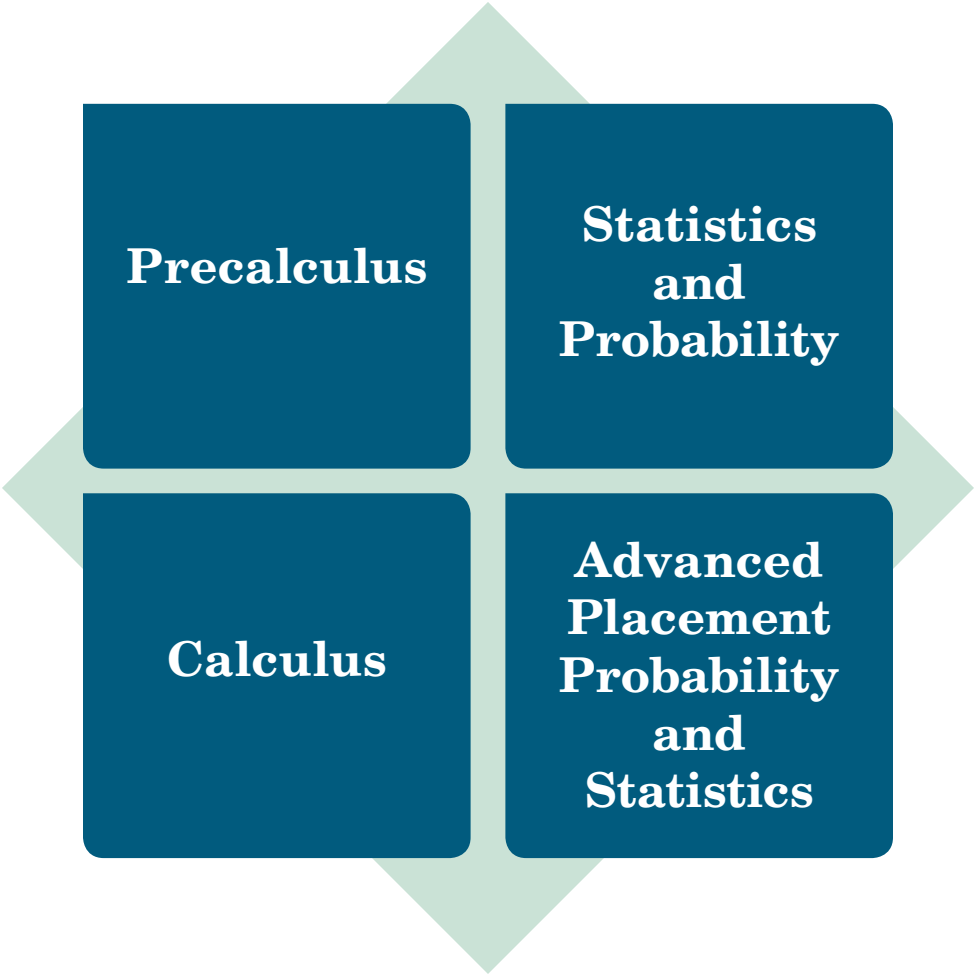
Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions. [Include more complex situations.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

Higher Mathematics Courses

Advanced Mathematics



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Precalculus

Precalculus combines concepts of trigonometry, geometry, and algebra that are needed to prepare students for the study of calculus. The course strengthens students' conceptual understanding of problems and mathematical reasoning in solving problems. Facility with these topics is especially important for students who intend to study calculus, physics, other sciences, and engineering in college. The main topics in the Precalculus course are complex numbers, rational functions, trigonometric functions and their inverses, inverse functions, vectors and matrices, and parametric and polar curves. Because the standards that comprise this course are mostly (+) standards, students who enroll in Precalculus should have met the college- and career-ready standards of the previous courses in the Integrated Pathway or Traditional Pathway. It is recommended that students complete Precalculus before taking an Advanced Placement calculus course.

What Students Learn in Precalculus

Students in Precalculus extend their work with complex numbers, which started in Mathematics III or Algebra II, to see that complex numbers can be represented in the Cartesian plane and that operations with complex numbers have a geometric interpretation. They connect their understanding of trigonometry and the geometry of the plane to express complex numbers in polar form.

Students begin working with vectors, representing them geometrically and performing operations with them. They connect the notion of vectors to complex numbers. Students also work with matrices and their operations, experiencing for the first time an algebraic system in which multiplication is not commutative. Additionally, they see the connection between matrices and transformations of the plane—namely, that a vector in the plane can be multiplied by a 2×2 matrix to produce another vector—and they work with matrices from the perspective of transformations. They also find inverse matrices and use matrices to represent and solve linear systems.

Students extend their work with trigonometric functions, investigating the reciprocal functions *secant*, *cosecant*, and *cotangent* and the graphs and properties associated with those functions. Students find inverse trigonometric functions by appropriately restricting the domains of the standard trigonometric functions and use them to solve problems that arise in modeling contexts.

Although students in Precalculus have worked previously with parabolas and circles, they now work with ellipses and hyperbolas. They also work with polar coordinates and curves defined parametrically and connect these to their other work with trigonometry and complex numbers.

Finally, students work with rational functions that are more complicated, graphing them and determining zeros, y-intercepts, symmetry, asymptotes, intervals for which the function is increasing or decreasing, and maximum or minimum points.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, useful, and logical subject. The Standards for Mathematical Practice represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. Table P-1 presents examples of how students can engage with each MP standard in Precalculus.

Table P-1. Standards for Mathematical Practice—Explanation and Examples for Precalculus

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students expand their repertoire of expressions and functions that can be used to solve problems. They grapple with understanding the connection between complex numbers, polar coordinates, and vectors.
MP.2 Reason abstractly and quantitatively.	Students understand the connection between transformations and matrices, seeing a matrix as an algebraic representation of a transformation of the plane.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students continue to reason through the solution of an equation and justify their reasoning to their peers. They defend their choice of a function to model a real-world situation.
MP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems. They also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MP.5 Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MP.6 Attend to precision.	Students make note of the precise definition of *complex number*, understanding that real numbers are a subset of complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying answers.
MP.7 Look for and make use of structure.	Students understand that matrices form an algebraic system in which the order of multiplication matters, especially when solving linear systems using matrices. They see that complex numbers can be represented by polar coordinates and that the structure of the plane yields a geometric interpretation of complex multiplication.
MP.8 Look for and express regularity in repeated reasoning.	Students multiply several vectors by matrices and observe that some matrices produce rotations or reflections. They compute with complex numbers and generalize the results to understand the geometric nature of their operations.

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a prominent place in instruction. Some standards are marked with a star symbol (★) to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards. Note that this does not preclude other standards from being taught with or through mathematical modeling.

In places where specific MP standards may be implemented with the Precalculus standards, the MP standards are noted in parentheses.

Precalculus Content Standards, by Conceptual Category

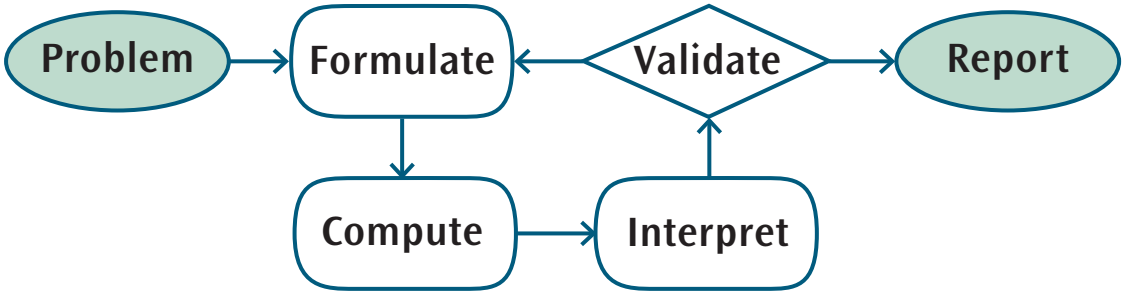
The Precalculus course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in Precalculus are described below, according to each conceptual category. Note that the standards are *not* listed in an order in which they should be taught.

Conceptual Category: Modeling

Throughout the California Common Core State Standards for Mathematics (CA CCSSM), specific standards for higher mathematics are marked with a ★ symbol to indicate they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise (e.g., Which of the quantities present in this situation are known and which are unknown?). Students need to decide on a solution path that may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new equation or function will apply. Additionally, students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, and the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure P-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Figure P-1. The Modeling Cycle



Readers are encouraged to consult appendix B (Mathematical Modeling) for further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

The standards of the Functions conceptual category can set the stage for students to learn other standards in Precalculus. At this level, expressions are often viewed as defining outputs for functions, and algebraic manipulations are then performed meaningfully with an eye toward what can be revealed about the function.

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★

   d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★

   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★

10. (+) **Demonstrate an understanding of functions and equations defined parametrically and graph them. CA** ★

11. (+) **Graph polar coordinates and curves. Convert between polar and rectangular coordinate systems. CA**

Building Functions F-BF

Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   b. (+) Verify by composition that one function is the inverse of another.
   c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
   d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Although many of the standards in the Interpreting Functions and Building Functions domains appeared in previous courses, students now apply them in cases of polynomial functions of degree greater than two, more complicated rational functions, reciprocal trigonometric functions, and inverse trigonometric functions. Students examine end behavior of polynomial and rational functions and learn how to find asymptotes.

Students further their understanding of inverse functions. Previously, students found inverse functions only in simple cases (e.g., solving for x, when $f(x)=c$, finding inverses of linear functions); in Precalculus they explore the relationship between two functions that are inverses of each other (i.e., that f and g are inverses if $(f\circ g)(x)=x$ and $(g\circ f)(x)=x$). They may also begin to use inverse function notation, expressing g as $g=f^{-1}$ (**MP.2, MP.6**). Students in Precalculus construct inverse functions by appropriately restricting the domain of a given function and use inverses in different contexts. They understand how a function and its domain and range are related to its inverse function. They realize that finding an inverse function is more than simply "switching variables" and solving an equation. They can even find simpler inverses mentally, such as when they reverse the "steps" for the equation $f(x)=x^3-1$ to realize that the inverse of f must be $f^{-1}(x)=\sqrt[3]{x+1}$ (**MP.7**).

Students in Precalculus study parametric functions, understanding that a curve in the plane that might describe the path of a moving object can be represented with such functions. Students also work with polar coordinates and graph polar curves. Connections should be made between polar coordinates and the polar representation of complex numbers (**N-CN.4–5**). Students also discover the important role that trigonometric functions play in working with polar coordinates. These standards are new for a typical Precalculus curriculum. Students can investigate these new concepts in modeling situations, such as by recording points on the curve along which a tossed ball travels, graphing the points as vectors and deriving equations for $x(t)$ and $y(t)$ (**MP.4**). They can also investigate the relationship between the graphs of the sine and cosine as functions of θ, as well as the graph of the curve defined by $x(\theta)=\cos\theta$, $y(\theta)=\sin\theta$, drawing connections between these graphs.

Trigonometric Functions	F-TF

Extend the domain of trigonometric functions using the unit circle.

4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. ★

Prove and apply trigonometric identities.

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

10. (+) **Prove the half angle and double angle identities for sine and cosine and use them to solve problems. CA**

These standards call for students to expand their understanding of the trigonometric functions by connecting properties of the functions to the unit circle. For example, students understand that since traveling 2π radians around the unit circle returns one to the same point on the circle, this must be reflected in the graphs of sine and cosine (**MP.8**). Students extend their knowledge of finding inverses to trigonometric functions and use these inverses in a wide range of application problems.

Students in Precalculus derive the addition and subtraction formulas for sine, cosine, and tangent, as well as the half angle and double angle identities for sine and cosine, and make connections among these. For example, students can derive from the addition formula for cosine $(\cos(x+y) = \cos x \cos y - \sin x \sin y)$ the double angle formula for cosine:

$$\cos 2x = \cos(x+x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$$

Another opportunity for connections arises here, as students can investigate the relationship between these formulas and complex multiplication.

Conceptual Category: Number and Quantity

The Number and Quantity standards in Precalculus represent a culmination of students' understanding of number systems. Students investigate the geometry of complex numbers more fully and connect it to operations with complex numbers. Additionally, students develop the notion of a vector and connect operations with vectors and matrices to transformations of the plane.

The Complex Number System — N-CN

Perform arithmetic operations with complex numbers.

3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,* $\left(-1+\sqrt{3}i\right)^3 = 8$ *because* $\left(-1+\sqrt{3}i\right)$ *has modulus 2 and argument 120°.*
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

As mentioned previously, complex numbers, polar coordinates, and vectors should be taught with an emphasis on connections between them. For instance, students connect the addition of complex numbers to the addition of vectors; they also investigate the geometric interpretation of multiplying complex numbers and connect that interpretation to polar coordinates by using the polar representation.

Vector and Matrix Quantities — N-VM

Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

4. (+) Add and subtract vectors.
   a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
   b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
   c. Understand vector subtraction $\boldsymbol{v}-\boldsymbol{w}$ as $\boldsymbol{v}+(-\boldsymbol{w})$, where $-\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
5. (+) Multiply a vector by a scalar.
   a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c\left(v_x, v_y\right) = \left(cv_x, cv_y\right)$.
   b. Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

Vector and Matrix Quantities *(continued)* N-VM

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is non-zero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Students investigate vectors as geometric objects in the plane that can be represented by ordered pairs, and they investigate matrices as objects that act on vectors. By working with vectors and matrices both geometrically and quantitatively, students discover that vector addition and subtraction behave according to certain properties, while matrices and matrix operations observe their own set of rules. Attending to structure, students discover with matrices a new set of mathematical objects and operations involving multiplication that is not commutative. They find inverse matrices by hand in 2×2 cases and use technology in other cases. Work with vectors and matrices sets the stage for solving systems of equations in the Algebra conceptual category.

Conceptual Category: Algebra

In the Algebra conceptual category, Precalculus students work with higher-degree polynomials and rational functions that are more complicated. As always, they attend to the meaning of the expressions they work with, and the expressions they encounter often arise in the context of functions. As in all other higher mathematics courses, students in Precalculus work with creating and solving equations and do so in contexts connected to real-world situations through modeling.

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* ★
2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

Arithmetic with Polynomials and Rational Expressions — A-APR

Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x)+r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

By the time students take Precalculus, they should have a well-developed understanding of the concept of a function. To make work with rational expressions more meaningful, students should be given opportunities to connect rational expressions to rational *functions* (whose outputs are defined by the expressions). For example, a traditional exercise with rational expressions might have the following form:

$$\text{Simplify } \frac{200}{x}+\frac{100}{x-10}$$

The intention here is that students will find a common denominator and transform the expression into $\frac{300x-2000}{x(x-10)}$. In contrast, students could view the two expressions as defining the outputs of two functions—f and g, respectively—where $f(x)=\frac{200}{x}$ and $g(x)=\frac{100}{x-10}$ (**MP.2**). In this case, f could be the function that represents the time it takes for a car to travel 200 miles at an average speed of x miles per hour, and g could be the function that represents the time it takes for the car to travel 100 miles at an average speed that is 10 miles per hour slower (**MP.4**). Students can be asked to consider the domains of the two functions, the domain on which the sum of the two functions defined by $(f+g)(x)=f(x)+g(x)$ makes sense, and what the sum denotes (total time to travel the 300 miles altogether). Furthermore, students can calculate tables of outputs for the two functions using a spreadsheet, add the outputs on the spreadsheet, and then graph the resulting outputs, discovering that the data fit the graph of the equation $y=\frac{300x-2000}{x(x-10)}$. Finally, if these expressions arise in a modeling context, students can interpret the results of studying these functions and their sum in the real-world context.

Creating Equations — A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★

Reasoning with Equations and Inequalities — A-REI

Solve systems of equations.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

Standards **A-CED.1–4** appear in most other higher mathematics courses, as they represent skills that are commonly employed when students work with equations. Students in Precalculus expand these skills into several areas: trigonometric functions, by setting up and solving equations such as $\sin 2\theta = \frac{1}{2}$; parametric functions, by making sense of the equations $x = 2t$; $y = 3t + 1$, and $0 \le t \le 10$; and rational expressions, by sketching a rough graph of equations such as $y = \frac{300x - 2000}{x(x - 10)}$ (**MP.7, MP.8**).

Students use matrix multiplication to connect their newfound knowledge of matrices to the representation of systems of linear equations. They can do this in modeling situations (e.g., those involving economic quantities or geometric elements).

Conceptual Category: Geometry

The standards of the Geometry conceptual category also connect to several other standards in the Precalculus curriculum. For example, students continue to work with conic sections (started in Mathematics III or Algebra II), and they view conic sections as examples of parametric functions (**F-IF.10**).

Similarity, Right Triangles, and Trigonometry — G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

3.1 Given a quadratic equation of the form $ax^2+by^2+cx+dy+e=0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola and graph the equation. CA

Students in Precalculus continue to study trigonometric functions by discovering that these functions can also be used with general (non-right) triangles through the use of appropriate auxiliary lines. The Laws of Sines and Cosines can be derived once these auxiliary lines are introduced into general triangles. Students can then use these laws to solve problems, and they connect the relationships described by the laws to the geometry of vectors.

Precalculus Overview

Number and Quantity

The Complex Number System

- Perform arithmetic operations with complex numbers.
- Represent complex numbers and their operations on the complex plane.

Vector and Matrix Quantities

- Represent and model with vector quantities.
- Perform operations on vectors.
- Perform operations on matrices and use matrices in applications.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.

Arithmetic with Polynomials and Rational Expressions

- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Solve systems of equations.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build new functions from existing functions.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Geometry

Similarity, Right Triangles, and Trigonometry

- Apply trigonometry to general triangles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

Number and Quantity

The Complex Number System — N-CN

Perform arithmetic operations with complex numbers.

3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,* $\left(-1+\sqrt{3}i\right)^3 = 8$ *because* $\left(-1+\sqrt{3}i\right)$ *has modulus 2 and argument 120°.*
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Vector and Matrix Quantities — N-VM

Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

4. (+) Add and subtract vectors.
   a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
   b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
   c. Understand vector subtraction $\boldsymbol{v}-\boldsymbol{w}$ as $\boldsymbol{v}+(-\boldsymbol{w})$, where $-\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
5. (+) Multiply a vector by a scalar.
   a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
   b. Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is non-zero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Algebra

Seeing Structure in Expressions — A-SSE

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of* P *and a factor not depending on* P. ★
2. Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

Arithmetic with Polynomials and Rational Expressions — A-APR

Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.* ★

Reasoning with Equations and Inequalities A-REI

Solve systems of equations.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★

10. (+) Demonstrate an understanding of functions and equations defined parametrically and graph them. CA ★

11. (+) Graph polar coordinates and curves. Convert between polar and rectangular coordinate systems. CA

Building Functions F-BF

Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   b. (+) Verify by composition that one function is the inverse of another.
   c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
   d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. ★

Prove and apply trigonometric identities.

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

10. (+) Prove the half angle and double angle identities for sine and cosine and use them to solve problems. CA

Geometry

Similarity, Right Triangles, and Trigonometry G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

3.1 Given a quadratic equation of the form $ax^2+by^2+cx+dy+e=0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola and graph the equation. CA

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Statistics and Probability

Statistics and Probability offers students an alternative to Precalculus as a fourth high school mathematics course. In the Statistics and Probability course, students continue to develop a more formal and precise understanding of statistical inference, which requires a deeper understanding of probability. Students learn that formal inference procedures are designed for studies in which the sampling or assignment of treatments was random, and these procedures may be less applicable to non-randomized observational studies. Probability is still viewed as long-run relative frequency, but the emphasis now shifts to conditional probability and independence, and basic rules for calculating probabilities of compound events. In the plus (+) standards are the Multiplication Rule, probability distributions, and their expected values. Probability is presented as an essential tool for decision making in a world of uncertainty.

The course may be taught as either a one-semester (half-year) course or a full-year course. Supplementing a one-semester course with additional modeling experiences can extend it to a full-year course.

What Students Learn in Statistics and Probability

Students extend their work in statistics and probability by applying statistics ideas to real-world situations. They link classroom mathematics and statistics to everyday life, work, and decision making by applying these standards in modeling situations. Students select and use appropriate mathematics and statistics to analyze and understand empirical situations and to improve decisions.

Students in Statistics and Probability take their understanding of probability further by studying expected values, interpreting them as long-term relative means of a random variable. They use this understanding to make decisions about both probability games and real-life examples using empirical probabilities.

The fact that numerous standards are repeated from previous courses does not imply that those standards should be omitted from those courses. In keeping with the California Common Core State Standards for Mathematics (CA CCSSM) theme that mathematics instruction should strive for depth rather than breadth, teachers should view this course as an opportunity to delve deeper into those repeated Statistics and Probability standards while addressing new ones.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject. The Standards for Mathematical Practice (MP) represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. Table SP-1 presents examples of how students can engage with the MP standards in the Statistics and Probability course.

Table SP-1. Standards for Mathematical Practice—Explanation and Examples for Statistics and Probability

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students correctly apply statistical concepts to real-world problems. They understand what information is useful and relevant and how to interpret the results they find.
MP.2 Reason abstractly and quantitatively.	Students understand that the outcomes in probability situations can be viewed as *random variables*—that is, functions of the outcomes of a random process, with associated probabilities attached to possible values.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students defend their choice of a function to model data. They pay attention to the precise definitions of concepts such as *causality* and *correlation* and learn how to discern between these two concepts, becoming aware of potential abuses of statistics.
MP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems. They also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MP.5 Use appropriate tools strategically.	Students continue to use spreadsheets and graphing technology as aids in performing computations and representing data.
MP.6 Attend to precision.	Students pay attention to approximating values when necessary. They understand margins of error and know how to apply them in statistical problems.
MP.7 Look for and make use of structure.	Students make use of the normal distribution when investigating the distribution of means. They connect their understanding of theoretical probabilities and find expected values in situations involving empirical probabilities, correctly applying expected values.
MP.8 Look for and express regularity in repeated reasoning.	Students observe that repeatedly finding random sample means results in a distribution that is roughly normal; they begin to understand this as a process for approximating true population means.

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a prominent place in instruction. Some standards are marked with a star (★) symbol to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards.

Statistics and Probability Content Standards, by Conceptual Category

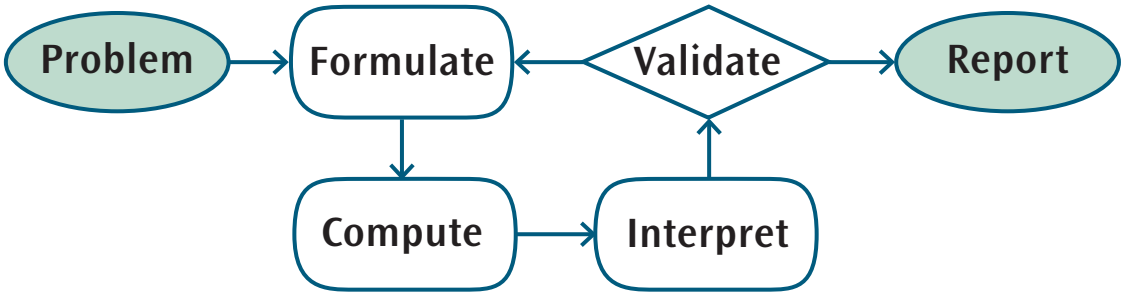
The Statistics and Probability course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in the Statistics and Probability course are described below, according to each conceptual category. Note that the standards are not listed in an order in which they should be taught.

Conceptual Category: Modeling

Throughout the CA CCSSM, specific standards for higher mathematics are marked with a ★ symbol to indicate that they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics to real-world problems. True modeling begins with students asking a question about the world around them, and mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: Which of the quantities present in this situation are known, and which are unknown? Students need to decide on a solution path that may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new equation or function will apply. Additionally, students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, or the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure SP-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Figure SP-1. The Modeling Cycle



Readers are encouraged to consult appendix B (Mathematical Modeling) for further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Statistics and Probability

All of the standards in the Statistics and Probability conceptual category are considered modeling standards, providing a rich ground for studying the content of this course through real-world applications. The first set of standards listed below deals with interpreting data, and although students have already encountered standards **S-ID.1–6**, there are opportunities to refine students' ability to represent data and apply their understanding to the world around them. For instance, students may examine news articles containing data and decide whether the representations used are appropriate or misleading, or they may collect data from students at their school and choose a sound representation for the data.

Interpreting Categorical and Quantitative Data — S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★
4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Summarize, represent, and interpret data on two categorical and quantitative variables.

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★
   b. Informally assess the fit of a function by plotting and analyzing residuals. ★
   c. Fit a linear function for a scatter plot that suggests a linear association. ★

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★

Students understand that the process of fitting and interpreting models for discovering possible relationships between variables requires insight, good judgment, and a careful look at a variety of options consistent with the questions being asked in the investigation. Students work more with the

correlation coefficient, which measures the "tightness" of data points about a line fitted to the data. Students understand that when the correlation coefficient is close to 1 or −1, the two variables are said to be *highly correlated*, and that high correlation does not imply causation (**S-ID.9**). For instance, in a simple grocery store experiment, students compare the cost of different types of frozen pizzas and the calorie content of each. They may find that a scatter plot of this data reveals a relationship that is nearly linear, with a high correlation coefficient. However, students learn to reason that an increase in the cost of a pizza does not necessarily *cause* the calories to increase, just as an increase in calories would not necessarily cause an increase in price. It is more likely that the addition of other expensive ingredients causes both the price and the calorie content to increase together (**MP.2, MP.3, MP.6**).

Making Inferences and Justifying Conclusions — S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Students have encountered standards **S-IC.1–3** in previous courses. However, in Statistics and Probability, students have an opportunity to build on these standards; they can use data from sample surveys to estimate attributes such as the population mean or proportion (**MP.2, MP.4**). With their understanding of the importance of random sampling (**S-IC.3**), students learn that running a simulation and obtaining multiple sample means will yield a roughly normal distribution when plotted as a histogram. They use this to estimate the true mean of the population and can develop a margin of error (**S-IC.4**).

Furthermore, students' understanding of random sampling can now be extended to the random assignment of treatments to available units in an experiment. For example, a clinical trial in medical research may have only 50 patients available for comparing two treatments for a disease. These 50 patients are the population, so to speak, and randomly assigning the treatments to the patients is the "fair" way to judge possible treatment differences, just as random sampling is a fair way to select a sample for estimating a population proportion.

Effects of Caffeine: There is little doubt that caffeine stimulates bodily activity, but how much caffeine does it take to produce a significant effect? This question involves measuring the effects of two or more treatments and deciding if the different interventions have different effects. To obtain a partial answer to the question on caffeine, it was decided to compare a treatment consisting of 200 milligrams (mg) of caffeine, with a control of no caffeine, in an experiment involving a finger-tapping exercise.

Twenty male students were randomly assigned to one of two treatment groups of 10 students each, one group receiving 200 milligrams of caffeine and the other group receiving no caffeine. Two hours later, the students were given a finger-tapping exercise. The response is the number of taps per minute, as shown in the table below.

Finger Taps per Minute in a Caffeine Experiment

	0 mg caffeine	200 mg caffeine
	242	246
	245	248
	244	250
	248	252
	247	248
	248	250
	242	246
	244	248
	246	245
	242	250
Mean	244.8	248.3

Source: Draper and Smith, *Applied Regression Analysis*, John Wiley and Sons, 1981.

The plot of the finger-tapping data shows that the two data sets tend to be somewhat symmetric and have no extreme data points (outliers) that would have undue influence on the analysis. The sample mean for each data set, then, is a suitable measure of center and will be used as the statistic for comparing treatments.

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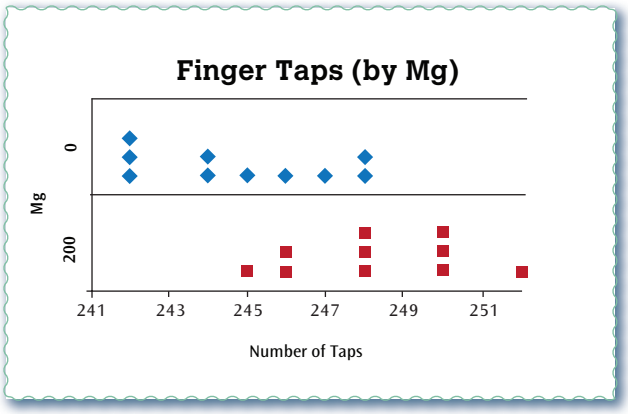
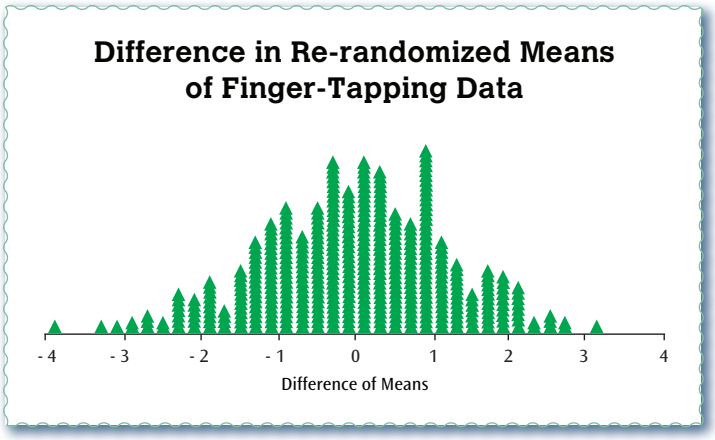


The mean for the 200-mg data is 3.5 taps larger than that for the 0-mg data. In light of the variation in the data, is that enough to be confident that the 200-mg treatment truly results in more tapping activity than the 0-mg treatment? In other words, could this difference of 3.5 taps be explained simply by randomization—the "luck of the draw," so to speak—rather than by any substantive difference in the treatments? An empirical answer to this question can be found by "re-randomizing" the two groups many times and studying the distribution of differences in sample means. If the observed difference of 3.5 occurs quite frequently, then it is safe to say the difference could be caused by the randomization process. However, if the difference does not occur frequently, then there is evidence to support the conclusion that the 200-mg treatment has increased the mean finger-tapping count.

The re-randomizing can be accomplished by combining the data in the two columns, randomly splitting them into two different groups of 10, each representing 0 and 200 mg, and then calculating the difference between the sample means. This can be expedited with the assistance of technology (such as a spreadsheet or statistical software).

The plot below shows the differences produced in 400 re-randomizations of the data for 200 mg and 0 mg. The observed difference of 3.5 taps is equaled or exceeded only once out of 400 times. Because the observed difference is reproduced only 1 time in 400 trials, the data provide strong evidence that the control and the 200-mg treatment do, indeed, differ with respect to their mean finger-tapping counts. In fact, there can be little doubt that the caffeine is the cause of the increase in tapping, because other possible factors should have been balanced out by the randomization (**S-IC.5**). Students should be able to explain the reasoning in this decision and the nature of the error that may have been made.



Adapted from University of Arizona (UA) Progressions Documents for the Common Core Math Standards 2012d, 10–11.

Conditional Probability and the Rules of Probability S-CP

Understand independence and conditional probability and use them to interpret data.

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*. ★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* ★

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, *P*(*A* or *B*) = *P*(*A*) + *P*(*B*) – *P*(*A* and *B*), and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, *P*(*A* and *B*) = *P*(*A*)*P*(*B*|*A*) = *P*(*B*)*P*(*A*|*B*), and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

Students can deepen their understanding of the rules of probability, especially when finding probabilities of compound events as called for in standards **S-CP.7–9**. Students can generalize from simpler events that exhibit independence (such as rolling number cubes) to understand that independence is often used as a simplifying assumption in constructing theoretical probability models that approximate real situations. For example, suppose a school laboratory has two smoke alarms as a built-in redundancy for safety. One alarm has a probability of 0.4 of going off when steam (not smoke) is produced by running hot water, and the other has a probability of 0.3 for the same event (**MP.2, MP.4**). The probability that both alarms go off the next time someone runs hot water in the sink can be reasonably approximated as the product $0.4 \times 0.3 = 0.12$, even though there may be some dependence between the two systems in the same room.

Using Probability to Make Decisions — S-MD

Calculate expected values and use them to solve problems.

1. (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions. ★
2. (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution. ★
3. (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.* ★
4. (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?* ★

Use probability to evaluate outcomes of decisions.

5. (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. ★
   a. Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.* ★
   b. Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.* ★
6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

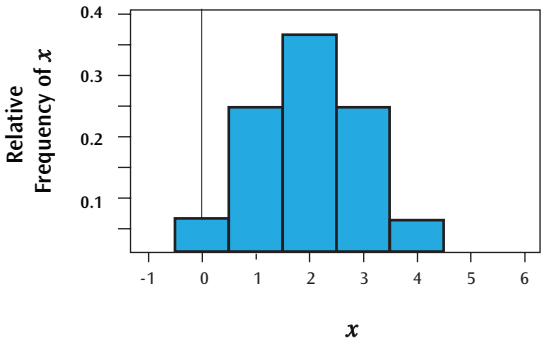
The standards of the **S-MD** domain allow students the opportunity to apply concepts of probability to real-world situations. For example, a political pollster will want to know how many people are likely to vote for a particular candidate, and a student may want to know the effectiveness of guessing on a true–false quiz. Students in Statistics and Probability begin to see the outcomes in such situations as *random variables*—functions of the outcomes of a random process, with associated probabilities attached to their possible values (**MP.2**).

For example, after students have calculated the probabilities of obtaining 0, 1, 2, 3, or 4 correct answers by guessing on a four-question true–false quiz, they can construct the following probability distribution with statistical software (**MP.5**).

True–False Quiz

Number Correct (x)	Probabliity
0	$\frac{1}{16}$
1	$\frac{4}{16}$
2	$\frac{6}{16}$
3	$\frac{4}{16}$
4	$\frac{1}{16}$

Number of Correct Answers



Students can consider the probabilities as long-run frequencies and average the probabilities to come up with a mean score:

$$0 \bullet \frac{1}{16} + 1 \bullet \frac{4}{16} + 2 \bullet \frac{6}{16} + 3 \bullet \frac{4}{16} + 4 \frac{1}{16} = 2$$

Students interpret this as saying that someone who guesses on four-question true–false quizzes can expect, over the long run, to get two correct answers per quiz. Students can generalize this example to develop the general rule that for any discrete random variable X, the *expected value of* X is given by:

$$E(X) = \sum (\text{value of } X)\,(\text{probability of that value}).$$

Students interpret the expected value of a random variable in situations such as games of chance or insurance payouts based on the probability of having an automobile accident. Although the probability distribution shown above comes from theoretical probabilities, students can also use probabilities based on empirical data to make similar calculations in applied problems.

For more information about this collection of standards and student learning expectations, readers should consult the University of Arizona Progressions document titled "High School Statistics and Probability": http://ime.math.arizona.edu/progressions/ (UA Progressions Documents 2012d [accessed April 6, 2015]).

Statistics and Probability ★

Overview

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.
- Summarize, represent, and interpret data on two categorical and quantitative variables.
- Interpret linear models.

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Conditional Probability and the Rules of Probability

- Understand independence and conditional probability and use them to interpret data.
- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- Calculate expected values and use them to solve problems.
- Use probability to evaluate outcomes of decisions.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★
4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Summarize, represent, and interpret data on two categorical and quantitative variables.

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★
   b. Informally assess the fit of a function by plotting and analyzing residuals. ★
   c. Fit a linear function for a scatter plot that suggests a linear association. ★

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★

4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Conditional Probability and the Rules of Probability S-CP

Understand independence and conditional probability and use them to interpret data.

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
3. Understand the conditional probability of *A* given *B* as $P(A \text{ and } B)/P(B)$, and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*. ★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* ★

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model. ★
7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. ★
8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model. ★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★

Using Probability to Make Decisions S-MD

Calculate expected values and use them to solve problems.

1. (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions. ★
2. (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution. ★

3. (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.* ★

4. (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?* ★

Use probability to evaluate outcomes of decisions.

5. (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. ★

   a. Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.* ★

   b. Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.* ★

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★

7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

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Calculus*

When taught in high school, calculus should be presented with the same level of depth and rigor as are entry-level college and university calculus courses. The content standards presented in this chapter outline a complete college curriculum in one-variable calculus. Many high school programs may have insufficient time to cover all of the following content in a typical academic year. For example, some districts may treat differential equations lightly and spend substantial time on infinite sequences and series; other districts may do the opposite. Consideration of the College Board syllabi for the Calculus AB and Calculus BC sections of the Advanced Placement Examination in Mathematics may be helpful in making curricular decisions.† Calculus is a widely applied area of mathematics and involves a beautiful intrinsic theory. Students who master this content will be exposed to both aspects of the subject.

The sample problems included in this chapter are meant to illustrate and clarify the content standards. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

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*This chapter is taken from the 2005 *Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* (CDE 2006). The standards presented in the chapter were first adopted in 1997 and were unchanged in the 2010 adoption of the California Common Core State Standards for Mathematics.

†Advanced Placement (AP) course descriptions are updated regularly. Please visit AP Central (http://apcentral.collegeboard.com/) to determine whether a more recent course description is available.

1.0 Students demonstrate knowledge of both the formal definition and the graphical interpretation of limit of values of functions. This knowledge includes one-sided limits, infinite limits, and limits at infinity. Students know the definition of convergence and divergence of a function as the domain variable approaches either a number or infinity:

1.1 Students prove and use theorems evaluating the limits of sums, products, quotients, and composition of functions.

1.2 Students use graphical calculators to verify and estimate limits.

1.3 Students prove and use special limits, such as the limits of $\frac{\sin(x)}{x}$ and $\frac{1-\cos(x)}{x}$ as x tends to 0.

Sample Problems:

Evaluate the following limits, justifying each step:

$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$

$\lim_{x\to 0} \frac{1-\cos(2x)}{\sin(3x)}$

$\lim_{x\to \infty} \left(x-\sqrt{x^2-x}\right)$

2.0 Students demonstrate knowledge of both the formal definition and the graphical interpretation of continuity of a function.

Sample Problem:

For what values of x is the function $f(x)=\frac{x^2-1}{x^2-4x+3}$ continuous? Explain.

3.0 Students demonstrate an understanding and the application of the intermediate value theorem and the extreme value theorem.

4.0 Students demonstrate an understanding of the formal definition of the derivative of a function at a point and the notion of differentiability:

4.1 Students demonstrate an understanding of the derivative of a function as the slope of the tangent line to the graph of the function.

4.2 Students demonstrate an understanding of the interpretation of the derivative as an instantaneous rate of change. Students can use derivatives to solve a variety of problems from physics, chemistry, economics, and so forth that involve the rate of change of a function.

4.3 Students understand the relation between differentiability and continuity.

4.4 Students derive derivative formulas and use them to find the derivatives of algebraic, trigonometric, inverse trigonometric, exponential, and logarithmic functions.

Sample Problem:

Find all the points on the graph of $f(x)=\frac{x^2-2}{x+1}$ where the tangent line is parallel to the tangent line at $x=1$.

5.0 Students know the chain rule and its proof and applications to the calculation of the derivative of a variety of composite functions.

6.0 Students find the derivatives of parametrically defined functions and use implicit differentiation in a wide variety of problems in physics, chemistry, economics, and so forth.

Sample Problem:

For the curve given by the equation $\sqrt{x}+\sqrt{y}=4$, use implicit differentiation to find $\frac{d^2y}{dx^2}$.

7.0 Students compute derivatives of higher orders.

8.0 Students know and can apply Rolle's theorem, the mean value theorem, and L'Hôpital's rule.

9.0 Students use differentiation to sketch, by hand, graphs of functions. They can identify maxima, minima, inflection points, and intervals in which the function is increasing and decreasing.

10.0 Students know Newton's method for approximating the zeros of a function.

11.0 Students use differentiation to solve optimization (maximum–minimum problems) in a variety of pure and applied contexts.

Sample Problem:

A man in a boat is 24 miles from a straight shore and wishes to reach a point 20 miles down shore. He can travel 5 miles per hour in the boat and 13 miles per hour on land. Find the minimal travel time for him to reach his destination and where along the shore he should land the boat to arrive as soon as possible.

12.0 Students use differentiation to solve related rate problems in a variety of pure and applied contexts.

13.0 Students know the definition of the definite integral by using Riemann sums. They use this definition to approximate integrals.

Sample Problem:

The following is a Riemann sum that approximates the area under the graph of a function $f(x)$, between $x=a$ and $x=b$. Determine a possible formula for the function $f(x)$ and for the values of a and b: $\sum_{i=1}^{n}\frac{2}{n}e^{1+\frac{2i}{n}}$.

14.0 Students apply the definition of the integral to model problems in physics, economics, and so forth, obtaining results in terms of integrals.

15.0 Students demonstrate knowledge and proof of the fundamental theorem of calculus and use it to interpret integrals as antiderivatives.

Sample Problem:

If $f(x)=\int_1^x \sqrt{1+t^3}\,dt$, find $f'(2)$.

16.0 Students use definite integrals in problems involving area, velocity, acceleration, volume of a solid, area of a surface of revolution, length of a curve, and work.

17.0 Students compute, by hand, the integrals of a wide variety of functions by using techniques of integration, such as substitution, integration by parts, and trigonometric substitution. They can also combine these techniques when appropriate.

Sample Problem:

Evaluate the following:

$\int \frac{\sin\left(1-\sqrt{x}\right)}{\sqrt{x}}dx$ $\int_1^e \frac{\ln x}{\sqrt{x}}dx$ $\int_0^1 \sqrt{1+\sqrt{x}}\,dx$

$\int \arctan x\,dx$ $\int \frac{\sqrt{x^2-1}}{x^3}dx$ $\int \frac{dx}{e^x\sqrt{1-e^{2x}}}$

18.0 Students know the definitions and properties of inverse trigonometric functions and the expression of these functions as indefinite integrals.

19.0 Students compute, by hand, the integrals of rational functions by combining the techniques in standard 17.0 with the algebraic techniques of partial fractions and completing the square.

20.0 Students compute the integrals of trigonometric functions by using the techniques noted above.

21.0 Students understand the algorithms involved in Simpson's rule and Newton's method. They use calculators or computers or both to approximate integrals numerically.

22.0 Students understand improper integrals as limits of definite integrals.

23.0 Students demonstrate an understanding of the definitions of convergence and divergence of sequences and series of real numbers. By using such tests as the comparison test, ratio test, and alternate series test, they can determine whether a series converges.

Sample Problem:

Determine whether the following alternating series converge absolutely, converge conditionally, or diverge:

$$\sum_{n=3}^{\infty}(-1)^n\left(\frac{2^n}{n!}\right) \qquad \sum_{n=3}^{\infty}\frac{(-1)^n}{n\ln n} \qquad \sum_{n=3}^{\infty}(-1)^n\left(\frac{1+n}{n+\ln n}\right)$$

24.0 Students understand and can compute the radius (interval) of the convergence of power series.

25.0 Students differentiate and integrate the terms of a power series in order to form new series from known ones.

26.0 Students calculate Taylor polynomials and Taylor series of basic functions, including the remainder term.

27.0 Students know the techniques of solution of selected elementary differential equations and their applications to a wide variety of situations, including growth-and-decay problems.

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Advanced Placement Probability and Statistics*

When taught in high school, Advanced Placement Probability and Statistics should be presented with the same level of depth and rigor as are entry-level college and university statistics and probability courses. The content standards presented in this chapter are technical and in-depth extensions of probability and statistics. In particular, mastery of academic content for advanced placement gives students the background to succeed in the Advanced Placement examination for this subject. Consideration of the College Board syllabi for the Statistics and Probability sections of the Advanced Placement Examination in Mathematics may be helpful in making curricular decisions.†

The sample problems in this chapter are meant to illustrate and clarify the content standards. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

*This chapter is taken from the 2005 *Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* (CDE 2006). The standards presented in the chapter were first adopted in 1997 and were unchanged in the 2010 adoption of the California Common Core State Standards for Mathematics.

†Advanced Placement (AP) course descriptions are updated regularly. Please visit AP Central (http://apcentral.collegeboard.com/) to determine whether a more recent course description is available.

1.0 Students solve probability problems with finite sample spaces by using the rules for addition, multiplication, and complementation for probability distributions and understand the simplifications that arise with independent events.

2.0 Students know the definition of *conditional probability* and use it to solve for probabilities in finite sample spaces.

Sample Problem:

You have 5 coins in your pocket: 1 penny, 2 nickels, 1 dime, and 1 quarter. If you pull out 2 coins at random and they are collectively worth more than 10 cents, what is the probability that you pulled out a quarter?

3.0 Students demonstrate an understanding of the notion of *discrete random variables* by using this concept to solve for the probabilities of outcomes, such as the probability of the occurrence of five or fewer heads in 14 coin tosses.

4.0 Students understand the notion of a *continuous random variable* and can interpret the probability of an outcome as the area of a region under the graph of the probability density function associated with the random variable.

Sample Problem:

Consider a continuous random variable x whose possible values are numbers between 0 and 2 and whose probability density function is given by $f(x) = 1 - \frac{1}{2}x$ for $0 \le x \le 2$. What is the probability that $x > 1$?

5.0 Students know the definition of the *mean of a discrete random variable* and can determine the mean for a particular discrete random variable.

6.0 Students know the definition of the *variance of a discrete random variable* and can determine the variance for a particular discrete random variable.

7.0 Students demonstrate an understanding of the standard distributions (normal, binomial, and exponential) and can use the distributions to solve for events in problems in which the distribution belongs to those families.

Sample Problem:

Suppose that X is a normally distributed random variable with mean $m = 0$. If $P(X < c) = \frac{2}{3}$, find $P(-c < X < c)$.

8.0 Students determine the mean and the standard deviation of a normally distributed random variable.

9.0 Students know the central limit theorem and can use it to obtain approximations for probabilities in problems of finite sample spaces in which the probabilities are distributed binomially.

10.0 Students know the definitions of the *mean*, *median*, and *mode* of distribution of data and can compute each of them in particular situations.

11.0 Students compute the variance and the standard deviation of a distribution of data.

12.0 Students find the line of best fit to a given distribution of data by using least squares regression.

13.0 Students know what the *correlation coefficient of two variables* means and are familiar with the coefficient's properties.

14.0 Students organize and describe distributions of data by using a number of different methods, including frequency tables, histograms, standard line graphs and bar graphs, stem-and-leaf displays, scatterplots, and box-and-whisker plots.

15.0 Students are familiar with the notions of a statistic of a distribution of values, of the sampling distribution of a statistic, and of the variability of a statistic.

16.0 Students know basic facts concerning the relation between the mean and the standard deviation of a sampling distribution and the mean and the standard deviation of the population distribution.

17.0 Students determine confidence intervals for a simple random sample from a normal distribution of data and determine the sample size required for a desired margin of error.

18.0 Students determine the *P*-value for a statistic for a simple random sample from a normal distribution.

19.0 Students are familiar with the *chi*-square distribution and *chi*-square test and understand their uses.

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Universal Access

The California Common Core State Standards for Mathematics (CA CCSSM) articulate rigorous grade-level expectations. These standards provide a historic opportunity to improve access to rigorous academic content for all students, including students with special needs. All students should be held to the same high expectations outlined in the mathematical practices and the content standards (both of which compose the CA CCSSM), although some students may require additional time, language support, and appropriate instructional support as they acquire knowledge of mathematics. Effective education of all students includes closely monitoring student progress, identifying student learning needs, and adjusting instruction accordingly. Regular and active participation in the classroom—not only solving problems and listening, but also discussing, explaining, reading, writing, representing, and presenting—is critical to each student's success in mathematics.

This chapter uses an overarching approach to address the instructional needs of students in California. Although suggestions and strategies for mathematics instruction are provided, they are not intended to—nor could they be expected to—offer teachers and other educators a road map for effectively meeting the instructional needs of every student. The instructional needs of each student are unique and change over time. Therefore, high-quality curriculum, purposeful planning, uninterrupted and protected instructional time, scaffolding, flexible grouping strategies, differentiation, and progress monitoring are essential components of ensuring universal access to mathematics learning.

The first sections in this chapter discuss planning for universal access, differentiation, Universal Design for Learning, the new language demands of the CA CCSSM, assessment for learning, and California's Multi-Tiered System of Supports (MTSS). Later sections focus on students with targeted instructional needs: students with disabilities, English learners, at-risk learners, and advanced learners.

Planning for Universal Access

The ultimate goal of mathematics programs in California is to ensure universal access to high-quality curriculum and instruction so that all students are prepared for college and careers. By carefully planning to modify curriculum, instruction, grouping, and assessment techniques, teachers can be well prepared to adapt to the diversity in their classrooms. *Universal access* in education is a concept that encompasses planning for the widest variety of learners from the beginning of the lesson design process; it should not be "added on" as an afterthought. Likewise, universal access is not a set of curriculum materials or specific time set aside for additional assistance; rather, it is a schema. For students to benefit from universal access, some teachers may need assistance in planning instruction, differentiating curriculum, utilizing flexible grouping strategies, and using the California English Language Development Standards (CA ELD standards) in tandem with the CA CCSSM. Teachers need to employ many different strategies to help all students meet the increased demands of the CA CCSSM.

For all students, it is important that teachers use a variety of instructional strategies—but this is *essential* for students with special needs. Below are some of the strategies that are important to consider when planning for universal access:

- Assess each student's mathematical skills and understandings at the start of instruction to uncover strengths and weaknesses.
- Assess or be aware of the English language development level of English learners.
- Differentiate instruction, focusing on the mathematical practice standards, the concepts within the content standards, and the needs of the students.
- Utilize formative assessments on an ongoing basis to modify instruction and reevaluate student placement or grouping.
- Create a safe environment and encourage students to ask questions.
- Draw upon students' literacy skills and content knowledge in their primary language.
- Engage in careful planning and organization with the various needs of all learners in mind and in collaboration with specialists (e.g., instructional coaches, teachers of special education, and so forth).
- Engage in backward and cognitive planning[1] to fill in gaps involving skills and knowledge and to address common misunderstandings.
- Use the principles of Universal Design for Learning (UDL) when modifying curriculum and planning lessons.
- Utilize the University of Arizona (UA) Progressions Documents for the Common Core Math Standards (UA 2011–13) to understand how mathematical concepts are developed at each grade level and to identify strategies to address individual student needs. The Progressions documents are available at http://ime.math.arizona.edu/progressions/ (accessed July 16, 2015).
- When necessary, organize lessons in a manner that includes sufficient modeling and guided practice before moving to independent practice. This is also known as *gradual release of responsibility*.
- Pre-teach routines to address changing seating arrangements (e.g., groups) and other classroom procedures.
- Use multiple representations (e.g., math drawings, manipulatives, and other forms of technology) to explain concepts and procedures.
- Allow students to demonstrate their understanding and skills in a variety of ways.
- Employ flexible grouping strategies.
- Provide frequent opportunities for students to collaborate and engage in mathematical discourse.

1. *Backward planning* identifies key areas such as prior knowledge needed, common misunderstandings, organizing information, key vocabulary, and student engagement. Backward planning is what will be included in a lesson or unit to support intended student learning. *Cognitive planning* focuses on how instruction will be delivered, anticipates potential student responses and misunderstandings, and provides opportunities to check for understanding and re-teaching during the delivery of the lesson. Backward planning determines what elements will be included; cognitive planning determines how those elements will be delivered.

- Include activities that allow students to discuss concepts and their thought processes.
- Emphasize and pre-teach (when necessary) academic and discipline-specific vocabulary.
- When students are learning to engage in mathematical discourse, provide them with language models and structures (such as sentence frames).
- Explore technology and consider using it along with other instructional devices.
- For advanced learners, deepen the complexity of lessons or accelerate the pace of student learning.

Additional suggestions to support students who have learning difficulties are provided in appendix E (Possible Adaptations for Students with Learning Difficulties in Mathematics). This list of possible adaptations addresses a range of students, some of whom may have identified instructional needs and others who are struggling unproductively for unidentified reasons. If a student has an individualized education program (IEP) or 504 Plan, the strategies, accommodations, or modifications in the plan guide the teacher on how to differentiate instruction. Additional adaptions should be used only when they are consistent with the IEP or 504 Plan.

Differentiation

Differentiated (or modified) instruction helps students with diverse academic needs master the same challenging grade-level academic content as students without special needs (California Department of Education [CDE] 2015b). In differentiated instruction, the method of delivery changes—not the topic of the instruction. Instructional decisions are based on the results of appropriate and meaningful student assessments. Differentiated instruction helps to provide a variety of ways for individual students to take in new information, assimilate it, and demonstrate what they have learned (CDE 2015b).

Differentiation is the foundation for universal access. As Carol Ann Tomlinson has written, "In a differentiated classroom, the teacher proactively plans and carries out varied approaches to content, process, and product in anticipation of and response to student differences in readiness, interest, and learning needs" (Tomlinson 2001, 7). For example, a teacher could differentiate content (*what* the student learns) based on readiness, interest, or learning profile. The same holds true for differentiating process (*how* the student learns) and product (the way the student communicates what he has learned) based on readiness, interest, or learning profile. These pieces of differentiation are all closely intertwined and often cannot be separated into individual practices.

> **"In a differentiated classroom, the teacher proactively plans and carries out varied approaches to content, process, and product in anticipation of and response to student differences in readiness, interest, and learning needs."**
>
> —Carol Ann Tomlinson (Tomlinson 2001, 7)

Research indicates that a student is most likely to learn content when the lesson presents tasks that may be "moderately challenging." When a student can complete an assignment independently, with little effort, new learning does not occur. On the other hand, when the material is presented in a manner that is too difficult, then "frustration, not learning, is the result" (Cooper 2006, 154). This idea is also at the heart of Vygotsky's "Zone of Proximal Development" (Vygotsky 1978). Advanced learners and students with

learning difficulties in mathematics often require systematically planned differentiation strategies to ensure that they experience appropriately challenging curriculum and instruction. This section looks at four modes of differentiation: depth, pacing, complexity, and novelty. Many of the strategies presented can benefit all students, not just those with special needs.

Depth

Depth of understanding refers to how concepts are represented and connected by learners. The greater the number and strength of the connections, the deeper the understanding is. In order to help students develop depth of understanding, teachers need to provide opportunities to build on students' current understanding and assist them in making connections between previously learned content and new content (Grotzer 1999).

Differentiation is achieved by increasing the depth to which a student explores a curricular topic. The CA CCSSM raise the level of cognitive demand through the Standards for Mathematical Practice (MP) as well as grade-level and course-level Standards for Mathematical Content. Targeted instruction is beneficial when it is coupled with adjustments to the level of cognitive demand (LCD). The LCD is the degree of thinking and ownership required in the learning situation. The more complex the thinking and the more ownership (invested interest) students have for learning, the higher the LCD. Likewise, a lower LCD requires straightforward, more simplistic thinking and less ownership by the students. Having high expectations for all students is critically important; however, posing a consistently high LCD can actually set up some students for failure. Similarly, posing a consistently low LCD for students is not pedagogically appropriate and is unlikely to result in new learning. To meet the instructional needs of the students, the LCD must be adjusted at the time of instruction (Taylor-Cox 2008). One strategy that teachers can use is tiered assignments with varied levels of cognitive demand to ensure that students explore the same essential ideas at a level that builds on their prior knowledge; this is appropriately challenging and prompts continual growth.

Pacing

Slowing down or speeding up instruction is referred to as *pacing*. This is perhaps the most common strategy that teachers employ for differentiation; it can be simple and inexpensive to implement, yet it can prove effective for many students with special needs (Benbow and Stanley 1996; Geary 1994). An example of pacing for advanced learners is to collapse a year's course into one semester by moving quickly through the material the students already know (curriculum compacting) without sacrificing either depth of understanding or application of mathematics to novel situations. Alternatively, students may move on to the content standards for the next grade level (accelerating). Caution is warranted to ensure that students are not placed in mathematics courses for which they are not adequately prepared—in particular, placing unprepared students in Mathematics I or Algebra I at middle school (see appendix D, Course Placements and Sequences, for additional information and guidance). Two recent studies on middle school mathematics report that grade-eight students are often placed in Mathematics I or Algebra I courses for which they are not ready, a practice that sets up many students for failure (Finkelstein et al. 2012; Williams et al. 2011).

For students whose achievement is below grade level in mathematics, an increase in instructional time may be appropriate. The amount of additional instructional time, in terms of both duration and frequency, depends on the unique needs of each student. Frequent use of formal and informal formative assessments of conceptual understanding, procedural skill and fluency, and application informs both the teacher and the student about progress toward instructional goals, and instructional pacing should be modified based on the student's progress (Newman-Gonchar, Clarke, and Gersten 2009).

Complexity

Understanding within and across disciplines is referred to as *complexity*. Modifying instruction by complexity requires teacher professional learning and collaboration and instructional materials that lend themselves to such variations. Complexity involves uncovering relationships between and among ideas, connecting other concepts, and using an interdisciplinary approach to the content. When students engage in a performance task or real-world problem, they must apply their mathematical knowledge and skills and knowledge of other subjects (Kaplan, Gould, and Siegel 1995).

For all students, but especially students who experience difficulty in mathematics, teachers should focus on the foundational skills, procedures, and concepts in the standards. Several studies have found that the use of visual representations and manipulatives can improve students' proficiency. Number lines, math drawings, pictorial representations, and other types of visual representations are effective scaffolds. However, if visual representations are not sufficient, concrete manipulatives should be incorporated into instruction (Gersten et al. 2009).

Teachers can differentiate the complexity of a task to maximize student learning outcomes. For students with special needs, differentiation is sometimes questioned by those who say that struggling students never progress to more interesting or complex assignments. It is important to focus on essential concepts embedded in the standards and on frequent assessment to ensure that students are prepared with the understanding and skills they will need to succeed in subsequent grades. Struggling students are expected to learn the concepts well so that they develop a foundation on which further mathematical understanding can be built; this can be accomplished through well-chosen and interesting tasks and problems. See the section on California's MTSS and Response to Instruction and Intervention (RtI^2) for additional information. Advanced students benefit from a combination of self-paced instruction and enrichment (National Mathematics Advisory Panel 2008).

Novelty

Keeping students engaged in learning is an ongoing instructional challenge that can be complicated by the varied instructional needs of students. Novelty is one differentiation strategy that is primarily student-initiated and can increase student engagement. Teachers can introduce novelty by encouraging students to re-examine or reinterpret their understanding of previously learned information. Students can look for ways to connect knowledge and skills across disciplines or between topics in the same discipline. Teachers can work with students to help them learn in more personalized, individualistic, and non-traditional ways. This approach may involve a performance task or real-world problem on a subject that interests the student and requires the student to use mathematics understandings and skills in new or more in-depth ways (Kaplan, Gould, and Siegel 1995).

Universal Design for Learning

As noted by Diamond (2004, 1), "*Universal access* refers to the teacher's scaffolding of instruction so all students have the tools they need to be able to access information. *Universal design* typically refers to those design principles and elements that make materials more accessible to more children—larger fonts, headings, and graphic organizers, for example." Diamond also comments that "[j]ust as designing entrance ramps into buildings makes access to individuals in wheelchairs easier, curriculum may also be designed to be easier to use. When principles of universal design are applied to curriculum materials, universal access is more likely" (Diamond 2004, 1).

Universal Design for Learning (UDL) is a framework for implementing the concepts of universal access by providing equal opportunities to learn for *all* students. Based on the premise that one-size-fits-all curricula create barriers to learning for many students, UDL helps teachers design curricula to meet the varied instructional needs of all of their students.

The purpose of UDL curricula is to help students become "expert learners" who are (a) strategic, skillful, and goal directed; (b) knowledgeable; and (c) purposeful and motivated to learn more (Center for Applied Special Technology [CAST] 2011, 7).

Goals of UDL

- Improve access, participation, and achievement for students.
- Eliminate or reduce physical and academic barriers.
- Value diversity through proactive design.

Source: CAST 2011.

The UDL guidelines developed by CAST are strategies to help teachers make curricula more accessible to all students. The guidelines are based on three primary principles of UDL and are organized under each of the principles as follows.[2]

Principle I: Provide Multiple Means of Representation (the "what" of learning)

Guideline 1: Provide options for perception.

Guideline 2: Provide options for language, mathematical expressions, and symbols.

Guideline 3: Provide options for comprehension.

The first principle allows flexibility so that mathematical concepts can be taught in a variety of ways to address the background knowledge and learning needs of students. For example, presentation of content for a geometry lesson could utilize multiple media that include written, graphic, audio, and interactive technology. Similarly, the presentation of content will include a variety of lesson formats, instructional strategies, and student grouping arrangements (Miller 2009, 493).

Principle II: Provide Multiple Means of Action and Expression (the "how" of learning)

Guideline 4: Provide options for physical action.

Guideline 5: Provide options for expression and communication.

Guideline 6: Provide options for executive functions.

2. For more information on UDL, including explanations of the principles and guidelines and the detailed checkpoints for each guideline, visit the National Center on Universal Design for Learning Web page at http://www.udlcenter.org/aboutudl/udlguidelines (CAST 2011).

The second principle allows for flexibility in how students demonstrate understanding of mathematical content. For example, when explaining the subtraction algorithm, students in grade four may use concrete materials, draw diagrams, create a graphic organizer, or deliver an oral report or a multimedia presentation (Miller 2009, 493).

Principle III: Provide Multiple Means of Engagement (the "why" of learning)

Guideline 7: Provide options for recruiting interest.

Guideline 8: Provide options for sustaining effort and persistence.

Guideline 9: Provide options for self-regulation.

The third principle aims to ensure that all students maintain their motivation to participate in mathematical learning. Alternatives are provided that are based upon student needs and interests, as well as "(a) the amount of support and challenge provided, (b) novelty and familiarity of activities, and (c) developmental and cultural interests" (Miller 2009, 493). Assignments provide multiple entry points with adjustable challenge levels. For example, students in grade six may gather, organize, summarize, and present data to describe the results of a survey of their own design. In order to develop self-regulation, students reflect upon their mathematical learning through a choice of journals, check sheets, learning logs, or portfolios and are provided with encouraging and constructive teacher feedback through a variety of formative assessment measures that demonstrate student strengths and areas where growth is still necessary.

Although it takes considerable time and effort to develop curriculum and plan instruction based on UDL principles, all students can benefit from an accessible and inclusive environment that reflects a universal design approach—and this type of environment is essential for learners with special needs. Teachers and other educators should be provided with opportunities for professional learning on UDL, time for curriculum development and instructional planning, and necessary resources (e.g., equipment, software, instructional materials) to effectively implement UDL. For example, interactive whiteboards can be a useful tool for providing universally designed instruction and engaging students in learning. Teachers and students can use these whiteboards to explain concepts or illustrate procedures. The large images projected onto whiteboards can be seen by most students, including those who have visual disabilities (DO-IT 2012).

New Language Demands of the CA CCSSM

Students who learn mathematics based on the CA CCSSM face increased language demands during mathematics instruction. Students are asked to engage in discussions about mathematics topics, explain their reasoning, demonstrate their understanding, and listen to and critique the reasoning of others. These increased language demands may pose challenges for all students and even greater challenges for both English learners and students who are reading or writing below grade level. These language expectations are made explicit in several of the standards for mathematical practice. Standard **MP.3**, "Construct viable arguments and critique the reasoning of others," states an expectation that students will justify their conclusions, communicate their conclusions to others, and respond to the arguments of others. It also states that students at all grade levels can listen to or read the arguments of others, decide whether those arguments make sense, and ask useful questions to clarify or improve

arguments. Standard **MP.6**, "Attend to precision," asks students to communicate precisely with each other, use clear definitions in discussions with others and in their own reasoning, and that beginning in the elementary grades, students offer carefully formulated explanations to each other. Standard **MP.1**, "Make sense of problems and persevere in solving them," states that students can explain correspondences between equations, verbal descriptions, tables, and graphs.

Standards that call for students to *describe*, *explain*, *demonstrate*, and *understand* provide opportunities for students to engage in speaking and writing about mathematics. These standards appear at all grade levels. For example, in grade two, standard **2.OA.9** asks students to explain why addition and subtraction strategies work. Another example occurs in the Algebra conceptual category of higher mathematics: standard **A-REI.1** requires students to explain each step in solving a simple equation and to construct a viable argument to justify a solution method.

To support students' ability to express their understanding of mathematics, teachers need to explicitly teach not only the language of mathematics, but also academic language for argumentation (*proof, theory, evidence, in conclusion, therefore*), sequencing (*furthermore, additionally*), and relationships (*compare, contrast, inverse, opposite*). Pre-teaching vocabulary and key concepts allows students to be actively engaged in learning during lessons. To help students organize their thinking, teachers may need to scaffold with graphic organizers and sentence frames (also called *communication guides*).

The CA CCSSM call for students to read and write in mathematics to support their learning. According to Bosse and Faulconer (2008), "Students learn mathematics more effectively and more deeply when reading and writing is directed at learning mathematics" (Bosse and Faulconer 2008, 8). Mathematics text is informational text that requires different skills to read than those used when reading narrative texts. The pages in a mathematics textbook or journal article can include text, diagrams, tables, and symbols that are not necessarily read from left to right. Students may need specific instruction on how to read and comprehend mathematics text.

Writing in mathematics also requires different skills than writing in other subjects. Students will need instruction in writing informational or explanatory text that requires facility with the symbols of mathematics and graphic representations, as well as understanding of mathematical content and concepts. Instructional time and effort focused on reading and writing in mathematics benefits students by "requiring them to investigate and consider mathematical concepts and connections" (Bosse and Faulconer 2008, 10), which supports the mathematical practices standards. Writing in mathematics needs to be explicitly taught, because skills do not automatically transfer from English language arts or English language development. Therefore, students benefit from modeled writing, interactive writing, and guided writing in mathematics.

As teachers and curriculum leaders design instruction to support students' reading, writing, speaking, and listening in mathematics, the California Common Core State Standards for English Language Arts and Literacy in History/Social Studies, Science, and Technical Subjects (CA CCSS for ELA/Literacy) and the California English Language Development Standards (http://www.cde.ca.gov/sp/el/er/eldstandards.asp [CDE 2013b]) are essential resources. The standards for reading informational text in the CA CCSS for ELA/Literacy specify the skills students must master in order to comprehend and apply what

they read. Writing Standard 2 of the CA CCSS for ELA/Literacy provides explicit guidance on writing informational or explanatory texts by clearly stating the expectations for students' writing according to grade level. Engaging in mathematical discourse can be challenging for students who have not had many opportunities to explain their reasoning, formulate questions, or critique the reasoning of others. Standard 1 in the Speaking and Listening strand of the CA CCSS for ELA/Literacy, as well as Part I of the CA ELD standards, calls for students to engage in collaborative discussions and set expectations for a progression in the sophistication of student discourse from kindergarten through grade twelve and from the emerging level to the bridging level for English learners. Teachers and curriculum leaders should utilize the CA CCSS for ELA/Literacy and the CA ELD standards in tandem with the CA CCSSM when planning instruction. In grades six through twelve, there are standards for literacy in science and technical subjects that include reading and writing focused on domain-specific content and that can provide guidance, as students are required to read and write more complex mathematics text.

It is a common misconception that mathematics is limited to numbers and symbols. Mathematics instruction is often delivered verbally or through text that is written in academic language, not everyday language. Francis et al. (2006a) note, "The skills and ideas of mathematics are conveyed to students primarily through oral and written language—language that is very precise and unambiguous" (Francis et al. 2006a, 35). Words that have one meaning in everyday language have a different meaning in the context of mathematics. Also, many individual words, such as *root*, *point*, and *table*, have technical meanings in mathematics that are different from what a student might use in other contexts. Reading a mathematics text can be difficult because of the special use of symbols and spatial aspects of notations (e.g., exponents and stacked fractions, diagrams, and charts), as well as the structural differences between informational and narrative text, with which students are often more familiar. For example, a student might misread 5^2 (five squared) as 52 (fifty-two). Language difficulties may also occur when students are translating a word problem into an algebraic or numeric expression or equation. As early as grade one, students will encounter phrases such as "seven less than 10"; and in grade eight, students are asked to translate "7 fewer than twice Ann's age is 16" into an equation. In higher mathematics, it is essential to understand the concept that the language is conveying.

Mathematics has specialized language that requires different interpretation than everyday language. Attention must be paid to particular terms that may be problematic. Table UA-1 provides examples of mathematical language that may cause difficulties for English learners, depending on context or usage.

As students explore mathematical concepts, engage in discussions about mathematics topics, explain their reasoning, and justify their procedures and conclusions, the mathematics classroom will be vibrant with conversation.

Table UA-1. Mathematical Language That May Cause Difficulties for English Learners

Words whose meanings are found only in mathematics (used only in academic English)	Hypotenuse, parallelogram, coefficient, quadratic, circumference, polygon, polynomial
Symbolic language (used almost universally)	$+, -, \times, \div, \pi, \frac{1}{2}$
Words with multiple meanings in everyday English	The floor is <u>even</u>. The picture is <u>even</u> with the window. Breathing develops an <u>even</u> rhythm during sleep. The dog has an <u>even</u> temperament. I looked sick and felt <u>even</u> worse. <u>Even</u> a three-year-old child knows the answer.
Words with multiple meanings in academic English	Number: <u>Even</u> numbers (e.g., 2, 4, 6, and so on) Number: <u>Even</u> amounts (e.g., even amounts of sugar and flour) Measurement: An <u>even</u> pound (i.e., an exact amount) Function: An <u>even</u> function (e.g., $f(x) = f(-x)$ or cosine function)
Phonologically similar words.	***tens*** versus ***tenths*** ***sixty*** versus ***sixteen*** ***sum*** versus ***some*** ***whole*** versus ***hole*** ***off*** versus ***of*** How many **halves** do you **have**? ***then*** versus ***than***

Adapted from Asturias 2010.

Helping all students meet mathematical language demands requires careful planning; attention to the language demands of each lesson, unit, and module; and ongoing monitoring of students' understanding and their ability to communicate what they know and can do. As students explore mathematical concepts, engage in discussions about mathematics topics, explain their reasoning, and justify their procedures and conclusions, the mathematics classroom will be vibrant with conversation.

Assessment for Learning

There are many types of assessment in education. This section focuses on assessment *for* learning: formative and diagnostic assessment. Teachers should determine their students' current achievement levels in mathematics so that each student or group of students can be offered mathematics instruction leading to mastery of all grade-level or course-level mathematics standards. Given the vertical alignment of the CA CCSSM, the concept that what students have already learned in mathematics should form the basis for further learning is particularly true. Assessments may help identify those students who are ready to move on or are ready for greater challenges. Assessments may also identify students' misconceptions, overgeneralizations, and overspecializations so that these types of errors can be corrected. (Refer to the Assessment chapter for additional information.)

Formative Assessment

Formative assessment is key to ensuring that all students are provided with mathematics instruction designed to help them progress at an appropriate pace from what they already know to higher levels of learning. Formative assessment is assessment *for* learning. Formative assessment allows the teacher to gather information about student learning as it is happening. Armed with this knowledge, teachers can alter their lesson or instructional strategies and offer academic support and enrichment to students who need it. The Glossary of Education Reform (Great Schools Partnership 2014) describes formative assessment in this way:

> Many educators and experts believe that formative assessment is an integral part of effective teaching. In contrast with most summative assessments, which are deliberately set apart from instruction, formative assessments are integrated into the teaching and learning process. For example, a formative-assessment technique could be as simple as a teacher asking students to raise their hands if they feel they have understood a newly introduced concept, or it could be as sophisticated as having students complete a self-assessment of their own writing (typically using a rubric outlining the criteria) that the teacher then reviews and comments on. While formative assessments help teachers identify learning needs and difficulties, in many cases the assessments also help students develop a stronger understanding of their own academic strengths and weaknesses. When students know what they do well and what they need to work harder on, it can help them take greater responsibility over their own learning and academic progress. (Great Schools Partnership 2014)

Diagnostic Assessment

Diagnostic assessment of students often reveals both strengths and weaknesses (sometimes referred to as *gaps*) in students' learning. Diagnostic assessment also may reveal learning difficulties and the extent to which limited English language proficiency is interfering with mathematics learning. When gaps are discovered, instruction can be designed to remediate specific weaknesses while taking into consideration identified strengths. With effective support, students' weaknesses can be addressed without slowing down the students' mathematics learning progression. For example, the development of fluency with division using the standard algorithm in grade six is an opportunity to identify and address learning gaps in place-value understanding. This approach, in which place-value instruction and learning support students' fluency with division, is more productive than postponing grade-level work to focus on earlier standards that address place value (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2012, 12). Additionally, assessments may indicate that a student

already possesses mathematical skills and conceptual understanding beyond that of his or her peers and requires a modified curriculum to remain engaged. For example, a more advanced student could be challenged to complete an investigation such as a "Problem of the Month" from the Inside Mathematics Web site (http://www.insidemathematics.org/ [Inside Mathematics 2015]).

If a student is struggling unproductively to complete grade-level tasks, the teacher needs to determine the cause of the student's lack of achievement. Contributing factors might include:

- a lack of content-area knowledge;
- limited English proficiency;
- inappropriate instructional pacing;
- learning difficulties;
- frequent absences from school;
- homelessness;
- family issues;
- reading difficulties.

Teachers need to know their students in order to address each student's instructional needs. Sometimes a student may have a persistent misunderstanding of a concept or skill in mathematics, or the student may have consistently repeated an error until it has become routine. These problems may affect the student's ability to understand and solve problems. Intervention may be necessary to help students with these types of difficulties.

Diagnostic testing may also uncover students who appear to be struggling, when in fact they have already mastered the content and need more of a challenge to remain engaged. These students also need creative intervention, such as investigations and challenging problems.

California's Multi-Tiered System of Supports and Response to Instruction and Intervention

The California Multi-Tiered System of Supports (MTSS) provides a basis for understanding how California educators can work together to ensure equitable access and opportunity for all students to master the CA CCSSM. California's MTSS includes Response to Instruction and Intervention (RtI^2) as well as additional philosophies and concepts.

In California, the MTSS is an integrated, comprehensive framework that focuses on the CA CCSS and other state-adopted content standards, core instruction, differentiated learning, student-centered learning, individualized student needs, and the alignment of systems necessary for all students' academic, behavioral, and social success. The MTSS offers the potential to create systematic change through intentional design, as well as redesign of services and supports that quickly identify and match the needs of all students.

Comparing the MTSS to RtI²

The CDE's RtI² processes focus on students who are struggling and provide a vehicle for teamwork and data-based decision making to strengthen student performance before and after educational and behavioral problems increase in intensity. For additional information, please visit the CDE's RtI² Resources Web page (http://www.cde.ca.gov/ci/cr/ri/rtiresources.asp [CDE 2015c]).

MTSS Differences with RtI²

The MTSS has a broader scope than does RtI². The MTSS also includes these elements:

- Focusing on aligning the entire system of initiatives, supports, and resources
- Promoting district, site, and grade-level participation in identifying and supporting systems for alignment of resources
- Systematically addressing support for all students, including gifted and high achievers
- Enabling a paradigm shift for providing support and setting higher expectations for all students through intentional design and redesign of integrated services and supports, rather than selection of a few components of RtI and intensive interventions
- Endorsing UDL instructional strategies so all students have opportunities for learning through differentiated content, processes, and product
- Integrating instructional and intervention support so that systemic changes are sustainable and based on CA CCSS–aligned classroom instruction
- Challenging all school staff members to change the ways in which they work across all school settings

The MTSS is not designed solely for consideration in special education placement; it focuses on all students.

MTSS Similarities to RtI²

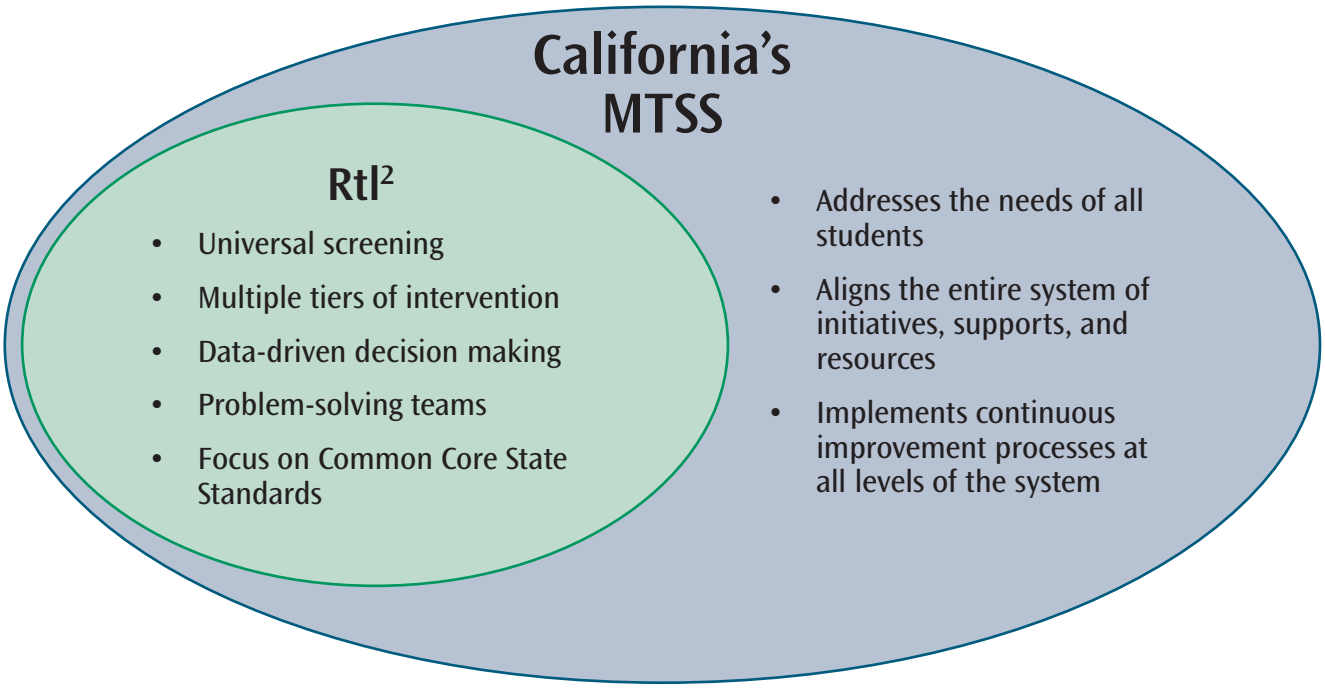
The MTSS incorporates many of the same components of RtI², such as these:

- Supporting high-quality standards and research-based, culturally and linguistically relevant instruction with the belief that every student can learn—including students who live in poverty, students with disabilities, English learners, and students from all ethnicities present in the school and district cultures
- Integrating a data collection and assessment system (including universal screening, diagnostics, and progress monitoring) to inform decisions appropriate for each tier of service delivery
- Relying on a problem-solving systems process and method to identify problems, develop interventions, and evaluate the effectiveness of interventions in a multi-tiered system of service delivery
- Seeking and implementing appropriate research-based interventions for improving student learning

- Using positive, research-based behavioral supports schoolwide and in classrooms to achieve important social and learning outcomes
- Implementing a collaborative approach to analyzing student data and working within the intervention process

Figure UA-1 provides a Venn diagram showing the similarities and differences between California's MTSS and RtI^2 processes. Both rely on RtI^2's data gathering through universal screening, data-driven decision making, and problem-solving teams, and both focus on the CA CCSS. However, the MTSS addresses the needs of all students by aligning the entire system of initiatives, supports, and resources and by implementing continuous improvement processes at all levels of the system.

Figure UA-1. Venn Diagram of the Similarities and Differences Between the MTSS and RtI^2



Tier 1, Tier 2, and Tier 3 Mathematics Interventions

With the caveat that there has been little research on effective RtI^2 interventions for mathematics, Gersten et al. (2009) provide eight recommendations (see table UA-2) to identify and support the needs of students who are struggling in mathematics.[3] The authors note that systematic and explicit instruction is a "recurrent theme in the body of scientific research." They cite evidence for the effectiveness of combinations of systematic and explicit instruction that include teacher demonstrations and think-alouds early in the lesson, unit, or module; student verbalization of how a problem was solved; scaffolded practice; and immediate corrective feedback (Gersten et al. 2009).

3. For additional information on the eight recommendations and detailed suggestions on implementing them in the classroom, see Gersten et al. 2009.

Table UA-2. Recommendations for Identifying and Supporting Students Who Are Struggling in Mathematics

Tier 1
Recommendation 1. All students should be screened to identify those at risk for potential mathematics difficulties and provide interventions to students identified as at risk.
Tiers 2 and 3
Recommendation 2. Instructional materials for students receiving interventions should focus intensely on in-depth treatment of whole numbers in kindergarten through grade five and on rational numbers in grades four through eight. These materials should be selected by committee.
Recommendation 3. Instruction during the intervention should be explicit and systematic. This includes providing models of proficient problem solving, verbalization of thought processes, guided practice, corrective feedback, and frequent cumulative review.
Recommendation 4. Interventions should include instruction on solving word problems that is based on common underlying structures.
Recommendation 5. Intervention materials should include opportunities for students to work with visual representations of mathematical ideas, and interventionists should be proficient in the use of visual representations of mathematical ideas.
Recommendation 6. Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.
Recommendation 7. Progress of students who receive supplemental instruction and other students who are at risk should be monitored.
Recommendation 8. Motivational strategies in Tier 2 and Tier 3 interventions should be included.

Adapted from Gersten et al. 2009.

With systematic instruction, concepts are introduced in a logical, coherent order, and students have many opportunities to apply each concept. As an example, students develop their understanding of place value in a variety of contexts before learning procedures for addition and subtraction of two-digit numbers. To help students learn to communicate their reasoning and the strategies they used to solve a problem, teachers model thinking aloud and ask students to explain their solutions. These recommendations fit within the overall framework of the MTSS described previously.

Planning Instruction for Students with Disabilities

Some students who receive their mathematics instruction in the general education classroom (Tier 1) or receive Tier 2 or Tier 3 interventions may also have disabilities that require accommodations or placements in programs other than general education. Students with disabilities who have difficulty remembering and retrieving basic mathematics facts may not be able to retain the information necessary to solve mathematics problems.

Students with disabilities are provided access to all the mathematics standards through a rich and supported program that uses instructional materials and strategies that best meet the students' needs.

A student's 504 accommodation plan or IEP often includes suggestions for a variety of teaching and learning techniques. This is to ensure that the student has full access to a program that will allow him or her to master the CA CCSSM, including the MP standards. Teachers must familiarize themselves with each student's 504 accommodation plan or IEP to help the student achieve mastery of the grade-level CA CCSSM.

Section 504 Plan

A Section 504 accommodation plan is typically produced by school districts in compliance with the requirements of Section 504 of the federal Rehabilitation Act of 1973. The plan specifies agreed-on services and accommodations for a student who, as a result of an evaluation, is determined to have a physical or mental impairment that substantially limits one or more major life activities. Section 504 allows a wide range of information to be contained in a plan: (1) the nature of the disability; (2) the basis for determining the disability; (3) the educational impact of the disability; (4) the necessary accommodations; and (5) the least restrictive environment in which the student may be placed.

Individualized Education Program (IEP)

An IEP is a comprehensive written statement of the educational needs of a child with a disability and the specially designed instruction and related services to be employed to meet those needs. An IEP is developed (and periodically reviewed and revised) by a team of individuals knowledgeable about the child's disability, including the parent(s) or guardian(s). The IEP complies with the requirements of the Individuals with Disabilities Education Act (IDEA) and covers items such as (1) the child's present level of performance in relation to the curriculum; (2) measurable annual goals related to the child's involvement and progress in the curriculum; (3) specialized programs (or program modifications) and services to be provided; (4) participation in general education classes and activities; and (5) accommodation and modification in assessments.

In recent years, five different meta-analyses of effective mathematics instruction for students with disabilities have been conducted. The studies included students who have learning disabilities, but also students with mild intellectual disabilities, attention deficit hyperactive disorder (ADHD), behavioral disorders, and students with significant cognitive disabilities (Adams and Carnine 2003; Baker, Gersten, and Lee 2002; Browder et al. 2008; Kroesbergen and Van Luit 2003; Xin and Jitendra 1999). These meta-analyses, along with the National Mathematics Advisory Panel (2008) report titled *Foundations for Success*, suggest that the following four methods of instruction show promise for improving mathematics achievement in students with disabilities:

1. **Systematic and explicit instruction.** Teachers guide students through a defined instructional sequence with explicit (direct) instructional practice. Teachers model a strategy for solving a particular type of problem so that students can see when and how to use the strategy and what they can gain by doing so. This type of instruction helps students learn to regularly apply strategies that effective learners use as a fundamental part of mastering concepts.

2. **Self-instruction.** Students manage their own learning through a variety of self-regulation strategies with specific prompting or solution-oriented questions.

3. **Peer tutoring.** This refers to many different types of tutoring arrangements, but most often involves pairing students together to learn or practice an academic task. Peer tutoring works best when students of different ability levels work together.

4. **Visual representation.** This type of instruction involves the use of manipulatives, pictures, number lines, and graphs of functions and relationships to teach mathematical concepts. The Concrete–Representational–Abstract (CRA) sequence of instruction is an evidence-based instructional practice involving manipulatives to promote conceptual understanding (Witzel, Riccomini, and Schneider 2008). It is the most common example of visual representation and shows promise for improving understanding of mathematical concepts for students with disabilities. The CRA instructional sequence consists of three tiers of learning: (1) concrete learning through hands-on instruction using actual manipulative objects; (2) representational learning through pictorial representations of the previously used manipulative objects during concrete instruction; and (3) learning through abstract notations such as operational symbols. Each tier is interconnected and builds upon the previous one, promoting conceptual understanding, procedural accuracy, and fluency and leading toward mathematical proficiency for students. The CRA sequence is built upon the premise of UDL, which calls for multi-modal forms of learning (e.g., seeing, hearing, moving muscles, and touching). This sequence allows learners to interact in multiple ways, which may increase student engagement and the desire to attend to the task at hand. Using manipulatives in concrete and representational ways helps learners to gain meaning from abstract mathematics by breaking down the steps into understandable concepts. To that end, the CRA instructional sequence provides a more meaningful and contextually relevant solution to rote memorization of algorithms and rules taught in isolation.

In order to improve mathematics performance in students with learning difficulties, Vaughn, Bos, and Schumm (2010) also suggest that when new mathematical concepts are introduced or when students have difficulty learning a concept, teachers need to "begin with the concrete and then move to the abstract" (Vaughn, Bos, and Schumm 2010, 385). Furthermore, these authors suggest that student improvement will occur when teachers provide:

- explicit instruction that is highly sequenced and indicates to students why the learning is important;
- assurance that students understand the teacher's directions as well as the demands of the task by closely monitoring student work;
- systematic use of learning principles such as positive reinforcement, varied practice, and student motivation;
- real-world examples that are understandable to students (Vaughn, Bos, and Schumm 2010, 385).

For students with significant cognitive disabilities, systematic instruction—which includes teacher modeling, repeated practice, and consistent prompting and feedback—was found to be an effective

instructional strategy. Studies focused on skills such as counting money and basic operations. Students also learned from instruction in real-world settings, such as a store or restaurant (Browder et al. 2008).

Although direct instruction has been shown to be an effective strategy for teaching basic mathematical skills, the CA CCSSM emphasize a *balance* of conceptual understanding, fluency with skills and procedures, and application of mathematics concepts to real-world contexts. This balance can be achieved by connecting mathematical practices to mathematical content. Helping students to develop mathematical practices, including analyzing problems and persevering in solving them, constructing arguments and critiquing others, and reasoning abstractly and quantitatively, requires a different approach. Based on their work with students who have disabilities and those working below grade level, Stephan and Smith (2012) offer suggestions for creating a standards-based learning environment. Three key components of this type of learning environment are the selection of appropriate problems, the role of the teacher(s), and the role of the students. The problems students are asked to solve must be engaging to students, open-ended, and rich enough to support mathematical discourse.

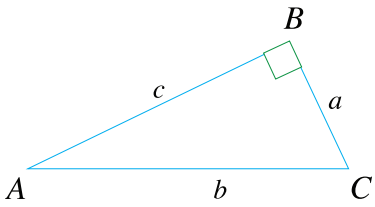
Stephan and Smith recommend that problems be "grounded in real-world contexts" (Stephan and Smith 2012, 174) and accessible to all students, and they should require little direct instruction to introduce. The teacher introduces the problem to be solved, reminds students of what they have already learned that may help them with the problem, and answers clarifying questions. The teacher does not provide direct instruction, but quickly sets the context for the students' work. To foster student discussion, the teacher takes the role of information gatherer and asks questions of the students that help them reason through a problem. If students are working in small groups, the teacher moves from group to group to ensure all students are explaining their reasoning and asking their peers for information and explanations. Students take on the role of active learners who must figure out how to solve the problem instead of being given the steps for solving it. They work with their peers to solve problems, analyze their own solutions, and apply previous learning to new situations. Depending on the problem posed, students find more than one possible answer and more than one way to solve the problem. When teachers utilize diverse pairings for group work (e.g., students working at or above grade level collaborate with students who are not), students can accomplish content- or language-task goals as well as mathematics goals. Collaborative work between the partners facilitates inclusion through the learning of mathematical content. Vaughn, Bos, and Schumm (2010) note that collaborative learning has proven to be an effective method of instruction for students with developmental disabilities in the general education classroom.

Patterns of Error in Computation

Vaughn, Bos, and Schumm (2010) indicate that many of the computation errors made by students fall into certain patterns. Ashlock (1998) theorizes that errors are generated when students "overgeneralize" during the learning process. On the other hand, other errors occur when students "overspecialize" during the learning process by restricting procedures in solving the problem (Ashlock 1998, 15). To diagnose the computational errors of students who are experiencing difficulty, assessment tools must alert the teacher to both overgeneralization and overspecialization. Teachers need to probe deeply as they examine written work—looking for misconceptions and erroneous procedures that form patterns across examples—and try to find out why specific procedures were learned. These discoveries will help teachers plan for and provide instruction to meet the needs of their students.

Errors also occur when students have not learned basic facts, perform an incorrect operation, do not complete the algorithm in the correct sequence, lack understanding of place value within the algorithm, or provide a random response. Figure UA-2 presents some examples of student errors.

Figure UA-2. Examples of Student Error Patterns

<table>
<tr><th colspan="5">Overgeneralization</th></tr>
<tr><td colspan="2">Because
$23 = 20 + 3$,
the student thinks that
$2y = 20 + y$
(so that if $y = 5$,
$2y = 25$)</td><td colspan="2"> 

$c^2 = a^2 + b^2$
The student cites the Pythagorean Theorem (ignoring the actual location of the hypotenuse).</td><td>$4 + 2 = 6$
$6 - 2 = 8$
The student does not pay attention to the addition and subtraction signs and thinks both answers are sums because they appear to the right side of the equal sign.</td></tr>
<tr><th colspan="5">Overspecialization</th></tr>
<tr><td colspan="2">The student writes
$100.36 + 12.57$ as
$100.36 + 125.70$
because the two addends must have the same number of digits on either side of the decimal point.</td><td colspan="3">The altitude of a triangle has to be contained within the triangle.
 </td></tr>
<tr><th colspan="5">Improper composing and decomposing</th></tr>
<tr><td>$\begin{array}{r} 47 \\ +34 \\ \hline 71 \end{array}$</td><td>$\begin{array}{r} 65 \\ +38 \\ \hline 93 \end{array}$</td><td>$\begin{array}{r} 37 \\ +25 \\ \hline 52 \end{array}$</td><td colspan="2">The composed ten is not added. The student may be composing the ten in her head and forgetting to add it, or she may be adding left to right and does not know what to do when the addition results in a two-digit answer, so she records only the ones digit. An interview with the student would provide further diagnostic information (Miller 2009, 230).</td></tr>
<tr><td>$\begin{array}{r} 45 \\ -37 \\ \hline 12 \end{array}$</td><td>$\begin{array}{r} 46 \\ -28 \\ \hline 22 \end{array}$</td><td>$\begin{array}{r} 36 \\ -17 \\ \hline 21 \end{array}$</td><td colspan="2">The student does not decompose the tens when needed. Instead, he subtracts the smaller “ones” number from the larger “ones” number (Miller 2009, 230).</td></tr>
<tr><td>$\begin{array}{r} 25 \\ \times 32 \\ \hline 50 \\ 75 \\ \hline 125 \end{array}$</td><td>$\begin{array}{r} 44 \\ \times 27 \\ \hline 308 \\ 88 \\ \hline 396 \end{array}$</td><td>$\begin{array}{r} 53 \\ \times 37 \\ \hline 371 \\ 159 \\ \hline 530 \end{array}$</td><td colspan="2">The student misaligns the second partial product (Miller 2009, 230).</td></tr>
</table>

As mentioned in figure UA-2, interviewing students to find out how they solved a problem can provide teachers with insights on students' misunderstandings or learning difficulties. Teachers can employ these and other remediation strategies:

- Returning to simpler problems
- Analyzing student errors and bringing to light students' misconceptions
- Estimating
- Demonstrating or providing students with concrete models to develop conceptual understanding (moving to a representational model and then abstract thinking as students progress)
- Using grid paper so students can align numbers by place value
- Designing graphic organizers and flowcharts
- Providing students with meaningful opportunities and sequential practice to learn basic facts for fluency

Students with disabilities can successfully study higher mathematics. They may require accommodations, such as access to a calculator or learning strategies that provide alternatives to memorizing computation facts, but no child should be denied the opportunity to study higher mathematics based on his or her disabilities.

Accommodations for Students with Disabilities

Accommodations support equitable instruction and assessment for students by lessening the effects of a student's disability. Without accommodations, students with disabilities may have difficulty accessing grade-level instruction and participating fully in assessments. When possible, accommodations should be the same or similar across classroom instruction, classroom tests, and state and district assessments. However, some accommodations may be appropriate only for instructional use and may not be appropriate for use on a standardized assessment. It is crucial for educators to be familiar with state policies regarding accommodations used for statewide assessment.

A small number of students with significant disabilities will struggle to achieve at or near grade level. These students, who will participate in alternative assessments, account for approximately 1 percent of the total student population. Substantial supports and accommodations are often necessary for these students to have meaningful access to academic content standards and to standards-aligned assessments that are appropriate for the students' academic and functional needs.

All students with disabilities can work toward grade-level academic content standards, and most of these students will be able to accomplish this goal when the following three conditions are met (Thompson et al. 2005):

1. Standards are implemented within the foundational principles of UDL.
2. A variety of evidence-based instructional strategies are considered to align materials, curriculum, and production to reflect the interests, preferences, and readiness of diverse learners maximizing students' potential to accelerate learning.
3. Appropriate accommodations are provided to help students access grade-level content.

Accommodations play an important role in helping students with disabilities to access the core curriculum and demonstrate what they know and can do. A student's IEP or 504 Plan team determines the appropriate accommodations for both instruction and state and district assessments. Decisions about accommodations must be made on an individual student basis, not on the basis of category of disability or administrative convenience. For example, rather than selecting accommodations from a generic checklist, IEP and 504 Plan team members (including families and the student) need to carefully consider and evaluate the effectiveness of accommodations for each student.

Accommodations are typically made in presentation, response, setting, and timing and scheduling so that learners are provided with equitable access during instruction and assessment.

> **"The [Common Core State Standards] should . . . be read as allowing for the widest possible range of students to participate fully from the outset and as permitting appropriate accommodations to ensure maximum participation of students with special education needs. For example, for students with disabilities *reading* should allow for the use of Braille, screen-reader technology, or other assistive devices, while *writing* should include the use of a scribe, computer, or speech-to-text technology. In a similar vein, *speaking* and *listening* should be interpreted broadly to include sign language."**
>
> —NGA/CCSSO 2010b

- **Presentation.** Accommodations in presentation allow students to access information in ways that do not require them to visually read standard print. These alternative modes of access are auditory, multi-sensory, tactile, and manual. For example, a student with a visual impairment may require that a test be presented in a different manner, such as in a digital format accompanied with a text-to-speech software application or with a braille test booklet.

- **Response.** Accommodations in response allow students to complete activities, assignments, and assessments in different ways or to solve or organize problems using some type of assistive device or organizer. For example, a student may require an alternative method of completing multi-step computational problems due to weak fine motor skills or physical impairments; such methods may include computer access with a specialized keyboard, a speech-to-text application, or other specialized software.

- **Setting.** Accommodations in setting allow for a change in the location where a test or assignment is given or in the conditions of an assessment setting. For example, a student may require that an assessment be administered in a setting appropriate to the student's individual needs (such as testing an individual student separately from the group to provide visual or auditory supports).

- **Timing and Scheduling.** Accommodations in timing and scheduling allow for an increase in the typical length of time to complete an assessment or assignment and perhaps change the way the time allotted is organized. For example, a student may take as long as reasonably needed to complete an assessment, including taking portions over several days to avoid fatigue caused by a chronic health condition.

The Council of Chief State School Officers provides guidance in its *Accommodations Manual: How to Select, Administer, and Evaluate Use of Accommodations for Instruction and Assessment of Students with Disabilities* (http://www.ccsso.org/Documents/2005/Accommodations_Manual_How_2005.pdf [Thompson et al. 2005]).

The selection and evaluation of accommodations for students with disabilities who are also English learners must include collaboration among educational specialists, the classroom teacher, teachers providing instruction in English language development, families, and the student. It is important to note that English learners are disproportionately represented (in high numbers) in the population of students who are identified as having disabilities. This suggests that some of these students may not have disabilities and that the identification process is inappropriate for English learners.

Accommodations are available to all students—those who have disabilities and those who do not. Accommodations do not reduce learning expectations; rather, they provide access. Accommodations can reduce or even eliminate the effects of a student's disability. It is important to note that although some accommodations may be appropriate for instructional use, they may not be appropriate for use on a standardized assessment.

Assistive Technology

A fundamental goal of the CA CCSSM is to promote a culture in which all students are challenged to meet high expectations. To ensure that all students have access to general education instruction, standards, and curriculum, students with disabilities may be provided additional supports and services, as appropriate. These supports are often provided through the use of assistive technology. Assistive technology is used by individuals to gain access and perform functions that might otherwise be difficult or impossible. Assistive technology is defined in federal law (the Individuals with Disabilities Education Improvement Act of 2004) as "any item, piece of equipment, or product system, whether acquired commercially off the shelf, modified, or customized, that is used to increase, maintain, or improve functional capabilities of a child with a disability" (Pub. L. No. 108–466, 118 Stat. 2652 [2004]). Assistive technology can include a wide variety of learning enhancements, including mobility devices, writing implements, communication boards, and grid paper, as well as hardware, software, and peripherals that assist in accessing lessons. For more information about assistive technology, visit https://www.washington.edu/accessit/ (National Center on Accessible Information Technology in Education 2015).

Teachers implement accommodations and modifications in mathematics instruction in numerous ways, including through the use of assistive technology. Students with physical, sensory, or cognitive disabilities may face additional learning challenges or may learn differently. For example, students with fine motor disabilities may not be able to hold a pencil to write answers on a test or use a standard calculator to solve mathematics problems. Students who have difficulty decoding text and symbols may struggle to comprehend text. When assistive technology is appropriately integrated into the classroom, students are provided with a variety of ways to access the information and to complete their work.

Disabilities vary widely, and accommodations must be tailored to each student's unique needs. Assistive technology helps teachers provide accommodations and modifications for students with

disabilities. Accommodations change how a student learns material, and modifications change what a student is taught or expected to learn.

- *Assistive technology accommodations.* Assistive technology provides access to the course curriculum. Students can receive assistance from a computer that scans and reads text or digital content to incorporate images, sound, video clips, and additional information. Students with visual impairments can gain access to instructional materials through digital large print with a contrasting background, the ability to change the font as it appears on the screen, or text-to-speech devices. Software that converts text to braille characters, using a refreshable display, provides students with access to printed information. Students can use mobile devices to create or record notes so that they can later print out assignments or use the notes to study for a test. A student with motor difficulties might use an enlarged or simplified computer keyboard, a talking computer with a joystick, or other modified input device such as a switch, headgear, or eye selection devices. The American Speech–Language–Hearing Association (ASHA) presents information on augmentative and alternative communication systems or applications[4] that help students with severe speech or language disabilities express thoughts, needs, or ideas. These and other types of assistance can provide access, but they do not change content and are therefore considered accommodations.

- *Assistive technology modifications.* Assistive technology provides additional help to students who otherwise would not be able learn a concept or show what they have learned. Examples of modifications provided by assistive technology include the use of speech-to-text devices, calculators, or other devices that provide information not otherwise available to students. Of course, there are many other types of modifications that do not involve the use of assistive technology.

Although assistive technology helps to level the playing field for students with special needs, many types of assistive technology (both software and hardware) are beneficial for all students. The flexibility of assistive technology allows a teacher to use tools and materials that support a student's individual strengths and also address his or her disability in the least restrictive environment.

The CDE provides information that clarifies basic requirements for consideration and provision of assistive technology and services to individuals with disabilities. Information is also available for local educational agencies, particularly members of IEP teams, to effectively address these requirements. For other examples of assistive technology, please visit the CDE Assistive Technology Checklist Web page at http://www.cde.ca.gov/sp/se/sr/atexmpl.asp (CDE 2015a).

Planning Instruction for California's English Learners

Students in California demonstrate a wide variety of skills, abilities, and interests as well as different levels of proficiency in English and other languages. California's students come from diverse cultural, linguistic, ethnic, and religious backgrounds, have different experiences, and live with different familial and socioeconomic circumstances. The greater the variation of the student population, the richer the learning experiences for all, and the more assets upon which teachers may draw. At the same time,

4. For more information about augmentative and alternative communication, visit http://www.asha.org/public/speech/disorders/AAC/ (ASHA 2015).

the teacher's role in providing high-quality curriculum and instruction that is sensitive to the needs of individuals becomes more complex. In diverse settings, the notion of *shared responsibility* is particularly crucial. Teachers need the support of one another, administrators, specialists, and the community in order to best serve all students.

Approximately 25 percent of California's public school students are learning English as an additional language. These students come to California schools from all over the world, but the majority were born in California. Schools and districts are responsible for ensuring that all English learners have full access to an intellectually rich and comprehensive curriculum, via appropriately designed instruction, and that they make steady—and even accelerated—progress in their English language development.

English learners come to school with a range of cultural and linguistic backgrounds; experiences with formal schooling; proficiency with mathematics, their native language, and English; migrant and socio-economic statuses; and interactions in the home, school, and community. All of these factors inform how educators support English learners to achieve school success through the implementation of the CA ELD standards in tandem with the CA CCSSM. Educators should not confuse students' language ability with their mathematical understanding.

Ethnically and racially diverse students make up approximately 74 percent of California's student population, making it the most diverse student population in the nation. In 2012–13, more than 1.3 million students—or roughly 25 percent of the California public school population—were identified as English learners. Of those English learners, 84.6 percent identified Spanish as their home language. The next largest group of English learners, 2.3 percent, identified Vietnamese as their home language (CDE 2013c). Given the large number of English learners in California's schools, it is essential to provide these students with effective mathematics instruction.

English learners face a significant challenge in learning subject-area content while simultaneously developing proficiency in English. Planning mathematical instruction for English learners is most effective when the instruction takes into consideration the students' mathematics skills and understandings as well as their assessed levels of proficiency in English and their primary language. Because of variations in academic background and age, some students may advance more quickly in mathematics or English language development than other students who require more support to make academic progress. Many districts use assessment tools such as the statewide assessment,[5] which measures the progress of English learners in acquiring the skills of listening, speaking, reading, and writing in English. The statewide assessment is designed to identify a student's proficiency level in English and to monitor the student's progress in English language development. Other tools for measuring progress in English language development are academic progress, teacher and parent evaluation, and tests of basic skills (such as district benchmarks).

The role of English language proficiency must be a consideration for English learners who experience difficulties in learning mathematics. Even students who have good conversational English skills may lack the academic language necessary to fully access mathematics curriculum (Francis et al. 2006a).

5. This statewide assessment was formerly known as the California English Language Development Test (CELDT) and will be replaced by the English Language Proficiency Assessments for California (ELPAC) in 2016.

Academic language, as described by Saunders and Goldenberg, "entails all aspects of language from grammatical elements to vocabulary and discourse structures and conventions" (Saunders and Goldenberg 2010, 106).

> **"[E]very teacher must incorporate into his or her curriculum instructional support for oral and written language as it relates to the mathematics standards and content. It is not possible to separate the content of mathematics from the language in which it is discussed and taught."**
>
> —Francis et al. 2006a, 38

Moschkovich (2012b) cautions that communicating in mathematics is more than a matter of learning vocabulary; students must also be able to participate in discussions about mathematical ideas, make generalizations, and support their claims. She states, "While vocabulary is necessary, it is not sufficient. Learning to communicate mathematically is not merely or primarily a matter of learning vocabulary" (Moschkovich 2012b, 18). Providing instruction that focuses on teaching for understanding, helping students use multiple representations to comprehend mathematical concepts and explain their reasoning, and supporting students' communication about mathematics is challenging (Moschkovich 2012a, 1). Moschkovich's recommendations for connecting mathematical content to language are provided in table UA-3.

Table UA-3. Recommendations for Connecting Mathematical Content to Language

1. Focus on students' mathematical reasoning, not accuracy in using language.
2. Shift to a focus on mathematical discourse practices; move away from simplified views of language.
3. Recognize and support students to engage with the complexity of language in math classrooms.
4. Treat everyday language and experiences as resources, not as obstacles.
5. Uncover the mathematics in what students say and do.

Source: Moschkovich 2012a, 5–8.

Teachers can take the following steps to support English learners in the acquisition of mathematical skills and knowledge as well as academic language:

- Explicitly teach academic vocabulary for mathematics, and structure activities in which students regularly employ key mathematical terms. Be aware of words that have multiple meanings (such as *root*, *plane*, *table*, and so forth).
- Provide communication guides, sometimes called *sentence frames*, as a temporary scaffold to help students express themselves not just in complete sentences but articulately within the MP standards.
- Use graphic organizers and visuals to help students understand mathematical processes and vocabulary.

For English learners who are of elementary-school age, progress in mathematics may be supported through intentional lesson planning for content, mathematical practice, and language objectives. Language objectives "articulate for learners the academic language functions and skills that they need to master to fully participate in the lesson and meet the grade-level content standards" (Echevarria, Vogt, and Short 2008). In mathematics, students' use of the MP standards requires students to translate between various representations of mathematics and to develop a command of receptive (listening, reading) and productive (speaking, writing) language. Language is crucial for schema-building; learners construct new understandings and knowledge through language, whether unpacking new learning for themselves or justifying their reasoning to a peer.

The following are examples of possible language objectives for a student in grade two:

- Read word problems fluently.
- Explain in writing the strategies used to solve addition and subtraction problems within 100.
- Describe orally the relationship between addition and subtraction.

Francis et al. (2006a) examined research on instruction and intervention in mathematics for English learners. The consensus among the researchers was that a lack of development of academic language is a primary cause of English learners' academic difficulties and that more attention needs to be paid to the development of academic language. Like Moschkovich, Francis et al. (2006a) make clear that understanding and using academic language involve many skills beyond merely learning new vocabulary words; these skills include using increasingly complex words, comprehending and using sentence structures and syntax, understanding the organization of text, and producing writing appropriate to the content and to the students' grade level.

One approach to improve students' academic language is to "amplify, rather than simplify" new vocabulary and mathematical terms (Wilson 2010). When new or challenging language is continually simplified for English learners, they cannot gain the academic language necessary to learn mathematics. New vocabulary, complex text, and the meanings of mathematical symbols need to be taught in context with appropriate scaffolding or amplified. Amplification helps increase students' vocabulary and makes mathematics more accessible to students with limited vocabulary. In the progression of rational-number learning throughout the grades, particularly relevant to upper elementary and middle school, students encounter increasingly complex uses of mathematical language (words, symbols) that may contradict student sense-making and associations of a term or phrase from earlier grades. For example, *half* is interpreted as either a call to divide a certain quantity by two, or to double that quantity, depending upon the context:

Half of 6 is _____?

Six (6) divided by one-half is _____?

The standards distinguish between *number* and *quantity*, where quantity is a numerical value of a specific unit of measure. By middle school, students are expected to articulate that a "unit rate for Sandy's bike ride is one-half mile per hour," based upon reading the slope of a distance-versus-time line graph

of a bike ride traveled at this constant rate. Here, "one-half" represents the distance traveled for each hour, rather than the equivalent ratio of one mile traveled for every two hours. The same symbols that students encountered in early elementary grade levels to represent parts of a whole—for example, partitioning in grade two, formalized unit fractions in grade three—are now attached to new language and concepts in upper elementary grade levels and middle school.

Mathematical Discourse

According to the New Zealand Council for Educational Research (2014), "Mathematical classroom discourse is about whole-class discussions in which students talk about mathematics in such a way that they reveal their understanding of concepts. Students also learn to engage in mathematical reasoning and debate." Teachers ask "strategic questions that elicit from students both how a problem was solved and why a particular method was chosen" (New Zealand Council for Educational Research 2014). Students learn to critique ideas (their own and those of other students), and they look for efficient mathematical solutions.

Researchers caution that focusing on academic language alone may promote teaching vocabulary without a context or lead to the misconception that students are lacking because of their inability to use academic language (Edelsky 2006; MacSwan and Rolstad 2003). It is essential for instruction to include teaching vocabulary in context so that the mathematical meaning can be emphasized. Classroom discourse is one instructional strategy that promotes the use of academic and mathematical language within a meaningful context. *Mathematics discourse* is defined as communication that centers on making meaning of mathematical concepts; it is more than just knowing vocabulary. It involves negotiating meanings by listening and responding, describing understanding, making conjectures, presenting solutions, challenging the thinking of others, and connecting mathematical notations and representations (Celedón-Pattichis and Ramirez 2012, 20).

Lesson plans that include objectives for language, mathematical content standards, and mathematical practice standards need to identify where these three objectives intersect and what specific scaffolds are necessary for English learners' mathematical discourse. As one example, a high school teacher of long-term English learners has planned a lesson that requires students to identify whether four points on a coordinate graph belong to a quadratic or an exponential function. Classroom routines for partner and group work have been established, and students know what "good listening" and "good speaking" look like and sound like. However, the teacher has also created bookmarks for students to use, with sentence starters and sentence frames to share their conjectures and rationales and to question the thinking of other students. The teacher is employing an instructional strategy called "Think-Write-Pair-Share" with scaffolds in the form of sentence frames. After a specified time for individual thinking and writing, students share their initial reasoning with a partner. A whole-class discussion ensues, with the teacher intentionally re-voicing student language and asking students to use their own words to share what they heard another student say. While the teacher informally assesses how students employ academic language in their oral statements, she also presses for "another way to say" or represent that thinking to amplify academic language.

Long-Term English Learners

The lack of English language proficiency and understanding of the language of mathematics is of particular concern for long-term English learners—students in grades six through twelve who have been enrolled in American schools for more than six years and have remained at the same English language proficiency level for two or more consecutive years, as determined by the state's annual English language development test. To address the instructional needs of long-term English learners, focused instruction such as instructed English language development (ELD) may be the most effective (Dutro and Kinsella 2010). Instructed ELD, as described by Dutro and Kinsella, focuses attention on language learning. Language skills are taught in a prescribed scope and sequence, ELD is explicitly taught, and there are many opportunities for student practice. Lessons, units, and modules are designed to build fluency and aim to help students achieve full English proficiency.

In addition to systematic ELD instruction, Dutro and Moran (2003) offer two recommendations for developing students' language in the content areas: front-loading and using "teachable moments."

> Front-loading of ELD describes a focus on language preceding a content lesson. The linguistic demands of a content task are analyzed and taught in an up-front investment of time to render the content understandable to the student. This front-loading refers not only to the vocabulary, but also to the forms or structures of language needed to discuss the content. The content instruction, like the action of a piston, switches back and forth from focus on language, to focus on content, and back to language. (Dutro and Moran 2003, 4)

The following example of Dutro and Moran's "piston" instructional strategy informally assesses and advances students' mathematical English language development.

List-Group-Label Activity

Purpose: Formative assessment of students' acquisition of academic language, as well as their ability to distinguish form and function of mathematical terms and symbols. For example, the term *polygon* reminds students of types of polygons (triangles, rectangles, rhombuses) or reminds students of components or attributes of polygons (angles, sides, parallel, perpendicular) or non-examples (circles).

Process: At the conclusion of instruction, the teacher posts a mathematical category or term that students encountered in the unit and asks students to generate as many mathematical words or symbols related to the posted term as they can.

Working with a partner or group, students compile lists of related words and agree how to best sort their lists into subgroups.

For each subgroup of terms or symbols, students must come to agreement on an appropriate label for the subgroup list and be prepared to justify their "List-Group-Label" to another student group.

Teachers also take advantage of teachable moments to expand and deepen language skills. Teachers must utilize opportunities "as they present themselves, to use precise language [**MP.6**] to fill a specific, unanticipated need for a word or a way to express a thought or idea. Fully utilizing the teachable

moment means providing the next language skill needed to carry out a task or respond to a stimulus" (Dutro and Moran 2003, 4).

M. J. Schleppegrell (2007) agrees that the language of mathematical reasoning differs from informal ordinary language. Traditionally, teachers have identified mathematics vocabulary as a challenge but are not aware of the grammatical patterning embedded in mathematical language that generates difficulties. Schleppegrell identifies these linguistic structures as "patterns of language that draw on grammatical constructions that create dense clauses linked with each other in conventionalized ways" (Schleppegrell 2007, 146) yet differ from ordinary use of language. Examples include the use of long, dense noun phrases such as *the volume of a rectangular prism with sides 8, 10, and 12 cm*; classifying adjectives that precede the noun (e.g., *prime number*, *right triangle*); and qualifiers that come after the noun (e.g., *a number that can be divided by 1 and itself*). Other challenging grammatical structures that may pose difficulty include signal words such as *if*, *when*, *therefore*, *given*, and *assume*, which are used differently in mathematics than in everyday language (Schleppegrell 2007, 143–146). Schleppegrell asserts that educators need to expand their knowledge of mathematical language to recognize when and how to include grammatical structures that enable students to participate in mathematical discourse.

Other work on mathematics discourse, such as from Suzanne Irujo (cited in Anstrom et al. 2010), provides concrete classroom applications for vocabulary instruction at the elementary and secondary levels. Irujo explains and suggests three steps for teaching mathematical and academic vocabulary (Anstrom et al. 2010, 23):

- The first suggested step is for educators to analytically read texts, tests, and materials to identify potential difficulties, focusing on challenging language.
- The second step follows Dutro and Moran's findings on pre-teaching with experiential activities in mathematics; only the necessary vocabulary and key concepts are taught to introduce the central ideas.
- The third and final step is integration of the learning process. New vocabulary is pointed out as it is encountered in context, its use is modeled frequently by the teacher, and the modeling cycle is repeated, followed by guided practice, small-group practice, and independent practice. Irujo also recommends teaching complex language forms (e.g., prefixes and suffixes) through mini-lessons.

Despite the importance of academic language for success in mathematics, "in mathematics classrooms and curricula the language demands are likely to go unnoticed and unattended to" (Francis et al. 2006a, 37). Both oral and written language need to be integrated into mathematics instruction. All students, not just English learners, must be provided many opportunities to engage in mathematics discourse—to talk about mathematics and explain their reasoning. The language demands of mathematics instruction must be noted and attended to. Mathematics instruction that includes reading, writing, and speaking enhances students' learning. As lessons, units, and modules are planned, both language objectives and content objectives should be identified. By focusing on and modifying instruction to address English learners' academic language development, teachers support their students' mathematics learning.

The CA ELD standards are an important tool for designing instruction to support students' reading, writing, speaking, and listening in mathematics. The CA ELD standards help guide curriculum, instruction,

and assessment for English learners who are developing the English language skills necessary to engage successfully with mathematics. California's English learners (ELs) are enrolled in a variety of school and instructional settings that influence the application of the CA ELD standards. The CA ELD standards are designed to be used by all teachers of academic content and of English language development in all settings, albeit in ways that are appropriate to each setting and to identified student needs. Additionally, the CA ELD standards are designed and intended to be used *in tandem with* the CA CCSSM to support ELs in mainstream academic content classrooms.

Neither the CA CCSSM nor the CA ELD standards should be treated as checklists. Instead, the CA ELD standards should be utilized as a tool to equip ELs to better understand mathematics concepts and solve problems. Factors affecting ELs' success in mathematics should also be taken into account. (See also the next section on Course Placement of English Learners.) There are a multitude of such factors that fall into at least one of seven characteristic types. These factors inform how educators can support ELs to achieve success in mathematics:

Van de Walle (2007) suggests specific strategies that teachers can incorporate into their mathematics instruction to support English learners:

- Let students know the purpose of the lesson and what they will accomplish during the lesson.
- Build background knowledge and link the lesson to what students already know.
- Encourage the use of each student's native language during group work while continuing to focus on English language development.
- Provide comprehensible input by simplifying sentence structure and limiting the use of non-essential vocabulary. Use visuals whenever possible.
- Explicitly teach vocabulary. Use a word wall and personal math dictionaries.
- Have students work in cooperative groups. This provides English learners with non-threatening opportunities to use language.

1. **Limited prior or background knowledge and experience with formal schooling**
   - Some ELs may lack basic mathematics skills. EL students with limited prior schooling may not have the basic computation skills required to succeed in the first year of higher mathematics. ELs who enter U.S. schools in kindergarten benefit from participation in the same instructional activities as their non-EL peers, along with additional differentiated support based on student needs. Depending upon the level and extent of previous schooling they have received, ELs who enter U.S. schools for the first time in high school may need additional support to master certain linguistic and cognitive skills and fully engage in intellectually challenging academic tasks. Regardless of their schooling background or exposure to English, all ELs should have full access to the same high-quality, intellectually challenging, and content-rich instruction and instructional materials as their non-EL peers, along with appropriate levels of scaffolding.
   - Some ELs may have prior or background knowledge, but it is important to avoid misconceptions of students' mathematics skill levels, especially when based upon their cultural background and upbringing.

2. **Cultural differences**

   - Mathematics is often considered a universal language in which numbers connect people regardless of culture, religion, age, or gender (NYU Steinhardt 2009). However, mathematics learning styles vary by country and culture, and by individual students.
   - The meanings of some symbols (such as commas and decimal points) and mathematical concepts differ according to culture and country of origin. This occurs frequently, especially when expressing currency values, measurement, temperature, and so on, and may impede an EL's understanding of the material being taught. Early on in the school year, teachers should survey their students to learn about the students' backgrounds and effectively address individual needs. It is important for teachers to inform themselves about particular aspects of their students' backgrounds, but also to see each student as an individual with distinct learning needs, regardless of cultural or linguistic influences.

3. **Linguistics**

   Everyday language is very different from academic language, and when students struggle to understand and apply these differences, they may experience difficulties in acquiring academic language. Teachers should develop all of their students' understandings of how, why, and when to use different registers and dialects of English. Some of these challenges may include understanding mathematics-specific vocabulary that is difficult to decode, associating mathematics symbols with concepts, as well as the language used to express those concepts, and grasping the complex and challenging structure of the passive voice.

4. **Polysemous words**

   Polysemous words have identical spellings and pronunciations, but different meanings that are based on context. For example, a *table* is a piece of furniture on which one can set food and dishes, but it is also a systematic arrangement of data or information. Similarly, an *operation* may be a medical procedure or a mathematical procedure; these meanings are different from each other in context, but they do have some relation to one another. The difference between *polysemes* and *homonyms* is subtle: polysemes have semantically related meanings, but homonyms do not.

5. **Syntactic features of word problems**

   - The arrangement of words in a sentence plays a major role in understanding phrases, clauses, or the entire sentence. Complex syntax is especially difficult in the reading, understanding, and solving of word problems in mathematics (NYU Steinhardt 2009). Extra support should be given to ELs regarding syntactic features.
   - Some algebraic expressions are troublesome for ELs, because if they attempt to translate the provided word order, the resulting equation may be inaccurate. For example: *A number* x *is 5 less than a number* y. It is logical to translate word for word when solving this problem, which would most likely result in the following translation: $x = 5 - y$. However, the correct equation would be $x = y - 5$.

6. **Semantic features**

   As shown in the following table (adapted from NYU Steinhardt 2009), many ELs may find semantic features challenging.

Feature	Examples
Synonyms	*add, plus, combine, sum*
Homophones	*sum/some, whole/hole*
Difficult expressions	If . . . then; given that . . .
Prepositions	*divided into* versus *divided by*; *above, over, from, near, to, until, toward, beside*
Comparative constructions	If Amy is taller than Peter, and Peter is taller than Scott, then Amy must be taller than Scott.
Passive structures	Five books were purchased by John.
Conditional clauses	Assuming x is true, then y . . .
Language function words	Words and phrases used to give instructions, to explain, to make requests, to disagree, and so on.

7. **Text analysis**

   Word problems often pose challenges because they require students to read and comprehend the text, identify the question, create a numerical equation, and then solve that equation. Reading and understanding written content in a word problem are often difficult for native speakers of English as well as ELs.

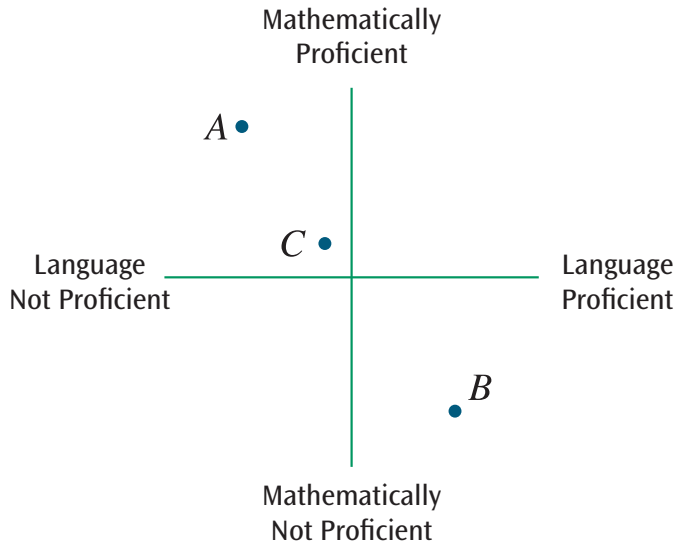
When addressing the factors that affect ELs in instruction, it is essential for teachers to know the ELD proficiency-level descriptor that applies to each student in their classroom. The *emerging*, *expanding*, and *bridging* levels identify what a student knows and can do at a particular stage of English language development and can help teachers differentiate their instruction appropriately. The seven factors discussed above remain barriers for EL students if they are not addressed by teachers. Schools and districts are responsible for ensuring that all ELs have full access to an intellectually rich and comprehensive curriculum, via appropriately designed instruction, and that they make steady and accelerated progress in English language development, particularly in secondary grades.

Course Placement of English Learners

Educators must pay careful attention to placement and assessment practices for students who have studied mathematics in other countries and may be proficient in higher-level mathematics but lack proficiency with the English language. Indeed, a student's performance on mathematics assessments may be affected by his or her language proficiency. For example, in figure UA-3, results for students A, B, and C on the same test may look very similar even though the students' language and mathematical proficiency levels vary considerably. The design of the assessment needs to be mindful of this problem,

and the results need to be interpreted with students' language proficiency factored in. If possible, mathematics assessments should be done in the student's primary language so that lack of English language proficiency does not affect the test results.

Figure UA-3. Intersecting Continua of Mathematics Proficiency with Language Proficiency



(Graphic adapted from Asturias 2010)

For English learners who may know the mathematical content but have difficulty on assessments due to lack of English language proficiency, Burden and Byrd (2009) list the following strategies for adapting assessments:

- *Level of support.* Increase the amount of scaffolding that is provided during the assessment.
- *Product.* Adapt the type of response to decrease reliance on academic language.
- *Participation.* Allow for cooperative group work and group self-assessment using student-created rubrics for performance tasks.
- *Range.* Decrease the number of assessment items.
- *Time.* Provide extra time for English learners to complete tasks.
- *Difficulty.* Adapt the problem, the task, or the approach to the problem.

Celedón-Pattichis (2004) advises that the initial placement of English learners is highly important because "these placements tend to follow students for the rest of their academic lives" (Celedón-Pattichis 2004, 188). When placement of highly proficient students is not based upon their mathematical competence, but rather on their language proficiency, the students may (1) lose academic learning time and the opportunity to continue with their study of higher-level mathematics; and (2) experience a decline in their level of mathematics achievement because of little practice. On the other hand, when low-performing students are placed in courses that are too difficult for their knowledge or language proficiency level, they are likely to become discouraged.

Similarly, students who have studied mathematics in other countries may experience significant differences in how mathematical concepts are represented in California classrooms. Notational differences include how students read and write numbers, use a decimal point, and separate digits in large numbers. There also may be differences in the designation of billions and trillions. For example:

> A student schooled in the United States will read 10,782,621,751 as "10 billion, 782 million, 621 thousand, 751." In some students' countries of origin, the number is read as "10 mil 782 *millones*, 621 mil, 751"; or it is read as "10 thousand 782 million, 621 thousand, 751." (Perkins and Flores 2002, 347)

Differences also occur in how students compute problems by algorithm. For example, students may mentally compute the steps in an algorithm and only write the answer or display the intermediate steps differently, as with long division. Additional difficulties occur as students confront U.S. currency (Perkins and Flores 2002).

These differences may become apparent when parents who have been educated in other countries assist their children at home. There is a strong need for a meaningful dialogue between parents and teachers in which learning about different learning methods and approaches can occur for all. For example, when students or parents possess different ways of performing arithmetic operations, teachers can use these different approaches as learning opportunities instead of dismissing them. This is particularly important for immigrant children (or children of immigrant parents), who are often navigating two worlds. As Cummins (2000) states, "Conceptual knowledge developed in one language helps to make input in the other language comprehensible" (Cummins 2000, 39).

Planning Instruction for Standard English Learners

The Los Angeles Unified School District (LAUSD) defines Standard English Learners (SELs) as "students for whom Standard English is not native and whose home language differs in structure and form from Standard and academic English" (LAUSD 2012, 83). The Academic English Mastery Program (AEMP) and the Multilingual and Multicultural Department of LAUSD have identified six access strategies to help SELs succeed:

1. *Making cultural connections* — the use of "cultural knowledge, prior experience, frames of reference and performance styles" of students to make learning more relevant, effective, and engaging (LAUSD 2012, 85).
2. *Contrastive analysis* — comparing and contrasting the linguistic features of the primary language and Standard English (LAUSD 2012, 162). During a content lesson, the teacher may demonstrate the difference in languages by repeating the student response in Standard English. This recasting then may be used at a later date as an exemplar to examine the differences. In the following example, note the differences in subject–verb agreement, plurals, and past tense:
   - **Non-Standard English.** *There was three runner. The winner finish the race in three minute.*
   - **Standard English.** *There were three runners. The winner finished the race in three minutes.*
3. *Cooperative learning* — working in pairs or small groups on tasks that are challenging enough to truly require collaboration, or as a way to provide strategic peer support to specific students.
4. *Instructional conversations* — academic conversations, often student-led, that allow students to use language to analyze, reflect, and think critically. These conversations may also be referred to as *accountable talk* or *handing off*.
5. *Academic language development* — explicit teaching of vocabulary and language patterns needed to express the students' thinking. Like English learners, SELs benefit from the use of sentence frames (communication guides); unlike the supports for English learners, the guides are based on Standard English and academic vocabulary and not on English language proficiency levels.
6. *Advanced graphic organizers* — visual representation to help students organize thoughts.

For additional guidance, see chapter 4, Theoretical Foundations and the Research Base of the California English Language Development Standards, in the California English Language Development Standards (CDE 2013b).

Planning Instruction for At-Risk Learners

Mathematical focus and in-depth coverage of the CA CCSSM are as necessary for students with mathematics difficulties as they are for more proficient students (Gersten et al. 2009). As soon as students begin to fall behind in their mastery of mathematics standards, immediate intervention is warranted. Interventions must combine practice in material not yet mastered with instruction in new skill areas. Students who are behind will find it challenging to catch up with their peers and stay current as new topics are introduced. The need for remediation is temporary and cannot be allowed to exclude these students from full instruction. In a standards-based environment, students who are struggling to learn or master mathematics need the richest and most organized type of instruction. For some students, Tier 3 interventions may be necessary.

Students who have fallen behind, or who are in danger of doing so, may need more than the normal schedule of daily mathematics. Systems must be devised to provide these students with ongoing tutorials. It is important to offer special tutorials during or outside of the regular school day; however, to ensure access for all students, extra help and practice should occur in additional periods of mathematics instruction during the school day. Instructional time might be extended in summer school, with extra support focused on strengthening and rebuilding gaps in foundational concepts and skills.

Requiring a student with intensive learning challenges to remain in a course for which he or she lacks the foundational skills to master major concepts is an inefficient use of student learning time. To ensure that students can successfully complete full courses, course and semester structures and class schedules should be re-examined and revised or re-created as needed. Targeted intervention, especially at the middle school level or earlier, can increase students' chances of being successful in higher mathematics. Early intervention in mathematics is both powerful and effective (Newman-Gonchar, Clarke, and Gersten 2009).

Grouping as an Aid to Instruction

As a tool, grouping should be used flexibly to ensure that all students master the standards—and instructional objectives should always be based on the CA CCSSM. Small-group instruction may be utilized as a temporary measure for students who have not learned the prerequisite content (Emmer and Evertson 2009). For example, a teacher may discover that some students are having trouble understanding and using the Pythagorean Theorem. Without this understanding, the students will have serious difficulties in higher-level mathematics. It is perfectly appropriate to group these students, find time to re-teach the concept or skill in a different way, and provide additional practice. These students should also participate with a more heterogeneous mix of students in other classroom activities and groups in which a variety of mathematics problems are discussed.

Teachers rely on their experiences and judgment to determine when and how to incorporate grouping strategies into the classroom. To promote maximum learning when grouping students, educators must

ensure that progress monitoring is ongoing, formative assessment is frequent, high-quality instruction is always provided to all students, and that students are frequently moved into appropriate instructional groups according to their needs.

Planning Instruction for Advanced Learners

In the context of this framework, *advanced learners* are students who demonstrate, or are capable of demonstrating, performance in mathematics at a level significantly above the performance that is typical for their age group. In California, each school district sets its own criteria for identifying gifted and talented students. The percentage of students identified varies, and each district may choose whether to identify students as "gifted" on the basis of their ability in mathematics and other subject areas. The criteria should take into account students who are struggling with language barriers. The criteria should also include alternative measures to identify students who are highly proficient in mathematics or have the capacity to become highly proficient in mathematics but may have a learning disability.

The National Mathematics Advisory Panel (2008) looked at research on effective mathematics instruction for gifted students and found only a few studies that met the panel's criteria for evaluating research. This lack of rigorous research limited the panel's findings and recommendations, and the panel called for more high-quality research to study the effectiveness of instructional programs and strategies for gifted students. Based on the research available, the panel reported the following findings.

National Mathematics Advisory Panel Recommendations for Gifted Students

- The studies that were reviewed provided some support for the value of differentiating the mathematics curriculum for students who have sufficient motivation, especially when acceleration is a component (i.e., pace and level of instruction are adjusted).
- A small number of studies indicated that individualized instruction, in which pace of learning is increased and often managed via computer by instructors, produces gains in learning.
- Gifted students who are accelerated by other means not only gained time and reached educational milestones (e.g., college entrance) earlier, but also appeared to achieve at levels at least comparable to those of their equally able same-age peers on a variety of indicators, even though they were younger when demonstrating their performance on various achievement benchmarks.
- Gifted students appeared to become more strongly engaged in science, technology, engineering, or mathematical areas of study. Additionally, there is no evidence in the research literature that gaps or holes in knowledge have occurred as a result of student acceleration.

Source: National Mathematics Advisory Panel 2008.

Based on these findings and general agreement in the field of gifted education, the panel stated, "combined acceleration and enrichment should be the intervention of choice" for mathematically gifted students (National Mathematics Advisory Panel 2008, 53). The panel recommended that mathematically gifted students be allowed to learn mathematics at an accelerated pace and encouraged schools to develop policies that support challenging work in mathematics for gifted students. (See appendix D, Course Placement and Sequences, for additional guidance.)

Several research studies have demonstrated the importance of setting high standards for all students, including advanced learners. The CA CCSSM provide students with goals worth reaching and identify the point at which skills and knowledge should be mastered. The natural corollary is that when standards are mastered, advanced students should either move on to standards at higher grade levels, be provided with enrichment activities that connect to or go beyond the standards, or delve deeper into mathematical concepts and connections across domains. Enrichment or extension leads students to complex, technically sound applications. Activities and challenging problems should be designed to contribute to deeper learning or new insights.

Accelerating the learning of advanced students requires the same careful, consistent, and continual assessment of their progress that is needed to support the learning of average and struggling students. Responding to the results of such assessments allows districts and schools to adopt innovative approaches to teaching and learning to best meet the instructional needs of their students.

In a classroom based on the CA CCSSM, the design of instruction demands dynamic, carefully constructed, mathematically sound lessons, units, and modules created by groups of teachers who pool their expertise to help all children learn. These teams must devise innovative methods for using regular assessments of student progress in conceptual understanding, procedural skill and fluency, and application to ensure that each student progresses toward mastery of the mathematics standards.

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Instructional Strategies

This chapter is intended to enhance teachers' repertoire, not prescribe the use of any particular instructional strategy. For any given instructional goal, teachers may choose among a wide range of instructional strategies, and effective teachers look for a fit between the material to be taught and strategies for teaching that material. (See the grade-level and course-level chapters for more specific examples.) Ultimately, teachers and administrators must decide which instructional strategies are most effective in addressing the unique needs of individual students.

In a standards-based curriculum, effective lessons, units, or modules are carefully developed and are designed to engage all members of the class in learning activities that aim to build student mastery of specific standards. Such lessons typically last at least 50 to 60 minutes daily (excluding homework). The goal that all students should be ready for college and careers by mastering the standards is central to the California Common Core State Standards for Mathematics (CA CCSSM) and this mathematics framework. Lessons need to be designed so that students are regularly exposed to new information while building conceptual understanding, practicing skills, and reinforcing their mastery of previously introduced information. The teaching of mathematics must be carefully sequenced and organized to ensure that all standards are taught at some point and that prerequisite skills form the foundation for more advanced learning. However, teaching should not proceed in a strictly linear order, requiring students to master each standard completely before they are introduced to another. Practice that leads toward mastery can be embedded in new and challenging problems that promote conceptual understanding and fluency in mathematics.

Before instructional strategies available to teachers are discussed, three important topics for the CA CCSSM will be addressed: Key Instructional Shifts, Standards for Mathematical Practice, and Critical Areas of Instruction at each grade level.

Key Instructional Shifts

Understanding how the CA CCSSM differ from previous standards—and the necessary shifts called for by the CA CCSSM—is essential to implementing California's newest mathematics standards. The three key shifts or principles on which the standards are based are *focus*, *coherence*, and *rigor*. Teachers, schools, and districts should concentrate on these three principles as they develop a common understanding of best practices and move forward with the implementation of the CA CCSSM.

Each grade-level chapter of the framework begins with the following summary of the principles.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Focus requires that the scope of content in each grade, from kindergarten through grade twelve, be significantly narrowed so that students experience more deeply the remaining content. Surveys suggest that postsecondary instructors value greater mastery of prerequisites over shallow exposure to a wide array of topics with dubious relevance to postsecondary work.

Coherence is about math making sense. When people talk about coherence, they often talk about making connections between topics. The most important connections are vertical: the links from one grade to the next that allow students to progress in their mathematical education. That is why it is critical to think across grades and examine the progressions in the standards to see how major content develops over time.

Rigor has three aspects: *conceptual understanding*, *procedural skill and fluency*, and *application*. Educators need to pursue, with equal intensity, all three aspects of rigor in the major work of each grade.

- The word *understand* is used in the standards to set explicit expectations for conceptual understanding. The word *fluently* is used to set explicit expectations for fluency.
- The phrase *real-world problems* (and the star [★] symbol) are used to set expectations and indicate opportunities for applications and modeling.

The three aspects of rigor are critical to day-to-day and long-term instructional goals for teachers. Because of this importance, they are described further below:

- *Conceptual understanding.* Teachers need to teach more than how to "get the right answer," and instead should support students' ability to acquire

Rigor in the Curricular Materials

"To date, curricula have not always been balanced in their approach to these three aspects of rigor. Some curricula stress fluency in computation without acknowledging the role of conceptual understanding in attaining fluency and making algorithms more learnable. Some stress conceptual understanding without acknowledging that fluency requires separate classroom work of a different nature. Some stress pure mathematics without acknowledging that applications can be highly motivating for students and that a mathematical education should make students fit for more than just their next mathematics course. At another extreme, some curricula focus on applications, without acknowledging that math doesn't teach itself. The standards do not take sides in these ways, but rather they set high expectations for all three components of rigor in the major work of each grade. Of course, that makes it necessary that we focus—otherwise we are asking teachers and students to do more with less."

—National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA/CCSSO) 2013, 4

concepts from several perspectives so that students are able to see mathematics as more than a set of mnemonics or discrete procedures. Students demonstrate solid conceptual understanding of core mathematical concepts by applying these concepts to new situations as well as writing and speaking about their understanding. When students learn mathematics conceptually, they understand *why* procedures and algorithms work, and doing mathematics becomes meaningful because it makes sense.

- *Procedural skill and fluency.* Conceptual understanding is not the only goal; teachers must also structure class time and homework time for students to practice procedural skills. Students develop fluency in core areas such as addition, subtraction, multiplication, and division so that they are able to understand and manipulate more complex concepts. Note that fluency is *not* memorization without understanding; it is the outcome of a carefully laid-out learning progression that requires planning and practice.
- *Application.* The CA CCSSM require application of mathematical concepts and procedures throughout all grade levels. Students are expected to use mathematics and choose the appropriate concepts for application even when they are not prompted to do so. Teachers should provide opportunities in all grade levels for students to apply mathematical concepts in real-world situations, as this motivates students to learn mathematics and enables them to transfer their mathematical knowledge into their daily lives and future careers. Teachers in content areas outside mathematics (particularly science) ensure that students use grade-level-appropriate mathematics to make meaning of and access content.

These three aspects of rigor should be taught in a balanced way. Over the years, many people have taken sides in a perceived struggle between teaching for conceptual understanding and teaching procedural skill and fluency. The CA CCSSM present a balanced approach: teaching *both*, understanding that each informs the other. Application helps make mathematics relevant to the world and meaningful for students, enabling them to maintain a productive disposition toward the subject so as to stay engaged in their own learning.

Throughout this chapter, attention will be paid to the three major instructional shifts (or principles). Readers should keep in mind that many of the standards were developed according to findings from research on student learning (e.g., on students' [in kindergarten through grade five] understanding of the four operations or on the learning of standard algorithms in grades two through six). The task for teachers, then, is to develop the most effective means for teaching the content of the CA CCSSM to diverse student populations while staying true to the intent of the standards.

Standards for Mathematical Practice

The Standards for Mathematical Practice (MP) describe expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" of longstanding importance in mathematics education. The first of these are the National Council of Teachers of Mathematics process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning; strategic compe-

tence; conceptual understanding (comprehension of mathematical concepts, operations, and relations); procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently, and appropriately); and productive disposition, which is the habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (NGA/CCSSO 2010q, 6).

Instruction must be designed to incorporate these standards effectively. Teachers should analyze their curriculum and identify where content and practice standards intersect. The grade-level chapters of this framework contain some examples where connections between the MP standards and the Standards for Mathematical Content are identified. Teachers should be aware that it is not possible to address every MP standard in every lesson and that, conversely, because the MP standards are themselves interconnected, it would be difficult to address only a single MP standard in a given lesson.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

The MP standards establish certain behaviors of mathematical expertise, sometimes referred to as "habits of mind" that *should be explicitly taught*. For example, students in third grade are not expected to know from the outset what a viable argument would look like (**MP.3**); the teacher and other students set the expectation level by critiquing reasoning presented to the class. The teacher is also responsible for creating a safe atmosphere in which students can engage in mathematical discourse that comes with rich tasks. Likewise, students in higher mathematics courses realize that the level of mathematical argument has increased: they use appropriate language and logical connections to construct and explain their arguments and communicate their reasoning clearly and effectively. The teacher serves as the guide in developing these skills. Later in this chapter, mathematical tasks are presented that exemplify the intersection of the mathematical practice and content standards.

Critical Areas of Instruction

At the beginning of each grade-level chapter in this framework, a brief summary of the Critical Areas of Instruction for the grade at hand is presented. For example, the following summary appears in the chapter on grade five:

> In grade five, instructional time should focus on three critical areas: (1) developing fluency with addition and subtraction of fractions and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); (2) extending division to two-digit divisors, integrating decimal fractions into the place-value system, developing understanding of operations with decimals to hundredths, and developing fluency with whole-number and decimal operations; and (3) developing understanding of volume (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010l). Students also fluently multiply multi-digit whole numbers using the standard algorithm.

The Critical Areas of Instruction should be considered examples of expectations of focus, coherence, and rigor for each grade level. The following points refer to the critical areas in grade five:

- Critical Area (1) refers to students using their understanding of equivalent fractions and fraction models to develop fluency with fraction addition and subtraction. Clearly, this is a major *focus* of the grade.
- Critical Area (1) is connected to Critical Area (2), as students relate their understanding of decimals as fractions to making sense out of rules for multiplying and dividing decimals, illustrating *coherence* at this grade level.
- A vertical example (i.e., one that spans grade levels) of *coherence* is evident by noticing that students have performed addition and subtraction with fractions with like denominators in grade four and reasoned about equivalent fractions in that grade; they further their understanding to add and subtract all types of fractions in grade five.
- Finally, there are several examples of *rigor* in grade five: in Critical Area (1), students apply their *understanding* of fractions and fraction models; also in Critical Area (1), students *develop fluency* in calculating sums and differences of fractions; and in Critical Area (3), students solve *real-world problems* that involve determining volumes.

Please see the CA CCSSM publication (CDE 2013a) for further explanation of these Critical Areas for each grade level. The publication is available at http://www.cde.ca.gov/re/cc/ (accessed September 1, 2015).

These are just a few examples of focus, coherence, and rigor from the Critical Areas of Instruction in grade five. Critical Areas of Instruction, which should be viewed by teachers as a reference for planning instruction, are listed at the beginning of each grade-level chapter. Additional examples of focus, coherence, and rigor appear throughout the grade-level chapters, and each grade-level chapter includes a table that highlights the content emphases at the cluster level for the grade-level standards. The bulk of instructional time should be given to "Major" clusters and the standards that are listed with them.

General Instructional Models

Teachers are presented with the task of effectively delivering instruction that is aligned with the CA CCSSM and pays attention to the Key Instructional Shifts, the Standards for Mathematical Practice, and the Critical Areas of Instruction at each grade level (i.e., instructional features). This section describes several general instructional models. Each model has particular strengths related to the aforementioned instructional features. Although classroom teachers are ultimately responsible for delivering instruction, research on how students learn in classroom settings can provide useful information to both teachers and developers of instructional resources.

Because of the diversity of students in California classrooms and the new demands of the CA CCSSM, a combination of instructional models and strategies will need to be considered to optimize student learning. Cooper (2006, 190) lists four overarching principles of instructional design for students to achieve learning with understanding:

1. Instruction is organized around the solution of meaningful problems.
2. Instruction provides scaffolds for achieving meaningful learning.

3. Instruction provides opportunities for ongoing assessment, practice with feedback, revision, and reflection.
4. The social arrangements of instruction promote collaboration, distributed expertise, and independent learning.

Mercer and Mercer (2005) suggest that instructional models may range from explicit to implicit instruction:

Explicit Instruction	Interactive Instruction	Implicit Instruction
Teacher serves as the provider of knowledge	Instruction includes both explicit and implicit methods	Teacher facilitates student learning by creating situations in which students discover new knowledge and construct their own meanings
Much direct teacher assistance	Balance between direct and non-direct teacher assistance	Non-direct teacher assistance
Teacher regulation of learning	Shared regulation of learning	Student regulation of learning
Directed discovery	Guided discovery	Self-discovery
Direct instruction	Strategic instruction	Self-regulated instruction
Task analysis	Balance between part-to-whole and whole-to-part	Unit approach
Behavioral	Cognitive/metacognitive	Holistic

Mercer and Mercer further suggest that the type of instructional models to be used during a lesson will depend on the learning needs of students and the mathematical content presented. For example, explicit instruction models may support practice to mastery, the teaching of skills, and the development of skills and procedural knowledge. On the other hand, implicit models link information to students' background knowledge, developing conceptual understanding and problem-solving abilities.

5E Model

Carr et al. (2009) link the 5E (interactive) model to three stages of mathematics instruction: introduce, investigate, and summarize. As its name implies, this model is based on a recursive cycle of five cognitive stages in inquiry-based learning: (a) engage, (b) explore, (c) explain, (d) elaborate, and (e) evaluate. Teachers have a multi-faceted role in this model. As a facilitator, the teacher nurtures creative thinking, problem solving, interaction, communication, and discovery. As a model, the teacher initiates thinking processes, inspires positive attitudes toward learning, motivates, and demonstrates skill-building techniques. Finally, as a guide, the teacher helps to bridge language gaps and foster individuality, collaboration, and personal growth. The teacher flows in and out of these various roles within each lesson.

The Three-Phase Model

The three-phase (explicit) model represents a highly structured and sequential strategy utilized in direct instruction. It has proved to be effective for teaching information and basic skills during whole-class instruction. In the first phase, the teacher introduces, demonstrates, or explains the new concept or strategy, asks questions, and checks for understanding. The second phase is an intermediate step designed to result in the independent application of the new concept or described strategy. When the teacher is satisfied that the students have mastered the concept or strategy, the third phase is implemented: students work independently and receive opportunities for closure. This phase also often serves, in part, as an assessment of the extent to which students understand what they are learning and how they use their knowledge or skills in the larger scheme of mathematics.

Singapore Math

Singapore math (an interactive instructional approach) emphasizes the development of strong number sense, excellent mental-math skills, and a deep understanding of place value. It is based on Bruner's (1956) principles, a progression from concrete experience using manipulatives, to a pictorial stage, and finally to the abstract level or algorithm. This sequence gives students a solid understanding of basic mathematical concepts and relationships before they start working at the abstract level. Concepts are taught to mastery, then later revisited but not retaught. The Singapore approach focuses on the development of students' problem-solving abilities. There is a strong emphasis on *model drawing*, a visual approach to solving word problems that helps students organize information and solve problems in a step-by-step manner. For additional information on Singapore math, please visit the National Center for Education Statistics Web site (https://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=WWCIRMSSM09 [accessed June 25, 2015]).

Concept Attainment Model

Concept attainment is an interactive, inductive model of teaching and learning that asks students to categorize ideas or objects according to critical attributes. During the lesson, teachers provide examples and non-examples, and then ask students to (1) develop and test hypotheses about the exemplars, and (2) analyze the thinking processes that were utilized. To illustrate, students may be asked to categorize polygons and non-polygons in a way that is based upon a pre-selected definition. Through concept attainment, the teacher is in control of the lesson by selecting, defining, and analyzing the concept beforehand and then encouraging student participation through discussion and interaction. This strategy may be used to introduce, strengthen, or review concepts, and as formative assessment (Charles and Senter 2012).

The Cooperative Learning Model

An important component of the mathematical practice standards is having students work together to solve problems. Students actively engage in providing input and assess their efforts in learning the content. They construct viable arguments, communicate their reasoning, and critique the reasoning of others (**MP.3**). The role of the teacher is to guide students toward desired learning outcomes. The cooperative learning model is an example of implicit instruction and involves students working either

as partners or in mixed-ability groups to complete specific tasks. It assists teachers in addressing the needs of diverse student populations, which are common in California's classrooms. The teacher presents the group with a problem or a task and sets up the student activities. While the students work together to complete the task, the teacher monitors progress and assists student groups when necessary (Charles and Senter 2012; Burden and Byrd 2010).

Cognitively Guided Instruction

The cognitively guided (implicit) instruction model calls for the teacher to have students consider different ways to solve a problem. A variety of student-generated strategies are used to solve a particular problem—for example, using plastic cubes to model the problem, counting on fingers, and using knowledge of number facts to figure out the answer. The teacher then asks the students to explain their reasoning process. They share their explanations with the class. The teacher may also ask the students to compare different strategies. Students are expected to explain and justify their strategies and, along with the teacher, take responsibility for deciding whether a strategy that is presented is viable.

This instructional model puts more responsibility on the students. Rather than being asked to simply apply a formula to virtually identical mathematics problems, students are challenged to use reasoning that makes sense to them in solving the problem and to find their own solutions. In addition, students are expected to publicly explain and justify their reasoning to their classmates and the teacher. Finally, teachers are required to open their instruction to students' original ideas and to guide each student according to his or her own developmental level and way of reasoning.

Expecting students to solve problems using mathematical reasoning and sense-making and then explain and justify their thinking has a major impact on students' learning. For example, students who develop their own strategies to solve addition problems are likely to intuitively use the commutative and associative properties of addition in their strategies. When students use their own strategies to solve problems and then justify these strategies, this contributes to a positive disposition toward learning mathematics (Wisconsin Center for Education Research 2007; National Center for Improving Student Learning and Achievement in Mathematics and Science 2000).

Problem-Based Learning

The MP standards emphasize the importance of making sense of problems and persevering in solving them (**MP.1**), reasoning abstractly and quantitatively (**MP.2**), and solving problems that are based upon "everyday life, society, and the workplace" (**MP.4**). Implicit instruction models, such as problem-based (interactive) learning, project-based learning, and inquiry-based learning, provide students with the time and support to successfully engage in mathematical inquiry by collecting data and testing hypotheses. Burden and Byrd (2010) attribute John Dewey's model of reflective thinking as the basis of the instructional model: "(a) Identify and clarify a problem; (b) Form hypotheses; (c) Collect data; (d) Analyze and interpret the data to test the hypotheses; and (e) Draw conclusions" (Burden and Byrd 2010, 145). These researchers suggest two approaches for problem-based learning: guided and unguided inquiry. During *guided inquiry*, the teacher provides the data and then questions the students so that they can arrive at a solution. Through *unguided inquiry*, students take responsibility for analyzing data and coming to conclusions.

In problem-based learning, students work either individually or in cooperative groups to solve challenging problems with real-world applications. The teacher poses the problem or question, assists when necessary, and monitors progress. Through problem-based activities, "students learn to think for themselves and show resourcefulness and creativity" (Charles and Senter 2012, 125). Martinez (2010, 149) cautions that when students engage in problem solving, they must be allowed to make mistakes: "If teachers want to promote problem solving, they need to create a classroom atmosphere that recognizes errors and uncertainties as inevitable accoutrements of problem solving." Through class discussion and feedback, student errors become the basis of furthering understanding and learning (Ashlock 1998). (For additional information, refer to appendix B [Mathematical Modeling].)

This is just a sampling of instructional models that have been researched across the globe. Ultimately, teachers and administrators must determine what works best for their student populations. Teachers may find that a combination of several instructional approaches is appropriate.

Strategies for Mathematics Instruction

As teachers progress through their careers, they develop a repertoire of instructional strategies. This section discusses several instructional strategies for mathematics instruction, but it is certainly not an exhaustive list. Teachers are encouraged to seek other mathematics teachers, professional learning from county offices of education, the California Mathematics Project, other mathematics education professionals, and Internet resources to continue building their repertoire.

Discourse in Mathematics Instruction

The MP standards call for students to make sense of problems (**MP.1**), construct viable arguments (**MP.3**), and model with mathematics (**MP.4**). Students are expected to communicate their understanding of mathematical concepts, receive feedback, and progress to deeper understanding. Ashlock (1998, 66) concludes that when students communicate their mathematical learning through discussions and writing, they are able to "relate the everyday language of their world to math language and to math symbols." Van de Walle (2007, 86) adds that the process of writing enhances the thinking process by requiring students to collect and organize their ideas. Furthermore, as an assessment tool, student writing "provides a unique window to students' thoughts and the way a student is thinking about an idea."

Number/Math Talks (Mental Math). Parrish (2010) describes number talks as:

> classroom conversations around purposefully crafted computation problems that are solved mentally. The problems in a number talk are designed to elicit specific strategies that focus on number relationships and number theory. Students are given problems in either a whole- or small-group setting and are expected to mentally solve them accurately, efficiently, and flexibly. By sharing and defending their solutions and strategies, students have the opportunity to collectively reason about numbers while building connections to key conceptual ideas in mathematics. A typical classroom number talk can be conducted in five to fifteen minutes. (Parrish 2010, xviii)

During a number talk, the teacher writes a problem on the board and gives students time to solve the problem mentally. Once students have found an answer, they are encouraged to continue finding efficient strategies while others are thinking. They indicate that they have found other approaches by raising another finger for each solution. This quiet form of acknowledgment allows time for students

to think, while the process continues to challenge those who already have an answer. When most of the students have indicated they have a solution and a strategy, the teacher calls for answers. All answers—correct and incorrect—are recorded on the board for students to consider.

Next, the teacher asks a student to defend her answer. The student explains her strategy, and the teacher records the student's thinking on the board exactly as the student explains it. The teacher serves as the facilitator, questioner, listener, and learner. The teacher then has another student share a different strategy and records his thinking on the board. The teacher is not the ultimate authority, but allows the students to have a "sense of shared authority in determining whether an answer is accurate" (Parrish 2010, 11).

Here are a few questions that teachers can ask:

- How did you solve this problem?
- How did you get your answer?
- How is one strategy similar to or different from another strategy?

Five Practices for Orchestrating Productive Mathematics Discussions. Smith and Stein (2011) identify five practices that assist teachers in facilitating instruction that advances the mathematical understanding of the class:

- Anticipating
- Monitoring
- Selecting
- Sequencing
- Connecting

Organizing and facilitating productive mathematics discussions for the classroom take a great deal of preparation and planning. Prior to giving a task to students, the teacher should anticipate the likely responses that students will have so that they are prepared to facilitate the lesson. Students usually come up with a variety of strategies, but it is helpful if teachers have already anticipated some of the strategies when leading the discussion. The teacher then poses the problem and gives the task to the students. The teacher monitors the responses while students work individually, in pairs, or in small groups. The teacher pays attention to the different strategies that students use. To conduct the "share and summarize" portion of the lesson, the teacher selects a student to present his or her mathematical work and sequences the sharing so that the various strategies are presented in a specific order, to highlight the mathematical goal of the lesson. As the teacher conducts the discussion, he or she deliberately asks questions to connect responses to the key mathematical ideas.

Student Engagement Strategies

Building a list of robust student engagement strategies is essential for all teachers. When students are engaged in the classroom, they remain focused and on task. Good classroom management and effective teaching and learning result from student engagement. The table below, provided by the Rialto Unified School District, illustrates several student engagement strategies for the mathematics classroom.

Student Engagement Strategy	Description	Math Example
Appointment Clock	Students partner to make appointments for discussions or work (a good grouping strategy).	Students are given a page with a clock printed on it. They use the clock to set appointments with other students to discuss math problems.
Carousel-Museum Walk	Each group posts sample work on the wall, and the leader for that group stands near the work while the rest of the group circulates around the room, looking at all the samples.	Each group is given a poster board and math problem to work on. When all groups have finished their work, each poster is affixed to the classroom walls. Each leader stays close to the poster created by his or her group and explains the work, while the other students walk around the room looking at other groups' work.
Charades	Students act out a scenario, individually or with a team.	Students work in teams to act out word problems while others try to solve the problems.
Clues (Barrier Games)	One partner has a picture of information that the other student does not have. Sitting back to back or using a visual barrier, students communicate to complete the task.	Working in pairs, each student communicates a different problem to the other student, who has to try to solve the problem from the information provided by the first student. The students sit with a barrier between them during the activity.
Coming to Consensus	Sharing their individual ideas, the group comes to a consensus and reveals that consensus to the entire class.	Each member of the group shares an answer to a given problem, the steps used, and so forth. When the group comes to a consensus, they reveal it to the entire class.
Explorers and Settlers	Assign half the class to be explorers and half to be settlers. Explorers seek a settler to discuss a question. Students may exchange roles and repeat the process.	Half of the students are designated as explorers who have a math term or problem. The other students are designated as settlers who have the definitions or answers. Explorers seek the settler with the correct answers and discuss the information.

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Student Engagement Strategy	Description	Math Example
Find My Rule	Students are given cards and must find the person who matches their card. One person has a card with a rule, and the other has an example of that rule. This is a great strategy for practicing inductive/deductive reasoning. It also works well for grouping students randomly and developing problem-solving skills.	Two types of cards are prepared: one with a problem and the other with the rule pertaining to that problem. Students circulate throughout the room to match the cards that are connected or related to the rule. Once all members of the group have been found, group members articulate the rule and how the group is connected.
Find Your Partner	Each student is given a card that matches another student's card in some way.	Example cards: Rectangle: Prime Number: 37 Problem $9+7$: Solution 16 Polynomial with degree 3: $4x^3-3x^2+5x-7$
Four Corners	Assign each corner of the room a category related to a topic. Students write which category they are most interested in, giving reasons, and then form groups in those corners. The activity could be adapted for different levels.	The corners of the room are numbered 2, 3, 4, and 5. Students are divided into four groups, and each group is sent to a corner. The teacher then poses a problem whose answer is a multiple of 2, 3, 4, or 5. Students in a corner that is a factor of that number will move to a different corner. If the teacher calls out 6, students in the corners labeled 2 and 3 will move. The activity ends with a prime-number answer, and students return to their seats.
Give One, Get One	After brainstorming ideas, students circulate among other students, giving one idea and receiving one. Students fold a piece of paper lengthwise to label the left side "Give one" and the right side "Get one."	The teacher gives the class a multi-step problem to solve within a specific time limit. On the "Give one" side of the paper, students name all the steps they know before finding a partner. Partner A gives an answer to partner B. If partner B has that answer, both students check it off. If partner B does not have the answer, partner B writes it on the "Get one" side. Students repeat the process with partner B going first. Once both partners have exchanged ideas, they raise their hands, find new partners, and continue until the teacher says to stop.

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Student Engagement Strategy	Description	Math Example
Inside/Outside Circle	Students stand or sit in two concentric circles, creating partners who face one another. The teacher poses a question to the class, and one partner responds. At a signal, the inner circle or outer circle rotates, and the conversation continues.	Students share information to solve problems. The teacher (or student) prepares question cards for each student. The inner-circle students ask a question from their card, the outer-circle students answer, and then these partners discuss the problem before switching roles. Once both students have asked and answered a question, the inner circle rotates clockwise to a new partner.
Jigsaw	A group of students is assigned a portion of text; these students then teach that portion to the remainder of the class.	"Factoring Jigsaw" is a game in which each student becomes an expert on a different concept or procedure in the factoring process and then teaches that concept to other students.
KWL	A cognitive graphic organizer sets the stage for learning. The teacher asks students to identify what they already **K**now, what they **W**ant to know, and what they need to do to **L**earn the skill or concept.	Math teachers use KWL as a diagnostic tool to determine student readiness, using pretest questions and a KWL chart.
Line Up (class building)	Students line up in a particular order given by the teacher (e.g., alphabetically by first name, by birth date, shortest to tallest, and so on).	Students line up in order by the number given to them: square root, fraction, decimal, or multiples of a given number. Once in line, they explain how they found their place. This is a good activity for the first day of class.
Making a List	Two students, using one word or phrase, add items to a list.	Students receive a multi-step or word problem and name the steps needed to solve the problem.

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Student Engagement Strategy	Description	Math Example
Numbered Heads Together	This is a cooperative learning strategy that holds each student accountable for learning the material. Students are placed in groups, and each person is given a number (from one to the maximum number in each group). The teacher poses a question, and students "put their heads together" to figure out the answer. The teacher calls a specific number to respond as spokesperson for the group. With students working together in a group, this strategy ensures that each member knows the answer to problems or questions asked by the teacher. Because no one knows which number will be called, all team members must be prepared.	Each group is given a problem to solve. The student whose number is called explains how the group came up with their answer.
Quiz, Quiz, Trade	Using two-sided cards prepared in advance by the teacher, students in pairs quiz each other, trade cards, and then find another partner.	May be used to help students review math vocabulary, discuss math facts, or improve their mental math skills.
Socratic Seminar	A group of students participate in a rigorous, thoughtful dialogue, seeking deeper understanding of complex ideas. Guidelines and language strategies are taught and followed during the seminar.	The teacher presents a distance-versus-time graph and asks students to describe what is happening. Alternatively, the teacher could present an action with four choices of graphs that depict the action. Students choose one of the graphs and explain and defend their choice.
Team Share	Teams take turns to share their final product.	Students work in teams on different math problems. Each team solves its assigned problem cooperatively. The team then has the opportunity to explain its answer with the entire class.
Think–Pair–Share	After the teacher poses a question, students are given time to **think** about their response. The teacher asks students to **pair** up in a specific way (e.g., elbow partners) and **share** their response only with their partner. This strategy helps students practice and refine their response.	What is the difference between prime numbers and composite numbers? Why is it sometimes a good idea to leave an expression in factored form? What is special about the number 1?

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Student Engagement Strategy	Description	Math Example
Think–Write–Pair–Share	This is a variation of think–pair–share. Students are asked to think about their response, write down their response, pair, and share. This strategy might be used when a more complicated response from students is required.	Students are given a word problem to solve. First, they are asked to think about what the problem is asking. Then they are asked to write down their idea. Finally, students share their idea with a partner.
Whiparound	The teacher poses a prompt that has multiple answers. Students write down as many responses as possible. Then the teacher "whips" around the room, calling on one student at a time. Each student shares one of his or her responses. When called on, students should not repeat a response; they must add something new.	What are examples of quadrilaterals? What are some tools you could use to help solve this problem?
Wraparound	After students write their ideas about a topic, each student shares one idea, repeating the statement of the previous student.	The teacher gives the entire class a problem and then allows time for students to write out the steps to solve the problem. Then each student describes one step in the process.

Tools for Mathematics Instruction

There are several instructional tools that teachers can use to make mathematics concepts more concrete for students. A repertoire of tools is especially important in classrooms with English learners or students with disabilities. This section highlights a small number of the tools that teachers can use with their students. (See the Universal Access chapter for more information.)

Visual representations. The MP standards suggest that students look for and make use of structure (**MP.7**), construct viable arguments (**MP.3**), model with mathematics (**MP.4**), and use appropriate tools strategically (**MP.5**). Visual representations can be used to help students achieve proficiency with these standards when used in alignment with the content standards.

To develop student understanding, meaningful relationships between mathematical concepts should be highlighted, not taught in isolation. Diagrams, concept maps, graphic organizers, math drawings, and flowcharts can be used to show relationships (Martinez 2010). Visual representations, such as graphic organizers, combine the use of words and phrases with symbols by using arrows to represent relationships (Burden and Byrd 2010). Ashlock (1998) posits that *concept maps* can be used as an overview of a lesson to summarize what has been taught and to inform instruction. A concept map is a visual organizer in which students place concepts, ideas, and algorithms in bubbles or boxes and connect the bubbles with lines or arrows and a description of how connected bubbles are related. Ashlock notes that these representations are well suited to depict computational procedures and can be created by teachers as well as by students. Visual representations may also be math drawings (e.g., students draw simple pictures to illustrate a story problem) and charts (e.g., fractions and decimals can be sorted and grouped into categories such as greater than one half, equal to one half, and less than one half).

Concrete models. The MP standards advocate the use of concrete models (also known as *manipulatives*) in order for students to make sense of problems and persevere in solving them (**MP.1**) and to use appropriate tools strategically (**MP.5**). Martinez (2010) suggests that learning that utilizes different modes of instruction is necessary to promote both student understanding and recall from long-term memory: "Good teachers know that presenting ideas in a variety of ways can make instruction more effective and more interesting, as well as better able to reach a variety of learners" (Martinez 2010, 229). Concrete models can be utilized to help students learn a wide range of mathematical concepts. For example, students create models to demonstrate the Pythagorean Theorem, they utilize tiles to demonstrate an algebraic expression, and they use base-ten models to demonstrate complex computational procedures.

Interactive technology. New teaching applications for tablet computers and laptops are being created continually. Teachers should feel comfortable about using such technology if it is available to them, but they should view teaching applications and software with a discerning eye to be sure that any technology used in the classroom adheres to the focus, coherence, and rigor of the CCSSM. (See the Technology in the Teaching of Mathematics chapter as well.)

A multitude of instructional resources are available for teachers of mathematics. It would not be possible to name them all in this chapter. Teachers are encouraged to seek multiple sources of information and research to build their instructional repertoire.

Examples of Tasks and Problems Incorporating the MP Standards

The following curricular examples illustrate the types of problems that incorporate the MP standards.

The problem below ("Marissa's Savings") addresses the grade-two standards **2.OA.1** and **2.MD.8**, as well as **MP.1**, **MP.4**, **MP.5**, and **MP.6**. The problem requires students to count a combination of coins and then demonstrate that they understand subtraction of monetary amounts; they do so by writing a story problem that shows how Marissa spends her money.

Marissa's Savings. Marissa has worked very hard to save money, and now she gets to go to the store. How much money does Marissa have? Write a story problem about how Marissa spends her money. Did she have any money left?

This problem demands that students work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them (**MP.1**) by choosing appropriate strategies to use. They apply the mathematics they know to solve problems that arise in everyday life (**MP.4**); utilize available tools, such as concrete models (**MP.5**); and use mathematically precise vocabulary to communicate their explanations by writing a story problem (**MP.6**).

Understanding Perimeter. The following hands-on activity illustrates the grade-three standard **3.MD.8**, as well as **MP.1**, **MP.3**, **MP.5**, and **MP.7**: Students will solve problems with a fixed area and perimeter and develop an understanding of the concept of perimeter by walking around the room, using rubber bands to represent the perimeter of a plane figure on a geoboard, or tracing around a shape on an interactive whiteboard. They find the perimeter of objects, use addition to find perimeters, and recognize the patterns that exist when finding the sum of the lengths and widths of rectangles.

Students use geoboards, tiles, and graph paper to find all the possible rectangles that have a given area (e.g., find the rectangles that have an area of 12 square units). Once students have learned to find the perimeter of a rectangle, they record all the possibilities by using dot or graph paper (**MP.1**); compile the possibilities into an organized list or a table, such as the one shown below (**MP.5**); and determine whether they have all the possible rectangles (**MP.3**). The patterns in the table allow the students to identify the factors of 12, connect the results to the commutative property (**MP.7**), and discuss the differences in perimeter within the same area (**MP.3**). This table can also be used to investigate rectangles with the same perimeter. (It is important to include squares in the investigation.)

Area (square inches)	Length (inches)	Width (inches)	Perimeter (inches)
12	1	12	26
12	2	6	16
12	3	4	14
12	4	3	14
12	6	2	16
12	12	1	26

Source: Kansas Association of Teachers of Mathematics (KATM) 2012, 3rd Grade Flipbook.

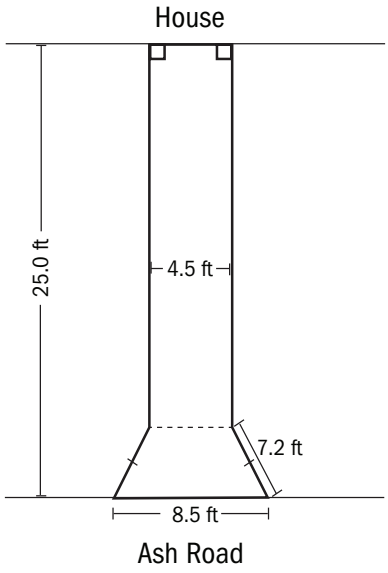
After-School Job. This problem addresses the grade-five standard **5.OA.3**, as well as **MP.1**, **MP.3**, **MP.4**, **MP.5**, and **MP.6**: Leonard needed to earn some money, so he offered to do some extra chores for his mother after school for two weeks. His mother was trying to decide how much to pay him when Leonard suggested the following idea: "You could pay me $1.00 every day for the two weeks, or you can pay me 1¢ for the first day, 2¢ for the second day, 4¢ for the third day, and so on, doubling my pay every day." Which of these two options does Leonard want his mother to choose? Write a letter to Leonard's mother suggesting the option that she should take. Be sure to include drawings that explain your mathematical thinking.

The problem requires students to generate two numerical patterns using two given rules ("add 1" and "double the sum"), generate terms in the resulting sequences over a 14-day period, and explain why the first option would cost Leonard's mother much less money. This problem demands that students

work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them by choosing the strategies to use (**MP.1**). They make conjectures and build a logical progression through careful analyses (**MP.3**); apply the mathematics they know to solve problems of interest to them that arise in everyday life (**MP.4**); utilize available tools, such as concrete models and calculators (**MP.5**); and use mathematically precise vocabulary to communicate their explanations through writing and through graphics, such as charts (**MP.6**).

The following problem ("Ms. Olsen's Sidewalk" [Smarter Balanced Assessment Consortium 2011, 111]) addresses the grade-seven standards **7.G.6** and **7.NS.3**, the grade-eight standard **8.G.7**, and MP standards **MP.1**, **MP.4**, and **MP.6**. In this task, students are given a real-world problem whose solution involves determining the areas of two-dimensional shapes as part of calculating the cost of a sidewalk.

Ms. Olsen's Sidewalk. Ms. Olsen is having a new house built on Ash Road. She is designing a sidewalk from Ash Road to her front door. Ms. Olsen wants the sidewalk to end in the shape of an isosceles trapezoid, as shown in this diagram:



The contractor charges a fee of $200 plus $12 per square foot of sidewalk. Based on the diagram, what will the contractor charge Ms. Olsen for her sidewalk? Show your work or explain how you found your answer.

A common problem in calculating the area of a trapezoid is the misuse of the length marked 7.2 feet. Students need to make use of this dimension, but they must avoid multiplying 8.5×7.2 in an attempt to find the area of the trapezoid. Once the decision has been made regarding how to best deconstruct the figure, the students need to apply the Pythagorean Theorem to calculate the length of the path connected to the trapezoid.

When this has been calculated, the remaining length and area calculations can be undertaken. The final stage of this multi-step problem is to calculate the cost of the paving based on the basic fee of $200 plus $12 per square foot. This task demands that students work across a range of mathematical practices. In particular, they need to make sense of the problem and persevere in solving it (**MP.1**), analyze the information given, and choose a solution pathway.

Furthermore, students need to attend to precision (**MP.6**) in their careful use of units in the cost calculations. In providing a written rationale of their work, both English learners and native speakers may experience linguistic difficulties in formulating their positions. Additional assistance from the teacher may be required.

The problem below ("Baseball Jerseys") comes from the Mathematics Assessment Resource Service (MARS) Web site (http://map.mathshell.org/stds.php [accessed September 1, 2015]). It addresses the grade-seven standards **7.EE.4** and **7.NS.3**, the grade-eight standards **8.EE.8** and **8.F.4**, and MP standards **MP.1**, **MP.4**, and **MP.7**.

Baseball Jerseys. Bill is going to order new jerseys for his baseball team. The jerseys will have the team logo printed on the front. Bill asks two local companies to give him a price for producing the jerseys. The first company, Print It, will charge $21.50 for each jersey. The second company, Top Print, has a setup cost of $70 and then charges $18 for each jersey. Figure out how many jerseys Bill would need to order so that the cost for Top Print would be less than the cost for Print It. Explain your answer.

Students may utilize the following approaches to solve this problem:

a. Using n for the number of jerseys ordered and c for the total cost in dollars, write an equation to show the total cost of jerseys from Print It.

b. Using n to stand for the number of jerseys ordered and c for the total cost in dollars, write an equation to show the total cost of jerseys from Top Print.

c. Use the two equations from questions (a) and (b) to figure out how many jerseys Bill would need to order so that the total cost for Top Print would be less than the total cost for Print It.

This problem considers the costing models of two print companies, and students should be able to produce two equations: $c = 21.5n$ and $c = 70 + 18n$. The third part of this task may be a bit more challenging. Students may construct the inequality $70 + 18n < 21.5n$ and then solve for n.

This problem also demands that students work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them (**MP.1**) by choosing what strategies to use. They also look for and make use of structure (**MP.7**) in that understanding the properties of linear growth leads to a solution for the problem. Finally, students practice modeling (**MP.4**) as they construct equations.

Several Internet resources provide grade-level curricular examples that are aligned with the CA CCSSM (including the MP standards). These include Department of Education Web sites from other states that have adopted the Common Core State Standards. References to these resources can be found throughout this framework. The MARS Web site (http://map.mathshell.org/stds.php) provides a multitude of exercises that focus specifically on the MP standards.

Real-World Problems

Teachers do not use real-world situations to serve mathematics; they use mathematics to serve and address real-world situations. Real-world problems provide opportunities for mathematics to be learned and engaged in context. Miller (2011) cautions that when students are assigned the task of performing real-world mathematics, the CA CCSSM do not simply want students to mimic real-world connections; the intent is for students to be able to successfully solve related mathematics problems. Students are already conditioned to do tasks. Even when a task might have strong connections to the real world, it can still be just that: a task to complete. Teachers need to keep this in mind when they ask students to perform real-world mathematics, just as the CA CCSSM suggest (Miller 2011).

In "Exploring World Maps," adapted from the California Mathematics Project (2012), students work toward mastery of standard **6.RP.3**, which calls for the use of ratio and rate reasoning to solve real-world and mathematical examples. Students are provided with a world map and are given Mexico's surface area (750,000 square miles). Then students are asked to use this information and other available tools, such as tracing paper and centimeter grids (**MP.5**), to estimate areas of several countries and continents. Finally, students are asked to provide short answers to the following questions:

a. Which area did you estimate to be larger—Mexico or Alaska?
b. Approximately how many times can Greenland fit into Africa?
c. Do you feel confident in your estimations?
d. What estimation methods did you use?
e. Now that you know the actual areas (students are provided with the actual areas prior to answering this question), what surprised you the most?
f. How does the location of the equator affect how this map is viewed?

Once again, teachers should be cognizant of potential linguistic difficulties that may affect English learners and native speakers alike. Schleppegrell (2007) notes that counting, measuring, and other "everyday" ways of doing mathematics draw on everyday language, but that the kind of mathematics that students need to develop through schooling uses language in new ways to serve new functions. It is the teacher's job to assist all students in acquiring this new language.

Supporting High-Quality Common Core Mathematics Instruction

Broad support is required to plan and implement effective and efficient mathematics instruction that meets the needs of every student. This is an important obligation shared by administrators, teacher leaders, college and university personnel, community members, parents, and other groups. The stakeholders at each school or school district form a support system that assists in the design, implementation, and evaluation of effective mathematics instructional programs. These stakeholders also serve an important function as advocates for a sustained focus on mastery of the California Common Core State Standards for Mathematics (CA CCSSM) by every student. This chapter addresses the roles and responsibilities of stakeholders in developing, implementing, and maintaining high-quality, standards-based mathematics instructional programs.

A comprehensive report titled *Greatness by Design: Supporting Outstanding Teaching to Sustain the Golden State* (Educator Excellence Task Force [EETF] 2012) calls for teachers, administrators, and other supervisors of mathematics instruction to take certain actions in response to the need for continual improvement of mathematics instruction in California. Recommendations from this report that are relevant to supporting high-quality mathematics instructional programs are summarized below (EETF 2012, 5–6):

- *Teacher education is uneven in duration and quality.* Some educators are given excellent preparation, while others receive minimal training. Education for and development of teacher leaders is more uneven in quality. Steps must be taken to ensure that every teacher participates in a high-quality preparation program and that mechanisms for developing leadership exist and are supported.

- *Mentoring for beginner teachers is decreasing.* Due to several factors, not the least of which is a decrease in funding, fewer and fewer teachers in California are receiving the benefits of high-quality mentoring. New teachers need to be supported through the difficult transition they experience in their first few years of teaching.

- *Professional learning time and opportunities are sorely underfunded.* California teachers have little time for collaboration or learning—usually only about three to five hours per week of individual planning time. Opportunities for professional learning and teacher collaboration must be seen as an integral part of the teaching profession.

- *Evaluation of teachers is inadequate.* Teacher evaluation is frequently spotty and rarely designed to give instructors and administrators the feedback and support that would help them improve or provide a fair and focused way to make personnel decisions. Evaluation efforts should be focused on helping teachers grow and improve, as opposed to being used to reprimand.

- *In most school districts, leadership pathways are poorly defined and inadequately supported.* There are few opportunities for expert teachers to share practices with their peers or to take

on leadership roles. Most teachers are still isolated from each other, teaching in self-contained classrooms and performing the same functions that they did when they first entered the profession. The spread of teaching expertise is uncommon, and this needs to change if high-quality mathematics instruction is to be available to every student in California. Instructional quality can improve when professionals work together.

It is evident that substantial professional learning for teachers will be needed to successfully implement the CA CCSSM. Although no single district or school has the absolute power or resources to address all of the concerns discussed in the *Greatness by Design* report, both the mathematics teaching community and stakeholders in mathematics instruction should consider these issues as major roadblocks to true progress. It is time for school and university educators throughout the state to combine their efforts and unite behind a common goal of improving mathematics instruction for all California students.

Administrative Role and Support

The role of school board members, district administrators, and school administrators is crucial to the success of any mathematics instructional program. Establishing and clearly articulating high expectations for all teachers and for every student are the foundation of a successful program. It is essential for administrators to express a positive attitude toward mathematics and an appreciation for the importance of mathematics in the future of every student. One of the most important jobs of principals and administrators is to help create a system of collaboration among teachers for developing CA CCSSM instructional practices—and to recognize that creating such a system will take time and support.

In order to effectively support programs of instruction, district and school administrators, as well as school board members, need to understand that high-quality mathematics instruction involves these elements:

- Knowledge of the Standards for Mathematical Practice (MP standards) contained in the CA CCSSM, as well as the Standards for Mathematical Content
- An understanding of the role of the MP standards and how they contribute to establishing effective mathematics learning environments
- An understanding that the MP standards are equal in importance to the content standards in the CA CCSSM—in particular, if students are not engaging in the MP standards, then the CA CCSSM are not being fully implemented

The following resources may help administrators understand the implications of the CA CCSSM for teaching:

1. Inside Mathematics—video excerpts of mathematics lessons highlighting the MP standards: http://www.insidemathematics.org/index.php/mathematical-practice-standards (accessed July 7, 2015)
2. The Mathematics Common Core Toolbox (PARCC prototyping project): http://www.ccsstoolbox.com (accessed July 7, 2015)

3. Implementing the Mathematical Practice Standards ("Browse Illustrations" tab): http://mathpractices.edc.org/browse-by-mps (accessed July 7, 2015)
4. Mathematics Assessment Project ("Standards" tab): http://map.mathshell.org/materials/stds.php (accessed July 7, 2015)
5. Kansas Association of Teachers of Mathematics (KATM) flipbooks: http://www.katm.org/baker/pages/common-core-resources.php (accessed August 28, 2015)

Together with their teaching staff, administrators may need to seek opportunities for learning more about the CA CCSSM through professional workshops, conferences, or professional learning. Administrators must become informed instructional leaders for mathematics education. They should also rely on teacher leaders at their school sites or within their districts to offer support and knowledge of such practices. Additionally, administrators must be aware of the assessment strategies that can be utilized in the mathematics classroom and have a balanced approach in assessing the effectiveness of mathematics instruction. They understand that the results of multiple assessment strategies—rather than a student's score on a single test—reflect an accurate understanding of student learning. In the same vein, a short walk-through of a classroom once a year is typically insufficient to accurately judge the effectiveness of instruction. To this end, district and school administrators should participate in ongoing professional learning on the topic of mathematics education and assessment of learning.

Administrators convey high expectations for mathematics instruction by supporting teachers with resources, including time for planning lessons, professional learning, and collaboration. Administrators also provide constructive, informative feedback while the teachers implement their plans. Frequent mathematics-lesson observations allow the school administrator to provide those teachers with relevant feedback regarding their instructional practices. Administrators engage with students and teachers to glean a full picture of the instructional practices used by the teacher and whether those practices are effective.

The MP standards play a crucial role in any CA CCSSM classroom. Administrators may be unfamiliar with these standards, and many would benefit from their own professional learning experiences that are centered on the CA CCSSM. The MP standards describe ways in which students engage in mathematics to develop deep conceptual understanding and procedural fluency. As students grow in mathematical maturity, the MP standards become evident in their classrooms. Students should be actively engaged in doing meaningful mathematics, discussing mathematical ideas and reasoning, applying mathematics in interesting situations, and discovering new mathematical ideas through modeling the world around them. The MP standards appear in different forms (depending on the grade level of the classroom), but in any classroom, they represent the ways in which students engage in *doing* mathematics and play a core role in instruction (adapted from Massachusetts Department of Elementary and Secondary Education [MDESE] 2011, 9). The MP standards are also described in the Overview of the Standards Chapters in this framework.

Table HQ-1 lists the MP standards and provides a few examples of what implementation of each practice may look like in the classroom.

Table HQ-1. Implementation of the Standards for Mathematical Practice

Standards for Mathematical Practice	Students	Teachers
MP.1 Make sense of problems and persevere in solving them.	• Analyze information and explain the meaning of the problem. • Actively engage in problem solving (develop, carry out, and refine a plan). • Show patience and positive attitudes. • Ask themselves if their answers make sense. • Check their answers with a different method.	• Pose rich problems and ask open-ended questions. • Provide wait-time for processing or finding solutions. • Circulate to pose probing questions and monitor student progress. • Provide opportunities and time for cooperative problem solving and reciprocal teaching.
MP.2 Reason abstractly and quantitatively.	• Represent a problem symbolically. • Explain their thinking. • Use numbers and quantities flexibly by applying properties of operations and place value. • Examine the reasonableness of answers and calculations.	• Ask students to explain their thinking regardless of accuracy. • Highlight flexible use of numbers. • Facilitate discussion through guided questions and representations. • Accept varied solutions or representations.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction.** **CA 3.1** (for higher mathematics only).	• Make conjectures to explore their ideas. • Justify solutions and approaches. • Listen to the reasoning of others, compare arguments, and decide whether the arguments make sense. • Ask clarifying and probing questions.	• Provide opportunities for students to listen to or read the conclusions and arguments of others. • Establish a safe environment for discussion. • Ask clarifying and probing questions. • Avoid giving too much assistance (e.g., providing answers or procedures).
MP.4 Model with mathematics.	• Apply prior knowledge to new problems and reflect. • Use representations to solve real-life problems. • Apply formulas and equations where appropriate. • Ask questions about the world around them and attempt to attach meaningful mathematics to the world.	• Pose problems connected to previous concepts. • Provide a variety of real-world contexts. • Use intentional representations. • Provide students the space to ask questions and pose problems about the world around them.

Table HQ-1 *(continued)*

MP.5 Use appropriate tools strategically.	• Select and use tools strategically (and flexibly) to visualize, explore, and compare information. • Use technological tools and resources to solve problems and deepen understanding.	• Make appropriate tools available for learning (e.g., calculators, concrete models, digital resources, pencils and paper, compasses, protractors, and the like). • Embed tools within instruction.
MP.6 Attend to precision.	• Calculate accurately and efficiently. • Explain thinking using mathematics vocabulary. • Use appropriate symbols and specify units of measure.	• Recognize and model efficient strategies for computation. • Use mathematics vocabulary precisely and consistently, and challenge students to do the same.
MP.7 Look for and make use of structure.	• Look for, develop, and generalize relationships and patterns. • Apply conjectures about patterns and properties to new situations.	• Provide time for applying and discussing properties. • Ask questions about the application of patterns. • Highlight different approaches for solving problems.
MP.8 Look for and make use of regularity in repeated reasoning.	• Look for methods and shortcuts through patterns in repeated calculations. • Evaluate the reasonableness of intermediate results and solutions.	• Provide tasks and problems with patterns. • Ask about possible answers before computations are made and inquire about reasonableness of answers after computations are made.

Adapted from Howard County Public School System 2011.

Administrators play an important role in supporting teachers during the transition to a CA CCSSM classroom and beyond. The MP standards represent a different vision of what students should be doing in classrooms. Students may investigate mathematical concepts with manipulatives for an entire class period or work on the same mathematics problem for a substantial amount of time. Parents may not understand this style of instruction or these new expectations for California students. Administrators will need to provide opportunities and support for teachers to introduce and explain the CA CCSSM during interactions with parents.

Mathematics Professional Learning for Teachers

For California mathematics teachers to provide highly effective mathematics instruction, there must be professional learning opportunities that deepen mathematics teachers' content knowledge and knowledge of effective instructional strategies. The content of such programs must be aligned with the goals and standards for teaching mathematics in California. As the *Greatness by Design* report notes, California must rebuild its professional learning system to make it "sustained, content-embedded, collegial and connected to practice; focused on student learning; and aligned with school improvement efforts" (EETF 2012, 16). Some of the important features of professional learning programs for teachers of mathematics are discussed in the following section.

Content of Professional Learning Programs

For a mathematics program to be effective, it must be taught by knowledgeable teachers. According to Liping Ma, "The real mathematical thinking going on in a classroom, in fact, depends heavily on the teacher's understanding of mathematics" (Ma 2010). A landmark study in 1996 found that students with initially comparable academic achievement levels had vastly different academic outcomes when teachers' knowledge of the subject matter differed (Milken 1999). The message from the research is clear: having knowledgeable teachers really does matter, and teacher expertise in a subject drives student achievement. "Improving teachers' content subject matter knowledge and improving students' mathematics education are thus interwoven and interdependent processes that must occur simultaneously" (Ma 2010, 125).

Professional learning for mathematics teachers must address the teachers' content knowledge of the topics taught at their grade level(s), as well as mathematics relevant to prior and later grade levels where appropriate. Research over the past decade has shown a positive correlation between teacher content knowledge and student learning (Hill and Lubienski 2007; Hill, Rowan, and Ball 2005). The content knowledge required of teachers at each grade level has changed significantly with the adoption of the CA CCSSM. These changes in content must be considered when professional learning programs for teachers are designed. Specific guidelines for the mathematics content knowledge at various grade spans that might appear in such programs are provided later in this chapter.

The MP standards represent a shift toward students "doing mathematics" in the classroom. As noted in *The Mathematical Education of Teachers II*, teachers "must not only understand the practices of the discipline, but how these practices can occur in school mathematics and be acquired by students" (The Conference Board of the Mathematical Sciences 2012, 8). To develop an understanding of the MP standards and the implications for mathematics instruction, teachers should engage in solving problems through the mathematical practices. Intensive, content-focused professional learning workshops—such as Saturday meetings or multi-day summer workshops—provide a forum where teachers can do this. For example, professional learning should accomplish the following results:

- Engage teachers in the posing and solving of problems, requiring teachers to make sense out of problems and learn to persevere in solving them (**MP.1**).
- Encourage teachers to explain their reasoning, make conjectures, and critique each other's reasoning in a safe environment (**MP.3**).
- Allow teachers to learn which tools are appropriate for the mathematics at hand and gather experience with the use of those tools in the classroom (**MP.5**).

Professional learning programs that incorporate teacher collaboration across schools or districts can draw on successful experiences of other teachers in teaching the MP standards.

In addition to a teacher's grade-level mathematics knowledge, contemporary mathematics education research points to the importance of teacher acquisition of a specific body of content knowledge for teaching mathematics, often referred to as *pedagogical content knowledge* (see Hill et al. [2007] for a comprehensive discussion of this idea). This body of knowledge includes understanding problem-solving strategies that arise through student thinking, knowledge of multiple representations of mathematical

concepts (e.g., multiple representations of fractions), comprehension of the relationships embedded in content areas, an understanding of common student thinking and misconceptions, knowledge of specific teaching strategies for different topics, and ways to differentiate instruction, among others.

Of note are strategies that involve students in classroom discourse as a means of implementing the MP standards. Also, formative assessment strategies can help inform teachers about the efficacy of lessons, units, or modules and the extent of student understanding. Finally, paying attention to the needs of certain populations, including students with disabilities and English learners, is crucial to providing high-quality mathematics instruction for all students in California. To the extent possible, mathematics professional learning for teachers should be attentive to these areas and rely on the most current materials and research.

Suggested mathematics content for teachers' professional learning is presented below, according to the domains and conceptual categories in the CA CCSSM. These suggestions are based on recommendations in two documents: *Gearing Up for the Common Core State Standards in Mathematics* (Institute for Mathematics and Education 2011) and *The Mathematical Education of Teachers II* (The Conference Board of the Mathematical Sciences 2012).

- Grades K–2: Counting and Cardinality; Number and Operations in Base Ten; Operations and Algebraic Thinking
- Grades 3–5: Number and Operations—Fractions; Number and Operations in Base Ten; Operations and Algebraic Thinking
- Grades K–5: Measurement and Data; Geometry
- Grades 6–8: Ratios and Proportional Relationships; The Number System; Geometry; Statistics and Probability
- Grades 9–12: Functions; Modeling; Transformational Geometry

The University of Arizona (UA) Progressions Documents for the Common Core Math Standards (http://ime.math.arizona.edu/progressions/ [UA 2011–13]) are useful tools for teachers exploring these topics. The documents can be used as starting points for content-based professional learning programs.

School administrators and teachers should strive to develop an understanding of students and youth culture to enhance mathematics instruction. Teachers have the potential to act as institutional agents with the capacity and commitment to provide institutional resources and opportunities to students (Stanton-Salazar 1997). Teacher–student relationships are potential *social capital*—that is, forms of support that help students become effective participants in the school system (Bourdieu 1977, 1986; Stanton-Salazar 1997). In the context of schools, teacher–student relationships include student learning and achievement (Katz 1999). Katz (1999) also states that two signs of productive teacher–student relationships are high expectations and caring for students. Many students value care and respect. When relationships between teachers and students become supportive, these relationships have the potential to alter students' lives in positive ways (Stanton-Salazar 2001). This notion of *teacher–student relationship* is derived from the social capital framework, which was cultivated by Bourdieu to examine the role of relationships between institutional agents and their students (Stanton-Salazar 2001).

Finally, supervisors who provide mathematics professional learning opportunities for teachers should be well versed in mathematics knowledge, knowledge of students and instructional strategies, and classroom issues that teachers face. Strong partnerships are encouraged between (1) schools, districts, and county offices of education; and (2) mathematics education faculty and mathematics faculty from nearby institutions of higher education. All have a stake in the mathematics instruction of California students, and all have something to offer to professional learning programs for teachers.

Forms of Mathematics Professional Learning Programs

The types of mathematics professional learning programs for teachers vary, but there are some common characteristics of effective professional learning programs that should be attended to when designing such programs. Professional learning programs for teachers should include mathematics content instruction for teachers, as well as effective and appropriate pedagogical strategies for the classroom. Programs for teachers should be sustained, with a focus on long-term goals. A one-shot, single-day workshop is unlikely to have a lasting effect on classroom instruction without consistent and long-term follow-up and support. Both research and the collective experience of thousands of teachers, administrators, and teacher educators in California confirm this (Darling-Hammond et al. 2009; Blank and de las Alas 2010).

Below are common models of lasting, supportive professional learning programs:

- *Summer intensive workshops or university courses for teachers.* One- or two-week summer professional learning institutes allow teachers to focus solely on the development of their knowledge of content and instructional strategies. Multimedia resources allow teachers to examine mathematics teaching in a collaborative environment and develop plans for implementation during the school year. Summer workshops, however, are most effective when paired with follow-up programs.
- *Teacher collaboration (coaching, math circles, professional learning communities).* Site-based professional learning engages teachers in real-time study of their practice. A lone teacher has a difficult road ahead if he or she wishes to implement new strategies in the classroom without the support and understanding of colleagues. Teacher collaboration has been a feature of successful professional learning programs that serve to help teachers make larger-scale changes in mathematics instruction at their schools. Such efforts are needed to implement the CA CCSSM.
- *Lesson study.* The challenges to collaboration include a tradition of autonomy in classrooms, time and scheduling constraints, lack of supportive leadership, and pressure for individual accountability. One innovative way that provides a structure for teacher collaboration is the Lesson Study Model. Lesson study, adapted from Japan, is a form of long-term professional development in which teams of teachers collaboratively plan, observe, analyze, and refine actual classroom lessons. Each lesson-study cycle consists of three phases: planning a lesson, observing student reactions to the lesson, and then analyzing those reactions. Because the focus is on the effectiveness of a lesson itself and what students learn rather than on an individual teacher's performance, the method helps reduce teachers' anxiety and resistance to being observed. To watch a full lesson-study cycle, visit https://www.collaborativeclassroom.org/lesson-study (Center for the Collaborative Classroom 2015).

- *Fostering of teacher leadership.* Teachers may be encouraged to use their expertise in formal or informal leadership roles. Teachers who attend workshops or conferences should be given the opportunity to share what they have learned with peer teachers. A teacher who shows commitment to professional learning can become a mathematics coach or start a lesson-study group at his or her school. Teachers may participate on a textbook committee, take a role in designing benchmark assessments, or be part of the school or district academic planning team. Many teachers are unaware of the leadership roles they can play in their school or district unless they are encouraged to take on such roles.

A final feature of effective mathematics professional learning is schoolwide administrative support. Teachers face many pressures in the classroom that may make them less willing to take risks when implementing new instructional techniques or using new materials. If principals and other administrators support teacher efforts to improve their instructional practices, then such changes are more likely to be integrated into classroom practice.

Induction and Support for New Teachers

Induction and support for new teachers should be given special attention in California schools. As of the writing of this document, the research of Ingersoll and Perda (2010) indicates that the recruitment and retention of mathematics teachers is of crucial importance nationwide. Data show that large numbers of teachers report dissatisfaction with their jobs because of feelings of isolation, a lack of schoolwide support and collaboration, and a lack of effective professional learning. Research indicates that this dissatisfaction can be alleviated to a large degree by the implementation of effective support programs tailored to new teachers (Ingersoll and Perda 2010). Features of such programs are similar to those described above, but also include these elements:

- Mentoring by knowledgeable, effective, reflective, and experienced teachers in the same grade level and content area as the novice teachers
- Content knowledge development to draw connections between the university mathematics courses that novice teachers just completed and the mathematics they are now required to teach
- Classroom strategies that address classroom management issues and difficulties with engaging students in the MP standards

Evaluation of Instruction

As described in the *Greatness by Design* report (EETF 2012), successful evaluation systems for teachers should provide useful feedback over time while also identifying those teachers who are struggling (and need intensive assistance) and removing those who do not improve. *Greatness by Design* (EETF 2012, 17) recommends that evaluation systems:

- *be based on the California Standards for the Teaching Profession* and assess an educator's practices—from pre-service preparation to induction—and throughout the remainder of the career;
- *tie evaluation to useful feedback and to professional learning opportunities* that are relevant to an educator's goals and needs;

- *assess the extent to which instruction aligns with the CA CCSSM*, including focus on both mathematics content and the MP standards;
- *combine data from a variety of sources*, including valid measures of educator practice, student learning, and professional contributions, which are examined in relation to one another;
- *include both formative and summative assessments*, providing information to both improve practice and support personnel decisions;
- *differentiate support* based on the educator's level of experience and individual needs;
- *build on successful Peer Assistance and Review models* for educators who need assistance in order to ensure intensive, expert support and well-grounded personnel decisions;
- *value and promote collaboration*, which supports improvement of the whole school;
- *be a priority in the district*, providing time, training, and support for evaluators and those who mentor educators needing assistance.

Expanded Learning Time

In 2012, the California Department of Education unveiled its *Common Core State Standards System Implementation Plan for California* (CDE 2012c). The plan recommends that districts "integrate the CCSS into programs and activities beyond the K–12 school setting" and suggests providing "professional development to district administrators, school principals, and after school program directors on how to collaborate to incorporate, into after school/extended day programs, activities that enrich the CCSS-related learning initiated during the regular day."[1]

Definition

"Expanded learning time" is an approach to enhance and integrate active learning experiences beyond the traditional school day—after school, before school, during summer, and with extended days, weeks, or school years—to reduce the achievement gap and improve student success.[2] These strategies utilize time outside the classroom as a unique opportunity to address the academic, social, emotional, and physical needs and interests of students through individualized and engaging learning that results in improved student achievement. Programs should be high-quality, include community partners, be results-driven, and flexible to student and community needs.

According to a report by the Forum for Youth Investment (Devaney and Yohalem 2012), traditional education and expanded learning partners may collaborate effectively on Common Core implementation by:

- increasing alignment and communication between the school staff and after school staff about learning supports and opportunities;
- increasing alignment of skills and knowledge emphasized in the Common Core standards;
- increasing awareness and sharing knowledge between school staff and after school staff;

1. The terms *extended learning programs* and *expanded learning programs* are used interchangeably and broadly refer to the learning times and experiences outside of the regular school day and year—for example, before school, after school, intersessions, and summer. The CDE has chosen to use the term *expanded learning time*.

2. This is the working definition of *expanded learning time* as of December 2012 and adopted by the California Department of Education's After School Division. This definition was developed in collaboration with the Partnership for Children and Youth.

- increasing shared professional development and planning time;
- supporting strategies on how to communicate the role and implementation of the California Common Core State Standards for Mathematics to parents and community partners.

As outlined in the definition of expanded learning time referenced above, the underlying principles explicitly reinforce and complement key aspects of the MP standards, such as making learning relevant, project-based, and engaging.

School staff and administrators may invite expanded learning providers to school and community meetings and trainings and then develop plans for more intentional alignment. This first step of collaboration may also include sharing resources and materials on Common Core implementation and the CA CCSSM. Additional resources and partners for schools and school staff include a range of providers of technical assistance in expanded learning, such as county offices of education and contracted entities that can facilitate local partnerships and share best practices for Common Core implementation. Because of the demands of Common Core implementation, the differences in programming and skill development between the traditional school day and expanded learning environments should be clearly understood by students, parents, teachers, and providers.

College, University, and Professional Support

The support of college and university personnel for high-quality mathematics instruction is also crucial. Personnel from institutions of higher education support K–12 mathematics education by joining in partnership with their local schools. By becoming more involved with other institutions of learning, college and university personnel become more aware of the research that needs to be done in the school settings. Armed with research conducted in their profession, college and university personnel can be strong advocates of high-quality mathematics instruction.

Teachers who are well prepared to teach mathematics are vital to the success of mathematics education in California. The adoption of new mathematics content standards and the forthcoming changes in assessment require many teachers to gain new knowledge and alter classroom practices. Even experienced teachers need support in learning and instituting new curriculum and instructional strategies, and new teachers and teacher candidates need even greater support in learning to teach mathematics as they acquire the fundamentals of teaching. Colleges and university personnel can provide support for those teachers through school visits and through the learning opportunities offered by higher education.

Additionally, the introduction of new mathematics standards means that the curriculum of college teacher-preparation courses that address mathematics must change to reflect new content and the MP standards. Developers of teacher-preparation programs must take the initiative to create programs that ensure knowledge of the CA CCSSM through appropriate course work and pedagogical preparation to teach higher-order thinking and performance skills for students, in addition to culturally and linguistically responsive pedagogy (EETF 2012, 29). Teacher credentialing programs should include a focus on implementing the CA CCSSM.

Local county offices of education are linked with the California Department of Education and can provide resources for the implementation of the CA CCSSM and professional learning for instruction.

County offices of education have access to the latest CA CCSSM resources and can provide support for administrators to understand the CA CCSSM and opportunities for collaboration among schools.

Finally, local, state, and national professional organizations can play a role in supporting schools, administrators, and teachers as they transition to the CA CCSSM. Examples of such organizations include the California Mathematics Project (http://csmp.ucop.edu/cmp [accessed July 9, 2015]), which has numerous regional sites; the California Mathematics Council (CMC); local affiliates of the CMC; the National Council of Teachers of Mathematics (NCTM); and the National Council of Supervisors of Mathematics (NCSM).

Community and Parent Support

Although schools are the primary learning environments for formal mathematics, students' homes and communities also play significant roles. When schools collaborate with students' family members and the community, students become fully prepared for a lifelong appreciation of mathematics. Mathematics can have a place outside the classroom: in mathematics clubs, through local and national mathematics competition teams, and through school mathematics activities that promote parent and family involvement (e.g., "Family Math Night").

Schools and districts can create formal and informal partnerships with a variety of public and private organizations, agencies, and businesses to seek support and participation in the mathematics education of children in California. Many private companies and organizations have education departments that seek opportunities to work with youngsters. Schools are encouraged to (1) use community resources to provide the additional adult support and instructional materials that students need to meet mathematics education requirements; and (2) start to develop students' ideas about the workforce, careers, and their relationship to mathematics and how to benefit their communities with that relationship.

Parental involvement in the mathematics education of children can take many different forms. Some parents may show support by voicing to their children consistent respect for the value of education in general and mathematics in particular. Parents help their children with homework or projects and take an active approach in their children's learning when they can. They mirror the appreciation for reasoning and the learning of mathematics that they hope to see in their children. Parents may volunteer in the classroom or serve in an advisory capacity on an appropriate committee. They may attend mathematics nights and workshops that are sponsored by the school or district. Regardless of how parents or family members support education, they are always made to feel welcome at their children's schools and know that their contributions are valued and appreciated.

Parents and families need to be advised of school district goals and plans for mathematics education programs. They need to be informed about the CA CCSSM and the grade- or course-level expectations for their children and how to support their children's mastery of the standards, including the habits of mind inculcated in the MP standards. In particular, parents must be aware that the CA CCSSM represent a change in mathematics instruction—a shift toward active student participation in the reasoning and discovery involved in learning mathematics. Community efforts are needed to propel California schools forward as they move to full implementation of the CA CCSSM.

Technology in the Teaching of Mathematics

The field of mathematics education has changed greatly because of technology. Educational technology can facilitate simple computation and the visualization of mathematics situations and relationships, allowing students to better comprehend mathematical concepts in practice. Technology can be a tool for students to model mathematical relationships in real-world situations. Technology is also an integral part of the Common Core State Standards Initiative and its emphasis on preparing students for college and twenty-first-century careers.

Technology pervades modern society. In such an environment, the question is not whether educational technology will be used in the classroom, but how best to use it (Cheung and Slavin 2011). Current-generation students are digital natives, and the generation of teachers who will enter the profession over the next few decades will likewise be the product of a culture in which technology is a constant presence and where the use of technology in education is a fundamental assumption. Training and supporting teachers in the use of technology are essential to the effective use of technology in the classroom.

Educational technology is a broad category that includes both a wide range of electronic devices and the applications that deliver content and support learning. Technology is an essential tool for learning mathematics in the twenty-first century, but it is only a tool; it cannot replace conceptual understanding, computational fluency, or problem-solving skills. Technological tools include both content-specific technologies (e.g., computer programs and computational devices) and content-neutral technologies, such as communication and collaboration tools (National Council of Teachers of Mathematics [NCTM] 2011a). According to guidelines adopted by the state of Massachusetts to help construct and evaluate curriculum, "Technology changes the mathematics to be learned, as well as when and how it is learned . . . Some mathematics becomes more important because technology requires it, some becomes less important because technology replaces it, and some becomes possible because technology allows it" (Massachusetts Department of Elementary and Secondary Education 2011).[1]

Research completed over the past decade has confirmed the potential benefits of educational technology applied to the teaching and learning of mathematics. When used effectively, educational technology can enhance student understanding of mathematical concepts, bolster student engagement, and strengthen problem-solving skills. Most of the recent meta-analyses of research studies in this area, however, note that these benefits depend on how educational technology is implemented, whether it is integrated with instruction, and the degree to which teachers are trained and interested in its use (Guerrero, Walker, and Dugdale 2004; Kahveci and Imamoglu 2007; Goos and Bennison 2007; Li and Ma 2010; Cheung and Slavin 2011). This chapter provides some suggestions and cautions on managing implementation to capitalize on the use of technology.

1. The excerpt from the Massachusetts Curriculum Frameworks is included by permission of the Massachusetts Department of Elementary and Secondary Education. The complete and current version of each Massachusetts curriculum framework is available at http://www.doe.mass.edu/frameworks/current.html (accessed September 2, 2015).

Educational Technology and the Common Core

The use of technology is directly integrated into the California Common Core State Standards for Mathematics (CA CCSSM). The mathematics content standards encourage the use of multiple representations and modeling to help students understand the mathematical concepts behind a problem. This is an area where the use of technology can be helpful. The standards specifically refer to using technology tools in a number of cases, especially in the middle grades and high school. For example, Geometry standard **7.G.2** states the following:

> Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle. (California Department of Education [CDE] 2013a, 50)

Similarly, the higher mathematics standards for algebra, functions, geometry, and statistics and probability include references to using technology to develop mathematical models, test assumptions, and conduct appropriate computations.

Technology is also an integral part of the Standards for Mathematical Practice (MP standards) that are emphasized throughout the CA CCSSM, starting in kindergarten and continuing through grade twelve. It is expected that students will be able to integrate technology tools into their mathematical work. For example, the descriptive text for standard **MP.5** (Use appropriate tools strategically) states the following:

> Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations . . . They are able to use technological tools to explore and deepen their understanding of concepts. (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010q)

Students who gain proficiency in the CA CCSSM are expected to know not only how to use technology tools, but also when to use them.

Technology and the Common Core: Illustrative Examples

Grade Level or Course	Content Standards	Practice Standards	Instructional Strategy Using Technology
Elementary Grades			
Kindergarten	K.CC.4	MP.2 MP.6 MP.7	Using a free application, such as "Concentration" or "Okta's Rescue" from the National Council of Teachers of Mathematics (NCTM) Illuminations[2] resources, students work in pairs to match number names with the corresponding numeral.
Grade One	1.OA.6	MP.2 MP.7 MP.8	Using a free application, such as "Deep Sea Duel" from NCTM Illuminations, students work in pairs to find various number combinations that sum to a particular number.
Grade Two	2.NBT.7	MP.1 MP.6 MP.7	Using a free application, such as "Grouping and Grazing" from NCTM Illuminations, students work on addition.
Grade Three	3.OA.7	MP.1 MP.6 MP.7	Using a free application, such as "Pick-a-Path" from NCTM Illuminations, the teacher assigns a group of students to solve problems on tablet computers while other students work directly with the teacher.
Middle Grades			
Grade Six	6.SP.3 6.SP.4	MP.3 MP.5	Using a computer, students find a data set online. They use a spreadsheet formula to calculate measures of center and variability, create a graphical representation, and write a description of the data based on the numerical and graphical evidence.
Grade Eight	8.SP.1 8.SP.2 8.SP.3	MP.4 MP.5	Students work in pairs, using two graphing calculators and one ultrasonic ranging device to collect data. The first student walks *toward* his or her partner, who uses the ranging device to record the distance between them. The two students attempt to produce a graph that is a straight line, repeating the measurements until both partners are happy with the result. The pair now reverses roles, but with the second student walking away from his or her recording partner. When the data are collected, the pair answer the following questions by manipulating the Time List and Distance List data stored in their two calculators: • How far away was your partner when he or she started? • How far away was your partner at the end of the experiment? • How long did the experiment last? • By computing $\frac{Dist\,(last) - Dist\,(first)}{Time\,(last) - Time\,(first)}$, calculate your velocity and your partner's velocity. How are these alike? How are these different? Explain your observations. • Compute your partner's velocity over the first half, second half, first quarter, second quarter, third quarter, and fourth quarter of the experiment to determine if your velocity was constant. How constant was the velocity? How do you know? • Manually or otherwise (e.g., using Median-Median or Least Squares), fit a line to your partner's data and obtain an equation for the line. What are the slope and y-intercept of your line? What do the slope and y-intercept represent in terms of the experiment? How do these compare to your earlier calculations of velocity?

Continued on next page

2 . The NCTM Illuminations resources are available at https://illuminations.nctm.org/Default.aspx (accessed September 2, 2015).

Technology and the Common Core: Illustrative Examples (*continued*)

Grade Level or Course	Content Standards	Practice Standards	Instructional Strategy Using Technology
Higher Mathematics			
Geometry Mathematics I Mathematics II	G-CO.9 G-CO.10 G-CO.11 G-CO.12 G-CO.13	MP.5 MP.7	Using dynamic geometry software and an interactive whiteboard, students investigate and create conjectures of geometric theorems and constructions.
Mathematics I Algebra I	S-ID.6 S-ID.7 S-ID.8 S-ID.9	MP.4 MP.5	Students use a computer to locate a bivariate data set. Then they use statistical software to create a scatter plot and calculate the least squares regression line. Students explore the properties of this line and use it to predict and interpret relevant results.
Mathematics I Algebra I	F-LE.3 S-ID.6a	MP.2 MP.4 MP.5	For a whole-class activity, the teacher needs a graphing calculator, one ultrasonic ranging device, a wooden plank ranging from 6 to 9 feet in length, and a large (family or industrial size) can of a non-liquid, such as refried beans or ravioli. The plank is raised to a small incline by propping up one end with one or two textbooks. The experiment consists of collecting data on the distance between the ranging device placed at the top of the ramp and the can placed at the bottom of the ramp. The can's position on the ramp and its velocity are recorded by the ultrasonic ranging device as the can is rolled up and allowed to roll back down the ramp. Preparing for the experiment, an assistant practices rolling the can up and down. From these practice rolls, the class decides on the length of the experiment (the number of trials), and students are asked to describe what they see. [The can's speed slows on the way up, there is an apparent pause at the top, and the can speeds up as it descends the ramp.] Having decided on the length of the experiment and possibly the rate of sampling, students then collect data. The rolling process is repeated until a clean run, one in which the can does not roll off the ramp, is obtained. Note that it is common for the can to roll off the ramp. The resulting graph is discussed. How close did the can get to the ranging device? The descending part of the graph corresponds to the ascent of the can. When did the can change direction (begin rolling *down* the ramp instead of *up*)? Students perform a quadratic regression and plot the resulting equation. Then they compute and examine the residuals, the number negative, the number positive, and the Mean Absolute Deviation to discuss the goodness of fit.

Technology is also an integral part of the assessment system used by the multi-state Smarter Balanced Assessment Consortium (Smarter Balanced), of which California is a governing member. Smarter Balanced has implemented computer-adaptive assessments that respond to a student's initial performance to more rapidly and accurately identify which skills the student has mastered. These assessments also allow for a quick turnaround of test results so that the results can be used to inform instruction. The Smarter Balanced test protocols allow the use of calculators on certain test items for middle and high school assessments, including integrating calculators directly into the assessment software. (For additional information, see the Assessment chapter.)

Educational Technology in the Classroom

Two basic types of educational technology are commonly used in mathematics lessons: handheld devices, which include calculators and wireless devices; and computer applications, which include software and online resources.

Handheld Devices

For decades, one of the biggest questions and controversies about educational technology was the use of calculators in classroom instruction and assessment. Previous studies (Hembree and Dessart 1986; Loveless 2004) raised cautions about the use of calculators in elementary grades, especially in terms of undermining students' skills at basic computation. Ellington (2003) found that calculators had the greatest benefit when used for both instruction and assessment. She noted, however, that "[s]tudents received the most benefit when calculators had a pedagogical role in the classroom and were not just available for drill and practice or checking work" (Ellington 2003, 456). Instruction needs to be structured to use technology in non-routine ways (i.e., not in drill and computational practice)—in other words, where students are using it to make decisions and solve problems (Guerrero, Walker, and Dugdale 2004). Graphing calculators can be used in conjunction with the technology emphasis in the CA CCSSM to allow students to actively participate in the process of developing mathematical ideas and solving problems. These devices can help students better understand spatial concepts, connect functions with their graphs, and visualize written problems to develop solutions (Ellington 2003).

In its 2011 research brief titled *Using Calculators for Teaching and Learning Mathematics*, the NCTM stated the following conclusion after a synthesis of nearly 200 studies conducted from 1976 through 2009:

> In general, we found that the body of research consistently shows that the use of calculators in the teaching and learning of mathematics does not contribute to any negative outcomes for skill development or procedural proficiency, but instead enhances the understanding of mathematics concepts and student orientation toward mathematics. (NCTM 2011b)

Although these findings do not prove that calculators enhance rather than supplant students' computational and procedural skills, they do provide reassurance that calculators can be integrated into instruction and assessment without harming student progress toward mathematical proficiency. It is important to remember that curriculum and instruction involving the use of calculators should be designed to emphasize the problem-solving and conceptual skills of students.

The next generation of handheld devices with networking capabilities also offers opportunities for using technology effectively in a classroom environment. Clark-Wilson (2010) conducted a study of seven teachers using handheld graphing computers connected via wireless network to the teacher's computer. These devices allowed teachers to monitor student work and provide live feedback and enabled students to lead the classroom discussion via a projector connected to the network. The advantages of using these devices included promoting a "collaborative classroom" where students were able to learn from each other. Clark-Wilson also noted the benefits of added student engagement, a finding that was duplicated in many other studies on the use of classroom technology.

Smartphones and tablet computers are other forms of handheld technology that are becoming increasingly common in schools. Likewise, educational applications (frequently referred to as "apps") that are designed to work with these devices are proliferating. Smartphones and tablet computers offer the advantages of built-in networking capability and access to the Internet, which enables immediate access to content and feedback from the teacher (as noted above). Tablet computers, with advantages in terms of weight and convenience, are being used by some school districts to provide delivery of instructional materials.

However, there are challenges associated with the use of handheld devices (including smartphones) to assist or provide instruction. Recent studies suggest that schools need to educate students as well as parents about the need for policies regarding student access to and use of technology at school, but comprehensive bans on cell phones may not be the most effective means of addressing these problems. Smartphones and tablet computers are easily lost or stolen and can be expensive to replace. Furthermore, having networked technology that is "always on" can also be detrimental. In a 2009 national survey of middle and high school students, 35 percent of the students admitted to using a cell phone to cheat, and 52 percent admitted to cheating by using the Internet. The survey indicated the cultural gaps involved with new technology; for example, many students did not consider texting a warning about a pop quiz to fellow students or copying text available online to turn in as their own work to be "cheating" (Common Sense Media 2009). This same survey also found that although 92 percent of parents indicated the belief that cheating through the use of cell phones happens at their child's school, only 3 percent of parents said that their child had engaged in such behavior (Challenge Success 2012). To discourage this type of cheating, teachers might include student assessments that support content mastery and require students to demonstrate their knowledge in multiple ways, explain how they solved a problem, or describe their reasoning behind a method.

Computers and Software

Computers have become a ubiquitous presence in schools over the past fifteen years. The number of computers has increased from approximately one computer for every 11 students in the late 1990s to one computer per 5.8 students in 2010–11 (Education Data Partnership 2011).

Research on the use of computer-mediated learning tools has demonstrated the potential for increased student achievement and proficiency in mathematical concepts. However, these benefits depend on the teaching approaches, types of programs, and the learners themselves (Li and Ma 2010). Kahveci and Imamoglu (2007) note that mathematics instruction is most effective when students frequently interact with the content and that the most successful instructional systems are those that adapt to students, allow them to work collaboratively, and give immediate feedback. Similarly, Li and Ma (2010) found that computer technology was most effective when used with constructivist teaching methods where students gain conceptual understanding through an inquiry-based, collaborative learning model.

As a more concrete example, Ruthven, Deaney, and Hennessy (2009) studied the use of graphing software in secondary schools for investigating algebraic equations. They found that the use of the software to enable students to graph linear and quadratic equations in the classroom had positive results in terms of efficiency of instruction, student engagement, and understanding. Students could

use the software to explore the topic and share their results—for example, by immediately seeing the effects on a graph of altering the coefficient in a formula-defining function. However, the authors of this study noted that despite the fact that the software had been "designed to do things easily," the teacher's role was still vital in structuring the activity and designing tasks that would help students master the mathematical content at the core of the lesson.

Online Learning

Online delivery of instruction is becoming increasingly popular. More than one million students in kindergarten through grade twelve enrolled in at least one online course in 2007–08 (United States Department of Education 2010). Online courses offer distinct advantages to school districts in terms of cost and convenience, especially for districts where students are distributed across a wide geographic area and challenges may exist in delivering instruction in particular content areas.

While more research is being conducted on the efficacy of online instruction, preliminary findings provide reason for optimism. In a 2009–10 study of online learning, the United States Department of Education found only five studies on K–12 education in its survey of research from 1996 through 2008. Of those five, only one dealt with mathematics, but in general, the study's authors found that the outcomes for online learning were not significantly different from those involving face-to-face instruction, and programs that combined online and face-to-face learning (a "blended" or "hybrid" model) could actually produce higher outcomes in terms of student performance. The study noted that newer online applications are able to combine delayed communication using asynchronous tools (e.g., e-mail, newsgroups, and discussion boards) with real-time communication using synchronous tools such as Web casting, chat, and video conferencing sessions. These combinations allow students to approach the subject with more interaction between the content, their peers, and their teacher, which is more conducive to the "deep learning" that is the goal of mathematics instruction. This interactive approach is consistent with a sociocultural perspective on learning, which holds that learning takes place in social environments where social activity provides support and assistance for learning (Vygotsky 1978; Cobb 1994). However, the relative newness of online learning and the limited number of studies available suggest that districts should approach online instruction with caution, especially when the material is intended to replace face-to-face instruction, rather than enhance it.

Professional Development and Teacher Support

The various research studies cited in this chapter share a consensus that educational technology cannot improve student outcomes without the classroom teacher playing a central role. The teacher must ensure that technological tools are used to support student understanding of mathematical concepts and practices; technology should not be used simply to entertain students or shield them from developing mathematical practices.

Moreover, teacher attitudes toward technology can affect its implementation and effectiveness. Guerrero, Walker, and Dugdale (2004) have noted that many teachers are cautious about technology and believe that its use may potentially hinder students' understanding and learning of mathematics. Some teachers fear, for example, that the inappropriate use of technology may interfere with students' ability to learn number facts or basic computational skills. Others are wary of the changes

to instruction that are necessary to make use of new technology in the classroom. In some cases, teachers are willing to use technology but face opposition in the form of reluctant administrators or an organizational culture that is resistant to change (Goos and Bennison 2007). It is also important for administrators to ensure that teachers have the technical support necessary to keep technology functioning and available.

Merely providing teachers with greater access to technology will not lead to its successful use (Goos and Bennison 2007; Walden University 2010). Using technology effectively requires changes in pedagogical approaches. The technological tools referenced in this chapter may involve changes to the working environment, to the format and timing of lessons and activities, and to the curriculum. Therefore, any innovation in technology must be accompanied by adaptations to the teacher's craft knowledge (Ruthven, Deaney, and Hennessy 2009).

A study by Walden University (2010) examined several myths about the relationship between educators, technology, and twenty-first-century skills. The study found that it is not necessarily true that newer teachers use technology more frequently than more experienced teachers do. The study also suggested that teachers and administrators often have very different ideas about classroom technology, with administrators more likely to assume that technology is used more often and is more effective than is actually the case. Teachers surveyed indicated that they did not feel particularly well prepared by their pre-service training programs for implementing technology and twenty-first-century skills. However, the study also reiterated the importance of the teacher's role in successful implementation of classroom technology.

These findings emphasize the critical importance of providing professional learning for teachers in the effective use of educational technology. Specifically, mathematics teachers need professional learning on how to use technology to enhance mathematics learning, not just how the tools work. This professional learning should be ongoing—not just a one-time event. Using technology to teach the same mathematical topics in fundamentally the same way does not take advantage of the capabilities of technology, and it may even be harmful in that it can show that technology is not worth the cost or effort of implementation (Garofalo et al. 2000).

The Digital Divide and the Achievement Gap

The term *digital divide* was coined in the 1990s to reference the gap in access to computers and to the Internet that separated different demographic and socioeconomic groups in the United States. The concept was popularized by a series of reports (titled *Falling Through the Net*) issued by the National Telecommunications and Information Administration (NTIA) [NTIA 1995, 1998, 1999, 2000]. These reports found that rural Americans, the socioeconomically disadvantaged, and ethnic minorities tended to have less access to modern information and communication technology and the benefits provided by those connections.

Although the gap in access has closed somewhat over the past two decades, especially in terms of access to broadband connections, it remains significant (Smith 2010). In 2009, 79.2 percent of white households had Internet access; the percentages for African American and Hispanic households were

60.0 and 57.4, respectively (United States Census Bureau 2009). Furthermore, there are concerns that minorities are less likely to be involved with social media and "Web 2.0" applications that include rich content and technologies for networking and collaboration online (Payton 2003; Trotter 2007). Given the overlap between the groups involved in the digital divide and the achievement gap in student performance, it is important that districts, schools, and teachers remain alert to the issue of equitable access to technology. Federal grants and other funding have helped to make the technology available to schools that have disproportionate populations of students from disadvantaged groups. However, despite these attempts to ensure that all students have equitable access to technology at school, many students still lack such access outside of their school environments. Studies have shown that gaps in access to reading material affect outcomes in reading achievement, and gaps in access to technology will have a similar impact on student success in twenty-first-century learning environments. The following solutions may help address these gaps (Davis et al. 2007):

- Giving students access to computer resources outside of school hours
- Issuing technology devices to students to take home
- Training teachers to be aware of access issues and providing them with strategies to address these issues as part of their professional development

Accessibility

Educational technology can help ensure that all children have access to the standards-based academic curriculum. Issues of universal access are discussed in more detail in the Universal Access chapter of this framework, but the specific ability of technology to support students with special needs should be addressed. One advantage of educational technology—the ability to differentiate instruction to meet varied learning needs—makes it a potentially effective tool to support the learning goals of these students.

Assistive technology can be used to help students with disabilities gain access to the core curriculum and perform functions that might otherwise be difficult or impossible. This technology may be a hardware device that helps a student overcome a physical disability or adaptive software that modifies content so that a student can access the curriculum. One example is a digital talking book that reads content for a student who has a visual handicap or a learning disability that affects his or her reading comprehension. Other examples include an enlarged, simplified computer keyboard; a talking computer with a joystick; headgear; or eye selection devices that could be used by students with motor difficulties. Li and Ma (2010) found that students in special education programs were a subgroup that tended to show higher gains than other students when computer technology was used to support instruction. Software that differentiates instruction can also be used to meet the needs of students who are below grade level in mathematics. The CDE's Clearinghouse for Specialized Media and Translations (http://www.cde.ca.gov/re/pn/sm/ [accessed September 2, 2015]) produces accessible versions of textbooks, workbooks, assessments, and ancillary student instructional materials. Accessible formats include braille, large print, audio, and digital files that may consist of Rich Text Format (RTF), HyperText Markup Language (HTML), the Digital Accessible Information System (DAISY), or Portable Document Format (PDF).

Educational technology may also be used to support English learners. Software that uses visual cues to assist in the teaching of mathematics concepts can help someone with limited English proficiency gain understanding. A 2010 study of one district's Digital Learning Classroom project found that interactive whiteboard technology used in grades three and five increased English learners' achievement and helped to close the achievement gap between English learners and students who are proficient in English (Lopez 2010).

Finally, educational technology can help to provide a challenging and interesting educational environment for advanced learners. Computer programs that include self-paced options and allow students to explore advanced concepts can keep these students engaged in the learning process. Technology that facilitates a collaborative learning environment can also help advanced students become involved in their peers' study of mathematics, which is a more useful outcome than simply giving advanced learners a longer list of problems to solve or sending them off to study independently. Adaptive-learning software provides individualized instruction that focuses on the needs of all students and challenges them to improve in mathematics achievement.

Assessment

Assessment plays a crucial role in delivering high-quality instruction and ensuring that all students learn. For assessment to be effective, teachers need to have sound reasons for selecting and using particular assessment tools. That is, assessment must have a clear purpose in instruction: to support and enhance student learning. Assessment activities should be embedded in instruction and provide opportunities for informative feedback to both students and teachers. A variety of assessment strategies need to be employed, as learning is multi-dimensional and cannot be adequately measured by a single instrument (Suurtamm, Koch, and Arden 2010, 400).

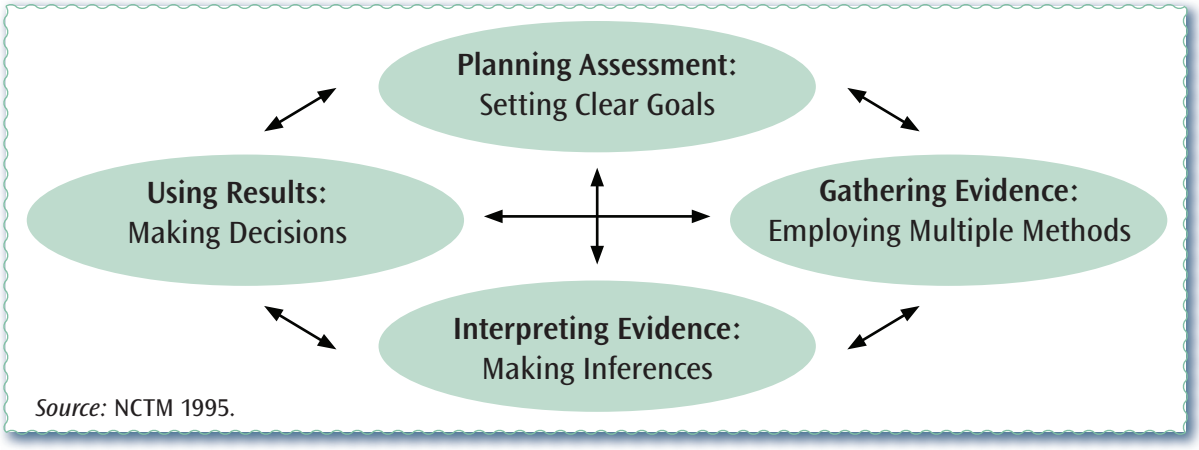
Assessment should be a major component of the learning process. It is essential to instruction because it provides:

- students with frequent and meaningful feedback on their performance;
- teachers with diagnostic tools for gauging students' depth of understanding;
- parents with information about their children's performance in the context of program goals;
- administrators with a means for measuring student achievement.

As students help identify goals for lessons or investigations, they gain greater awareness of what they need to learn and how they will demonstrate that learning. Engaging students in this kind of goal setting can help them reflect on their work, understand the standards to which they are held accountable, and take ownership of their learning.

According to the National Council of Teachers of Mathematics (NCTM), "Assessment is the process of gathering evidence about a student's knowledge of, ability to use, and disposition towards mathematics and of making inferences from the evidence for a variety of purposes" (NCTM 1995, 3). As shown in figure AS-1, the NCTM suggests that there are four interrelated phases of developing assessment and analyzing results.

Figure AS-1. Four Phases of Assessment



Source: NCTM 1995.

The purpose of this chapter is to elucidate some of the key ideas of assessment and provide examples of how to implement them.

Purposes of Assessment

As stated previously, the purpose of assessment should be to support and enhance student learning. A particular assessment may be designed to support the students in an entire school or district, the students in a single classroom, or individual students. Evidence gathered from assessments—regardless of the type of assessment involved—should be used to inform instructional decisions. For example, a teacher may use a mathematics portfolio project to measure students' long-term learning and understanding of the connections among big ideas in a unit, and then use inferences derived from the results to decide how to fill apparent gaps in student understanding before a major summative test. A district or school may use interim assessments (sometimes known as *benchmark assessments*) to track the progress of all grade-five students and then identify schools or classrooms that seem to need the most support in improving student learning. A district may collect statewide testing data and use it to identify populations of students that need support and areas where professional development is needed. If an assessment is implemented without a clearly identified goal for its use or results, then the assessment practice in question should be re-examined; resources may need to be redirected to create an assessment that is more purposeful, or the assessment may need to be eliminated altogether.

At the classroom, department, and possibly school levels, the purpose of assessment of individual students should be more than simply measuring "what students know." Traditional paper-and-pencil and "high-stakes" tests have prompted teachers to emphasize basic, factual information and to provide few opportunities for students to learn how to apply knowledge (Fuchs et al. 1999, 611). Assessment in mathematics must go beyond focusing on how well a student uses a memorized algorithm or procedure and must also elicit, assess, and respond to students' mathematical understandings (NCTM 1995; Suurtamm, Koch, and Arden 2010, 401). This change is essential in light of the Standards for Mathematical Practice, which require students to persevere through solving difficult problems, communicate mathematical thinking, use tools and model with mathematics, use quantities appropriately and attend to precision, and transfer patterns in reasoning and structure to new problems. The focus of assessment must then shift toward assessing content knowledge *and* practices rather than simply assessing content ("what students know how to do"). To help identify learning in both mathematics content and mathematical practice, assessments should ask for variety in what students produce—for example, answers and solutions, arguments and explanations, diagrams and mathematical models.

On a larger scale, assessments may be used to track progress toward long-term learning goals for groups of students or for schools receiving instructional support. Large-scale assessments can help indicate the effectiveness of a professional development program or new instructional materials. Data from statewide summative assessments can be used to identify schools that are performing well in an area or district and those where additional resources can be provided to support improvements in instruction. At the school and district levels, administrators should carefully measure the impact of chosen assessment practices on the classroom; if teachers are constantly under pressure to assess their students, then instruction will often reflect this, and the phenomenon of "teaching to the test" may

emerge. Both anecdotal and research evidence show that the undesirable outcome of teaching to the test can and does occur (Fuchs et al. 1999).

Types of Assessment

Current mathematics education literature recognizes two major forms of assessment practices: *formative* and *summative*. The distinction between these types of assessment is based on how they are used, and many forms of assessment can be used both formatively and summatively. Additionally, *diagnostic assessments* are used frequently as tools to place students in courses or identify which students might benefit from an intervention program.

Formative Assessment. Formative assessment is a systematic process to continually gather evidence and provide feedback about learning while instruction is under way. Formative assessment may span a fifteen-minute time period with an individual student, a weeklong unit, or an entire school year. The key feature of formative assessment is that action is taken to close an identified gap in students' learning based on evidence elicited from the assessment practice. As Paul Black and Dylan Wiliam (2001) state in their seminal work on the topic, "assessment becomes 'formative assessment' when the evidence is actually used to adapt the teaching work to meet the needs [of students]" (Black and Wiliam 2001, 2). If an assessment tool is used to gather information and there is no responsive change in instruction to address student misunderstandings, then the tool is not being used formatively. The four phases of assessment come into play with formative assessment, as teachers are often involved in the creation of the assessment tool, the alignment with specific goals, the administration of the tool, and reflection on the results (refer to figure AS-1).

The primary purpose of formative assessment is not merely to audit learning, but to improve it. This is assessment *for* learning rather than assessment *of* learning. Formative assessment is both an instructional tool that teachers and their students "use while learning is occurring" and "an accountability tool to determine if learning has occurred" (National Education Association 2003, 3). In other words, to be *formative*, assessments must inform the decisions that teachers and their students make minute by minute in the classroom. For example, a mid-chapter quiz is usually considered a formative assessment. However, if the result of the quiz is merely recorded in a grade book to serve the purpose of accountability or to certify competence, it cannot be considered a formative assessment. Table AS-1 shows some of the key components of formative assessment in greater detail.

Table AS-1. The Interrelated Dimensions of Formative Assessment

Shared learning targets and criteria for success. A vision of the end point makes the journey possible. Students who have a clear picture of the learning goals and of the criteria for success are likely to have a sense of what they can and should do to make their work measure up to those criteria and goals. They also have some sense of control over their work and are poised to be strategic self-regulators.
Feedback that promotes further learning. The power of formative assessment lies in its double-barreled approach, addressing both cognitive and motivational factors. To be effective, feedback comments should identify what has been done well, point out what still needs improvement, and give guidance on how to make improvements. As part of the overall learning process, teachers should plan for students to have opportunities to respond to comments. Feedback given to any pupil should be about the particular qualities of his or her work and should avoid comparisons with other pupils.
Self-assessment and peer assessment. Many successful innovations have developed self- and peer assessment by pupils as a way of enhancing formative assessment. The main problem for self-assessment is not reliability or trustworthiness; in fact, it is found that pupils are generally honest and reliable in assessing both themselves and one another. They may be too hard on themselves as often as they are too kind. The primary challenge is that pupils can assess themselves only when they have a sufficiently clear picture of the learning targets they are meant to attain. When pupils do acquire such an overview, they become more committed and more effective as learners. Their own assessments become an object of discussion with their teachers and with one another, which promotes learning.
Adapted from Black et al. 2004.

Not every formative assessment tool is appropriate for every student, goal, or topic area; therefore, teachers need to differentiate their formative assessment practices based on their experiences of using the tool(s) with their students. Furthermore, formative assessment practices are not necessarily independent of one another; several are often built into the lesson of the day or into the weekly unit.

Summative Assessment. Summative assessment refers to the assessment of learning at a particular point in time; it is meant to summarize a learner's development. Summative assessments frequently take the form of chapter or unit tests, weekly quizzes, or end-of-term tests. In contrast to formative assessment, summative assessment represents the state of a student's skills and knowledge at a particular point in time and is meant to evalute the effectiveness of instruction and a student's learning progress. Such assessments are not necessarily used to inform instruction, but they can be used to measure the effectiveness of an instructional program.

Both summative and formative assessments are essential. However, the crucial distinction between these assessment types is that one aims to determine a student's learning status and the other aims to promote greater learning. Some distinguishing characteristics of formative and summative assessment are provided in table AS-2.

Table AS-2. Characteristics of Formative and Summative Assessment

Formative Assessment (Assessment *for* Learning)	Summative Assessment (Assessment *of* Learning)
Purpose: To improve learning and achievement	**Purpose:** To measure or audit attainment of learning goals
Carried out while learning is in progress—day to day, minute by minute	Carried out from time to time to create snapshots of what has happened
Focused on the learning process and on learning progress	Focused on the products of learning
Viewed as an integral part of the teaching–learning process	Viewed as a separate activity performed after the teaching–learning cycle
Collaborative—Teachers and students know where they are headed, understand the learning needs, and use assessment information as feedback to guide and adapt what they do to meet those needs.	*Teacher directed*—Teachers assign what the students must do and then evaluate how well the students completed the assignment.
Fluid—An ongoing process influenced by student needs and teacher feedback	*Rigid*—An unchanging measure of what the students achieved
Teachers and students adopt the role of intentional learners.	Teachers adopt the role of auditors, and students assume the role of those who are being audited.
Teachers and students use the evidence they gather to make adjustments for continuous improvement.	Teachers use the results to make final "success or failure" decisions about a relatively fixed set of instructional activities.

Adapted from Moss and Brookhart 2009.

Assessment Tools

Many of the assessment tools and strategies listed in this section can be used both formatively and summatively. The list is by no means exhaustive. Furthermore, the tools listed may be administered in a formal way, such as with a checklist of skills for student observation that is filled out for every student throughout a week; or informally, such as with a "ticket-out-the-door" mini-assessment question that is used to gauge student understanding of that day's or week's major concept.

- ***Student observation*** refers to in-classroom observation of students working on mathematics tasks, either independently or in groups. Many teachers already do this by walking around the classroom, actively listening to students, asking questions, directing discourse, and helping students where needed, although they may not see this as a form of assessment. The instantaneous feedback provided to students—suggesting where to go next, recommending questions to ask to gain insight into a problem, correcting computational errors, and so on—results in this practice being a type of formative assessment. Teachers may focus their observations by using checklists that are based on specific skills and concepts.

- ***Graphic organizers*** such as flowcharts and concept maps may be used to assess students' understanding of mathematical concepts and connections between ideas. For instance, a teacher may post several terms in the classroom and ask students to (a) define the terms in their own words; and (b) connect each term to as many other terms as they can, indicating connections with an arrow and providing an explanation of why the terms are related. Teachers can ask students to provide examples of terms or concepts, to explain how and why a certain algorithm or skill works, or describe situations in which a given concept applies.

- ***Student interviews*** can help teachers gain insight into student thinking and guide teachers in providing differentiated instruction. When teachers formally or informally discuss mathematics with students, checking for understanding of concepts or procedures, there is potential to gain a much better understanding of a student's current ability than through the information provided by a paper-and-pencil test. Teachers could use such interviews as a means for assessing student progress on mastering a given standard, and the results of interviews could be factored into grading policies.

- ***Journals and learning logs*** allow students to do mathematical writing that illuminates their current understandings. For example, a teacher may provide each student with a journal—kept in the classroom—that is used for students to solve an "exit problem" of the day. Or students may be asked to explain what they learned that day or what they think the major idea of the lesson was. Such journals have a variety of uses, but teachers should not feel required to grade everything in a math journal; in fact, doing so may diminish its use, as students may feel compelled to write a "correct response." Instead, teachers can periodically read some or all of their students' journals to get feedback on student understanding.

- ***Mathematics portfolios*** are a way to assess students' understanding of important ideas, connections between ideas, procedural knowledge, and the Standards for Mathematical Practice. A project can be explored in groups over several class periods, and a "portfolio" of all the students' relevant work is submitted at the conclusion of the project. Given the nature of the California Common Core State Standards for Mathematics (CA CCSSM) and their emphasis on mathematical *practices*, tools such as portfolios will be necessary to assess students' development as problem solvers and can be used to document students' learning over time.

- ***Self- and peer evaluation*** give students ownership of their learning and provide teachers with insight into students' recognition of their own progress.

- ***Short tests and quizzes*** are used to inform instruction, and small-scale tests and quizzes can be used as formative assessments when integrated as part of a unit or chapter. Such tests and quizzes may involve several different problem types and may or may not contribute to a student's overall course grade. However, if the results of such tests and quizzes are not used to inform future instruction, then these tools are *not* being used formatively.

- ***Performance tasks*** consist of problems or scenarios that require students to think about a problem, encourage students to justify their thinking, and often require students to engage with other students. Administered to individual students or to groups, performance tasks are often complex problem-solving activities that require students to apply prior knowledge in a given situation or to extend current knowledge in new directions. The term *performance task* is broad; it may refer to in-classroom tasks or even to assessment items (see Smarter Balanced Assessment Consortium [Smarter Balanced] 2012c, 31]). Teachers may monitor students' progress on the task and give them immediate feedback as part of a larger formative assessment program.

The CA CCSSM require students to acquire a deep conceptual understanding of mathematics. The introduction of the Standards for Mathematical Practice increases the complexity of gathering evidence to determine student proficiency. Often referred to as *projects*, *oral presentations*, or *written responses to open-ended real-world problems*, performance tasks require a student to demonstrate mathematical learning across several content and practice standards that are considered prerequisite skills for college and career readiness (Smarter Balanced 2012c, 31). Various approaches can be used to determine student proficiency through performance tasks, including rating scales such as rubrics, checklists, and anecdotal records (Burden and Byrd 2010).

On Using Rubrics: A *rubric* is a type of rating scale that allows the teacher to determine mathematical learning along a continuum. By utilizing rubrics, teachers can quantify student learning while focusing upon the predetermined key components of the performance task. Popham (2010) suggests that scoring rubrics have three key features:

1. Evaluative criteria (usually three or four) that indicate the quality of the student's response
2. Descriptions of the qualitative differences in student performance for the evaluative criteria
3. Strategy for scoring, such as whether the performance task will be scored holistically (e.g., a single overall score) or analytically (e.g., points are awarded for each of the performance indicators to provide students with more specific feedback)

Van de Walle (2007) provides an example of a generic four-point rubric (see below) that can be used to sort student responses into two categories before assigning a point designation on a four-point scale. Van de Walle suggests that sharing the rubric with students ahead of time "clearly conveys what is valued" in completing the performance task (Van de Walle 2007, 84).

Scoring with a Four-Point Rubric			
Got It Evidence shows that the student understands the target concept or idea.		**Not Yet** Student shows evidence of major misunderstanding, an incorrect concept or procedure, or failure to engage in the task.	
4 Excellent: Full Accomplishment	**3** Proficient: Substantial Accomplishment	**2** Marginal: Partial Accomplishment	**1** Unsatisfactory: Little Accomplishment
Strategy and execution meet the content, process, and qualitative demands of the task. Communication is judged by effectiveness, not length. May have minor errors.	Could work to full accomplishment with minimal feedback. Errors are minor, so teacher is confident that understanding is adequate to accomplish the objective.	Part of the task is accomplished, but there is a lack of evidence of understanding or evidence of not understanding. Direct input or further teaching is required.	The task is attempted, and some mathematical effort is made. There may be fragments of accomplishment, but there is little or no success.

The Smarter Balanced Assessment Consortium provides examples of rubrics that are based upon the CCSSM, such as the following grade-six problem and scoring rubric that demonstrate student learning for standard **6.EE.5** and mathematical practice standards **MP.1**, **MP.2**, and **MP.4**.

Smarter Balanced Sample Performance Task and Scoring Rubric

Part A

Ana is saving to buy a bicycle that costs $135. She has saved $98 and wants to know how much more money she needs to buy the bicycle.

The equation $135 = x + 98$ models this situation, where x represents the additional amount of money Ana needs to buy the bicycle.

- When substituting for x, which value(s), if any, from the set $\{0, 37, 08, 135, 233\}$ will make the equation true?
- Explain what this means in terms of the amount of money needed and the cost of the bicycle.

Part B

Ana considered buying the $135 bicycle, but then she decided to shop for a different bicycle. She knows the other bicycle she likes will cost more than $150.

This situation can be modeled by the following inequality:

$$x + 98 > 150$$

- Which values, if any, from -250 to 250 will make the inequality true? If more than one value make the inequality true, identify the least and greatest values that make the inequality true.
- Explain what this means in terms of the amount of money needed and the cost of the bicycle.

Sample Top-Score Response:

Part A

The only value in the given set that makes the equation true is 37. This means that Ana will need exactly $37 more to buy the bicycle.

Part B

The values from 53 to 250 will make the inequality true. This means that Ana will need from $53 to $250 to buy the bicycle.

Scoring Rubric: Responses to this item will receive 0–3 points, based on the following descriptions.

3 points: The student shows a thorough understanding of equations and inequalities in a contextual scenario, as well as a thorough understanding of substituting values into equations and inequalities to verify whether they satisfy the equation or inequality. The student offers a correct interpretation of the equality and the inequality in the correct context of the problem. The student correctly states that 37 will satisfy the equation and that the values from 53 to 250 will satisfy the inequality.

2 points: The students shows a thorough understanding of substituting values into equations and inequalities to verify whether they satisfy the equation or inequality, but limited understanding of equations or inequalities in a contextual scenario. The student correctly states that 37 will satisfy the equation and that the values from 53 to 250 will satisfy the inequality, but the student offers an incorrect interpretation of the equality or the inequality in the context of the problem.

1 point: The student shows a limited understanding of substituting values into equations and inequalities to verify whether they satisfy the equation or inequality and demonstrates a limited understanding of equations and inequalities in a contextual scenario. The student correctly states that 37 will satisfy the equation, does not state that the values from 53 to 250 will satisfy the inequality, and offers incorrect interpretations of the equality and the inequality in the context of the problem. **OR** The student correctly states that the values from 53 to 250 will satisfy the inequality, does not state that 37 satisfies the equation, and offers incorrect interpretations of the equality and the inequality in the context of the problem.

0 points: The student shows little or no understanding of equations and inequalities in a contextual scenario and little or no understanding of substituting values into equations and inequalities to verify whether they satisfy the equation or inequality. The student offers incorrect interpretations of the equality and the inequality in the context of the problem, does not state that 37 satisfies the equation, and does not state the values from 53 to 250 will satisfy the equation.

- ***Unit or chapter assessments*** measure student learning of the content and skills in a unit or chapter. Such tests should include items that are linked to specific learning goals, be connected to the CA CCSSM, and pay attention to the Standards for Mathematical Practice. To effectively assess such goals, such tests should include various types of tasks, including multiple choice, selected response (possibly more than one correct response), short answer, and short performance tasks.

- ***Diagnostic assessments*** are often broad in scope, containing a range of topics that are prerequisites for success in a particular unit, class, or grade level. Such assessments may identify specific areas of difficulty that need to be addressed through intervention and can inform the placement of students into intervention programs.

- ***Interim assessments*** can be administered on a relatively frequent basis and are used to measure the incremental learning of students throughout a given period of time. These tests identify specific performance standards students have or have not achieved and often reveal possible reasons why students have not yet progressed in certain areas. Interim assessments are frequently used as formative assessments as well.

- ***State or national assessments*** are large-scale assessments used to gather information about the progress of academic systems and entire bodies of students. (See the section on the Smarter Balanced Assessment Consortium's assessments that appears later in this chapter.)

Thoughts On Grading. Although a classroom grading policy is ultimately a local decision, a message is presented here about the overall purpose and direction of a grading policy. In a chapter titled "The Last Frontier: Tackling the Grading Dilemma" (O'Connor 2007), author Ken O'Connor provides several guidelines for designing grading policies:

- Rather than determining one final grade based only on assessment methods (quizzes, tests, homework, and so forth), teachers should issue grades that are based on and provided for intended learning goals linked to the CA CCSSM.
- Individual achievement should be the primary attribute included in a student's grade. Other aspects such as effort and participation can be graded, but these should not impact measures of achievement.
- Grading should be flexible enough to provide for a sampling of student performance, rather than including every activity and assignment in a grade, and quality assessments with proper recording of student achievement should determine that performance.
- Finally, teachers should discuss assessment with students and involve students in assessment throughout the learning process.

Thoughts on Homework. As with grading policies, whether and how to use homework as an instructional tool and an assessment tool is a local decision. However, if homework is used in a course, it should have clear, standards-based goals that students can achieve on their own. Homework should also promote student ownership of their learning, instill a sense of competence, and be clear and accessible to students. Some reasons for assigning homework include pre-learning of concepts, checking for understanding of classroom work, practice of skills and procedures, and processing of concepts developed in class. Appropriate homework feedback can serve a formative purpose if it provides students and the teacher with direction for learning. As an example, teachers may indicate to students that they should work on problems 1 through 5 first; if these problems are not difficult, then students can move on. However, if a student has difficulty with these first five problems, then that should serve as a warning sign that the student needs to see the teacher for further instruction. Regardless, teachers and administrators should consider a clear purpose for homework as a means for assessment and learning (Van de Walle and Folk 2005).

Smarter Balanced Assessment Consortium, Common Core Assessments

California's participation in the Smarter Balanced Assessment Consortium has resulted in a statewide assessment program designed to measure students' and schools' progress toward meeting the goals of the CA CCSSM for grades three through eight and in grade eleven. Smarter Balanced assessments require students to think critically, solve problems, and show a greater depth of knowledge. They are aligned with the following four claims:

Claim 1	**Concepts and Procedures: Students can explain and apply mathematical concepts and interpret and carry out mathematical procedures with precision and fluency.** This claim addresses procedural skills and the conceptual understanding on which the development of skills depends. It is important to assess students' knowledge of how concepts are linked and why mathematical procedures work the way they do. Central to understanding this claim is making the connection to elements of these mathematical practices as stated in the CA CCSSM: **MP.5**, **MP.6**, **MP.7**, and **MP.8**.
Claim 2	**Problem Solving: Students can solve a range of complex, well-posed problems in pure and applied mathematics, making productive use of knowledge and problem-solving strategies.** Assessment items and tasks focused on Claim 2 include problems in pure mathematics and problems set in context. Problems are presented as items and tasks that are well posed (that is, problem formulation is not necessary) and for which a solution path is not immediately obvious. These problems require students to construct their own solution pathway rather than follow a solution pathway that has been provided for them. Such problems are therefore unstructured, and students will need to select appropriate conceptual and physical tools to solve them.
Claim 3	**Communicating Reasoning: Students can clearly and precisely construct viable arguments to support their own reasoning and to critique the reasoning of others.** Claim 3 refers to a recurring theme in the CA CCSSM content and practice standards: the ability to construct and present a clear, logical, and convincing argument. For older students this may take the form of a rigorous deductive proof based on clearly stated axioms. For younger students this will involve justifications that are less formal. Assessment tasks that address this claim typically present a claim and ask students to provide a justification or counterexample.
Claim 4	**Modeling and Data Analysis: Students can analyze complex, real-world scenarios and can construct and use mathematical models to interpret and solve problems.** Modeling is the bridge between "school math" and "the real world"—a bridge that has been missing from many mathematics curricula and assessments. Modeling is the twin of mathematical literacy, which is the focus of international comparison tests in mathematics given by the Programme for International Student Assessment (PISA). The CA CCSSM feature modeling as both a mathematical practice at all grade levels and a content focus in higher mathematics courses.

Some of the features of the Smarter Balanced assessment program are listed below. Additional information about the assessment program is available at http://www.smarterbalanced.org/ (accessed September 4, 2015).

California now uses a statewide assessment system known as the California Assessment of Student Performance and Progress (CAASPP). Information on the CAASPP system is available at http://www.cde.ca.gov/ta/tg/ca/ (accessed September 4, 2015).

- ***Computer-based testing.*** Schools with the capability to administer tests electronically do so for every student in their purview. Computer-based testing allows for smoother test administration, faster reporting of results, and the utilization of computer-adaptive testing.
- ***Computer-adaptive testing.*** The Smarter Balanced assessments use a system that monitors a student's progress as he or she is taking the assessment and presents the student with harder or easier problems depending on the student's performance on the current item. In this way, the computer system can make adjustments to more accurately assess the student's knowledge and skills.
- ***Varied items.*** The Smarter Balanced tests allow for several types of items that are intended to measure different learning outcomes. For instance, a selected response item may have two correct choices out of four; a student who selects only one of those correct items would indicate a different understanding of a concept than a student who selects both of the correct responses. Constructed-response questions are featured, as well as performance assessment tasks (which include extended-response questions) that measure students' abilities to solve problems and use mathematics in context, thereby measuring students' progress toward employing the mathematical practice standards and demonstrating their knowledge of mathematics content. Finally, the assessments feature technology-enhanced items that aim to provide evidence of mathematical practices.

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Instructional Materials to Support the California Common Core State Standards for Mathematics

Although instructional resources have changed over the years from slate boards and chalk to interactive whiteboards, one thing remains true: high-quality instructional resources help teachers to teach and students to learn. Instructional resources are an important component in the implementation of the California Common Core State Standards for Mathematics (CA CCSSM). They should be selected with great care and with the instructional needs of all students in mind.

Instructional resources for mathematics include a variety of instructional materials—tools such as connectable cubes, rulers, protractors, graph paper, calculators, and objects to count; and technology such as interactive whiteboards and student-response devices. The term *instructional materials* is broadly defined to include textbooks, technology-based materials, other educational materials, and tests. This chapter provides guidance on the selection of instructional materials, including the state adoption of instructional materials, guidance for local districts on the adoption of instructional materials for students in grades nine through twelve, the social content review process, supplemental instructional materials, and accessible instructional materials.

State Adoption of Instructional Materials

The California State Board of Education (SBE) adopts instructional materials for use by students in kindergarten through grade eight. Under current state law, local educational agencies (LEAs)—school districts, charter schools, and county offices of education—are not required to purchase state-adopted instructional materials. LEAs have the authority and the responsibility to conduct their own evaluation of instructional materials and to adopt the materials that best meet the needs of their students. Additionally, there is no state-level adoption of instructional materials for use by students in grades nine through twelve; LEAs have the sole responsibility and authority to adopt instructional materials for those students.

The primary source of guidance for the selection of instructional materials is the *Criteria for Evaluating Mathematics Instructional Materials for Kindergarten Through Grade Eight (Criteria)*, adopted by the SBE on January 16, 2013 (see next page). The *Criteria* document provides a comprehensive description of effective instructional programs that are aligned with the CA CCSSM and support the principles of focus, coherence, and rigor. The *Criteria* document was the basis for the 2014 Primary Adoption of Mathematics Instructional Materials and is a useful tool for LEAs that conduct their own evaluations of instructional materials.

Criteria for Evaluating Mathematics Instructional Materials for Kindergarten Through Grade Eight

Adopted by the California State Board of Education on January 16, 2013

Instructional materials that are adopted by the state help teachers to present and students to learn the content set forth in the *Common Core State Standards for Mathematics with California Additions* (Standards)[1]; this refers to the content standards and the standards for mathematical practice, as revised pursuant to California *Education Code* Section 60605.11 (added by Senate Bill 1200, Statutes of 2012). To accomplish this purpose, this document establishes criteria for evaluating instructional materials for the eight-year adoption cycle beginning with the primary adoption in 2013–14. These criteria serve as evaluation guidelines for the statewide adoption of mathematics instructional materials for kindergarten through grade eight, as called for in *Education Code* Section 60207.

The Standards require focus, coherence, and rigor, with content and mathematical practice standards intertwined throughout. The Standards are organized by grade level in kindergarten through grade eight and by conceptual categories for higher mathematics. For this adoption, the standards for higher mathematics are organized into model courses and are assigned to a first course in a traditional or an integrated sequence of courses. There are a number of supportive and advisory documents that are available for publishers and producers of instructional materials that define the depth of instruction necessary to support the focus, coherence, and rigor of the standards. These documents include the *Progressions Documents for Common Core Math Standards* (http://ime.math.arizona.edu/progressions/); the PARCC *Model Content Frameworks* (available at http://www.parcconline.org/); Smarter Balanced Test Specifications (available at http://www.smarterbalanced.org/); the Illustrative Mathematics project (http://illustrativemathematics.org/); and California's mathematics framework. Overall, the Standards do not dictate a singular approach to instructional resources—to the contrary, they provide opportunities to raise student achievement through innovations.

It is the intent of the State Board of Education that these criteria be seen as neutral on the format of instructional materials in terms of digital, interactive online, and other types of curriculum materials.

I. Focus, Coherence, and Rigor in the Common Core State Standards for Mathematics

With the advent of the Common Core, a decade's worth of recommendations for greater focus and coherence finally have a chance to bear fruit. Focus and coherence are the two major evidence-based design principles of the Standards. These principles are meant to fuel greater achievement in a rigorous curriculum, in which students acquire conceptual understanding, procedural skill and fluency, and the ability to apply mathematics to solve problems. Thus, the implications of the standards for mathematics education could be summarized briefly as follows:

Focus: Place strong emphasis where the Standards focus.

Coherence: Think across grades, and link to major topics in each grade.

1. As of 2014, the Standards are now called the California Common Core State Standards for Mathematics (CA CCSSM).

Rigor: In major topics, pursue with equal intensity:

- conceptual understanding;
- procedural skill and fluency;
- applications.

Focus

Focus requires that we significantly narrow the scope of content in each grade so that students more deeply experience that which remains.

The overwhelming focus of the Standards in early grades is arithmetic, along with the components of measurement that support it. That includes the **concepts** underlying arithmetic, the **skills** of arithmetic computation, and the ability to **apply** arithmetic to solve problems and put arithmetic to engaging uses. Arithmetic in the K–5 standards is an important life skill, as well as a thinking subject and a rehearsal for algebra in the middle grades.

Focus remains important through the middle and high school grades in order to prepare students for college and careers; surveys suggest that postsecondary instructors value greater mastery of prerequisites over shallow exposure to a wide array of topics with dubious relevance to postsecondary work.

Both of the assessment consortia have made the focus, coherence, and rigor of the Standards central to their assessment designs.[2] Choosing materials that also embody the Standards will be essential for giving teachers and students the tools they need to build a strong mathematical foundation and succeed on standards-aligned assessments.

Coherence

Coherence is about making math make sense. Mathematics is not a list of disconnected tricks or mnemonics. It is an elegant subject in which powerful knowledge results from reasoning with a small number of principles such as place value and properties of operations.[3] The standards define progressions of learning that leverage these principles as they build knowledge over the grades.[4]

When people talk about coherence, they often talk about making connections between topics. The most important connections are vertical: the links from one grade to the next that allow students to progress in their mathematical education. That is why it is critical to think across grades and examine the progressions in the standards to see how major content develops over time.

Connections at a single grade level can be used to improve focus, by tightly linking secondary topics to the major work of the grade. For example, in grade three, bar graphs are not "just another topic to cover." Rather, the standard about bar graphs asks students to use information presented in bar graphs

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2 . See the Smarter Balanced content specifications and item development specifications, as well as the PARCC Model Content Framework and item development ITN. Complete information about the consortia can be found at http://www.smarterbalanced.org/ and http://www.parcconline.org/.

3. For some remarks by Phil Daro on this theme, see the video at https://vimeo.com/45730600 (accessed September 3, 2015).

4. For more information on progressions in the Standards, visit http://ime.math.arizona.edu/progressions/ (accessed September 3, 2015).

to solve word problems using the four operations of arithmetic. Instead of allowing bar graphs to detract from the focus on arithmetic, the Standards are showing how bar graphs can be positioned in support of the major work of the grade. In this way coherence can support focus.

Materials cannot match the contours of the Standards by approaching each individual content standard as a separate event. Nor can materials align with the Standards by approaching each individual **grade** as a separate event: "The standards were not so much assembled out of topics as woven out of progressions. Maintaining these progressions in the implementation of the standards will be important for helping all students learn mathematics at a higher level . . . For example, the properties of operations, learned first for simple whole numbers, then in later grades extended to fractions, play a central role in understanding operations with negative numbers, expressions with letters, and later still the study of polynomials. As the application of the properties is extended over the grades, an understanding of how the properties of operations work together should deepen and develop into one of the most fundamental insights into algebra. The natural distribution of prior knowledge in classrooms should not prompt abandoning instruction in grade-level content, but should prompt explicit attention to connecting grade-level content to content from prior learning. To do this, instruction should reflect the progressions on which the CCSSM [Common Core State Standards for Mathematics] are built."[5]

Rigor

To help students meet the expectations of the Standards, educators will need to pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, procedural skill and fluency, and applications. The word *understand* is used in the Standards to set explicit expectations for conceptual understanding; the word *fluently* is used to set explicit expectations for fluency; and the phrase *real-world problems* and the star (★) symbol are used to set expectations and flag opportunities for applications and modeling (which is a standard for mathematical practice as well as a content category in high school). Real-world problems and standards that support modeling are also opportunities to provide activities related to careers and the work world.

To date, curricula have not always been balanced in their approach to these three aspects of rigor. Some curricula stress fluency in computation without acknowledging the role of conceptual understanding in attaining fluency. Some stress conceptual understanding without acknowledging that fluency requires separate classroom work of a different nature. Some stress pure mathematics without first acknowledging that applications can be highly motivating for students and, moreover, that a mathematical education should prepare students for more than just their next mathematics course. At another extreme, some curricula focus on applications without acknowledging that math does not teach itself.

The Standards do not take sides in these ways, but rather they set high expectations for all three components of rigor in the major work of each grade. Of course, that makes it necessary that we first follow through on the focus in the Standards—otherwise we are asking teachers and students to do more with less.

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5. See "Appendix: The Structure of the Standards" in *K–8 Publishers' Criteria for the Common Core State Standards for Mathematics*, p. 21 (http://www.corestandards.org/assets/Math_Publishers_Criteria_K-8_Summer%202012_FINAL.pdf [accessed September 3, 2015]).

II. Criteria for Materials and Tools Aligned with the Standards

Three Types of Programs

Three types of programs will be considered for adoption: basic grade-level for kindergarten through grade eight, Algebra I, and Integrated Mathematics I (hereafter referred to as Mathematics I). All three types of programs must stand alone and will be reviewed separately. Publishers may submit programs for one grade or any combination of grades. In addition, publishers may include intervention and acceleration components to support students.

Basic Grade-Level Program

The basic grade-level program is the comprehensive curriculum in mathematics for students in kindergarten through grade eight. It provides the foundation for instruction and is intended to ensure that all students master the *Common Core State Standards for Mathematics with California Additions*.

Common Core Algebra I and Common Core Mathematics I

When students have mastered the content described in the *Common Core State Standards for Mathematics with California Additions* for kindergarten through grade eight, they will be ready to complete Common Core Algebra I or Common Core Mathematics I. The course content will be consistent with its high school counterpart and will articulate with the subsequent courses in the sequence.

Criteria for Materials and Tools Aligned with the Standards

The criteria for the evaluation of mathematics instructional resources for kindergarten through grade eight are organized into six categories:

1. **Mathematics Content/Alignment with the Standards.** Content as specified in the *Common Core State Standards for Mathematics with California Additions*, including the Standards for Mathematical Practices, and sequence and organization of the mathematics program that provide structure for what students should learn at each grade level.
2. **Program Organization.** Instructional materials support instruction and learning of the standards and include such features as lists of the standards, chapter overviews, and glossaries.
3. **Assessment.** Strategies presented in the instructional materials for measuring what students know and are able to do.
4. **Universal Access.** Access to the standards-based curriculum for all students, including English learners, advanced learners, students below grade level in mathematical skills, and students with disabilities.
5. **Instructional Planning.** Information and materials that contain a clear road map for teachers to follow when planning instruction.
6. **Teacher Support.** Materials designed to help teachers provide effective standards-based mathematics instruction.

Materials that fail to meet the criteria for category 1 (Mathematics Content/Alignment with the Standards) will not be considered suitable for adoption. The criteria for category 1 must be met in the core materials or via the primary means of instruction, rather than in ancillary components. In addition, programs must have strengths in each of categories 2 through 6 to be suitable for adoption.

Category 1: Mathematics Content/Alignment with the Standards

Mathematics materials should support teaching to the *Common Core State Standards for Mathematics with California Additions*. Instructional materials suitable for adoption must satisfy the following criteria:

1. **The mathematics content is correct, factually accurate, and written with precision. Mathematical terms are defined and used appropriately. Where the standards provide a definition, materials use that as their primary definition to develop student understanding.**

2. **The materials in basic instructional programs support comprehensive teaching of the *Common Core State Standards for Mathematics with California Additions* and include the standards for mathematical practice at each grade level or course.** The standards for mathematical practice must be taught in the context of the content standards at each grade level or course. The principles of instruction must reflect current and confirmed research. The materials must be aligned with and support the design of the *Common Core State Standards for Mathematics with California Additions* and address the grade-level content standards and standards for mathematical practice in their entirety.

3. **In any single grade in the kindergarten-through-grade-eight sequence, students and teachers using the materials as designed spend the large majority of their time on the major work of each grade.** The major work (major clusters) of each grade is identified in the Content Emphases by Cluster documents for K–8.[6] In addition, major work should especially predominate in the first half of the year (e.g., in grade 3 this is necessary so that students have sufficient time to build understanding and fluency with multiplication). Note that an important **subset** of the major work in grades K–8 is the progression that leads toward Algebra I and Mathematics I (see table IM-1 on the next page). Materials give especially careful treatment to these clusters and their interconnections. Digital or online materials that allow navigation or have no fixed pacing plan are explicitly designed to ensure that students' time on task meets this criterion.

6. For cluster-level emphases in grades K–8, see http://www.achievethecore.org/downloads/Math%20Shifts%20and%20Major%20Work%20of%20Grade.pdf (accessed September 4, 2015).

Table IM-1. Progression to Algebra I and Mathematics I in Kindergarten Through Grade Eight

Kindergarten	Grade One	Grade Two	Grade Three	Grade Four	Grade Five	Grade Six	Grade Seven	Grade Eight
Know number names and the count sequence Count to tell the number of objects Compare numbers Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from Work with numbers 11–19 to gain foundations for place value	Represent and solve problems involving addition and subtraction Understand and apply properties of operations and the relationship between addition and subtraction Add and subtract within 20 Work with addition and subtraction equations Extend the counting sequence Understand place value Use place-value understanding and properties of operations to add and subtract Measure lengths indirectly and by iterating length units	Represent and solve problems involving addition and subtraction Add and subtract within 20 Understand place value Use place-value understanding and properties of operations to add and subtract Measure and estimate lengths in standard units Relate addition and subtraction to length	Represent and solve problems involving multiplication and division Understand properties of multiplication and the relationship between multiplication and division Multiply and divide within 100 Solve problems involving the four operations, and identify and explain patterns in arithmetic Develop understanding of fractions as numbers Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects Geometric measurement: understand concepts of area, and relate area to multiplication and to addition	Use the four operations with whole numbers to solve problems Generalize place-value understanding for multi-digit whole numbers Use place-value understanding and properties of operations to perform multi-digit arithmetic Extend understanding of fraction equivalence and ordering Build fractions from unit fractions by applying and extending previous understandings of operations Understand decimal notation for fractions, and compare decimal fractions	Understand the place-value system Perform operations with multi-digit whole numbers and decimals to hundredths Use equivalent fractions as a strategy to add and subtract fractions Apply and extend previous understandings of multiplication and division to multiply and divide fractions Geometric measurement: understand concepts of volume, and relate volume to multiplication and to addition Graph points in the coordinate plane to solve real-world and mathematical problems*	Apply and extend previous understandings of multiplication and division to divide fractions by fractions Apply and extend previous understandings of numbers to the system of rational numbers Understand ratio concepts and use ratio reasoning to solve problems Apply and extend previous understandings of arithmetic to algebraic expressions Reason about and solve one-variable equations and inequalities Represent and analyze quantitative relationships between dependent and independent variables	Apply and extend previous understanding of operations with fractions to add, subtract, multiply, and divide rational numbers Analyze proportional relationships and use them to solve real-world and mathematical problems Use properties of operations to generate equivalent expressions Solve real-life and mathematical problems using numerical and algebraic expressions and equations	Work with radicals and integer exponents Understand the connections between proportional relationships, lines, and linear equations Analyze and solve linear equations and pairs of simultaneous linear equations Define, evaluate, and compare functions Use functions to model relationships between quantities

Adapted from Achieve the Core 2012.

*Indicates a cluster that is well thought of as part of a student's progress to algebra, but that is currently not designated as Major by one or both of the assessment consortia (PARCC and Smarter Balanced) in their draft materials. Apart from the one exception marked by an asterisk, the clusters listed here are a subset of those designated as Major in both of the assessment consortia's draft documents.

4. **Focus:** In aligned materials there are no chapter tests, unit tests, or other assessment components that make students or teachers responsible for any topics before the grade in which they are introduced in the Standards. (One way to meet this criterion is for materials to omit these topics entirely prior to the indicated grades.) If the materials address topics outside of the *Common Core State Standards for Mathematics with California Additions*, the publisher will provide a mathematical and pedagogical justification.

5. **Focus and Coherence Through Supporting Work: Supporting clusters do not detract from focus, but rather enhance focus and coherence simultaneously by engaging students in the major clusters of the grade.** For example, materials for K–5 generally treat data displays as an occasion for solving grade-level word problems using the four operations.[7]

6. **Rigor and Balance: Materials and tools reflect the balances in the Standards and help students meet the Standards' rigorous expectations, by all of the following:**

    a. **Developing students' conceptual understanding of key mathematical concepts, where called for in specific content standards or cluster headings, including connecting conceptual understanding to procedural skills.** Materials amply feature high-quality conceptual problems and questions that can serve as fertile conversation starters in a classroom if students are unable to answer them. In addition, group discussion suggestions include facilitation strategies and protocols. In the materials, conceptual understanding is not a generalized imperative applied with a broad brush, but is attended to most thoroughly in those places in the content standards where explicit expectations are set for understanding or interpreting. (Conceptual understanding of key mathematical concepts is thus distinct from applications or fluency work, and these three aspects of rigor must be balanced as indicated in the Standards.)

    b. **Giving attention throughout the year to individual standards that set an expectation of fluency.** The Standards are explicit where fluency is expected. In grades K–6, materials should help students make steady progress throughout the year toward fluent (accurate and reasonably fast) computation, including knowing single-digit products and sums from memory (see, for example, standards **2.OA.2** and **3.OA.7**). The word *fluently* in particular as used in the Standards refers to fluency with a written or mental method, not a method using manipulatives or concrete representations. Progress toward these goals is interwoven with developing conceptual understanding of the operations in question.[8]

    Manipulatives and concrete representations such as diagrams that enhance conceptual understanding are closely connected to the written and symbolic methods to which they refer (see, for example, standard **1.NBT**). As well, purely procedural problems and exercises are present. These include cases in which opportunistic strategies are valuable—for example, the sum

7. For more information about this example, see Table 1 in the *Progression* for K–3 Categorical Data and 2–5 Measurement Data (https://commoncoretools.files.wordpress.com/2011/06/ccss_progression_md_k5_2011_06_20.pdf). More generally, the PARCC *Model Content Frameworks* give examples in each grade of how to improve focus and coherence by linking supporting topics to the major work.

8. For more about how students develop fluency in tandem with understanding, see the *Progressions* for Operations and Algebraic Thinking (https://commoncoretools.files.wordpress.com/2011/05/ccss_progression_cc_oa_k5_2011_05_302.pdf) and for Number and Operations in Base Ten (https://commoncoretools.files.wordpress.com/2011/04/ccss_progression_nbt_2011_04_073.pdf).

$698 + 240$ or the system $x + y = 1$, $2x + 2y = 3$ — as well as an ample number of generic cases so that students can learn and practice efficient algorithms (e.g., the sum $8767 + 2286$). Methods and algorithms are general and based on principles of mathematics, not mnemonics or tricks.[9] Materials do not make fluency a generalized imperative to be applied with a broad brush, but attend most thoroughly to those places in the content standards where explicit expectations are set for fluency. In higher grades, algebra is the language of much of mathematics. Like learning any language, we learn by using it. Sufficient practice with algebraic operations is provided so as to make realistic the attainment of the Standards as a whole; for example, fluency in algebra can help students get past the need to manage computational details so that they can observe structure (**MP.7**) and express regularity in repeated reasoning (**MP.8**).

c. **Allowing teachers and students using the materials as designed to spend sufficient time working with engaging applications, without losing focus on the major work of each grade.** Materials in grades K–8 include an ample number of single-step and multi-step contextual problems that develop the mathematics of the grade, afford opportunities for practice, and engage students in problem solving. Materials for grades 6–8 also include problems in which students must make their own assumptions or simplifications in order to model a situation mathematically. Applications take the form of problems to be worked on individually, as well as classroom activities centered on application scenarios. Materials attend thoroughly to those places in the content standards where expectations for multi-step and real-world problems are explicit. Applications in the materials draw only on content knowledge and skills specified in the content standards, with particular stress on applying major work, and a preference for the more fundamental techniques from additional and supporting work. Modeling builds slowly across K–8, and applications are relatively simple in early grades. Problems and activities are grade-level appropriate, with a sensible tradeoff between the sophistication of the problem and the difficulty or newness of the content knowledge the student is expected to bring to bear.[10]

Additional aspects of the Rigor and Balance Criterion:

(1) *The three aspects of rigor are not always separate in materials.* (Conceptual understanding needs to underpin fluency work; fluency can be practiced in the context of applications; and applications can build conceptual understanding.)

(2) *Nor are the three aspects of rigor always together in materials.* (Fluency requires dedicated practice to that end. Rich applications cannot always be shoehorned into the mathematical topic of the day. And conceptual understanding will not come along for free unless explicitly taught.)

(3) Digital and online materials with no fixed lesson flow or pacing plan are not designed for superficial browsing, but rather instantiate the Rigor and Balance criterion and promote depth and mastery.

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9. Non-mathematical approaches (such as the "butterfly method" of adding fractions) compromise focus and coherence and displace mathematics in the curriculum (see **5.NF.1**). For additional background on this point, see the remarks by Phil Daro at https://vimeo.com/45730600 (accessed September 3, 2015).

10. See *Common Core State Standards for Mathematics* (CCSSM, 84) at http://www.corestandards.org/the-standards (accessed September 4, 2015). Also note that modeling is a mathematical practice in every grade, but in high school it is also a content category (CCSSM, 72–73); therefore, modeling is generally enhanced in high school materials, with more elements of the modeling cycle (CCSSM, 72).

7. **Consistent Progressions: Materials are consistent with the progressions in the Standards, by (all of the following):**

   a. **Basing content progressions on the grade-by-grade progressions in the Standards.**

      Progressions in materials match closely with those in the Standards. This does not require the table of contents in a book to be a replica of the content standards; but the match between the Standards and what students are to learn should be close in each grade. Discrepancies are clearly aimed at helping students meet the Standards as written, rather than effectively re-writing the standards. Comprehensive materials do not introduce gaps in learning by omitting content that is specified in the Standards.

      The basic model for grade-to-grade progression involves students making tangible progress during each given grade, as opposed to substantially reviewing and then marginally extending from previous grades. Remediation may be necessary, particularly during transition years, and resources for remediation may be provided, but review is clearly identified as such to the teacher, and teachers and students can see what their specific responsibility is for the current year.

      Digital and online materials that allow students and/or teachers to navigate content across grade levels promote the Standards' coherence by tracking the structure and progressions in the Standards. For example, such materials might link problems and concepts so that teachers and students can browse a progression.

   b. **Giving all students extensive work with grade-level problems.**

      Differentiation is sometimes necessary, but materials often manage unfinished learning from earlier grades inside grade-level work, rather than setting aside grade-level work to re-teach earlier content. Unfinished learning from earlier grades is normal and prevalent; it should not be ignored nor used as an excuse for cancelling grade-level work and retreating to below-grade work. (For example, the development of fluency with division using the standard algorithm in grade six is the occasion to surface and deal with unfinished learning about place value; this is more productive than setting aside division and backing up.) Likewise, students who are "ready for more" can be provided with problems that take grade-level work in deeper directions, not just exposed to later-grades' topics.

   c. **Relating grade-level concepts explicitly to prior knowledge from earlier grades.**

      The materials are designed so that prior knowledge becomes reorganized and extended to accommodate the new knowledge. Grade-level problems in the materials often involve application of knowledge learned in earlier grades. Although students may well have learned this earlier content, they have not learned how it extends to new mathematical situations and applications. They learn basic ideas of place value, for example, and then extend them across the decimal point to tenths and beyond. They learn properties of operations with whole numbers and then extend them to fractions, variables, and expressions. The materials make these extensions of prior knowledge explicit. Note that cluster headings in the Standards sometimes signal key moments where reorganizing and extending previous knowledge is important in order to accommodate new knowledge (e.g., see the cluster headings that use the phrase "Apply and extend previous understanding").

8. **Coherent Connections: Materials foster coherence through connections at a single grade, where appropriate and where required by the Standards, by (all of the following):**

   a. **Including learning objectives that are visibly shaped by CCSSM cluster headings, with meaningful consequences for the associated problems and activities.** While some clusters are simply the sum of their individual standards (e.g., Grade 8, Expressions and Equations, Cluster C: Analyze and solve linear equations and pairs of simultaneous linear equations), many are not (e.g., Grade 8, Expressions and Equations, Cluster B: Understand the connection between proportional relationships, lines, and linear equations). In the latter cases, cluster headings function like topic sentences in a paragraph in that they state the point of, and lend additional meaning to, the individual content standards that follow. Cluster headings can also signal multi-grade progressions by using phrases such as "Apply and extend previous understandings of $[X]$ to do $[Y]$." Hence an important criterion for coherence is that some or many of the learning objectives in the materials are visibly shaped by CCSSM cluster headings, with meaningful consequences for the associated problems and activities. Materials do not simply treat the Standards as a sum of individual content standards and individual practice standards.

   b. **Including problems and activities that serve to connect two or more clusters in a domain, or two or more domains in a grade, in cases where these connections are natural and important.** If instruction only operates at the individual standard level, or even at the individual cluster level, then some important connections will be missed. For example, robust work in standard **4.NBT** should sometimes or often synthesize across the clusters listed in that domain; robust work in grade four should sometimes or often involve students applying their developing computation NBT skills in the context of solving word problems detailed in OA. Materials do not invent connections not explicit in the standards without first attending thoroughly to the connections that are required explicitly in the Standards (e.g., standard **3.MD.7** connects area to multiplication, to addition, and to properties of operations; standard **A-REI.11** connects functions to equations in a graphical context; proportion connects to percentage, similar triangles, and unit rates). Not everything in the standards is naturally well connected or needs to be connected (e.g., Order of Operations has essentially nothing to do with the properties of operations, and connecting these two things in a lesson or unit title is actively misleading). Instead, connections in materials are mathematically natural and important (e.g., base-ten computation in the context of word problems with the four operations), reflecting plausible, direct implications of what is written in the Standards without creating additional requirements. Instructional materials include problems and activities that connect to real-world and career settings, where appropriate.

9. **Practice-to-Content Connections: Materials meaningfully connect content standards and practice standards.** The National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA/CCSSO) states, "Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction" (NGA/CCSSO 2010c, 8). Over the course of any given year of instruction, each mathematical practice standard is meaningfully present in the form of activities or problems that stimulate students to develop the habits of mind described in the practice standards. These

practices are well grounded in the content standards. Materials are accompanied by an analysis, aimed at evaluators, of how the authors have approached each practice standard in relation to content within each applicable grade or grade band. Materials do not treat the practice standards as static across grades or grade bands, but instead tailor the connections to the content of the grade and to grade-level-appropriate student thinking. Materials also include teacher-directed materials that explain the role of the practice standards in the classroom and in students' mathematical development.

10. **Focus and Coherence via Practice Standards: Materials promote focus and coherence by connecting practice standards with content that is emphasized in the Standards.** Content and practice standards are not connected mechanistically or randomly, but instead support focus and coherence. Examples: Materials connect looking for and making use of structure (**MP.7**) with structural themes emphasized in the Standards such as properties of operations, place-value decompositions of numbers, numerators and denominators of fractions, numerical and algebraic expressions, and so forth; materials connect looking for and expressing regularity in repeated reasoning (**MP.8**) with major topics by using regularity in repetitive reasoning as a tool with which to explore major topics. (In K–5, materials might use regularity in repetitive reasoning to shed light on, for example, the 10×10 addition table, the 10×10 multiplication table, the properties of operations, the relationship between addition and subtraction or multiplication and division, and the place-value system; in 6–8, materials might use regularity in repetitive reasoning to shed light on proportional relationships and linear functions; in high school, materials might use regularity in repetitive reasoning to shed light on formal algebra as well as functions, particularly recursive definitions of functions.)

11. **Careful Attention to Each Practice Standard: Materials attend to the full meaning of each practice standard.** For example, standard **MP.1** does not say "Solve problems" or "Make sense of problems" or "Make sense of problems and solve them." It says, "Make sense of problems and persevere in solving them." Thus, students using the materials as designed build their perseverance in grade-level-appropriate ways by occasionally solving problems that require them to persevere to a solution beyond the point when they would like to give up. Standard **MP.5** does not say "Use tools" or "Use appropriate tools." It says, "Use appropriate tools strategically." Thus, materials include problems that reward students' strategic decisions about how to use tools or about whether to use them at all. Standard **MP.8** does not say "Extend patterns" or "Engage in repetitive reasoning." It says, "Look for and express regularity in repeated reasoning." Thus, it is not enough for students to extend patterns or perform repeated calculations. Those repeated calculations must lead to an insight (e.g., "When I add a multiple of 3 to another multiple of 3, then I get a multiple of 3"). The analysis for evaluators explains how the full meaning of each practice standard has been attended to in the materials.

12. **Emphasis on Mathematical Reasoning: Materials support the Standards' emphasis on mathematical reasoning, by all of the following:**

    a. **Prompting students to construct viable arguments and critique the arguments of others concerning key grade-level mathematics that is detailed in the content standards (see standard MP.3).** Materials provide sufficient opportunities for students to reason mathematically in independent thinking and express reasoning through classroom discussion and written work. Reasoning is not confined to optional or avoidable sections of the materials but is inevitable

when using the materials as designed. Materials do not approach reasoning as a generalized imperative, but instead create opportunities for students to reason about key mathematics detailed in the content standards for the grade. Materials thus attend first and most thoroughly to those places in the content standards setting explicit expectations for explaining, justifying, showing, or proving. Students are asked to critique given arguments, for example, by explaining under what conditions, if any, a mathematical statement is valid. Materials develop students' capacity for mathematical reasoning in a grade-level-appropriate way, with a reasonable progression of sophistication from early grades up through high school.[11] Teachers and students using the materials as designed spend classroom time communicating reasoning (by constructing viable arguments and explanations and critiquing those of others concerning key grade-level mathematics)—recognizing that learning mathematics also involves time spent working on applications and practicing procedures. Materials provide examples of student explanations and arguments (e.g., fictitious student characters might be portrayed).

b. **Engaging students in problem solving as a form of argument.** Materials attend thoroughly to those places in the content standards that explicitly set expectations for multi-step problems; multi-step problems are not scarce in the materials. Some or many of these problems require students to devise a strategy autonomously. Sometimes the goal is the final answer alone (see standard **MP.1**); sometimes the goal is to show work and lay out the solution as a sequence of well-justified steps. In the latter case, the solution to a problem takes the form of a cogent argument that can be verified and critiqued, instead of a jumble of disconnected steps with a scribbled answer indicated by drawing a circle around it (see standard **MP.6**). Problems and activities of this nature are grade-level-appropriate, with a reasonable progression of sophistication from early grades up through high school.

c. **Explicitly attending to the specialized language of mathematics.** Mathematical reasoning involves specialized language. Therefore, materials and tools address the development of mathematical and academic language associated with the Standards. The language of argument, problem solving, and mathematical explanations are taught rather than assumed. Correspondences between language and multiple mathematical representations including diagrams, tables, graphs, and symbolic expressions are identified in material designed for language development. Note that variety in formats and types of representations—graphs, drawings, images, and tables in addition to text—can relieve some of the language demands that English learners face when they have to show understanding in math.

d. **Materials help English learners access challenging mathematics, learn content, and develop grade-level language.** For example, materials might include annotations to help with comprehension of words, sentences, and paragraphs, and give examples of the use of words in other situations. Modifications to language do not sacrifice the mathematics, nor do they put off necessary language development.

11. As students progress through the grades, their production and comprehension of mathematical arguments evolves from informal and concrete toward more formal and abstract. In early grades, students employ imprecise expressions which, with practice over time, become more precise and viable arguments in later grades. Indeed, the use of imprecise language is part of the process in learning how to make more precise arguments in mathematics. Ultimately, conversation about arguments helps students transform assumptions into explicit and precise claims.

Category 2: Program Organization

The organization and features of the instructional materials support instruction and learning of the Standards. Teacher and student materials include such features as lists of the standards, chapter overviews, and glossaries. Instructional materials must have strengths in these areas to be considered suitable for adoption.

1. A list of *Common Core State Standards for Mathematics with California Additions* is included in the teacher's guide together with page-number citations or other references that demonstrate alignment with the content standards and standards for mathematical practice. All standards must be listed in their entirety with their cluster heading included.

2. Materials drawn from other subject-matter areas are consistent with the currently adopted California standards at the appropriate grade level, including the *California Career Technical Education Model Curriculum Standards* where applicable.

3. Intervention components, if included, are designed to support students' progress in mathematics and develop fluency. Intervention materials should provide targeted instruction on standards from previous grade levels and develop student learning of the standards for mathematical practice.

4. Middle school acceleration components, if included, are designed to support students' progress beyond grade-level standards in mathematics. Acceleration materials should provide instruction targeted toward readiness for higher mathematics at the middle school level.

5. Teacher and student materials contain an overview of the chapters, clearly identify the mathematical concepts, and include tables of contents, indexes, and glossaries that contain important mathematical terms.

6. Support materials are an integral part of the instructional program and are clearly aligned with the *Common Core State Standards for Mathematics with California Additions*.

7. The grade-level content standards and the standards for mathematical practice demonstrating alignment with student lessons shall be explicitly stated in the student editions.

Category 3: Assessment

Instructional materials should contain strategies and tools for continually measuring student achievement. Formative assessment is a systematic process to continuously gather evidence and provide feedback about learning while instruction is under way. Formative assessments can take multiple forms and occur over varied durations of time. They are to be used to gather information about student learning and to address student misunderstandings. Formative assessments are to provide guidance for the teacher in determining whether the student needs additional materials or resources to achieve grade-level standards and conceptual understanding. Instructional materials in mathematics must have strengths in these areas to be considered suitable for adoption:

1. Not every form of assessment is appropriate for every student or every topic area, so a variety of assessment types need to be provided for formative assessment. Some of these could include (but are not limited to) graphic organizers, student observation, student interviews, journals and learning logs, exit ticket activities, mathematics portfolios, self- and peer evaluations, short tests and quizzes, and performance tasks.

2. *Summative assessment* is the assessment of learning at a particular time point and is meant to summarize a learner's skills and knowledge at a given point in time. Summative assessments frequently come in the form of chapter or unit tests, weekly quizzes, end-of-term tests, or diagnostic tests.

3. All assessments should have content validity and measure individual student progress both at regular intervals and at strategic points of instruction. The assessments should be designed to:
   - monitor student progress toward meeting the content and mathematical practice standards;
   - assess all three aspects of rigor—conceptual understanding, procedural skill and fluency, and applications;
   - provide summative evaluations of individual student achievement;
   - provide multiple methods of assessing what students know and are able to do, such as selected response, constructed response, real-world problems, performance tasks, and open-ended questions;
   - assist the teacher in keeping parents and students informed about student progress.

4. Intervention aspects of mathematics programs should include initial assessments to identify areas of strengths and weaknesses, formative assessments to demonstrate student progress toward meeting grade-level standards, and a summative assessment to determine student preparedness for grade-level work.

5. Suggestions on how to use assessment data to guide decisions about instructional practices and how to modify instruction so that all students are consistently progressing toward meeting or exceeding the standards should be included.

6. Assessments that ask for variety in what students produce, answers and solutions, arguments and explanations, diagrams, mathematical models.

7. Assessment tools for grades six through eight help to determine student readiness for Common Core Algebra I and Common Core Mathematics I.

8. Middle school acceleration aspects of mathematics programs include an initial assessment to identify areas of strengths and weaknesses, formative assessments to demonstrate student progress toward exceeding grade-level standards, and a summative assessment to determine student preparedness for above-grade-level work.

Category 4: Universal Access

Students with special needs must be provided access to the same standards-based curriculum that is provided to all students, including both the content standards and the standards for mathematical practice. Instructional materials should provide access to the standards-based curriculum for all students, including English learners, advanced learners, students below grade level in mathematical skills, and students with disabilities. Instructional materials in mathematics must have strengths in these areas to be considered suitable for adoption:

1. Comprehensive guidance and differentiation strategies, based on current and confirmed research, to adapt the curriculum to meet students' identified special needs and to provide effective, efficient instruction for all students. Strategies may include:
   - working with students' misconceptions to strengthen their conceptual understanding;
   - intervention strategies that describe specific ways to address the learning needs of students using rich problems that engage them in the mathematics reviewed and stress conceptual development of topics rather than focusing only on procedural skills;
   - suggestions for reinforcing or expanding the curriculum;
   - additional instructional time and additional practice, including specialized teaching methods or materials and accommodations for students with special needs;
   - help for students who are below grade level, including more explicit explanations with ample and different opportunities for review and practice of both content and mathematical practices standards, or other assistance that will help to accelerate student performance to grade level;
   - technology that may be used to aid in the implementation of these strategies.
2. Strategies for English learners that are consistent with the English Language Development Standards adopted under *Education Code* Section 60811. Materials incorporate strategies for English learners in both lessons and teachers' editions, as appropriate, at every grade level and course level.
3. Materials incorporate instructional strategies to address the needs of students with disabilities in both lessons and teachers' editions, as appropriate, at every grade level and course level, pursuant to *Education Code* section 60204(b)(2).
4. Teacher and student editions include thoughtful and well-conceived alternatives for advanced students and that allow students to accelerate beyond their grade-level content (acceleration) or to study the content in the *Common Core State Standards for Mathematics with California Additions* in greater depth or complexity (enrichment).
5. Materials should help students understand and use appropriate academic language and participate in discussions about mathematical concepts and reasoning. Materials should include content that is relevant to English learners, advanced learners, students below grade level in mathematical skills, and students with disabilities.
6. Materials help English learners access challenging mathematics, learn content, and develop grade-level language. For example, materials might include annotations to help with comprehension

of words, sentences and paragraphs, and give examples of the use of words in other situations. Modifications to language do not sacrifice the mathematics, nor do they put off necessary language development.

7. Materials are consistent with the strategies found in Response to Instruction and Intervention (http://www.cde.ca.gov/ci/cr/ri/).

8. The visual design of the materials does not distract from the mathematics, but instead serves to support students in engaging thoughtfully with the subject.

Category 5: Instructional Planning

Instructional materials must contain a clear road map for teachers to follow when planning instruction. Instructional materials in mathematics must have strengths in these areas to be considered suitable for adoption:

1. A teacher's edition with ample and useful annotations and suggestions on how to present the content in the student edition and in the ancillary materials, including modifications for English learners, advanced learners, students below grade level in mathematical skills, and students with disabilities.

2. A list of program lessons in the teacher's edition, cross-referencing the standards covered and providing an estimated instructional time for each lesson, chapter, and unit.

3. Unit and lesson plans, including suggestions for organizing resources in the classroom and ideas for pacing lessons.

4. A curriculum guide for the academic instructional year.

5. All components of the program are user friendly and, in the case of electronic materials, platform neutral.

6. Answer keys for all workbooks and other related student activities.

7. Concrete models, including manipulatives, support instruction of the *Common Core State Standards for Mathematics with California Additions* and include clear instructions for teachers and students.

8. A teacher's edition that explains the role of the specific grade-level mathematics in the context of the overall mathematics curriculum for kindergarten through grade twelve.

9. Technical support and suggestions for appropriate use of audiovisual, multimedia, and information technology resources.

10. Homework activities, if included, that extend and reinforce classroom instruction and provide additional practice of mathematical content, practices, and applications that have been taught.

11. Strategies for informing parents or guardians about the mathematics program and suggestions for how they can help support student progress and achievement.

Category 6: Teacher Support

Instructional materials should be designed to help teachers provide mathematics instruction that ensures opportunities for all students to learn the essential skills and knowledge specified in the *Common Core State Standards for Mathematics with California Additions*. Instructional materials in mathematics must have strengths in these areas to be considered suitable for adoption:

1. Clear, grade-appropriate explanations of mathematics concepts that teachers can easily adapt for instruction of all students, including English learners, advanced learners, students below grade level in mathematical skills, and students with disabilities.
2. Strategies to identify, address, and correct common student errors and misconceptions.
3. Suggestions for accelerating or decelerating the rate at which new material is introduced to students.
4. Different kinds of lessons and multiple ways in which to explain concepts, offering teachers choice and flexibility.
5. Materials designed to help teachers identify the reason(s) that students may find a particular type of problem(s) more challenging than another (e.g., identify skills not mastered) and point to specific remedies.
6. Learning objectives that are explicitly and clearly associated with instruction and assessment.
7. A teacher's edition that contains full, adult-level explanations and examples of the more advanced mathematics concepts in the lessons so that teachers can improve their own knowledge of the subject, as necessary.
8. Explanations of the instructional approaches of the programs and identification of the research-based strategies.
9. Explanations of the mathematically appropriate use of manipulatives or other visual and concrete representations.

Guidance for Instructional Materials for Grades Nine through Twelve

The *Criteria* document (above) is intended to guide publishers in the development of instructional materials for students in kindergarten through grade eight. It also provides guidance for selection of instructional materials for students in grades nine through twelve. The six categories in the *Criteria* document are an appropriate lens through which to view any instructional materials a district or school is considering purchasing. Additional guidance for evaluating instructional materials for grades nine through twelve is provided in the *High School Publishers' Criteria for the Common Core State Standards for Mathematics* (http://www.corestandards.org/assets/Math_Publishers_Criteria_HS_Spring%202013_FINAL.pdf [NGA/CCSSO 2013]).

The major points from the NGA/CCSSO's criteria are presented here. For the complete NGA/CCSSO criteria and in-depth explanations of the major points, see the *High School Publishers' Criteria for the Common Core State Standards for Mathematics* (NGA/CCSSO 2013).

Focus, Coherence, and Rigor

Focus: Place strong emphasis where the Standards focus.

Coherence: Think across grades, and link to major topics in each grade.

Rigor: Pursue with equal intensity:

- conceptual understanding;
- procedural skill and fluency;
- applications.

Focus

In high school, *focus* is important in order to prepare students for college and careers. A college-ready high school curriculum that includes all of the standards without a (+) symbol should devote the majority of students' time to building the particular knowledge and skills that are most important as prerequisites for a wide range of college majors, postsecondary programs, and careers.

Coherence

Coherence is about making math make sense. Taking advantage of coherence can reduce clutter in the curriculum. For example, if students can see that both the distance formula and the trigonometric identity $\sin^2(t)+\cos^2(t)=1$ are manifestations of the Pythagorean Theorem, they have an understanding that helps them reconstruct these formulas and not just memorize them temporarily.

Rigor

To help students meet the expectations of the standards, educators need to pursue, with equal intensity, three aspects of *rigor*: (1) conceptual understanding, (2) procedural skill and fluency, and (3) applications. The word *rigor* isn't a code word for just one of these three aspects; rather, it means equal intensity in all three. The word *understand* is used in the standards to set explicit expectations for conceptual understanding, and the phrase *real-world problems* and the star (★) symbol are used to set expectations and flag opportunities for applications and modeling.

Criteria for Materials and Tools Aligned with the Standards

The following criteria were adapted from the *High School Publishers' Criteria for the Common Core State Standards for Mathematics* (NGA/CCSSO 2013).

1. Focus on Widely Applicable Prerequisites: In any single course, students using the materials as designed spend the majority of their time developing knowledge and skills that are widely applicable as prerequisites for postsecondary education.

2. Rigor and Balance: Materials and tools reflect the balances in the standards and help students meet the standards' rigorous expectations, by (all of the following, in the case of comprehensive materials; at least one of the following for supplemental or targeted resources):
   a. Developing students' conceptual understanding of key mathematical concepts, especially where called for in specific content standards or cluster headings.
   b. Giving attention throughout the year to procedural skill and fluency.
   c. Allowing teachers and students using the materials as designed to spend sufficient time working with engaging applications and modeling.

   Additional aspects of the Rigor and Balance Criterion:

   1) *The three aspects of rigor are not always separate in materials.* (Conceptual understanding and fluency go hand in hand; fluency can be practiced in the context of applications; and brief applications can build conceptual understanding.)
   2) *Nor are the three aspects of rigor always together in materials.* (Fluency requires dedicated practice to that end. Rich applications cannot always be shoehorned into the mathematical topic of the day. And conceptual understanding will not always come along for free unless explicitly taught.)
   3) Digital and online materials with no fixed lesson flow or pacing plan are not designed for superficial browsing, but rather should be designed to instantiate the Rigor and Balance criterion.

3. Consistent Content: Materials are consistent with the content in the standards, by (all of the following):
   a. Basing courses on the content specified in the standards.
   b. Giving all students extensive work with course-level problems.
   c. Relating course-level concepts explicitly to prior knowledge from earlier grades and courses.

4. Coherent Connections: Materials foster coherence through connections in a single course, where appropriate and where required by the standards, by (all of the following):
   a. Including learning objectives that are visibly shaped by CA CCSSM cluster and domain headings.
   b. Including problems and activities that serve to connect two or more clusters in a domain, two or more domains in a category, or two or more categories, in cases where these connections are natural and important.
   c. Preserving the focus, coherence, and rigor of the standards even when targeting specific objectives.

5. Practice-Content Connections: Materials meaningfully connect content standards and practice standards.

6. Focus and Coherence via Practice Standards: Materials promote focus and coherence by connecting practice standards with content that is emphasized in the standards.

7. Careful Attention to Each Practice Standard: Materials attend to the full meaning of each practice standard.

8. Emphasis on Mathematical Reasoning: Materials support the standards' emphasis on mathematical reasoning, by (all of the following):
   a. Prompting students to construct viable arguments and critique the arguments of others concerning key course-level mathematics that is detailed in the content standards (see standard **MP.3**).
   b. Engaging students in problem solving as a form of argument.
   c. Explicitly attending to the specialized language of mathematics.

Indicators of Quality in Instructional Materials and Tools for Mathematics

In addition to the major points listed above, the NGA/CCSSO criteria suggest indictors of quality that instructional resources and tools should exhibit. The overarching indicators are listed below without their full explanations. For more detailed information, see the *High School Publishers' Criteria for the Common Core State Standards for Mathematics* (NGA/CCSSO 2013).

Quality Indicators (adapted from NGA/CCSSO 2013):

- Problems in the materials are worth doing.
- There is variety in the pacing and grain size of content coverage.
- There is variety in what students produce.
- Lessons are thoughtfully structured and support the teacher in leading the class through the learning paths at hand, with active participation by all students in their own learning and in the learning of their classmates.
- There are separate teacher materials that support and reward teacher study.
- The use of manipulatives follows best practices (see, for example, National Research Council 2001).
- The visual design is not distracting, chaotic, or aimed at adult purchasers, but instead serves only to support young students in engaging thoughtfully with the subject.
- Materials are carefully reviewed in an effort to ensure:
  - ✓ Freedom from mathematical errors
  - ✓ Age appropriateness
  - ✓ Freedom from bias—for example, problem contexts that use culture-specific background knowledge do not assume readers from all cultures have that knowledge; simple explanations, illustrations, or hints scaffold comprehension
  - ✓ Freedom from unnecessary language complexity
- Support for English learners is thoughtful and helps those learners to meet the same standards as all other students.

The process of selecting instructional materials at the district or school level usually begins with the appointment of a committee of educators, including teachers and curriculum specialists, who determine what instructional materials are needed, develop evaluation criteria and rubrics for reviewing materials, and establish a review process that involves teachers and content-area experts on review committees. After the review committee develops a list of instructional materials that are being considered for adoption, the next step is to pilot the instructional materials. An effective piloting process helps determine if the materials provide teachers with the resources necessary to implement an instructional program based on the CA CCSSM. One resource on piloting is the SBE policy document "Guidelines for Piloting Textbooks and Instructional Materials," which is available through the California Department of Education (CDE) Web site (http://www.cde.ca.gov/); enter "Guidelines for Piloting Textbooks" in the Search box to access a link to the document.

Selection of instructional materials at the local level is a time-consuming but very important process. Poor instructional materials that are not fully aligned with the principles of focus, coherence, and rigor and the CA CCSSM waste precious instructional time. High-quality instructional materials support effective instruction and student learning.

Social Content Review

To ensure that instructional materials reflect California's multi-cultural society, avoid stereotyping, and contribute to a positive learning environment, instructional materials used in California public schools must comply with the state laws and regulations that involve social content. Instructional materials must conform to *Education Code* sections 60040–60045, as well as the SBE's *Standards for Evaluating Instructional Materials for Social Content* (available through the CDE Web site at http://www.cde.ca.gov/ci/cr/cf/lc.asp). Instructional materials that are adopted by the SBE meet the social content requirements. The CDE conducts social content reviews of a range of instructional materials and maintains a searchable database of the materials that meet these social content requirements; the database is available at http://www.cde.ca.gov/ci/cr/cf/ap2/search.aspx.

If an LEA intends to purchase instructional materials that have not been adopted by the state or are not included on the list of instructional materials that meet the social content requirements maintained by the CDE, then the LEA must complete its own social content review. Information about the review process is posted on the CDE's Social Content Review Web page at http://www.cde.ca.gov/ci/cr/cf/lc.asp.

Supplemental Instructional Materials

The SBE traditionally adopts only basic instructional materials programs,[12] but has occasionally adopted supplemental instructional materials. LEAs adopt supplemental materials for local use more frequently. Supplemental instructional materials are defined in California *Education Code* section 60010(l) and are generally designed to serve a specific purpose, such as providing more complete coverage of a topic or subject, meeting the instructional needs of groups of students, and providing current, relevant technology to support interactive learning.

12. These programs are designed for use by students and their teachers as a principal learning resource and meet, in organization and content, the basic requirements of a full course of study (generally, one school year in length).

With the adoption of the CA CCSSM, there was a demand from educators for instructional materials to help schools transition from the previously adopted mathematics standards to the CA CCSSM. In response to this demand for CA CCSSM–aligned instructional materials, the CDE conducted a supplemental instructional materials review (SIMR). The SIMR was a two-phase review of supplemental instructional materials that bridge the gap between the CA CCSSM and programs being used by LEAs that were aligned with the previously adopted mathematics standards. At the recommendation of the CDE, the SBE approved seven mathematics supplemental instructional programs in November 2012 and an additional four programs in July 2013. Additional information on the supplemental review process and approved materials is available at http://www.cde.ca.gov/ci/cr/cf/simrmathprograms.asp.

Open Educational Resources

Open educational resources (OERs) are online instructional materials and resources that are available to teachers, students, and parents free of charge. OERs include a range of offerings, from full courses to quizzes, classroom activities, tasks, and games. Students may create OERs to fulfill an assignment. Teachers may work together to develop curriculum, lesson plans, or projects and assignments and make them available for others as OERs. OERs offer the promise of more engaging and more relevant instructional content, variety, and up-to-the-minute information. However, they should be subjected to the same type of evaluation as other instructional materials used in schools and reviewed to determine (a) if they are aligned with the content that students are expected to learn, and (b) whether they are at an appropriate level for intended students. Furthermore, OERs need to be reviewed with the social content requirements in mind to ensure that students are not inadvertently exposed to name brands, corporate logos, or materials that demean or stereotype people.

The California Learning Resource Network (CLRN) reviews supplemental electronic learning resources by applying review criteria and using a process approved by the SBE. A complete explanation of the process is provided in the document titled *California Learning Resource Network (CLRN) Supplemental Electronic Learning Resources Review Criteria and Process* (http://www.clrn.org/info/criteria/Criteria.pdf [CLRN 2000]). This document was produced before the CA CCSSM were adopted and refers to the previously adopted California standards, but it still serves as a general resource for guiding selection of supplemental electronic resources. Below is a short checklist to consider when reviewing electronic instructional materials.

Minimum Requirements

1. The resource addresses standards as evidenced in the CLRN standards match, provides for a systematic approach to the teaching of the standard(s), and contains no material contrary to any of the other California student content standards.

2. Instructional activities (sequences) are linked to the stated objectives for this electronic learning resource (ELR).

3. Reading and/or vocabulary levels are commensurate with the skill levels of intended learners.

4. The ELR exhibits correct spelling, punctuation, and grammar, unless it is a primary source document.

5. Content is current, accurate, and scholarly; this includes material taken from other subject areas.

6. The presentation of instructional content must be enhanced and clarified by the use of technology through approaches that may include access to real-world situations (graphics, video, audio); multi-sensory representations (auditory, graphic, text); independent opportunities for skill mastery; collaborative activities and communication; access to concepts through hypertext, interactivity, or customization features; use of the tools of scholarship (research, experimentation, problem solving); and simulated laboratory situations.

7. The resource is user friendly, as evidenced by the use of features such as effective help functions, clear instructions, a consistent interface, and intuitive navigational links.

8. Documentation and instruction on how to install and operate the ELR are provided and are clear and easy to use.

9. The model lesson or unit plan demonstrates effective use of the ELR in an instructional setting.

The NGA/CCSSO criteria provide the following guidance on the selection of digital and online instructional materials:

> Digital materials offer substantial promise for conveying mathematics in new and vivid ways and customizing learning. In a digital or online format, diving deeper and reaching back and forth across the grades is easy and often useful. That can enhance focus and coherence. But if such capabilities are poorly designed, focus and coherence could also be diminished. In a setting of dynamic content navigation, the navigation experience must preserve the coherence of Standards clusters and progressions while allowing flexibility and user control: Users can readily see where they are with respect to the structure of the curriculum and its basis in the Standards' domains, clusters and standards.
>
> Digital materials that are smaller than a course can be useful. The smallest granularity for which they can be properly evaluated is a cluster of standards. These criteria can be adapted for clusters of standards or progressions within a cluster, but might not make sense for isolated standards. (NGA/CCSSO 2013)

Three OER Web sites that support instruction and learning of the CA CCSSM and offer high-quality resources for use in the classroom and for professional learning are listed below:

- **Illustrative Mathematics** (https://www.illustrativemathematics.org/). An initiative of the Institute for Mathematics and Education, Illustrative Mathematics provides tasks, videos, lesson plans, and curriculum modules for teachers; mathematics content for teachers and instructional leaders; and a forum for educators to share information and expertise.

- **Inside Mathematics** (http://www.insidemathematics.org/). This site features classroom examples, tools for instruction, and problems designed for schoolwide participation.

- **The Mathematics Assessment Project** (http://map.mathshell.org/index.php). This site provides tools for both formative and summative assessment, including tasks for middle and high school students and lessons for middle and high school teachers.

Accessible Instructional Materials

The CDE's Clearinghouse for Specialized Media and Translations (CSMT) provides instructional resources in accessible and meaningful formats to students with disabilities, including students who have hearing or vision impairments, severe orthopedic impairments, or other print disabilities. The CSMT produces accessible versions of textbooks, workbooks, literature books, and assessment books. Specialized instructional materials include braille, large print, audio recordings, digital talking books, electronic files, and American Sign Language video books. Local assistance funds finance the conversion and production of these specialized materials. The distribution of various specialized media to public schools provides general education curricula to students with disabilities. Information about accessible instructional materials and other resources, including what is available and how to order, is posted on the CSMT's Media Ordering Guide page (http://csmt.cde.ca.gov/).

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Appendix A

Financial Literacy and Mathematics Education

Financial literacy is defined as the knowledge, tools, and skills that are essential for effective management of personal fiscal resources and financial well-being. Gaining mathematical knowledge is the first step toward developing financial literacy, which in turn provides early opportunities for meaningful mathematical modeling. The global economic downturn that occurred in the late 2000s highlighted the need for increased financial education for school-age students as well as adults. A 2009 survey conducted by the Financial Industry Regulatory Authority (FINRA) showed that about half of the Americans surveyed had trouble keeping up with their monthly expenses. Members of the same survey group were unable to save a portion of their income for emergencies or retirement (FINRA Investor Education Foundation 2009). This inability to save money has even greater implications for the future as the average life expectancy increases and people need more money to sustain themselves throughout their lives.

The President's Advisory Council on Financial Capability (2012) states, "Research shows that low levels of financial literacy are associated with high levels of indebtedness, lower wealth accumulation, and less retirement savings." Individuals with low levels of financial literacy are also particularly vulnerable to predatory lending. In response to these troubling social trends, there have been movements in many states across the country to increase the financial education of Americans, beginning in elementary school and continuing through postsecondary education.

California has not adopted its own standards for financial literacy; however, there are two sets of national standards that teachers may use to influence their instruction. The Jump$tart Coalition for Personal Financial Literacy created and maintains the *National Standards in K–12 Personal Finance Education,* available at http://jumpstart.org/assets/files/standard_book-ALL.pdf (accessed May 28, 2014). These standards describe financial knowledge and skills that students should be able to exhibit. The Jump$tart standards are organized under six major categories of personal finance:

- Financial Responsibility and Decision Making
- Income and Careers
- Planning and Money Management
- Credit and Debt
- Risk Management and Insurance
- Saving and Investing

The second set of national standards available to teachers is the *National Standards for Financial Literacy* published by the Council for Economic Education (CEE). The CEE standards are available at http://www.councilforeconed.org/wp/wp-content/uploads/2013/02/national-standards-for-financial-literacy.pdf (accessed May 28, 2014) and, like the Jump$tart standards, are organized under six major categories of personal finance:

- Earning Income
- Buying Goods and Services
- Saving
- Using Credit
- Financial Investing
- Protecting and Insuring

The standards in each category provide expectations for students' financial knowledge and skills at each grade level, but leave it to stakeholders to determine the methods for delivery. For example, under Jump$tart's Planning and Money Management category, the standards call for students to develop a plan for spending and saving, keep and use a system for financial records, apply consumer skills to purchasing decisions, and use other important money-management tools. A recent study supported by the National Endowment for Financial Education and the Citi Foundation shows that students who receive cumulative (repeated) financial education demonstrate more positive financial behaviors as adults. In addition, the study documents that early exposure to financial education has a positive impact on people's lives (Serido and Shim 2011). In some cases, students have very positive role models at home when it comes to financial decision making; however, other students may greatly benefit from learning about these concepts and tools at school.

There are numerous Web resources available to teachers free of charge to support financial education in schools. Links to these resources can be found at the end of this appendix, as well as on the CDE's K–12 Financial Literacy Resources Web page at http://www.cde.ca.gov/eo/in/fl/finlitk12.asp (accessed May 28, 2014).

Time constraints in the regular school day often make it impossible to offer a separate course in financial education; however, financial literacy concepts can be integrated into other core content areas. Mathematics courses are often considered a natural fit for the integration of financial literacy exercises and skills. The California Common Core State Standards for Mathematics (CA CCSSM) provide multiple entry points for the incorporation of problems or exercises that teach important financial literacy concepts and skills—and the opportunity to teach financial literacy concepts is even more evident in the real-world problems emphasized by the CA CCSSM. This includes the use of mathematical modeling, which is included in every grade level in the standards (see appendix B for more information). In addition, the Standards for Mathematical Practice (MP) emphasize the analytical skills that students will use when solving problems that involve financial literacy concepts. Although financial literacy has a place in the mathematics classroom, analyzing financial situations and making decisions based on the analysis is not pure mathematics per se. There are *computational* aspects of finance

problems and the various terms, definitions, and mathematical meanings that constitute the pure mathematics in these problems. In addition, financial literacy problems provide students with rich opportunities to hone their mathematical problem-solving skills within real-world contexts.

Teachers must find an appropriate balance when considering the integration of financial literacy into the mathematics classroom. Standard **2.OA.1** asks students to "Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions." Students in second grade may have some experience with doing chores at home to earn an allowance. They also may have had to make choices about how to spend their money. Together with students' experiences, standard **2.OA.1** provides opportunities to discuss concepts in the CEE's financial literacy standards (Earning Income and Buying Goods and Services) as well as Jump$tart's financial literacy categories (Income and Careers and Planning and Money Management). Consider the following word problem:

Lucy earns an allowance of $5 per week. She also walks her neighbor's dog every day, earning $15 per week. In addition, Lucy received a birthday gift of $20 from her aunt. In all, Lucy has three different sources of income this week. Use column A in the following table to input Lucy's income.

Column A: Income		Column B: Expenses	
Source	**Amount ($)**	**Item**	**Amount ($)**
Allowance	$5		
Dog walking	$15		
Birthday money	$20		
		Savings	$5
	Total: $40		Total:
How much is left over? (Column A total minus Column B total): $ _____			

Lucy must decide how to spend her money. At the end of the week, she would also like to have $5 left over to deposit into her savings account. She comes up with a list of possible ways to spend her money:

- Trip to the movies: $14
- Birthday gift for her brother: $10
- Favorite magazine: $4
- Donation to the local food bank: $5
- Materials for a school assignment: $7
- Money owed to her sister for a previous loan: $6

Follow-up financial literacy questions:

1. Does Lucy have enough money for all of these things?
2. How would you suggest that she spend her money?

Use column B in the previous table to show your suggestions for Lucy's expenses (adapted from Federal Reserve Bank of Cleveland 2007).

The standards for mathematics in middle school allow for more in-depth exercises that address financial literacy concepts. For example, the seventh-grade standard **7.EE.4b** reads as follows:

> Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. *For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.*

This standard has the potential to address several financial literacy categories in the Jump$tart standards (Financial Responsibility and Decision Making; Income and Careers; Planning and Money Management; and Saving and Investing) as well as three of the CEE's financial literacy topics (Earning Income; Buying Goods and Services; and Saving). Consider the following problem:

Darla works as a salesperson. Her monthly salary is $1000, and she can make an additional $25 per sale. In order to pay her monthly bills, she must make at least $2750 per month. How many sales will she need to make per month to meet her monthly financial obligations? The following table lists Darla's monthly expenses:

Expenses	
Item	**Amount ($)**
Rent	$1200
Utility bills	$280
Cell phone bill	$75
Entertainment	$150
Bus fares	$105
Groceries	$450
Credit card payments	$240
Charitable contributions	$100
Clothing purchases	$150
	Total: $2750

[Write an inequality for the number of sales she will need to make, and describe the solutions.]

$$1000 + 25x \geq 2750$$

Darla would like to save an additional $200 per month over the next year for a down payment on a car. Considering the previous information, how many sales will she need to make per month? [Write an inequality for the number of sales she will need to make, and describe the solutions.]

$$1000 + 25x \geq 2750 + 200$$

Once Darla purchases the car, her monthly car payment will be $300. How many sales will she need to make each month in order to meet her monthly financial obligations, including the car payment? [Write an inequality for the number of sales she will need to make, and describe the solutions.]

$$1000 + 25x \geq 2750 + 300$$

Follow-up financial literacy questions:

1. What additional costs associated with car ownership must Darla consider?
2. What risk is Darla taking when basing the total income she needs on the number of sales she makes beyond her base salary?
3. Using the previous table that shows Darla's monthly expenses, if her sales are lower than expected for any given month, where might she cut some costs in order to afford her car?

In higher mathematics, the CA CCSSM allow for a more sophisticated discussion of financial management and decision making. For instance, the early use of recursively defined sequences (standard **F-IF.3**) allows for a simpler and more intuitive discussion of compound interest (i.e., $a_0 = P$ (principal), $a_{n+1} = a_n \bullet (1+r)$ for $n \geq 0$); discount (i.e., $a_0 = P$ (principal), $a_{n+1} = a_n \bullet (1-r)$ for $n \geq 0$); and amortization of debt (i.e., $a_0 = P$ (principal), $a_{n+1} = a_n \bullet (1+r) -$(payment) for $n \geq 0$). These types of interest and payment calculations can help students understand the origin of common formulas such as $A(t) = P_0(1+r)^t$.

In the conceptual category of Functions, standard **F-BF.1** calls for students to "Build a function that models a relationship between two quantities" (cluster heading) and specifically to "Determine an explicit expression, a recursive process, or steps for calculation from a context" (**F-BF.1a**). Consider the following problem concerning the cost of credit:

> *James arrived at college and was given two credit cards. He didn't really know much about managing his money, but he did understand how to use the cards—so he bought a few things for his dorm room, including a television for $1200 and a microwave for $200. Each of the items was purchased with a different credit card, and each card had a different interest rate. The television was purchased with a card that had an 18% annual interest rate; the microwave was purchased with a card that had a 28% annual interest rate. James has a job. He earns $1500 per month and spends $900 per month on school-related and living expenses.*
>
> 1. What questions do you have about each credit card that would help you advise James on how to pay off each of his debts? (For example, students might ask about the minimum payments required for each card, late charges, and so forth.)

2. If James takes the amount of money he has left after paying his other expenses and splits it between the two cards, how long would it take him to pay off each account?
3. What other options does James have for paying off the debts?
4. Which option would result in James paying the least amount of interest?
   a) Write one or more equations to model the situation and support your answer.
   b) What is the total amount of interest James will end up paying for each credit card?

CA CCSSM Alignment

Building Functions (F-BF)

Build a function that models a relationship between two quantities.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.

When students are introduced to financial literacy education early in their academic lives, they can develop a lifelong foundation for making intelligent decisions about how to earn, save, and invest money. In many cases there is not enough time in the regular school day to offer a course in financial literacy, and thus it is important to leverage opportunities in other core subject areas to include financial literacy lessons where appropriate. The CA CCSSM open many doors for examining and practicing financial literacy topics, especially through the application of the Standards for Mathematical Practice and real-world problems. It is important for mathematics instructors to find the appropriate balance between teaching the mathematical standards and concepts and financial literacy skills.

The following table provides information about financial literacy word problems created by the Math Forum @ Drexel that align with both sets of national financial literacy standards, as well as the content and mathematical practices standards of the CA CCSSM. To view the actual word problems, please visit http://mathforum.org/pow/financialed/ (accessed May 28, 2014).

Grade Level or Course	Name of Word Problem	Standards for Mathematical Practice	Standards for Mathematical Content[1]	CEE National Standard	Jump$tart Standard
Kindergarten	The Yard Sale	**MP.1:** Make sense of problems and persevere in solving them. **MP.3:** Construct viable arguments and critique the reasoning of others.	**Operations and Algebraic Thinking:** Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem. (**K.OA.2**)	**Standard II:** Buying Goods and Services	**Planning and Money Management, Standard 4:** Apply consumer skills to purchase decisions.[2]
Grade 1	Jordan's Jobs	**MP.1:** Make sense of problems and persevere in solving them. **MP.3:** Construct viable arguments and critique the reasoning of others.	**Operations and Algebraic Thinking:** Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem. (**1.OA.2**)	**Standard I:** Earning Income	**Planning and Money Management, Standard 4:** Apply consumer skills to purchase decisions.
Grade 2	Money Matters	**MP.1:** Make sense of problems and persevere in solving them. **MP.3:** Construct viable arguments and critique the reasoning of others.	**Operations and Algebraic Thinking:** Represent and solve problems involving addition and subtraction. (**2.OA** cluster heading)	**Standard II:** Buying Goods and Services	**Planning and Money Management, Standard 4:** Apply consumer skills to purchase decisions.
Grade 3	Miracle Miranda and the Mascot	**MP.1:** Make sense of problems and persevere in solving them. **MP.3:** Construct viable arguments and critique the reasoning of others. **MP.5:** Use appropriate tools strategically. **MP.7:** Look for and make use of structure.	**Operations and Algebraic Thinking:** Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding. (**3.OA.8**)	**Standard I:** Earning Income	**Income and Careers, Standard 2:** Identify sources of personal income.

1. For some examples in this column, cluster headings are listed rather than standards.

2. The correct term is *purchasing decisions,* but the Jump$tart standards use the term *purchase decisions*.

Grade Level or Course	Name of Word Problem	Standards for Mathematical Practice	Standards for Mathematical Content[1]	CEE National Standard	Jump$tart Standard
Grade 4	Building Bouquets	**MP.1:** Make sense of problems and persevere in solving them. **MP.3:** Construct viable arguments and critique the reasoning of others.	**Operations and Algebraic Thinking:** Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite. (**4.OA.4**)	**Standard I:** Earning Income	**Income and Careers, Standard 2:** Identify sources of personal income.
Grade 5	Super Salsa Deal	**MP.1:** Make sense of problems and persevere in solving them. **MP.3:** Construct viable arguments and critique the reasoning of others. **MP.5:** Use appropriate tools strategically. **MP.6:** Attend to precision.	**Number and Operations in Base Ten:** Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. (**5.NBT.7**)	**Standard II:** Buying Goods and Services	**Planning and Money Management, Standard 4:** Apply consumer skills to purchase decisions.
Grade 6	Buying Cola	**MP.1:** Make sense of problems and persevere in solving them. **MP.3:** Construct viable arguments and critique the reasoning of others.	**Ratios and Proportional Relationships:** Understand the concept of a unit rate $\frac{a}{b}$ associated with a ratio $a:b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is $\frac{3}{4}$ cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."* (**6.RP.2**)	**Standard II:** Buying Goods and Services	**Planning and Money Management, Standard 4:** Apply consumer skills to purchase decisions.

Grade Level or Course	Name of Word Problem	Standards for Mathematical Practice	Standards for Mathematical Content[1]	CEE National Standard	Jump$tart Standard
Grade 7	That's Interesting!	**MP.1:** Make sense of problems and persevere in solving them. **MP.3:** Construct viable arguments and critique the reasoning of others. **MP.4:** Model with mathematics. **MP.5:** Use appropriate tools strategically.	**Ratios and Proportional Relationships:** Use proportional relationships to solve multi-step ratio and percent problems. *Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.* (**7.RP.3**)	**Standard III** (Saving) and **Standard IV** (Financial Investing)	**Saving and Investing, Standard 3:** Evaluate investment alternatives.
Grade 8	Saving Your Raise	**MP.1:** Make sense of problems and persevere in solving them. **MP.2:** Reason abstractly and quantitatively. **MP.3:** Construct viable arguments and critique the reasoning of others. **MP.4:** Model with mathematics. **MP.5:** Use appropriate tools strategically.	**Expressions and Equations:** Know and apply the properties of integer exponents to generate equivalent numerical expressions. *For example,* $3^2 \times 3^{-5} = 3^{-3} = \frac{1}{3^3} = \frac{1}{27}$. (**8.EE.1**)	**Standard III:** Saving	**Saving and Investing, Standard 2:** Explain how investing builds wealth and helps meet financial goals.
Algebra I, Mathematics I	Dinner at Pepe's	**MP.1:** Make sense of problems and persevere in solving them. **MP.3:** Construct viable arguments and critique the reasoning of others. **MP.4:** Model with mathematics.	**Algebra, Reasoning with Equations and Inequalities:** Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. (**A-REI.1**) **Algebra, Creating Equations:** Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA★** (**A-CED.1**)	**Standard II:** Buying Goods and Services	**Financial Responsibility and Decision Making, Standard 5:** Develop communication strategies for discussing financial issues.

Grade Level or Course	Name of Word Problem	Standards for Mathematical Practice	Standards for Mathematical Content[1]	CEE National Standard	Jump$tart Standard
Algebra I, Mathematics II	Credit Card Payoff Options	**MP.1:** Make sense of problems and persevere in solving them. **MP.2:** Reason abstractly and quantitatively. **MP.3:** Construct viable arguments and critique the reasoning of others. **MP.4:** Model with mathematics. **MP.5:** Use appropriate tools strategically.	**Algebra, Seeing Structure in Expressions:** Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $\left(1.15^{1/12}\right)^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*★ (**A-SSE.3c**) **Algebra, Building Functions:** Build a function that models a relationship between two quantities. (**F-BF** cluster heading)	**Standard IV:** Using Credit	**Credit and Debt, Standard 1:** Identify the costs and benefits of various types of credit.
Statistics and Probability	A Great Way to Make Money	**MP.1:** Make sense of problems and persevere in solving them. **MP.3:** Construct viable arguments and critique the reasoning of others.	**Statistics and Probability:** Using Probability to Make Decisions: (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution. ★ (**S-MD.2**)	**Standard I:** Earning Income	**Saving and Investing, Standard 3:** Evaluate investment alternatives.
Calculus	College Savings	**MP.1:** Make sense of problems and persevere in solving them. **MP.2:** Reason abstractly and quantitatively. **MP.3:** Construct viable arguments and critique the reasoning of others. **MP.4:** Model with mathematics.	**Calculus:** Students demonstrate an understanding of the interpretation of the derivative as an instantaneous rate of change. Students can use derivatives to solve a variety of problems from physics, chemistry, economics, and so forth that involve the rate of change of a function. (**Calculus 4.2**)	**Standard III:** Saving	**Saving and Investing, Standard 2:** Explain how investing builds wealth and helps meet financial goals.

Grade Level or Course	Name of Word Problem	Standards for Mathematical Practice	Standards for Mathematical Content[1]	CEE National Standard	Jump$tart Standard
The following lesson plan was produced by the Mathematics Assessment Resource Service (MARS). See http://map.mathshell.org/materials/download.php?fileid=1250 (accessed July 24, 2014) for more information.					
Algebra II, Mathematics II/III	Comparing Investments	**MP.1:** Make sense of problems and persevere in solving them. **MP.2:** Reason abstractly and quantitatively. **MP.4:** Model with mathematics. **MP.7:** Look for and make use of structure.	**Algebra:** Seeing Structure in Expressions **Functions:** Linear, Quadratic, and Exponential Models		

Financial Literacy Curriculum Resources

California Department of Education. 2014. Grades K–12 Financial Literacy Resources. http://www.cde.ca.gov/eo/in/fl/finlitk12.asp (accessed May 28, 2014).

Council for Economic Education. 2014. Online Economic Lessons. http://www.econedlink.org/lessons/economic-lesson-search.php?type=educator (accessed May 28, 2014).

Curators of the University of Missouri. 2008. *Money Math: Lessons for Life.* http://www.treasurydirect.gov/indiv/tools/tools_moneymath.pdf (accessed May 28, 2014).

Federal Reserve. 2014. Lesson Plans. http://federalreserveeducation.org/resources/classroom/lesson-plans/ (accessed May 28, 2014).

Federal Reserve Bank of Atlanta. 2014. Education Resources. http://www.frbatlanta.org/pubs/extracredit/lessons_activities.cfm (accessed May 28, 2014).

Federal Reserve Bank of Cleveland. 2007. *Great Minds Think: A Kid's Guide to Money.* http://www.clevelandfed.org/Learning_Center/Online_Activities/great_minds_think/Great_Minds_Think.pdf (accessed May 28, 2014).

Federal Reserve Bank of Dallas. 2013. *Building Wealth: A Beginner's Guide to Securing Your Financial Future.* http://www.dallasfed.org/assets/documents/cd/wealth/wealth.pdf (accessed May 28, 2014).

Federal Reserve Bank of New York. 2009. *It's All About Your $: A Teacher's Guide.* http://www.newyorkfed.org/education/its_all_about_your_money.pdf (accessed May 28, 2014).

_____. n.d. *Econ Explorers: Journal for Students.* http://www.newyorkfed.org/education/econ_student.pdf (accessed May 28, 2014).

Federal Reserve Bank of Philadelphia. 2014. Lesson Plans for Teachers. http://www.philadelphiafed.org/education/teachers/lesson-plans/ (accessed May 28, 2014).

Mathematics Assessment Resource Service (MARS). 2014. Mathematics Assessment Project. http://map.mathshell.org/materials/index.php (accessed June 26, 2014).

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Appendix B

Mathematical Modeling

The California Common Core State Standards for Mathematics (CA CCSSM) include mathematical modeling as a Standard for Mathematical Practice (**MP.4,** Model with mathematics), which should be learned by students at every grade level. In higher mathematics, modeling is established as a conceptual category. Additionally, modeling standards are spread throughout other conceptual categories, with a star (★) symbol indicating that they are modeling standards. This appendix serves to clarify the meaning of mathematical modeling and the role of modeling in teaching the CA CCSSM.

What Mathematical Modeling *Is Not*

The terms *model* and *modeling* have several connotations, and although the term *model* has a general definition of "using one thing to represent something else," *mathematical modeling* is a more specific term. Below is a list of some things that do *not* constitute mathematical modeling in the context of the CA CCSSM.

- Telling students, "I do this; now you do the same."
- Using manipulatives to represent mathematical concepts; this might instead be referred to as "using concrete representations."
- Using a graph, equation, or function and calling it a model. True modeling is a process.
- Starting with a real-world situation and solving a math problem. Modeling returns students to a real-world situation and uses mathematics to inform their understanding of the world (i.e., contextualizing and de-contextualizing; see standard **MP.2**).
- Beginning with the mathematics and then moving to the real world. Modeling begins with real-world situations and represents them with mathematics.

What Mathematical Modeling *Is*

Mathematical modeling is the process of using mathematical tools and methods to ask and answer questions about real-world situations (Abrams 2012). Modeling will look different at each grade level, and success with modeling is based on students' mathematical background knowledge as well as their ability to ask modeling questions. However, as discussed below, all mathematical modeling situations share similar features. For example, at a very basic level, grade-four students might be asked to find a way to organize a kitchen schedule to serve a large family holiday meal based on factors such as cooking times, oven availability, cleanup times, equipment use, and so forth (English 2007). The students engage in modeling when they construct their schedule based on non-overlapping time periods for equipment, paying attention to time constraints. When high school students participate in a discussion to evaluate the "efficiency" of the packaging for a 12-pack of juice cans, and then use formulas for area and volume, calculators, dynamic geometry software, and other tools to create their own packaging (making it as efficient as possible), they are also engaged in modeling.

Example of Mathematical Modeling

"Giant's Feet." At Fairytale Town in Sacramento, California, there is a model of the foot of the giant from the story "Jack and the Beanstalk." The foot measures 1.83 meters wide, 4.27 meters long, and 1.27 meters high. If a giant person had feet this large, approximately how tall would he or she be? Explain your solution.

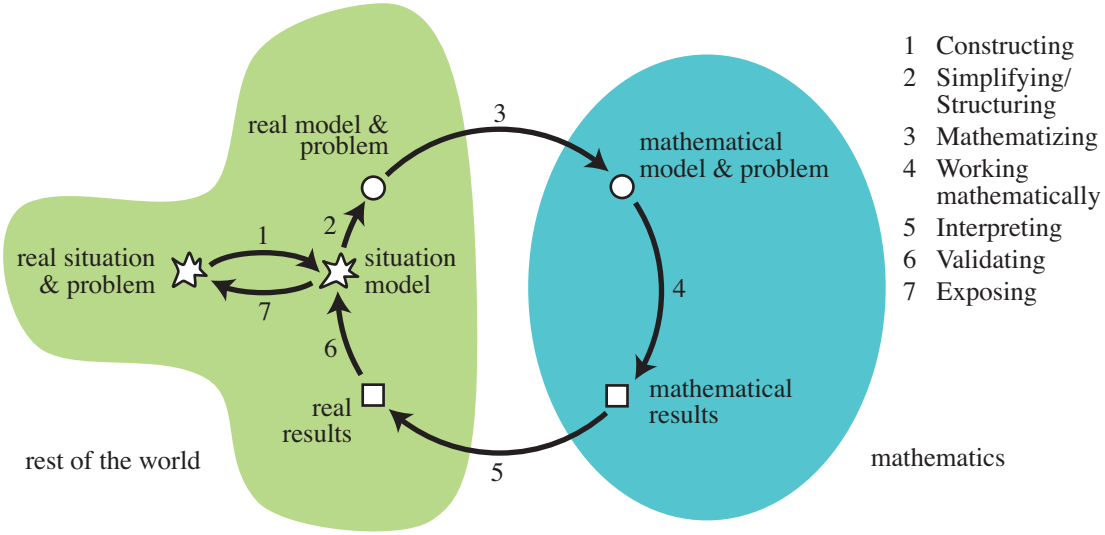
Photo reprinted by permission from Fairytale Town, Sacramento, California.

Mathematical modeling plays a part in many different professions, including engineering, science, economics, and computer science. Professional mathematical modeling often involves looking at a novel real-world problem or situation, asking questions about the situation, creating mathematical representations ("models") that describe the situation (e.g., equations, functions, graphs of data, geometric models, and so on), computing with or extending these representations to learn something new about the situation, and then reflecting on the information found. Students, even those in lower grade levels, can be encouraged to do the same: when presented with a real-world situation, they can ask questions that lead to applying mathematics to new and interesting situations and lead to new mathematical ideas: *How could we measure that? How will that change? Which is more cost-effective, and why?*

Mathematical modeling may be seen as a multi-step process: posing the real-world question, developing a model, solving the problem, checking the reasonableness of the solution, and reporting results or revising the model. These steps all work together, informing one another, until a satisfactory solution is found. Thus, the parameters in a linear model such as $f(x) = 0.8x + 1.5$ may need to be altered to better predict the growth of the supply of a product over time based on initial calculations. Or, a simplification that was made previously in the model formation may need to be revisited to develop a more accurate model.

As shown in figure B-1, Blum and Ferri (2009) offer a schematic that describes a typical modeling process.

Figure B-1. A Typical Modeling Process



Source: Blum and Ferri 2009, 46.

In this cycle, the first step is examining the real world and constructing a problem, typically by asking a question. Second, the important objects or aspects of the problem are identified and, if necessary, simplifications are made (e.g., ignoring that a juice can is not exactly a cylinder). Next, the situation is "mathematized": quantities are identified through measurement, relationships among quantities are described mathematically, or data are collected. This is the step of creating a "mathematical model." Next, the modeler works with his or her model—solving an equation, graphing data, and so forth—and then interprets and validates results in the context of the problem. At this step, the modeler may need to return to his or her model and refine it, creating a looping process. Finally, the results of modeling the problem are disseminated.

The Role of Modeling in Teaching the CA CCSSM

Modeling supports the CA CCSSM goals of preparing all students for college and careers, teaching students that mathematics is a part of their world and can describe the world in surprising ways. Modeling supports the learning of useful skills and procedures, helps develop logical thinking, problem solving, and mathematical habits of mind, and promotes student discourse and reflective discussion. Modeling also allows students to experience the beauty, structure, and usefulness of mathematics.

In contrast with the typical "problem solving" encountered in schools, modeling problems have important mathematical ideas and relationships embedded within the problem context, and students elicit these as they work through the problem (English 2007, 141). In a modeling situation, the exact solution path is often unclear and may involve making assumptions that lead students to use a mathematical skill and reflect on whether they were justified in doing so; this is much different from a word problem in which students are simply required to apply a mathematical skill they have just learned in a new context. Additionally, modeling problems "necessitate the use of important, yet underrepresented, mathematical processes such as constructing, describing, explaining, predicting, and representing, together with organizing, coordinating, quantifying, and transforming data" (English 2007, 141–42). These are some of the same mathematical processes encapsulated in many of the Standards for Mathematical Practice (MP standards). Modeling problems "are also multifaceted and multidisciplinary: students' final products encompass a variety of representational formats, including written text, graphs, tables, diagrams, spreadsheets, and oral reports; the problems also cut across several disciplines including science, history, environmental studies, and literature (English 2007, 141–42).

Current mathematics education literature points to two main uses of modeling in teaching: "modeling as vehicle" and "modeling as content" (see Galbraith 2012).

- **Modeling as vehicle.** According to this perspective, modeling is a way to provide an alternative setting in which students can learn mathematics. This perspective views modeling as a way to motivate and introduce students to new mathematics or to practice and refine their understanding of mathematics they have already learned. When modeling is seen as a vehicle for teaching mathematics, emphasis is not placed on students becoming proficient modelers themselves.
- **Modeling as content.** According to this perspective, modeling is experienced as its own *content*. Specific attention is placed on the development of students' skills as modelers as well as mathematical goals. With modeling as content, mathematical concepts or procedures are not the sole outcome of the modeling activity. As Galbraith (2012) states, "When included as content, mod-

eling sets out to enable students to use their mathematical knowledge to solve real problems, and to continue to develop this ability over time" (Galbraith 2012, 13).

Both of these perspectives on modeling can be included in school mathematics curricula to achieve the complementary goals of having students learn mathematics content and learn how to be modelers. However, the modeling-as-content approach has the additional goal of specifically helping students develop their ability to address problems in their world, which is an important aspect of college and career readiness.

As noted by Burkhardt (2006), people model with mathematics from a very early age: "Children estimate the amount of food in their dish, comparing it with their siblings' portions. They measure their growth by marking their height on a wall. They count to make sure they have a 'fair' number of sweets" (Burkhardt 2006, 181). Zalman Usiskin (2011) notes that the grading system is a stark example of mathematical modeling in many classrooms: "A student obtains a score on a test, typically a single number. This score is on some scale, and that scale is a mathematical model that ostensibly describes how much the student knows . . . the problem to model is that we want to know how much the student knows" (Usiskin 2011, 2). Usiskin also notes that another common example of modeling—determining how big something is—does not appear to be so: "Consider an airplane. We might describe its size by its length, its wingspan, its height off the ground, its weight, the maximum weight it can handle, and the maximum number of passengers it can handle . . . We recognize that [one] cannot describe an airplane's size by a single number" (Usiskin 2011, 3). Still another example of mathematical modeling involves a class of students that will cast votes to elect a new class president. Some voting systems allow each voter to rank the top three candidates and assign different values based on placement (e.g., First = 5 points, Second = 3 points, Third = 1 point). Is this a fair way to determine a winner? These and many other examples show that mathematical modeling occurs from very early on and that modeling questions can arise in many different situations. Thus, there is a unique opportunity in mathematics education to build on this seemingly innate tendency to use modeling to understand the world.

Bringing modeling to the classroom can be a challenging task. The fact that the CA CCSSM focus on *depth* rather than the amount of material covered is an advantage for teachers; having to cover fewer concepts in each grade level or course may allow for more time for modeling experiences that allow students to learn concepts at a deep level. Several challenges to teaching mathematical modeling will arise, not the least of which are understanding the role of the teacher as well as the role of the students, the availability of modeling curriculum, and support for teachers. Each of these issues is discussed in greater detail below, but it is clear that modeling with mathematics will be new to many teachers and students—and therefore it requires care and patience to introduce modeling in a classroom.

Example: Modeling in the Classroom (Grades Four Through Six)

"Holiday Dinner." The three Thompson children—Dan, Sophie, and Eva—want to organize and cook a special holiday dinner for their parents, who will be working at the family store from 7 a.m. until 7 p.m. The children will decorate the house and prepare, cook, and serve the holiday dinner. They know that they need to carefully plan a schedule to get everything done on time. The last time they tried something like this, for their parents' wedding-anniversary dinner, they created an activity list and a schedule for preparing and cooking the meal. Unfortunately, the previous schedule made by the Thompson children did not work very well; they found that they stumbled around the kitchen and wanted to use the same equipment at the same time. They also realized that they had not thought of all the things they needed to include in their schedule.

The children decided on the following menu for their holiday dinner:

- Appetizers (cheese, dip, carrot sticks, and crackers)
- Baked turkey as the main course, served with roasted vegetables and steamed vegetables
- Pavlova,[1] ice-cream, and fresh strawberries for dessert

Dan, Sophie, and Eva know their parents will be home at 7 p.m., and they are all available to begin preparing the dinner at 2:30 p.m. They have four and a half hours to get everything ready. All they need to do is organize a schedule that works better than the wedding-anniversary schedule.

Here are some things the children need to consider:

- How long will it take to cook the turkey?
- What other items can be cooked in the oven with the turkey?
- When should the table be decorated and set?
- When should they make the pavlova, and how long it will take?
- How often do they need to clean in between the cooking?
- How much counter space do they have for food preparation?
- What food needs to be ready first?
- Who will use the equipment, and when?
- How will the tasks be divided among the children?

In the kitchen, there are two counters to work on, a double sink, a microwave oven, and a stove with four top burners and an oven. The oven is large enough to fit the turkey and one other item at the same time.

Dan, Sophie, and Eva need help! They have numerous tasks to complete in order to surprise their parents, and they need a reliable schedule. Students are asked to help the Thompson children in the following ways:

1. Make a preparation and cooking schedule. Chart what each person will do and when, including the use of kitchen equipment.
2. Write an explanation of how you developed the schedule. The children plan to have other surprise celebrations for their parents, and they hope to use your explanation as a guide for making future schedules.

Adapted from English 2007.

1. Pavlova is a dessert consisting of a meringue cake and shell usually topped with whipped cream and fruit.

The Role of the Teacher

The image of students working feverishly in a classroom to solve a real-world problem that resulted from a question they asked paints a different picture of the role of the teacher. When teaching modeling, the teacher is seen as a guide or facilitator who allows students to follow a solution path that they have come up with, making suggestions and asking questions when necessary. Teachers in the modeling classroom are aware of suitable contexts so that their students have an entry point and can ask appropriate questions to attempt to solve the problem. When using modeling to teach certain mathematical concepts, teachers guide the class discussion toward their instructional goal. Teachers in a modeling classroom move away from a role of manager, explainer, and task setter and toward a role of counselor, fellow mathematician, and resource (Burkhardt 2006, 188).

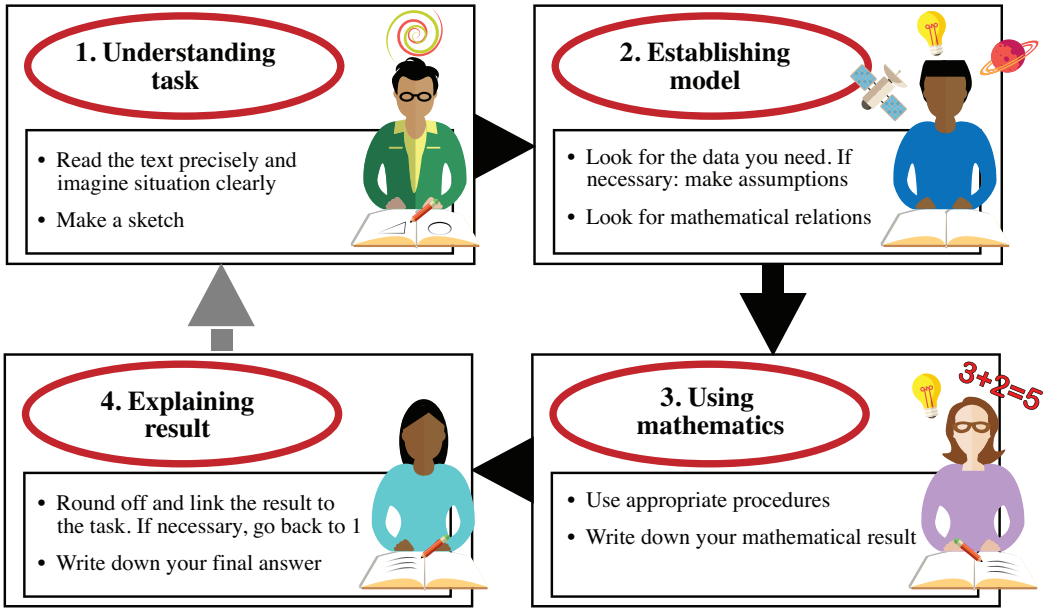
Teachers who are new to modeling may have difficulty allowing their students to grapple with difficult mathematical situations. Modeling involves problem solving, and, as Abrams (2001) states, "Problem solving involves being stuck. If a task does not puzzle us at all, then it is not a problem. It is merely an exercise" (Abrams 2001, 20). Teachers need to remember that learning occurs as the result of struggling with difficult concepts, and thus a certain amount of productive struggle is necessary and desirable.

Blum and Ferri (2009) posit some general implications for teaching modeling based on empirical findings. They note that teachers:

- must provide appropriate modeling tasks for students, and a balance between maximum student independence and minimal teacher guidance should be found;
- should be familiar enough with assigned tasks so that they can support students' individual modeling routes and encourage multiple solutions;
- must be aware of different means of strategic intervention during modeling activities;
- must be aware of ways to support student strategies for solving problems.

As shown in figure B-2, Blum and Ferri (2009) suggest a four-step schematic—simplified from the seven-step process shown in figure B-1—for guiding students' strategies.

Figure B-2. Four Steps for Solving a Modeling Task



Source: Blum and Ferri 2009.

The Role of Students

The transition to modeling in the classroom may prove difficult for students as well as teachers. It is no secret that many mathematics lessons involve a teacher explaining and demonstrating steps while students imitate the teacher. As Burkhardt (2006) notes:

> Most school mathematics curricula are fundamentally imitative—students are only asked to tackle tasks that are closely similar to those they have been shown exactly how to do. This is no preparation for practical problem solving or, indeed, non-routine problem solving in pure mathematics or any other field; in life and work, you meet new situations so you need to learn how to handle problems that are *not* just like those you have tackled before. (Burkhardt 2006, 182)

The transition from passive learner to active learner will pose a major challenge to students who are accustomed to simply mimicking their teacher's actions. However, this transition is empowering for students; they become effective problem solvers when they combine reasoning and persistence to solve problems where the outcome is meaningful to them.

Teachers can help facilitate this transition for students by starting with manageable modeling situations and gradually increasing the complexity of tasks. Students need to learn that in modeling situations, the teacher is a *resource* and not simply a person who provides answers; the students are responsible for doing the hard work. Through entry-level modeling tasks, students can learn to be investigators, managers, and explainers, and they become responsible for their reasoning and its correctness. Eventually, students can pose their own questions and fully carry out the modeling process. Abrams (2012) suggests a "Spectrum of Applied Mathematics" that teachers can follow when providing tasks; this spectrum will allow students to ramp up to full modeling (Abrams 2012, 46). The following spectrum is derived from Abrams's work, and it should be viewed in that context: as a spectrum, not a *ladder*, in the sense that teachers can enter the spectrum in various places according to the needs and abilities of their students.

Spectrum of Mathematical Modeling (Examples Suitable for Upper Middle School and High School)
Level 9 (highest level): Students choose the context and the question. They experience the entire modeling process while confronting two or more iterations. The question may be practical or may concern something about which the student is curious.
Level 8: The context is provided by the teacher. Students determine a meaningful question related to the context and use the modeling process to determine an answer. *Example: Presented with a 12-pack of juice cans (or water bottles), what questions could be asked that would lead to a practical solution?*
Level 7: The teacher determines the context and poses the question to be answered. Students determine the relevant variables, make assumptions, and choose to simplify or ignore some of the variables. Students will need to justify their decision when making presentations. *Example: Find a better way to package juice cans.*
Level 6: Same as level 7, with the exception that the teacher guides students through the process of making assumptions and simplifications. Students develop and apply mathematical models and determine the reasonableness of the solutions. *Example: Find a better (more efficient) way to package juice cans. The discussion will determine that "efficient" means the ratio of the space used to the space available in the package. Students and teachers will assume the cans are perfect cylinders, restrict the package to the height of a single can, orient all cans in the same direction, and use a package that is a prism with congruent polygon bases (no shrink-wrap).*
Level 5: The teacher provides a simplified version of a real-life question and context. The problem is rich enough to allow for several solution paths and allow for access to various levels of mathematical background. *Example: Which package uses the highest percentage of space—a rectangular 12-pack, a triangular 10-pack, a trapezoidal 9-pack, or a hexagonal 7-pack? (All are prisms with a height equal to one can or bottle). Students make accurate representations of these packages and may determine the use of space through measurement, algebraic manipulation applied to polygons, or by using geometric sketchpad software.*
Level 4: Students are guided through the solution process that starts with a real-life context and question. The series of questions ensures that students will follow a particular path and use expected mathematics to solve the problem. The reasonableness of the solution is analyzed. *Example: Which is more efficient—the hexagonal 7-pack or the triangular 10-pack? Determine the percentage of space used in a hexagonal 7-pack.*
Level 3: A context and question are given. This is a real-world context with a mathematical focus. *Example: Six cans (circles) are placed together to form a triangle shape. The design engineer needs to find the height of the configuration. Determine the distance from the bottom can to the top can.*
Level 2: The context or real-world nature is incidental to the problem. The problem may even be contrived. *Example: Three circles are placed tangent to one another. Calculate the area bounded by the three circles.*
Level 1: There is no real-world context; the question is purely mathematical. *Example: Calculate the area of a circle with diameter equal to 2.5 inches.*

The Modeling Curriculum

As noted previously, much of the current mathematics curriculum involves students imitating what their teachers show them. Real-world situations are often employed only as exercises for students to practice mathematics they are currently learning, often in the form of word problems. The preceding discussion already pointed out some features of a modeling curriculum, which includes open-ended tasks, complex problems, student independence, and multiple means of sharing results. Abrams (2012) suggests some differences between true mathematical modeling situations and mathematical exercises (Abrams 2012, 40); see table B-1.

Table B-1. Mathematical Modeling Versus Mathematical Exercises

Mathematical Modeling	Mathematical Exercises
Unfamiliar	Familiar
Memorable	Forgettable
Relevant	Irrelevant
Many possible correct answers	One right answer
Lengthy	Brief
Complex	Simple
Students discover processes	Students follow instructions
Open-ended	Closed (goals chosen by teacher)
Cyclic—constant refining	Linear
Does not appear on a particular page	Appear too often, and then not enough

Modeling often involves project-based work, which can take place over days, weeks, or even months. Modeling problems are unfamiliar and original to students; they are memorable because students must take an active role in the learning process. Modeling problems can be whimsical and clever, with the potential of being extended to the real world. Modeling tasks have an inherent relevance, as students clearly see applications to the real world. Tasks are not predetermined and often have messier endings than traditional problems, as students must sometimes decide whether they have enough information to make a decision. Modeling situations offer great opportunities for cross-disciplinary work and may include problems drawn from the sciences that require students to write reports as a summative activity. Finally, real modeling problems do not come with instructions. Students may take a certain solution path only to find that it did not shed much light on the situation, and then realize that they need to start over along a different path.

Teachers should use their newfound knowledge of the features of modeling to scrutinize instructional materials that are aligned with the CA CCSSM. Teachers may find it necessary to supplement their curricula with rich modeling tasks; see the Related Resources at the end of this appendix for ideas.

Teachers can even make use of problems from traditional curricula in modeling; a traditional word problem can often be changed into a modeling task by asking what would happen if something about the original problem were changed. Teachers can use their own experience to experiment with developing their own modeling tasks for students.

When the goal is the learning or application of particular mathematics content, the challenge for teachers and curriculum developers is to select appropriate investigative tasks that use the modeling process. The tasks must involve question formulation based upon authentic, real-world contexts that are likely to introduce, develop, or apply the desired mathematics content. Because rich real-world contexts are often complicated, simplification in the modeling process is a critical step. Care must be taken to avoid contrived or overly simplified problems. As students develop a deeper understanding of the mathematical modeling process, they should be involved more and more in the stages of formulating questions and simplification. In addition students should experience the latter steps in the mathematical modeling process. Real-world questions should lead to real-world solutions. The solutions are examined for

reasonableness (e.g., *Does the answer make sense?*) and usefulness (e.g., *Is the solution applicable to the original situation, or is it necessary to revisit and reformulate the model?*).

Supporting Teachers and Students

With regard to mathematical modeling, many teachers will benefit from their own professional learning, just as they would when dealing with any change in instruction. Teachers in a professional learning setting should experience the process of modeling themselves. By practicing modeling, teachers can get a feel for looking at the world through a mathematical lens; they begin to ask questions, notice mathematically interesting situations, and recognize the usefulness of mathematics in the world. Such exposure is certainly the first step for teachers to take to employ modeling in their classrooms.

Teachers also need experience recognizing, creating, and modifying good modeling problems. Consider Usiskin's "reverse given–find" problems (Usiskin 2011). Typically, mathematics word problems are of the "given–find" variety, wherein certain information is known (i.e., *given*) and students are asked to derive some unknown information (e.g., *find* x). For example, given the sides of a triangle, students are asked to find its perimeter; given the side lengths of a rectangle, students find its area; given a polynomial, students find its roots; and so on. Usiskin suggests reversing these questions: A triangle has perimeter 12 units; what are the possible whole-number side lengths of this triangle? A rectangle has an area of 24 square units; how many rectangles with whole-number side lengths have this area? A polynomial has the following roots . . . can you determine the polynomial? (Usiskin 2011, 5). Of course, this is only one way to develop simple, open-ended situations that are mathematical in nature. However, teachers can start in this way and expand to more complex examples and real-world situations.

Enhancing the Modeling Process

The modeling process is enriched by the following elements:

1. *The facilitative skill of the teacher.* The teacher must create a safe, positive environment in which student ideas and questions are honored and constructive feedback is given by the teacher and by other students. Students do the thinking, problem solving, and analyzing.
2. *The content knowledge of the teacher.* The teacher understands the mathematics relevant to the context well enough to guide students through questioning and reflective listening.
3. *Teacher and student access to a variety of representations and mathematical tools.* Examples include manipulatives and technological tools (dynamic geometry software, spreadsheets, Internet resources, graphing calculators, and so on).
4. *Teacher and student understanding of the modeling process.* Teachers and students who have had prior experience with the modeling process have better understanding of the process and the use of models.
5. *Teacher and student understanding of the context.* Background information or experience may be needed and gained through Internet searches, print media, videos, photographs and drawings, samples, field trips, guest speakers, and so forth.
6. *Richness of the problem to invite open-ended investigation.* Some problems invite a variety of viable answers and multiple ways to represent and solve them. Some contrived problems may appear to be "real-world" but are not realistic or cognitively demanding.
7. *The context of the problem.* Selecting real-world problems is important, and real-world problems that tap into student experiences (prior and future) and interests are preferred.

Teachers need to experience modeling firsthand and develop a different set of skills that they may not possess yet. For example, although the ability to conduct student discourse—allowing discussions to unfold in a non-directive but supportive way and allowing students the time to discuss their ideas—is critical for teaching many of the MP standards, it is of crucial importance for teaching modeling. Teachers also must develop knowledge of tasks, knowledge of the steps of the modeling process so as to identify student difficulties, and knowledge of intervention strategies.

Teachers' belief systems concerning the teaching of mathematics also need to undergo a shift. As mentioned previously, the goals of modeling are not strictly mathematical. Modeling can help students better understand the world and help create a more rounded picture of mathematics for them, in addition to teaching mathematical content. In order to have the time and space needed to implement modeling, teachers need to reflect on their instructional goals and must be willing to practice patience and understanding (with students and themselves), because the modeling process represents a rather large instructional shift.

As noted in the chapter on Supporting High-Quality Common Core Mathematics Instruction, administrators must allow teachers the time and space to implement new CA CCSSM teaching strategies in the classroom. Many of the MP standards are encompassed in the modeling process itself (e.g., standards **MP.1**, **MP.2**, **MP.3**, **MP.5**, and **MP.6**), and therefore teaching with modeling, both as vehicle and as content, supports teaching the CA CCSSM. Administrators must be aware of this and should be supportive of teacher efforts to include modeling in their instruction, especially in middle and high school classrooms. Administrators should be aware that classrooms engaged in modeling tasks are noisy and messy, with students often working excitedly in groups while some students choose to work independently. Teachers with backgrounds in engineering or the sciences are a good resource for implementing modeling.

Finally, the role of parents should also be highlighted. Many great ideas for improving mathematics education have been resisted because of the misunderstanding of parents. In the case of mathematical modeling, problems are messier and teachers are not simply showing students what to do and then having them practice. If parents are informed of the goals and process of modeling, then they can better understand their child's response to the question, "What did you do in math class today?" Parents must be included in the support structure if the CA CCSSM are to be successfully implemented.

Modeling in Higher Mathematics

In the CA CCSSM, modeling is considered a conceptual category for higher mathematics. By the time students have gained proficiency with the K–8 standards, their understanding of number and operations, equations, functions, graphing, and geometry is quite solid. Students further develop these ideas in the higher mathematics courses—especially the notion of *function*, which can play an important role in modeling. While the function concept is developed more fully, and as students' repertoire of expressions and equations increases, students are able to work with more challenging modeling situations.

It is notable that this question may be considered "whimsical": it is a fun question that results in a real-world answer, but it may not be important in the grand scheme of things. However, many mathematical discoveries and questions were discovered by beginning with a similarly whimsical problem, and therefore questions such as this should be explored and encouraged.

The first course in higher mathematics (i.e., Mathematics I or Algebra I) is the first place where mathematical modeling is introduced as a conceptual category. Standard **MP.4** (Model with mathematics) should have been a common experience for students in previous grade levels. Students should have had numerous opportunities to apply mathematical models to solve real-world problems. In Mathematics I or Algebra I, explicit attention is given to teaching the *process* of mathematical modeling. Students learn and practice *all* the steps in the process; they come to understand that the modeling process is seldom linear and that it often involves revisiting steps to formulate a model that is both useful and solvable. The model developed must authentically approximate the real-world context and, at the same time, provide access to students in terms of the mathematics needed to understand the situation or answer the question.

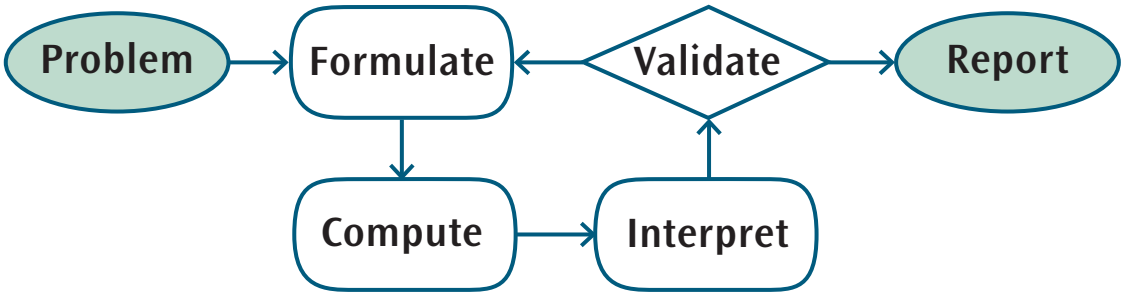
Problems that arise from the real world seldom involve a single content standard. Given enough of their own positive experiences with application problems, teachers and curriculum developers will find that real-life contexts can truly bring out the mathematics that is supposed to be taught in many courses.

Abrams (2012) describes a modeling problem in which students asked themselves, "How can you eat a peanut-butter-cup candy in more than one bite and ensure that each bite has the same ratio of chocolate to peanut butter?" Abrams (2012, 43). After simplifying the peanut-butter cup to two cylinders, with a peanut-butter cylinder embedded in a chocolate cylinder, and simplifying a bite into an arc of a circle intersecting these two cylinders, students tried to discover a formula for the volumes of both chocolate and peanut butter in each bite. The students eventually derived an equation that involves complicated rational expressions contained in square roots, inverse trigonometric functions, and both variables and parameters. They found the ratio in question as a function of the size of the first bite. As Abrams admits, although the problem is not the most important question facing humankind, his students were completely engaged with it and quickly discovered how difficult it was to solve.

It is notable that this question may be considered "whimsical": it is a fun question that results in a real-world answer, but it may not be important in the grand scheme of things. However, many mathematical discoveries and questions were discovered by beginning with a similarly whimsical problem, and therefore questions such as this should be explored and encouraged.

As shown in figure B-3, the National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA/CCSSOO) developed a schematic for modeling at the higher mathematics level.

Figure B-3. The Modeling Cycle



According to the authors:

> The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. (NGA/CCSSO 2010a)

The vision of modeling in the higher mathematics standards of the CA CCSSM aligns with much of the discussion in this appendix.

Table B-2 presents some examples of modeling problems suitable for upper middle school and higher mathematics courses.

Table B-2. Modeling in Upper Middle School and Higher Mathematics Courses

Example: Linear Functions. There are numerous real-world contexts that can be modeled with linear functions. Situations involving repetitive addition of a constant amount are plentiful. Common contexts such as comparing cost, revenue, and profit for a business or any context with a fixed and variable component—such as a membership fee combined with a monthly maintenance fee, a down payment followed by monthly payments, or a beginning amount with constant growth, such as simple interest—are opportunities to apply linear models. Students may be asked to determine the feasibility of starting a business—for example, selling hot dogs. A teacher may facilitate a class discussion to identify relevant factors, make assumptions, and gather necessary information. The class might survey the market to determine a reasonable price, such as $2.25 per hot dog. The cost factors, such as ingredients and paper goods, could be simplified and condensed into a single cost per hot dog, such as $1.10. The total cost usually includes fixed costs such as rent, licensing or permit requirements, and so forth. Many towns have street fairs or "Market Night" where a space may be rented for a cost (such as $50 for a four-hour period). Tables, graphs, or equations may be used as models to answer teacher or student questions such as these: *How many hot dogs do I need to sell if I want to make a profit of $400? How many hot dogs do I have to sell each hour to break even?* Graphing calculators, computer applications, or spreadsheets could also provide powerful models to generalize or extend the investigation.
Example: Exponential Functions. Exponential functions model situations representing a constant multiplier, such as population growth or decay, the elimination of a therapeutic drug in the body, the filtering of harmful pollutants in air or water, compound interest, or cell division. The current population of a town is 18,905. If the population is growing at an average rate of 3 percent each year, when should the population be expected to reach 20,000? What community services might need to increase and thus be reflected in the town's budget? The use of an average rate would be a simplification. Perhaps students would rather investigate a high and low rate to project a range of possible population projections.

Continued on next page

Table B-2 (*continued*)

Example: Juice-Can Packaging. A problem adapted from the National Council of Teachers of Mathematics Illuminations Web site (NCTM Illuminations 2015) involves the packaging of juice cans. [*Note:* water bottles may be substituted for juice cans.] Typically, juice cans are packaged into rectangular prisms in quantities of 6, 12, or 24 cans. The cans are situated in a rectangular array; space between the cans is wasted. Students are challenged to design a new package that will minimize the amount of space that is wasted.

Students identify variables and assumptions; the most important aspects are used to focus the problem. Students or the teacher may decide to limit the number of cans to a minimum of 4 and a maximum of 12 with the knowledge that any simplifications may be revisited when the reasonableness of solutions, based on the models, is analyzed. Additional restrictions may include these: the packages are prisms with polygon bases, all cans are situated in the same direction, the cans are perfect cylinders, and the cans are not stacked. The last restriction allows students to simplify the models to two dimensions—circles within polygons—because the dimension of height is held constant for all designs. Students begin the investigation by moving cans around to visualize different arrangements. They represent their designs with careful drawings (circular discs) or by using geometric sketchpad technology. Decisions are made about how to determine whether the wasted space is calculated as an absolute area or volume or a percentage of the available space. Students incorporate the Pythagorean Theorem, equilateral and 30-60-90 right triangles, similar triangles, area of polygons, area of circles, tangents to circles, volume of cylinders, and ratios and proportions.

Creating a Mathematical Modeling Course for High School

A course in mathematical modeling should build upon modeling experiences from previous mathematics courses. The course should allow students to deepen their understanding of the modeling process, apply in new contexts mathematical models they have already learned, and learn new mathematics content to solve unique real-world problems.

Students with a strong background in mathematical modeling should be able to apply mathematics to understand or solve novel problems in career and college settings. A modeling course should allow students to experience all stages of the modeling process, including problem formation; model building that incorporates a variety of mathematical models, skills, and tools for solving the problems; and sufficient analysis to determine if the solution is reasonable or if the model should be revised.

The goal of mathematical modeling is to answer a question, solve a problem, understand a situation, design or improve a product or plan, or make a decision. Mathematical modeling in the school setting includes the additional expectation that students will learn or apply particular mathematical content at a particular grade level. If the learning or application of content standards is the goal, then teachers need to select real-world problems that are likely to have the desired mathematics embedded in them.

Most teachers and students have experienced a single path for learning higher mathematics: it is logical, builds concept upon concept, increases in complexity, and aims to accumulate tools that may be used to solve problems. Rarely, if ever, are students given the opportunity to solve real-world problems in the way that those problems are actually encountered in life. Typically, "application" problems in textbooks are formulated and presented in the form of exercises with the hope that students will buy into the importance of mathematics.

There is another way to learn mathematics that almost no one has experienced in the classroom but most have experienced in everyday life or in a career: Start with a real-world problem or question, apply mathematics already learned to a novel situation, or learn new mathematics that can be applied to solve the problem.

Courses in mathematical modeling may be developed to serve a variety of curricular goals. One course may revisit or build upon modeling standards from previous course work. The emphasis would be on students deepening their understanding and skill by applying previous learning in novel, unique, and unfamiliar situations. Another modeling course may be designed to learn new mathematics (not addressed in previous courses). It should also be noted that real-world problems are not constrained by content standards and often incorporate multiple standards with varied depth. A course in mathematical modeling should extend or supplement—not replace—the integration of modeling into all higher-level mathematics courses and pathways.

Mathematical modeling should be experienced by students and teachers in several ways. Some examples are provided below:

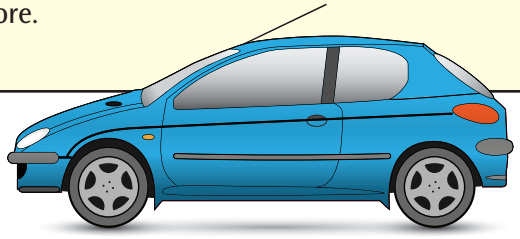
- Short, simple, real-world problems, questions, or "I wonder . . . ?" scenarios that can be solved in just a few minutes by using mathematics
- More complex problems that require additional information acquired through research while also filtering out extraneous information through the process of simplifying and making assumptions. The search for a problem solution would involve more than a day and more than one iteration through the modeling cycle.
- Extensive problems requiring the development of new models and the learning of new mathematics that might constitute an entire unit of study of relevant math content arising from the single real-world scenario. The unit of exploration can extend the mathematics learned to other contexts so that students begin to see the universality of some mathematical models and processes.
- An entire course developed from extensive problem units. The units may or may not be related to one another, but they evolve from real-world situations. The less guidance and scaffolding that is offered by the teacher or the curriculum, the more authentic the modeling experience will be.

Financial literacy is a topic that can find a place in the teaching of the CCSSM in higher mathematics. Topics for a mathematical modeling course could include those dealing with simple cost analysis using linear functions, finding simple and compound interest using exponential functions, finding total cost of payments on a loan, and so forth. Clearly, such topics are relevant to students' future lives and are therefore an important application of modeling.

Example: Owning a Used Car

The teacher poses the following question to a high school class: *How old a car should you buy, and when should you sell it?* The teacher invites students to research several variables online, including options for financing, total cost of the car, depreciation, gas mileage, and the like. Students organize their information and use mathematics to create an argument for why they would buy a car of the model year they have chosen. A modeling situation such as this one could involve proportions, percentages, rates, units, linear functions, exponential functions, and more.

Adapted from Burkhardt 2006, 184.



A Course in Mathematical Modeling Should:

- Be in addition to, not a replacement for, the incorporation of mathematical modeling into the fabric of all higher mathematics courses. The Modeling conceptual category was not intended as a separate course that students may or may not encounter in high school.
- Deepen a student's understanding of, and experience with, all stages of the mathematical modeling process. The course should be about modeling as well as mathematics, and the relationship between the two.
- Allow for sufficient opportunities for students to apply mathematical content they have already learned to unique problems and contexts.
- Challenge and motivate students to recognize the **need** to learn and apply new mathematics and related models. When the models and the mathematics are introduced, students are challenged to find other contexts in which they could apply the same model (or a similar one).
- Progressively allow students more freedom and opportunities to formulate their own questions; develop, apply, and justify their own mathematical models; and analyze and defend their own conclusions through collaboration and dialogue with peers and teachers. (It is challenging for teachers to provide the right balance of freedom and support. Students need to struggle in order for learning to take place, but they should not become so discouraged that they feel like quitting. Teachers need to know which scaffolds to use and should develop open-ended questions to support and sometimes guide students' thinking.)
- Help students and teachers recognize that, to some extent, all people engage in mathematical modeling every day. This can be accomplished by fostering two related dispositions: (1) the ability to look at a life situation and wonder how mathematics might be applied to understand or solve the situation; and (2) the ability to look at a mathematical concept and wonder how it might be applied to life experiences.
- Provide opportunities for students to tackle real-world problems of different complexity. This includes ordinary tasks such as figuring out which coupon to use, which phone plan to choose, or how much of a tip to leave, as well as more complex decisions involving how to prepare for a natural disaster, how to solve a crime, how to rate products, or how to balance the need for increased energy supplies with the need to protect the environment.
- Allow for the learning of mathematical principles, "big ideas," concepts, procedures, standard models, and skills in a meaningful setting, to establish meaning and relevance before teaching mathematics whenever possible.

Sample Topic Areas in an Applied Mathematical Modeling Course

Each starred (★) standard from the higher mathematics conceptual categories of the CA CCSSM could be considered part of a modeling course and may be combined with other higher mathematics standards when creating a course. Table B-3 offers sample topic areas that might be explored in an applied modeling course. All of the starred (★) standards are listed in table B-4 at the end of this appendix.

Table B-3. Sample Topics for a Mathematical Modeling Course

Topic Area	Sample Contexts or Problems	Intended Mathematics Content
Linear Functions—Part 1		
• Making Money • Membership • Choosing Plans	• Simple business models of cost, revenue, and profit are excellent contexts for modeling with linear functions. A real-life business incorporates several variables that will have to be simplified. Sample questions: *How many units of a particular item must I sell to break even or reach a target profit? Which item should I sell if I want to make the largest profit?* • Which plan should I choose? (Consider phone plans, data plans, health clubs, music clubs, membership benefits, and so forth.) • Should I buy or lease a car? • Should I take a job that pays a regular salary, commission, or a blend of the two?	• Linear functions expressed in tabular, graphical, and symbolic forms • Systems of linear functions and equations • Recursive forms with constant addition
Linear Functions—Part 2		
• Line of Best Fit	• Begin with a set of data that are approximately linear and formulate questions. OR • Begin with a real-world question that is likely to have a linear relationship, and then collect the data through simulation, survey, or activity or search for data on the Internet. • For a particular event, how many people should we plan for and/or how much food should we buy? • Based on past trends, how many tickets or programs should we print for this event? • How many schools will we need? How many people might attend? What will my car's value be in seven years? What do we expect to be the total?	• Linear functions derived from approximately linear data; use residuals to determine appropriateness of a model

Table B-3 (*continued*)

Table B-3. Sample Topics for a Mathematical Modeling Course		
Topic Area	**Sample Contexts or Problems**	**Intended Mathematics Content**
Exponential Functions	• Contexts related to growth of populations (people, animal, bacteria, disease) or money (compound interest). Make predictions and/or plans based on anticipated growth of the population or money. • Contexts related to decline or decay of populations (such as half-life of over-the-counter or prescription drugs or depreciation of money) • Contexts related to filtration (such as fans to exhaust particulates from a room or filters that remove pollutants from a water supply) • Whimsical problems that have characteristics that are similar to real-world problems	• Exponential functions expressed in tabular, graphical, and symbolic forms • Equations derived from exponential functions • Decisions to make about limiting the domain to whole numbers, integers, or rational numbers for the base and/or the exponent • Recursive forms involving a constant multiplier • Inverse of exponential function (introduced informally and identified as a logarithm)
Quadratic Functions	• Contexts related to the Pythagorean Theorem and distance. Possibly explore parabolic presence in satellite dishes, telescopes, searchlights, covert listening devices, and solar cookers. • Contexts related to the sum of a series (e.g., carpet rolls and paper rolls) • Contexts related to projectile motion • Contexts related to area • Contexts related to cost, revenue, and profit where price is a linear function	• Quadratic functions expressed in tabular, graphical, and symbolic forms • Equations derived from quadratic functions • Other content that can be connected to geometric contexts
Polynomials	• Problems related to volume (e.g., Produce a box with maximum volume from a flat piece of card stock, but cut out the corners)	• Add, subtract, multiply, and divide polynomials. • Construct polynomials from a real-world situation. • Relate polynomials to geometry.
Absolute Value	• In a town with parallel and perpendicular streets, what is the best location for a new school, hospital, or mall? • Contexts related to tolerance (e.g., factory specifications for a door indicate the door should be 36” wide with a tolerance of 1/32”)	• Absolute value in the real world

Table B-3 (*continued*)

Table B-3. Sample Topics for a Mathematical Modeling Course		
Topic Area	**Sample Contexts or Problems**	**Intended Mathematics Content**
Probabilistic	• Drug testing and determining the cost to test a pool of samples versus testing individual blood samples. For example, if 10 blood samples are pooled and tested as one and the results are negative, then you have saved the cost of testing nine other samples. If the result is positive, then you have to re-test the samples individually or in a smaller group. • Genetic combinations • Fingerprint and DNA testing	• Expected values for false positives and false negatives
Mixed	• How is mathematics used to build the code for representing the movement of objects on a screen or within a video game? • How do blood-spatter patterns help in a crime scene investigation? • Will an asteroid collide with Earth?	• Quadratic functions (for movement of projectiles affected by gravity, such as basketballs) • Parametric equations involving time and location in two or three dimensions • Movements driven or altered by forces and represented by vectors that would then involve trigonometry ratios and possible law of sines and cosines
Polygons	• How do you accurately enlarge or reduce an object? How does scaling affect surface area and weight? • What is the most efficient package design for the packing of cylinders into right prisms with polygonal bases? • Where should sprinklers be placed to optimize water coverage for a lawn or crops? • Any packaging or tiling context using polygon-shaped objects either as the content objects or as the package	• Similarity, scale factors, and dilations • Perimeter and area • Tessellations (rotations, translations, reflections)
Trigonometry • Right Triangles	• How can you estimate the height of a tall object, such as a tower or mountain, when you are prevented from finding the direct distance along the ground? • Using a device to measure angle of inclination or angle of depression, how do you find (indirectly) the height of an object? • How do you render accurately in a drawing or within a video game the height of an object that is tilted at an angle to the viewing plane?	• Right-triangle ratios (tangent, sine, and cosine)

Table B-3 (*continued*)

Table B-3. Sample Topics for a Mathematical Modeling Course		
Topic Area	**Sample Contexts or Problems**	**Intended Mathematics Content**
Circles	• Which pizza size gives consumers the best deal for the money they will spend? • Contexts involving circular motion (including wheels, gears, belt-driven motors, and so forth)	• Area and circumference • Tangents to circles
Volume and Surface Area	• Maximizing the volume of a container while minimizing the surface area (amount of materials needed to make the container) • Pistons and displacement in an internal-combustion engine • Is King Kong possible? How are surface area, weight, and volume affected by enlargement or reduction due to scale factor? • Whimsical: How large is the giant that would fit into the world's largest pair of shoes? • How much liquid would it take to fill the giant cola bottle that is displayed in Las Vegas?	• Prisms, cylinders, cones, spheres • Scale factors (length, area, volume ratios)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
Number and Quantity	
N-Q.1	Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
N-Q.2	Define appropriate quantities for the purpose of descriptive modeling. ★
N-Q.3	Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★
Algebra	
A-SSE.1	Interpret expressions that represent a quantity in terms of its context. ★ a. Interpret parts of an expression, such as terms, factors, and coefficients ★ b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of* P *and a factor not depending on* P. ★
A-SSE.3	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★ a. Factor a quadratic expression to reveal the zeros of the function it defines. ★ b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. ★ c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* ★

Table B-4 (*continued*)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
A-SSE.4	Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★
A-CED.1	Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. **CA** ★
A-CED.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
A-CED.3	Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
A-CED.4	Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★
A-REI.11	Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★
Functions	
F-IF.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
F-IF.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
F-IF.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★
F-IF.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★ a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★ b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★ c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★ d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★ e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
F-IF.10	(+) **Demonstrate an understanding of functions and equations defined parametrically and graph them. CA** ★

Table B-4 (*continued*)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
F-BF.1	Write a function that describes a relationship between two quantities. ★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★ b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★ c. (+) Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.* ★
F-BF.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★
F-LE.1	Distinguish between situations that can be modeled with linear functions and with exponential functions. ★ a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★ b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★ c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★
F-LE.2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★
F-LE.3	Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★
F-LE.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★
F-LE.4.1	**Prove simple laws of logarithms. CA** ★
F-LE.4.2	**Use the definition of logarithms to translate between logarithms in any base. CA** ★
F-LE.4.3	**Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA** ★
F-LE.5	Interpret the parameters in a linear or exponential function in terms of a context. ★
F-LE.6	**Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity. CA** ★
F-TF.5	Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★
F-TF.7	(+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. ★
Geometry	
G-SRT.8	Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★
G-GPE.7	Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★
G-GMD.3	Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★

Table B-4 (*continued*)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
G-GMD.5	**Know that the effect of a scale factor k greater than zero on length, area, and volume is to multiply each by k, k^2, and k^3, respectively; determine length, area and volume measures using scale factors. CA ★**
G-MG.1	Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
G-MG.2	Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
G-MG.3	Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★
Statistics and Probability	
S-ID.1	Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
S-ID.2	Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
S-ID.3	Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★
S-ID.4	Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★
S-ID.5	Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
S-ID.6	Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★ a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★ b. Informally assess the fit of a function by plotting and analyzing residuals. ★ c. Fit a linear function for a scatter plot that suggests a linear association. ★
S-ID.7	Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
S-ID.8	Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
S-ID.9	Distinguish between correlation and causation. ★
S-IC.1	Understand statistics as a process for making inferences about population parameters based on a random sample from that population. ★
S-IC.2	Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★
S-IC.3	Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
S-IC.4	Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
S-IC.5	Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★

Table B-4 (*continued*)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
S-IC.6	Evaluate reports based on data. ★
S-CP.1	Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
S-CP.2	Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
S-CP.3	Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B. ★
S-CP.4	Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
S-CP.5	Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* ★
S-CP.6	Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model. ★
S-CP.7	Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. ★
S-CP.8	(+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B\|A) = P(B)P(A\|B)$, and interpret the answer in terms of the model. ★
S-CP.9	(+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★
S-MD.1	(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions. ★
S-MD.2	(+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution. ★
S-MD.3	(+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.* ★
S-MD.4	(+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?* ★

Table B-4 (*continued*)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
S-MD.5	(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. ★ a. Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.* ★ b. Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.* ★
S-MD.6	(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
S-MD.7	(+) Analyze decisions and strategies using probability concepts (e.g. product testing, medical testing, pulling a hockey goalie at the end of a game). ★

Related Resources

- Consortium for Mathematics and Its Applications (COMAP): http://www.comap.com/ (accessed September 9, 2015)

  Modeling problems are available at http://www.mathmodels.org/problems/ (accessed September 9, 2015).

- Mathematics TEKS Toolkit, Charles A. Dana Center: http://www.utdanacenter.org/mathtoolkit/instruction/activities/models.php (accessed September 9, 2015)

- Illuminations, National Council of Teachers of Mathematics (NCTM): http://illuminations.nctm.org/ (accessed September 9, 2015)

- Illustrative Mathematics: https://www.illustrativemathematics.org/ (accessed September 9, 2015)

- Learning and Education in and through Modeling and Applications (LEMA): http://www.lema-project.org/web.lemaproject/web/eu/tout.php (accessed September 9, 2015)

- Shell Center for Mathematical Education, Mathematics Assessment Project: http://map.mathshell.org.uk/materials/tasks.php (accessed September 9, 2015)

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Appendix C

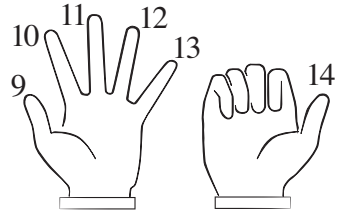
Methods Used for Solving Single-Digit Addition and Subtraction Problems

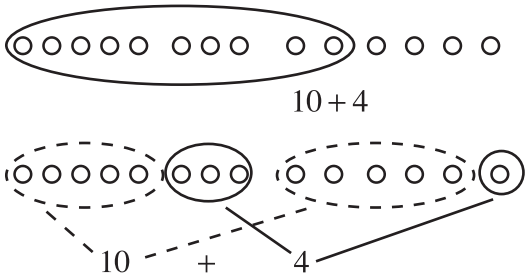
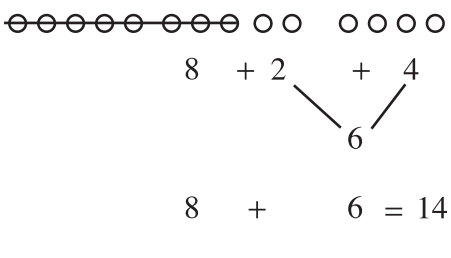
This appendix was adapted from the University of Arizona (UA) Draft K–5 Progression on Counting and Cardinality and Operations and Algebraic Thinking (UA Progressions Documents 2011a). It discusses various computational methods (levels 1, 2, and 3) that students might use to solve addition and subtraction problems. Each framework chapter for kindergarten through grade two also includes a table of "Methods Used for Solving Single-Digit Addition and Subtraction Problems" that summarizes these three methods. Additionally, the grade-level chapters provide examples and explanations of how students might use these methods to solve grade-appropriate addition and subtraction problems.

Level 1: Direct Modeling by Counting All or Taking Away

Represent the situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

- Adding ($8+6=\square$): Represent each addend by a group of objects. Put the two groups together. Count the total. Use this strategy for Add To/Result Unknown and Put Together/Total Unknown problems.
- Subtracting ($14-8=\square$): Represent the total by a group of objects. Take the known addend number of objects away. Count the resulting group of objects to find the unknown addend. Use this strategy for Take From/Result Unknown problems.

Level	$8+6=14$	$14-8=6$
Level 1: Count all	Count All a b 1 2 3 4 5 6 7 8 1 2 3 4 5 6 ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 c	Take Away a 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ~~○ ○ ○ ○ ○ ○ ○ ○~~ ○ ○ ○ ○ ○ ○ 1 2 3 4 5 6 7 8 1 2 3 4 5 6 b
Level 2: Count on	Count On 8 ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ 8 9 10 11 12 13 14	To solve $14-8$, I count on $8+?=14$ I took away 8  8 to 14 is 6, so $14-8=6$

Level 3: Re-compose Make a ten (general): one addend breaks apart to make 10 with the other addend. Make a ten (from 5's within each addend)	Re-compose: Make a Ten  	$14 - 8$: I make a ten for $8 + ? = 14$ 
Doubles = *n*	$6 + 8$ $= 6 + 6 + 2$ $= 12 + 2 = 14$	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Level 2: Counting On

Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total, but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. The count is tracked and monitored in some way (e.g., with fingers, objects, mental images of objects, body motions, or other count words).

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as an unknown addend).

Counting on may be used to find the total or to find an addend. These look the same to an observer. The difference is what is monitored: the total or the known addend. Some students count down to solve subtraction problems, but this method is less accurate and more difficult than counting on. Counting on is not a rote method. It requires several connections between cardinal and counting meanings of the number words, as well as extended experience with Level 1 methods in kindergarten.

- Adding (e.g., $8 + 6 = \square$) uses counting on to find a total: One counts on from the first addend (or the larger number is taken as the first addend). Counting on is monitored so that it stops when the second addend has been counted on. The last number word is the total.
- Finding an unknown addend (e.g., $8 + \square = 14$): One counts on from the known addend. The "keeping track" method is monitored so that counting on stops when the known total has been reached. The "keeping track" method tells the unknown addend.
- Subtracting ($14 - 8 = \square$): One thinks of subtracting as finding the unknown addend, as $8 + \square = 14$, and uses counting on to find an unknown addend (as above).

In the Glossary of this framework, table GL-4 includes problems that can be solved with Level 1 methods in kindergarten or by using Level 2 methods: counting on to find the total (adding) or counting on to find the unknown addend (subtracting). Level 2 and 3 methods are generally used in grades one and two.

Finding an unknown addend (e.g., $8 + \square = 14$) is used for Add To/Change Unknown problems, Put Together/Take Apart/Addend Unknown problems, and Compare/Difference Unknown problems. It is also used for Take From/Change Unknown ($14 - \square = 8$) problems after a student has decomposed the total into two addends, which means they can represent the situation as $14 - 8 = \square$.

Adding or subtracting by counting on is used by some students for each of the kinds of Compare problems (see the equations in table GL-4 of the Glossary). Students in grade one do not necessarily master the Compare Bigger Unknown or Smaller Unknown problems that use misleading language (such as the words *fewer* or *more than*). These problem types appear in the bottom row of table GL-4 of the Glossary.

Solving an equation such as $6 + 8 = \square$ by counting on from 8 relies on the understanding that $8 + 6$ gives the same total—an implicit use of the commutative property without the accompanying written representation $6 + 8 = 8 + 6$.

Level 3: Convert to an Easier Equivalent Problem

Decompose an addend and compose a part with another addend.

The following methods can be used to add or to find an unknown addend (and thus to subtract). The methods implicitly use the associative property.

Adding

Make a ten. For example, for $8 + 6 = \square$,

$$8 + \underline{6} = 8 + \underline{2 + 4} = 10 + 4 = 14,$$

so $8 + 6$ becomes $10 + 4$.

Doubles plus or minus 1. For example, for $6 + 7 = \square$,

$$6 + \underline{7} = 6 + \underline{6 + 1} = 12 + 1 = 13,$$

so $6 + 7$ becomes $12 + 1$.

Finding an unknown addend

Make a ten. For example, for $8 + \square = 14$,

$8 + \underline{2} = 10$ and $\underline{4}$ more makes 14

$$\underline{2 + 4} = 6$$

So $8 + \square = 14$ is done as two steps: how many up to 10 and how many over 10 (which can be seen in the ones place of 14).

Doubles plus or minus 1. For example, for $6+\square=13$,

$$6+\underline{6+1}=12+1$$

$$\underline{6+1}=7$$

So $6+\square=13$ is done as two steps: how many up to $12\left(6+6\right)$ and how many from 12 to 13.

Subtracting

Thinking of subtracting as finding an unknown addend.

For example, using the methods shown above, solve $14-8=\square$ or $13-6=\square$ as $8+\square=14$ or $6+\square=13$ (make a ten or use doubles plus or minus 1).

Thinking of subtraction as "Make a ten first" or "Breaking down to 10"

For example, $15-8=\square$ can be done in two steps:

$$15-8=\left(\underline{15-5}\right)-3=\underline{10}-3=7$$

Students think how to make a $10\left(15-5\right)$ and then subtract what remains from the subtrahend (the number being subtracted—3 in the example).

The Level 1 and Level 2 problem types can be solved by using these Level 3 methods.

Level 3 problem types can be solved by representing the situation with an equation or drawing, then re-representing to create a situation solved by adding, subtracting, or finding an unknown addend (as shown above) by methods at any level, but usually at Level 2 or 3. Many students show in their writing only part of this multi-step process of re-representing the situation.

- Students re-represent Add To/Start Unknown situations by using the commutative property (formally or informally). For example, $\square+6=14$ is re-represented as $6+\square=14$.
- Students re-represent Take From/Start Unknown situations by reversing them. For example, $\square-8=6$ is re-represented as $6+8=\square$, which may then be solved by counting on from 8 or using a Level 3 method.

At Level 3, the Compare problems with misleading language can be solved by representing the known quantities in a diagram that shows the bigger quantity in relation to the smaller quantity. The diagram allows the student to find a correct situation by representing the difference between quantities and seeing the relationship among the three quantities. Such diagrams are the same as those used for the other versions of Compare situations; focusing on which quantity is bigger and which is smaller helps students to overcome the misleading language.

Some students may solve Level 3 problem types by re representing (as described above), but use Level 2 counting on.

As students move through levels of solution methods, they increasingly use equations to represent problem situations as situation equations and then to re-represent the situation with a solution equation or a solution computation. They relate equations to diagrams, which facilitates the process of re-representing. Labeling diagrams may help connect the parts of the diagram to the corresponding parts of the situation. However, students may know and understand things that they may not use for a given solution of a problem, as they increasingly do various representing and re-representing steps mentally.

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Appendix D

Course Placement and Sequences

Increased Rigor of Grade Eight and Algebra I/Mathematics I Standards

Success in Algebra I or Mathematics I is crucial to students' overall academic success, their continued interest and engagement in mathematics, and the likelihood of their meeting California's a–g requirements. The California Common Core State Standards for Mathematics (CA CCSSM) represent a tight progression of skills and knowledge that is inherently rigorous and designed to provide a strong foundation for success in the new, more advanced Algebra I and Mathematics I courses that are typically taken by most students in grade nine.

Development of these skills and knowledge depends on students being placed in appropriate courses, with emphasis on foundational concepts at the appropriate time, throughout their K–8 sequence and beyond. With the help of diagnostic information that is based upon rich common assessments, placement decisions should be reviewed by a team of stakeholders that includes teachers and instructional leadership (Massachusetts Department of Elementary and Secondary Education [MDESE] 2012).

Unfortunately, misplacement of students is common, with negative consequences for students who are unable to keep pace with the incremental difficulty of mathematics content; students' weaknesses in key foundational areas that support algebra readiness frequently translate into substantial difficulty reaching proficiency in higher-level mathematics while in high school (Finkelstein et al. 2012). At the same time, students need to be appropriately challenged and engaged in order to maintain their interest and skill development in mathematics throughout high school and beyond. Some students will take college-level courses (e.g., Advanced Placement Calculus, Statistics, or International Baccalaureate) as high school seniors, and the course sequences of the earlier grade levels need to support this level of course-taking. Therefore, one particular placement consideration, discussed later in this appendix, examines when and under what conditions to accelerate students in their mathematics sequence to reach the advanced courses while in high school.

Course Sequencing Challenges Involving the Transition to the CA CCSSM

Implementation of the CA CCSSM comes with many transitions over the next several years—new instructional approaches, new instructional materials, professional support for teachers, and technology readiness, among others. Furthermore, the transition from existing course sequences to new course sequences will inevitably provide challenges at both the school-district and school-site levels. Although the fundamental design of new courses presents its own immediate challenges, so too does the linkage between courses to ensure vertical articulation between grade levels and even between school systems. For example, some K–8 school districts feed into high school–only districts. In the particular case of mathematics, there is a "vocabulary" around the names of mathematics courses that is likely to cause confusion not only for educators, but also for parents. Prior to the development of the CA CCSSM, "Algebra I" was taught in grade eight to an increasing number of students. That same course

name will be the default for grade nine, as most students who move forward will complete the CA CCSSM for grade eight—and the new version of Algebra I is more rigorous and more demanding than previous versions of Algebra I. Even so, the changes are expected to cause confusion. The most practical solution is to describe the course content, in addition to giving course names, as a way to eliminate confusion until "Algebra I," as commonly used, now refers to a ninth-grade and not an eighth-grade course.

Research on Course Placement and Mathematics

The mathematics courses that students take greatly affect student achievement. The research studies briefly described below provide some additional context for the tradeoffs that are inherent in deciding how best to organize CA CCSSM course sequences and place students accordingly. "Algebra I" refers to courses that were in place under the 1997 California mathematics standards, prior to the adoption of the CA CCSSM. A big difference is that the CA CCSSM have rigorous grade-eight standards, but the California standards adopted in 1997 did not have specific standards for grade eight. Over the past decade, there has been a dramatic increase in the number and proportion of eighth-grade students enrolled in Algebra I in California.[1] Williams et al. (2011) reported that, between 2003 and 2009, the percentage of grade-eight students taking Algebra I increased from 32 percent to 54 percent. Although the increase in grade-eight enrollment in Algebra I resulted in greater percentages of grade-eight students achieving either "Proficient" or "Advanced" on the Algebra I California Standards Test, it also led to larger numbers of grade-eight students achieving "Far Below Basic" or "Below Basic" on the test (Williams et al. 2011). Williams et al. (2011) concluded that the practice of placing all eighth-graders in Algebra I, regardless of their preparation, sets up many students to fail. Kurlaender, Reardon, and Jackson (2008) looked at students in San Francisco, Fresno, and Long Beach and found that students' grade point average in grade seven and course failures in grade eight were predictive of the students' high school completion. These authors also found that the timing of when students take algebra is a strong predictor of students' high school success. In two of the three districts that they analyzed, students who completed algebra by grade eight were 30 percent more likely to graduate from high school than students who had not completed algebra by grade eight.

As expected and known for some time, course work in middle school relates closely to course work in high school. Findings from 20 years ago show that course-taking patterns in middle school are highly predictive of course-taking patterns in high school. Oakes, Gamoran, and Page (1992) stated that the courses students take in junior high school are "scholastically consequential, as the choice predicts later placement in high track classes in senior high school" (Oakes, Gamoran, and Page 1992, 574). More recently, Wang and Goldschmidt (2003) concluded that middle school mathematics achievement is significantly related to high school mathematics achievement, and "mathematics preparedness is vitally important when one enters high school—where courses begin to 'count' and significantly affect postsecondary opportunities" (Wang and Goldschmidt 2003, 15). In a study examining the National Education Longitudinal Study, Stevenson et al. (1994) found that the level of mathematics that students take in eighth grade is closely related to what they take in high school. They conclude that "students who are in an accelerated mathematics sequence beginning in eighth grade are likely to maintain that position in high school" (Stevenson et al. 1994, 196).

1. This increase was not confined to California. Similar increases in grade-eight Algebra I enrollment have occurred across the country (Walston and McCarroll 2010; Stein et al. 2011).

However, many students who finish middle school are not actually prepared to succeed in a rigorous sequence of college-preparatory mathematics courses in high school (Balfanz, McPartland, and Shaw 2002). Therefore, it is not surprising that previous research found that in the high school grades, ninth grade is a key year for students in terms of future academic success. Choi and Shin (2004) examined student transcripts from a large, urban school district in California. The authors found that most students fall off track for college eligibility in grade nine. Similarly, Finkelstein and Fong (2008) found that more than 40 percent of the students did not meet the California State University requirement of completing two semesters of college-preparatory mathematics in the ninth grade. They concluded that students who fall off the college-preparatory track early in high school tend to fall farther behind and are less likely to complete a college-preparatory program as they progress through high school. Neild, Stoner-Eby, and Furstenberg (2008) further conclude that the experience of the ninth-grade year contributes substantially to the probability of dropping out of high school, even after controlling for eighth-grade academic performance and pre–high school attitudes and ambitions.

The grade-eight standards in the CA CCSSM are significantly more rigorous than the Algebra I course that many students took in eighth grade. The CA CCSSM for grade eight address the foundations of algebra by including content that was previously part of the Algebra I course—such as more in-depth study of linear relationships and equations, a more formal treatment of functions, and the exploration of irrational numbers. For example, by the end of the CA CCSSM for grade eight, students will have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. The CA CCSSM for grade eight also include geometry standards that relate graphing to algebra in a way that was not explored previously. Additionally, the statistics presented in the CA CCSSM for grade eight are more sophisticated than those previously included in middle school and connect linear relations with the representation of bivariate data.

The new Algebra I and Mathematics I courses build on the CA CCSSM for grade eight and are therefore more advanced than the previous courses. Because many of the topics included in the former Algebra I course are in the CA CCSSM for grade eight, the new Algebra I and Mathematics I courses typically start in ninth grade with more advanced topics and include more in-depth work with linear functions and exponential functions and relationships, and they go beyond the previous high school standards for statistics. Mathematics I builds directly on the continuation of the CA CCSSM for grade eight and provides a seamless transition of content through an integrated curriculum.

Because of the rigor that has been added to the CA CCSSM for grade eight, course sequencing needs to be recalibrated to ensure students are able to master the additional content. Specifically, today's students, who are similar to those who previously may have been able to master an Algebra I course in grade eight, may find the new CA CCSSM for grade-eight content significantly more difficult. This transition to the CA CCSSM provides an opportunity to strengthen conceptual understanding by encouraging students—even strong mathematics students—to take the grade-eight CA CCSSM course instead of skipping ahead to Algebra I in grade eight.

Recalibrating the course placement process will require school-district personnel, including teachers, counselors, and instructional specialists, to rethink the information they use for assigning students to courses, particularly in middle school mathematics. Many variations may exist in the mathematics

sequence from grade six to grade eight. As the CA CCSSM are implemented during the next several years, steps need to be taken at the school-district and school-site levels to ensure that the sequence of courses guides students to CA CCSSM mastery by the end of grade eight.

Mathematics Course Design and Placement Under the CA CCSSM

Designing CA CCSSM–aligned mathematics courses in middle school requires careful planning to ensure that all content and practice standards are fully addressed. Some students may move through the standards more quickly than other students. Getting the pacing right will require implementation of new courses and analysis of students' progress. As noted previously, placing students in a course pathway for which they are not adequately prepared can have negative consequences. A recent longitudinal analysis based on California statewide assessment data revealed that students who fail the state exam for algebra in grade eight have a greater chance of repeating the course and failing the exam again in grade nine compared with their peers who pass the state exam for general mathematics in grade eight (Liang, Heckman, and Abedi 2012). Similarly, Finkelstein et al. (2012) reported that as many as 33 percent of students in a representative sample of California repeated algebra between grades seven and twelve (most often from grade eight to grade nine), and most of those students did not improve their demonstrated mastery following the repeated course. In essence, under standards that were adopted prior to the more rigorous CA CCSSM, California's eighth-graders who were underprepared for algebra were still underprepared in ninth grade.

In light of these findings, school systems across the nation and in California are revising the criteria used to determine mathematics placement and the different weights assigned to each criterion. Most districts typically rely on teacher recommendations and course grades to determine course placement (Bitter and O'Day 2010, 6), with standardized mathematics test scores, student or parent preferences, and counselor recommendations considered as additional factors in the decision (Hallinan 2003). As Hallinan (1994) notes, "[s]chools vary in the constellation of factors on which they rely to assign students to tracks and in the weight they attach to each factor" (Hallinan 1994, 80). Similarly, Oakes, Muir, and Joseph (2000) note, "Increasingly, school systems do not use fixed criteria to assign students to particular course levels" (Oakes, Muir, and Joseph 2000, 16). Rather, teacher and counselor placement recommendations are used; these include subjective judgments about "students' personalities, behavior, and motivation" as well as test-score performance (Oakes, Muir, and Joseph 2000, 16).

Research has also shown discrepancies in the placement of students in "advanced" classes by race, ethnicity, or socioeconomic background. Although the decision to accelerate is almost always a joint decision between the school and the family, serious efforts must be made to consider solid, objective evidence of student learning in order to avoid unwittingly depriving particular groups of students of opportunities. Among the considerations is the need to assess near-term mathematics readiness with the student's longer-term prospects for mastering advanced mathematics content. The consideration for school districts is: When, and under what circumstances, will placing students in the grade-eight CA CCSSM course transfer to greater mathematics understanding throughout high school?

In developing a policy on course sequences and student placement at the district level, districts may also turn to guidance from other education agencies. For example, as described in Appendix A of the national Common Core State Standards for Mathematics document (National Governors Association

Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO 2010a, 81]), the Achieve Pathways Group developed guidelines on how placement decisions and course sequences should be evaluated:

1. **Compacted courses should include the same Common Core State Standards as the non-compacted courses.** "Learning the mathematics prescribed by CA CCSSM requires that all students, including those most accomplished in mathematics, rise to the challenge by spending the time to learn each topic with diligence and dedication. Skimming over existing materials in order to rush ahead to more advanced topics will no longer be considered good practice" (Wu 2012). When accelerated pathways are considered, it is recommended that three years of material be compacted into two years, rather than compacting two years into one. The rationale is that mathematical concepts are likely to be omitted when two years of material are squeezed into one. This practice is to be avoided, as the standards have been carefully developed to define clear learning progressions through the major mathematical domains. Moreover, the compacted courses should not sacrifice attention to the Standards for Mathematical Practice.

2. **Decisions to accelerate students into the Common Core State Standards for higher mathematics before ninth grade should not be rushed.** Premature placement of students into an accelerated pathway should be avoided at all costs. In order to ensure that students are developmentally ready for accelerated content, it is not recommended to compact the standards before grade seven. In Appendix A of the national Common Core State Standards for Mathematics document (NGA/CCSSO 2010a, 81]), it is understood that compaction begins in seventh grade for both the traditional and integrated sequences.

3. **Decisions to accelerate students into higher mathematics before ninth grade should be based on solid evidence of student learning.** "Mathematics is by nature hierarchical. Every step is a preparation for the next one. Learning it properly requires thorough grounding at each step, and skimming over any topics will only weaken one's ability to tackle more complex material down the road" (Wu 2012). Before a student is placed on an accelerated pathway, serious efforts must be made to consider solid evidence of the student's conceptual understanding, knowledge of procedural skills, fluency, and ability to apply mathematics.

4. **A menu of challenging options should be available for students after their third year of mathematics—and all students should be strongly encouraged to take mathematics in all years of high school.** Traditionally, students taking higher mathematics in the eighth grade are expected to take Precalculus in their junior year and then Calculus in their senior year. This is a good and worthy goal, but it should not be the only option for students. Advanced courses may also include Statistics, Discrete Mathematics, or Mathematical Decision Making via mathematical modeling. An array of challenging options will keep mathematics relevant for students and give them a new set of tools for their future in college and careers.

Students Who May Be Ready for Acceleration

Although the CA CCSSM are more rigorous than California's previous standards for mathematics, there will still be some students who are able to move through the mathematics quickly. Those students may choose to take an accelerated or enhanced mathematics program beginning in eighth grade (or even earlier) so they can take college-level mathematics in high school. However, the previous course sequences for acceleration will need to be updated because of the increased rigor of the CA CCSSM. Students who are capable of moving more quickly deserve thoughtful attention, both to ensure that they are challenged and that they master the full range of mathematical content and skills—without omitting critical concepts and topics. Care must be taken to ensure that students fully understand all important topics in the mathematics curriculum, and that the continuity of the mathematics learning progression is not disrupted. There should be a variety of opportunities for students to advance to mathematics courses beyond those included in this publication (NGA/CCSSO 2010a).

Maintaining motivation and engagement in advanced mathematics is essential for some students who enjoy work in mathematics and excel in mathematics and consequently in school. Slowing down instruction or restricting access to accelerated sequences may discourage and disengage some students from their progress in math, and potentially other courses as well. Therefore, some students may look forward to Advanced Placement (AP) Calculus or Multivariate Calculus as real options for their senior year of high school. For high schools that do not offer these courses on a regular basis, concurrent enrollment in local colleges and universities may provide some students with an alternative to high school courses.

Districts are encouraged to work with mathematics leadership, teachers, parents, and curriculum coordinators to design pathways that best meet the needs of students. Enrichment opportunities should allow students to increase their depth of understanding by developing expertise in the modeling process and applying mathematics to novel and complex contexts (MDESE 2012).

In the CA CCSSM, students begin preparing for algebra in kindergarten, as they start learning about the properties of operations. Furthermore, much of the content central to Algebra I courses of the past—namely, linear equations, inequalities, and functions—is now found in the grade-eight CA CCSSM. Mastery of the algebra content, including the Standards for Mathematical Practice, is fundamental for success in further mathematics and on college entrance examinations. Skipping over material to get students to a particular point in the curriculum will create gaps in the students' mathematical background. To accelerate, students must prove that they are proficient in the CA CCSSM for kindergarten through grade eight (NGA/CCSSO 2010a).

It is essential that multiple measures are used to determine a student's readiness for acceleration. Districts should create a system for gathering evidence of a student's readiness for an accelerated pathway. Placement assessments that include constructed responses should be used to determine a student's conceptual understanding. The assessments should incorporate performance items that address multiple domains. Additionally, the assessments should measure a student's ability to demonstrate the skills included in the Standards for Mathematical Practice. Many schools and districts in California use commercially produced assessments; however, others use valid and reliable exams created by districts themselves. A portfolio of student work may be collected as evidence of readiness, in addition to student grade reports and assessment data from their previous mathematics courses.

One example of a widely available cognitive diagnostic assessment is the Mathematics Diagnostic Testing Project (MDTP), a statewide effort involving the California State University, the University of California, California Community Colleges, and California K–12 mathematics teachers to develop readiness tests and constructed-response materials. The MDTP provides students and teachers with diagnostic information about student readiness for a broad range of math courses from Prealgebra through Calculus. This information can help students identify specific areas where additional study or review is needed and can help teachers identify topics and skills that need more attention in courses. The MDTP readiness tests can be administered online, and the results are immediately available after test completion. In addition to using MDTP test results formatively to adapt instruction, some districts use the test results to assist with course placement decisions.

Examples of Accelerated Middle School Pathways

If the precautions noted above are considered, a middle school acceleration pathway could compact grade seven, grade eight, and Algebra I or Mathematics I in middle school. The term *compacted* means to compress content, which requires a faster pace to complete; it does not involve skipping content. To prepare eighth-grade students for higher mathematics, districts are encouraged to have a well-crafted sequence of compacted courses. The Achieve Pathways Group has provided "compacted" pathways in which the standards from grade seven, grade eight, and the Algebra I or Mathematics I course could be compressed into an accelerated pathway for students in grades seven and eight, allowing students to enter the Geometry (or Mathematics II) course in grade nine. Details of this "Compacted Pathway" example can be found in Appendix A of the national Common Core State Standards for Mathematics document (NGA/CCSSO 2010a). The appendix is posted at http://www.corestandards.org/Math/ (accessed September 30, 2015).

Examples of Accelerated High School Pathways

Because of the importance of middle school mathematics, districts may choose to offer high school acceleration options instead of, or in addition to, an accelerated pathway that begins in middle school. Some students may not have the necessary preparation to enter a Compacted Pathway but may still develop an interest in taking advanced mathematics, such as AP Calculus or AP Statistics in their senior year. Districts are encouraged to work with mathematics leadership, teachers, and curriculum coordinators to design pathways that best meet the abilities and needs of students. For students who study the eighth-grade standards in grade eight, there are pathways that will lead them to advanced mathematics courses in high school (e.g., Calculus). As shown with the numbered list that follows, compressed and accelerated pathways for high school students may follow a range of models. Note that the accelerated high school pathways delay decisions about which students to accelerate while still allowing access to advanced mathematics in grade twelve (MDESE 2012); see the course-sequence illustrations at the end of this appendix.

1. Students could "double up" by enrolling in the Geometry course during the same year that they take Algebra I or Algebra II.
2. Allow students in schools with block scheduling to take a mathematics course in both semesters of the same academic year.

3. Offer summer courses that are designed to provide the equivalent experience of a full course in all aspects, including attention to the Standards for Mathematical Practice.[2]
4. Create different compaction ratios, compressing four years of high school content into three years, beginning in ninth grade.
5. Create a hybrid Algebra II/Precalculus or Mathematics III/Precalculus course that allows students to go straight to Calculus in grade twelve (see the Enhanced Pathway).
6. Standards that focus on a sub-topic such as trigonometry or statistics could be pulled out and taken alongside the traditional or integrated courses so that students would only need to "double up" for one semester.
7. Standards from Mathematics I, Mathematics II, and Mathematics III courses could be compressed into an accelerated pathway for students for two years, allowing students to enter the Precalculus course in the third year.

A combination of these methods and the suggested compacted sequences in Appendix A of the national Common Core State Standards for Mathematics (NGA/CCSSO 2010a) would allow for the most mathematically inclined students to take advanced mathematics courses during high school.

Students Who May Need Additional Support

It is expected that students across the state will find the CA CCSSM challenging at all grade levels. For students who needed additional support to meet previously adopted mathematics standards, the CA CCSSM will likely provide even greater teaching and learning challenges. A common structural solution in California's public schools has been to encourage students to repeat courses where they have not demonstrated mastery of content. This has been done frequently between eighth and ninth grade, when concerns about the mastery of pre-algebraic and algebraic content have arisen. Under the CA CCSSM, it is intended that course repetition be reduced. An alternative is to rethink the content of existing courses in grades six, seven, and eight. Alignment with earlier grade levels in elementary school is essential, as is the need to examine how mathematics standards from early grade levels are mastered.

Some school districts in California have developed course structures that allow mathematics content to be reinforced over multiple years through *expansion*—the opposite of compaction. Under the CA CCSSM, it is possible that this approach will be helpful, particularly with the assistance of formative testing under the Smarter Balanced Assessment Consortium and other diagnostic testing. Districts should consider how scheduling within the school day, within the school year, and across school years might facilitate increased mastery on the *combined* CA CCSSM from grades six through eight.

Support for K–12 Teachers

The increased rigor of the CA CCSSM and the demands of fully addressing the MP standards create additional opportunities and challenges for California's K–12 teachers. Accelerating students who are prepared for advanced course work will add a new layer to this set of challenges. Students who follow a compacted pathway undertake advanced work at an accelerated pace. This creates a great challenge

2. As with other methods of accelerating students, enrolling students in summer courses should be handled with care, as the pace of the courses will likely be fast.

for those students as well as their teachers, who will teach eighth-grade standards and Algebra I or Mathematics I standards that are significantly more rigorous than in the past and within a compressed timeframe. Teachers must be prepared not only to address new and more challenging content, but they will also need to build on their repertoire of acceleration strategies. Teacher-preparation programs must respond to this call for additional training and support for teachers. Support and professional learning for experienced teachers should be provided by school districts, county offices of education, and the California Mathematics Project.

Acceleration

Figures D-1 through D-5 show some possibilities for accelerating students so that they may take an Advanced Placement mathematics course in grade twelve (e.g., Calculus or Statistics and Probability). Readers should note that the decision points for acceleration vary with each plan; some decision points occur as early as grade six. These are merely examples; there are other ways to accelerate students. The important thing to keep in mind is to avoid skipping any content.

Figure D-1. No Acceleration

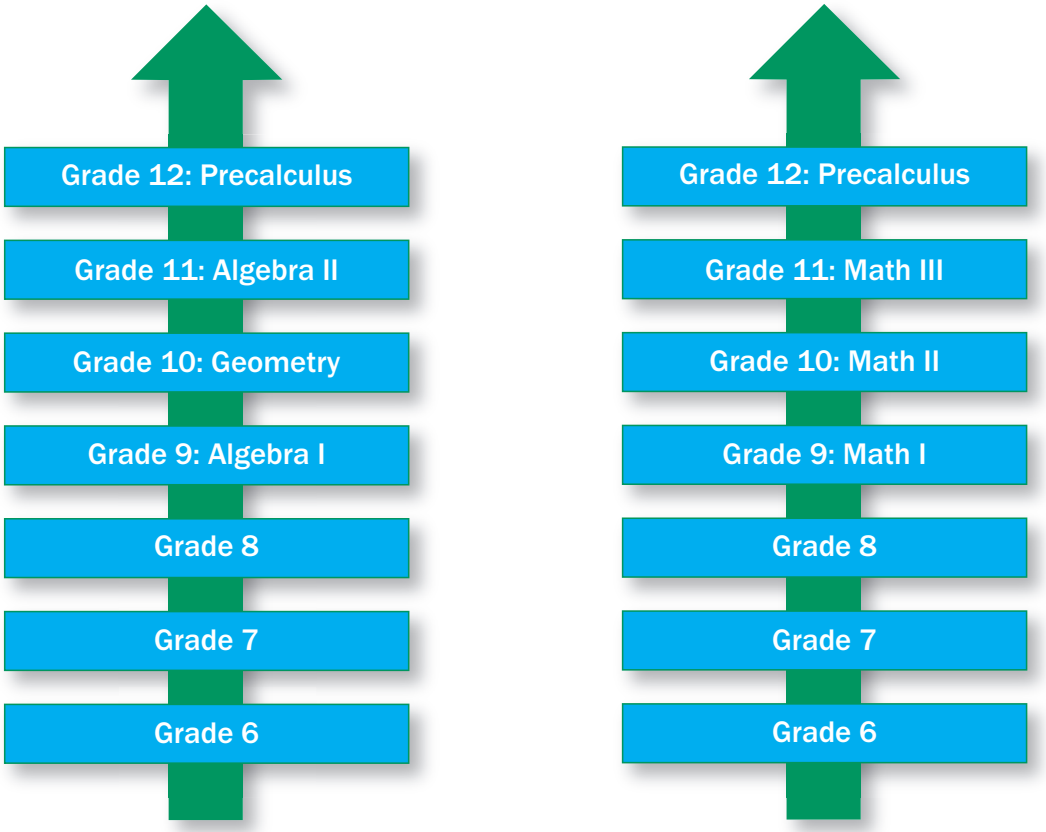


Figure D-2. Acceleration in Middle School

The decision point is at the end of grade six.

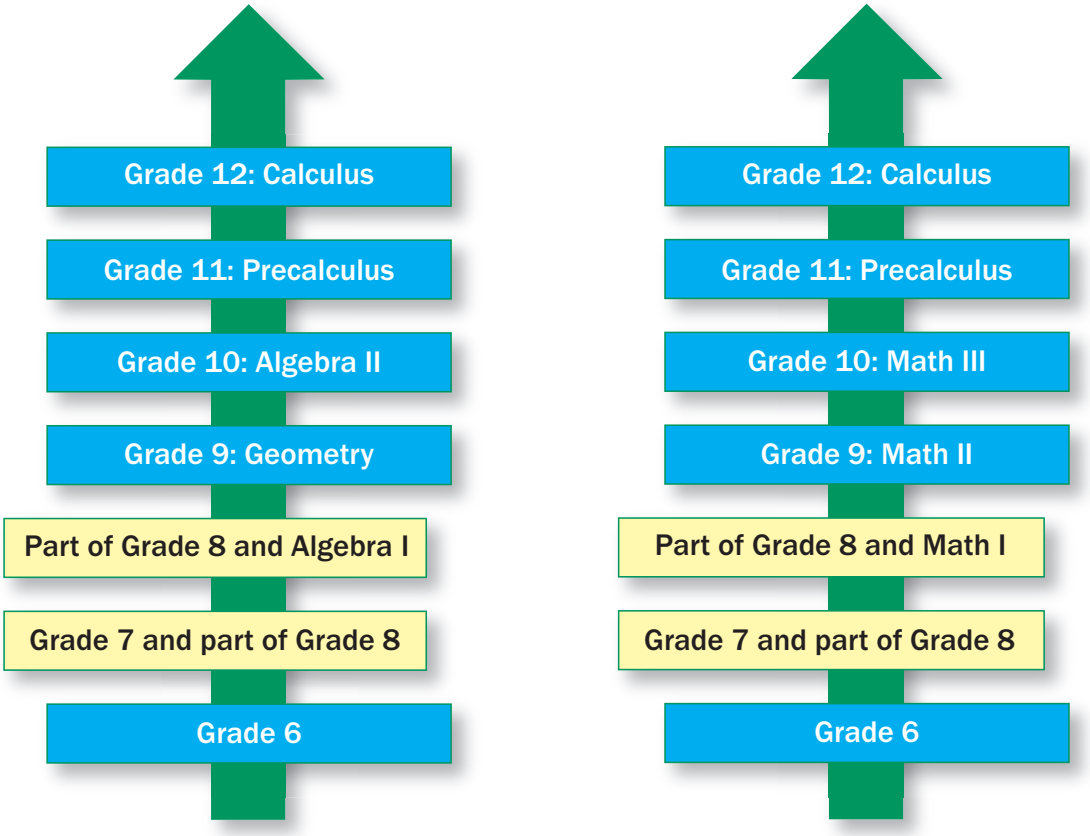


Figure D-3. Acceleration in High School

The decision point is at the end of grade eight.

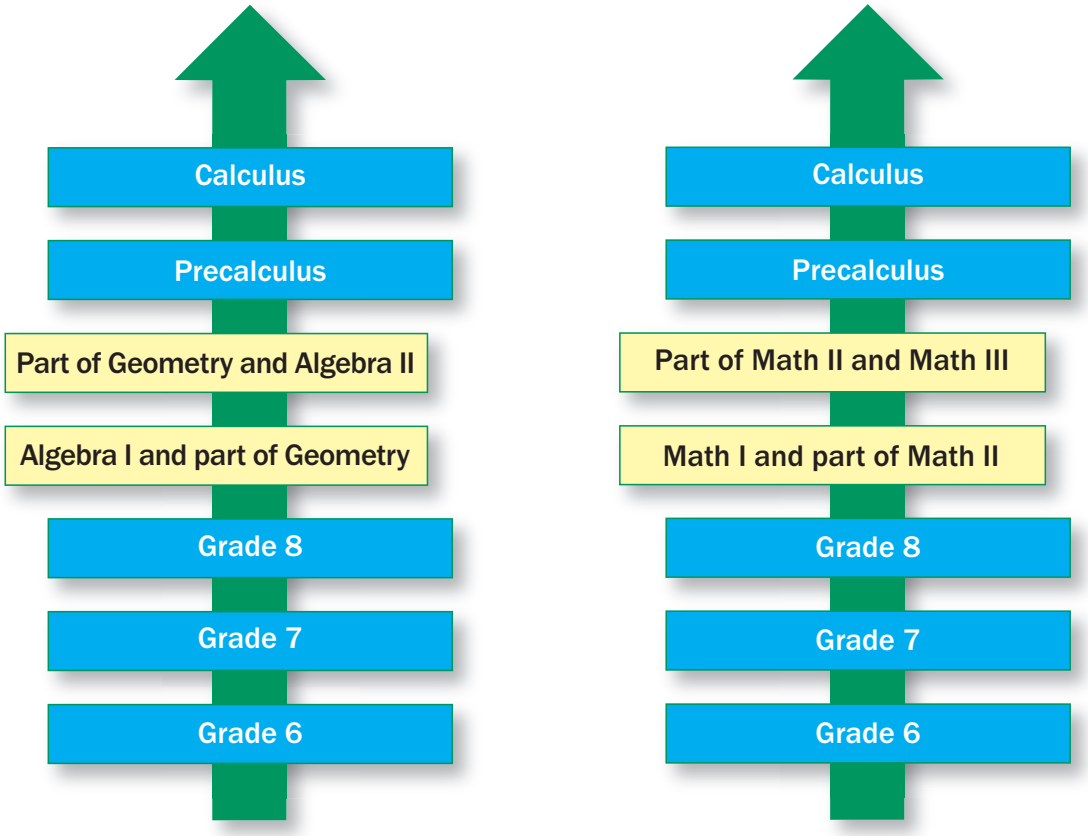
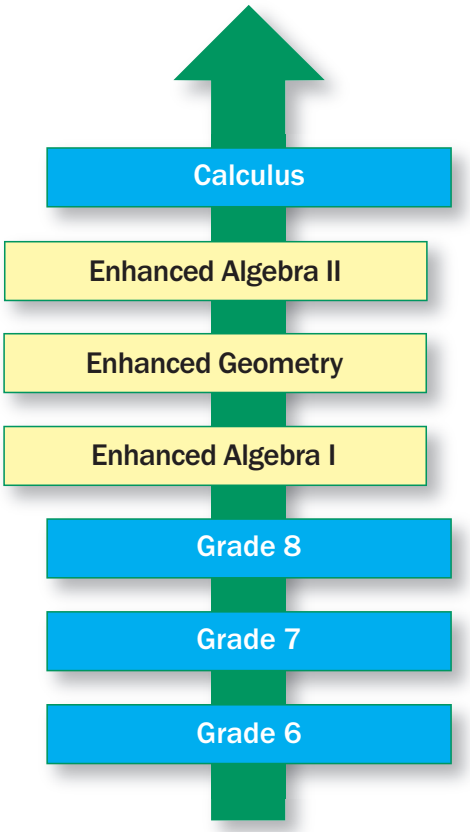
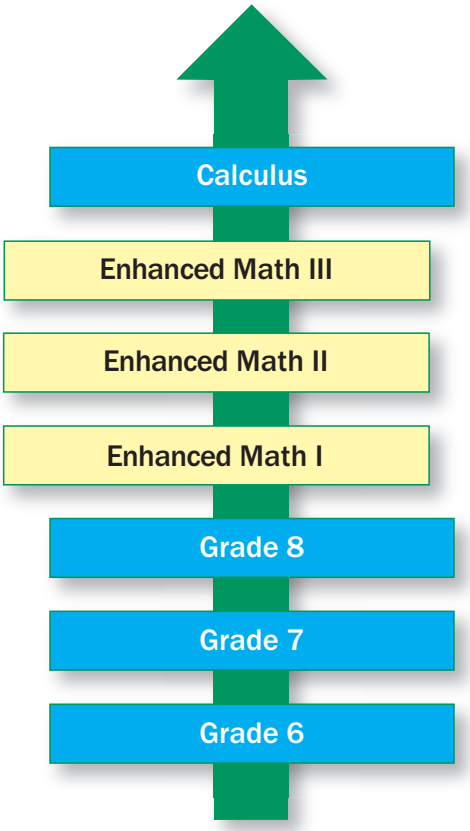


Figure D-4. Enhanced High School Sequence

The decision point is at the end of grade eight.



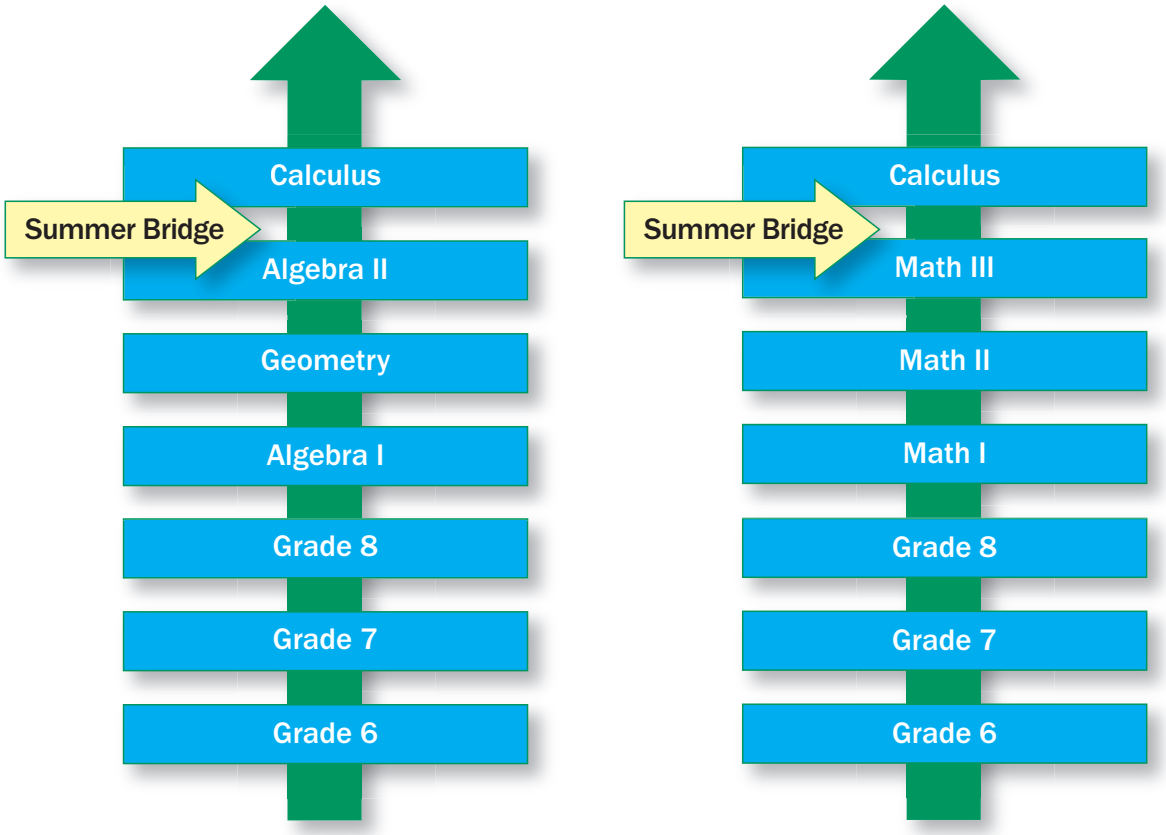
Traditional Pathway



Integrated Pathway

Figure D-5. Summer Bridge Sequence

The decision point is at the end of grade eleven.



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Appendix E

Possible Adaptations for Students with Learning Difficulties in Mathematics

This appendix presents suggested adaptations that teachers can use when planning instruction for students who have learning difficulties in mathematics. For additional information on meeting the instructional needs of all students, see the Universal Access chapter. It should be noted that individualized education programs (IEPs)—for students with disabilities who receive special education services—provide specific guidance for supplementary aids and services (including accommodations, modifications, and assistive technology) that are tailored to address the unique needs of each student.

Possible Adaptations for Students with Visual and Auditory Difficulties

- The student is located close to where the teacher provides instruction and is able to receive peer assistance.
- The student's desk is free of distractions.
- Visual cues are provided on the wall.
- The teacher previews the content and makes key concepts explicit to students with review and frequent checks for understanding.
- Students are provided with study guides.
- The teacher uses consistent routines.
- When presenting material, the teacher utilizes a moderate tone of voice, clearly enunciates words, and often repeats the lesson's key ideas.
- The teacher decreases visual complexity by presenting one key idea or problem at a time on the overhead or projector screen. Similarly, templates are used to block out all of the problems on a worksheet page except for the one that the student is completing.
- To maintain respect for student and teacher ideas, there is a rule in the classroom that only one person may talk at any given time. This helps students to remain focused.
- The teacher uses methods of organizing written assignments (e.g., completing computations on centimeter grid paper). Templates are drawn for traditional algorithms.
- The teacher uses concrete models instead of pictures.
- The student is provided with audio or video lessons (or both).
- To assist with reading assignments and problems, the student is provided with access to text according to his or her preferences (e.g., through peer assistance, specialized software and computer access, audio recordings, and so forth).

- Reading tasks are shortened.
- Frequent connections are made between what is happening in class and real-life situations outside of class.

Possible Adaptations for Students with Memory Difficulties

- The teacher provides only one instruction at a time.
- After giving instructions, the teacher asks students to repeat the instructions in their own words. The teacher also writes the instructions on the board.
- The teacher provides frequent reviews (distributed practice).
- Students are able to use calculators.
- Additional time is provided for students to complete assignments and assessments.
- Assignments and a calendar of due dates are available electronically (e.g., on the teacher's Web site).

Possible Adaptations for Students with Integrative Difficulties (e.g., abstract thinking and conceptualization)

- Teachers utilize concrete models and multimedia for an extended period of time.
- Students communicate what they are doing through words, pictures, and numbers.
- Students are encouraged to justify their thinking.
- New conceptual ideas are repeated and practiced.
- Students are encouraged to re-state word problems in their own words.
- Students are provided with opportunities to teach concepts to each other.
- An abstract concept is represented in a variety of ways—for example, through concrete examples, words, symbols, and drawings, or by acting it out.
- Students are placed in heterogeneous groups for peer assistance and modeling (Vaughn, Bos, and Schumm 2010, 168; Hoover et al. 2008; Van de Walle 2007).
- The teacher scaffolds open-ended inquiries.

Possible Adaptations for Students with Attention Deficit Hyperactivity Disorder (ADHD)

- Novelty in instruction and directions is supported by students who highlight important instructions and key points. For example, students may highlight the operations signs on a math page.
- Classroom schedules and routines are well established and maintained.
- Students are prepared ahead of time for transitions and are provided with support in completing transitions.
- The teacher emphasizes time limits for completing assignments.
- Positive feedback about students' performance and behavior is provided consistently and often.

- Teacher instructions are brief and clear.
- Assignments and classwork allow for movement and postures other than sitting.
- The classroom environment is arranged to facilitate attention and minimize distractions.
- The teacher uses effective questions to promote active participation by all students and to develop the students' critical-thinking skills.
- Students are given the option to complete fewer problems than are assigned. When using this adaptation, teachers must be careful not to lower their expectations or standards for students.
- Multiple forms of assessment are utilized to determine student learning (Vaughn, Bos, and Schumm 2010, 168; Hoover et al. 2008; Van de Walle 2007).

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Appendix F

Higher Mathematics Pathways Standards Table

	Traditional Pathway			Integrated Pathway		
	Algebra I	Geometry	Algebra II	Mathematics I	Mathematics II	Mathematics III
Number and Quantity						
The Real Number System	N-RN.1 N-RN.2 N-RN.3				N-RN.1 N-RN.2 N-RN.3	
The Complex Number System			N-CN.1 N-CN.2 N-CN.7 N-CN.8 N-CN.9		N-CN.1 N-CN.2 N-CN.7 N-CN.8 N-CN.9	N-CN.8 N-CN.9
Quantities	N-Q.1 N-Q.2 N-Q.3			N-Q.1 N-Q.2 N-Q.3		
Algebra						
Seeing Structure in Expressions	A-SSE.1a A-SSE.1b A-SSE.2 A-SSE.3a A-SSE.3b A-SSE.3c		A-SSE.1a A-SSE.1b A-SSE.2 A-SSE.4	A-SSE.1a A-SSE.1b	A-SSE.1a A-SSE.1b A-SSE.2 A-SSE.3a A-SSE.3b A-SSE.3c	A-SSE.1a A-SSE.1b A-SSE.2 A-SSE.4
Arithmetic with Polynomials and Rational Expressions	A-APR.1		A-APR.1 A-APR.2 A-APR.3 A-APR.4 A-APR.5 A-APR.6 A-APR.7		A-APR.1	A-APR.1 A-APR.2 A-APR.3 A-APR.4 A-APR.5 A-APR.6 A-APR.7
Creating Equations	A-CED.1 A-CED.2 A-CED.3 A-CED.4		A-CED.1 A-CED.2 A-CED.3 A-CED.4	A-CED.1 A-CED.2 A-CED.3 A-CED.4	A-CED.1 A-CED.2 A-CED.4	A-CED.1 A-CED.2 A-CED.3 A-CED.4
Reasoning with Equations and Inequalities	A-REI.1 A-REI.3 **A-REI.3.1** A-REI.4a A-REI.4b A-REI.5 A-REI.6 A-REI.7 A-REI.10 A-REI.11 A-REI.12		A-REI.2 **A-REI.3.1** A-REI.11	A-REI.1 A-REI.3 **A-REI.3.1** A-REI.5 A-REI.6 A-REI.10 A-REI.11 A-REI.12	A-REI.4a A-REI.4b A-REI.7	A-REI.2 A-REI.11

Higher Mathematics Pathways Standards Table (*continued*)

	Traditional Pathway			Integrated Pathway		
	Algebra I	Geometry	Algebra II	Mathematics I	Mathematics II	Mathematics III
Functions						
Interpreting Functions	F-IF.1 F-IF.2 F-IF.3 F-IF.4 F-IF.5 F-IF.6 F-IF.7a F-IF.7b F-IF.7e F-IF.8a F-IF.8b F-IF.9		F-IF.4 F-IF.5 F-IF.6 F-IF.7b F-IF.7c F-IF.7e F-IF.8 F-IF.9	F-IF.1 F-IF.2 F-IF.3 F-IF.4 F-IF.5 F-IF.6 F-IF.7a F-IF.7e F-IF.9	F-IF.4 F-IF.5 F-IF.6 F-IF.7a F-IF.7b F-IF.8a F-IF.8b F-IF.9	F-IF.4 F-IF.5 F-IF.6 F-IF.7b F-IF.7c F-IF.7e F-IF.8 F-IF.9
Building Functions	F-BF.1a F-BF.1b F-BF.2 F-BF.3 F-BF.4a		F-BF.1b F-BF.3 F-BF.4a	F-BF.1a F-BF.1b F-BF.2 F-BF.3	F-BF.1a F-BF.1b F-BF.3 F-BF.4a	F-BF.1b F-BF.3 F-BF.4a
Linear, Quadratic, and Exponential Models	F-LE.1a F-LE.1b F-LE.1c F-LE.2 F-LE.3 F-LE.5 **F-LE.6**		F-LE.4 **F-LE.4.1** **F-LE.4.2** **F-LE.4.3**	F-LE.1a F-LE.1b F-LE.1c F-LE.2 F-LE.3 F-LE.5	F-LE.3 **F-LE.6**	F-LE.4 **F-LE.4.1** **F-LE.4.2** **F-LE.4.3**
Trigonometric Functions			F-TF.1 F-TF.2 **F-TF.2.1** F-TF.5 F-TF.8		F-TF.8	F-TF.1 F-TF.2 **F-TF.2.1** F-TF.5
Geometry						
Congruence		G-CO.1 G-CO.2 G-CO.3 G-CO.4 G-CO.5 G-CO.6 G-CO.7 G-CO.8 G-CO.9 G-CO.10 G-CO.11 G-CO.12 G-CO.13		G-CO.1 G-CO.2 G-CO.3 G-CO.4 G-CO.5 G-CO.6 G-CO.7 G-CO.8 G-CO.12 G-CO.13	G-CO.9 G-CO.10 G-CO.11	

Higher Mathematics Pathways Standards Table (*continued*)

	Traditional Pathway			Integrated Pathway		
	Algebra I	Geometry	Algebra II	Mathematics I	Mathematics II	Mathematics III
Similarity, Right Triangles, and Trigonometry		G-SRT.1a G-SRT.1b G-SRT.2 G-SRT.3 G-SRT.4 G-SRT.5 G-SRT.6 G-SRT.7 G-SRT.8 **G-SRT.8.1** G-SRT.9 G-SRT.10 G-SRT.11			G-SRT.1a G-SRT.1b G-SRT.2 G-SRT.3 G-SRT.4 G-SRT.5 G-SRT.6 G-SRT.7 G-SRT.8 **G-SRT.8.1**	G-SRT.9 G-SRT.10 G-SRT.11
Circles		G-C.1 G-C.2 G-C.3 G-C.4 G-C.5			G-C.1 G-C.2 G-C.3 G-C.4 G-C.5	
Expressing Geometric Properties with Equations		G-GPE.1 G-GPE.2 G-GPE.4 G-GPE.5 G-GPE.6 G-GPE.7	**G-GPE.3.1**	G-GPE.4 G-GPE.5 G-GPE.7	G-GPE.1 G-GPE.2 G-GPE.4 G-GPE.6	**G-GPE.3.1**
Geometric Measurement and Dimension		G-GMD.1 G-GMD.3 G-GMD.4 **G-GMD.5** **G-GMD.6**			G-GMD.1 G-GMD.3 **G-GMD.5** **G-GMD.6**	G-GMD.4
Modeling with Geometry		G-MG.1 G-MG.2 G-MG.3				G-MG.1 G-MG.2 G-MG.3
Statistics and Probability						
Interpreting Categorical and Quantitative Data	S-ID.1 S-ID.2 S-ID.3 S-ID.5 S-ID.6a S-ID.6b S-ID.6c S-ID.7 S-ID.8 S-ID.9		S-ID.4	S-ID.1 S-ID.2 S-ID.3 S-ID.5 S-ID.6a S-ID.6b S-ID.6c S-ID.7 S-ID.8 S-ID.9		S-ID.4

Higher Mathematics Pathways Standards Table (*continued*)

	Traditional Pathway			Integrated Pathway		
	Algebra I	Geometry	Algebra II	Mathematics I	Mathematics II	Mathematics III
Making Inferences and Justifying Conclusions			S-IC.1 S-IC.2 S-IC.3 S-IC.4 S-IC.5 S-IC.6			S-IC.1 S-IC.2 S-IC.3 S-IC.4 S-IC.5 S-IC.6
Conditional Probability and the Rules of Probability		S-CP.1 S-CP.2 S-CP.3 S-CP.4 S-CP.5 S-CP.6 S-CP.7 S-CP.8 S-CP.9			S-CP.1 S-CP.2 S-CP.3 S-CP.4 S-CP.5 S-CP.6 S-CP.7 S-CP.8 S-CP.9	
Using Probability to Make Decisions		S-MD.6 S-MD.7	S-MD.6 S-MD.7		S-MD.6 S-MD.7	S-MD.6 S-MD.7

Glossary: Mathematical Terms, Tables, and Illustrations

This glossary was adapted from the *Massachusetts Curriculum Framework for Mathematics: Grades Pre-Kindergarten to 12* (March 2011). Excerpts from the Massachusetts curriculum framework are included by permission of the Massachusetts Department of Elementary and Secondary Education. The complete and current version of each Massachusetts curriculum framework is available at http://www.doe.mass.edu/frameworks/current.html (accessed May 15, 2014).

The glossary also includes terms defined in the Common Core State Standards Initiative's Mathematics Glossary (available at http://www.corestandards.org/ [accessed May 15, 2014]), as well as many additional terms.

AA similarity (angle–angle similarity). When two angles of one triangle are congruent to two angles of a second triangle, the triangles are similar.

absolute value. The absolute value of a number x is the non-negative number that represents its distance from 0 on a number line. Equivalently, $|x| = x$ if $x \geq 0$, or $-x$ *if* $x < 0$.

addition and subtraction within 5, 10, 20, 100, or 1000. Addition or subtraction of two whole numbers with whole-number answers, and with sum or minuend in the range 0–5, 0–10, 0–20, or 0–100, respectively. *Example:* $8 + 2 = 10$ is an addition within 10; $14 - 5 = 9$ is a subtraction within 20; and $55 - 18 = 37$ is a subtraction within 100.

additive identity property of 0. See table GL-1 in this glossary.

additive inverses. Two numbers whose sum is 0 are additive inverses of one another.

Example: $\frac{3}{4}$ and $-\frac{3}{4}$ are additive inverses of one another because $\frac{3}{4} + \left(-\frac{3}{4}\right) = \left(-\frac{3}{4}\right) + \frac{3}{4} = 0$

Source: National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA/CCSSO) 2010c.

algorithm. A set of predefined steps applicable to a class of problems that give the correct result in every case when the steps are carried out correctly.

analog. Having to do with data represented by continuous variables—for example, a clock with hour, minute, and second hands (Merriam-Webster 2013).

analytic geometry. The branch of mathematics that uses functions and relations to study geometric phenomena (e.g., the description of ellipses and other conic sections in the coordinate plane by quadratic equations).

argument of a complex number. The angle θ when a complex number is represented in polar form, as in $r(\cos\theta + i\sin\theta)$.

ASA congruence (angle–side–angle congruence). When two triangles have corresponding angles and the included side that are congruent, the triangles themselves are congruent (Mathwords 2013).

associative property of addition. See table GL-1 in this glossary.

associative property of multiplication. See table GL-1 in this glossary.

assumption. A fact or statement (as a proposition, axiom, postulate, or notion) accepted as true.

attribute. A common feature of a set of figures.

benchmark fraction. A common fraction against which other fractions can be measured, such as $\frac{1}{2}$.

binomial theorem. The theorem that gives the polynomial expansion of each whole-number power of a binomial.

bivariate data. Pairs of linked numerical observations. An example is a list of the height and weight for each player on a football team.

box plot. A graphic that shows the distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50 percent of the data. (*Source:* Wisconsin Department of Public Instruction [WDPI] 2013.)

calculus. The mathematics of change and motion. The main concepts of calculus are limits, instantaneous rates of change, and areas enclosed by curves.

capacity. The maximum amount or number that can be contained or accommodated. *Examples:* The jug has a one-gallon *capacity*; the auditorium was filled to *capacity*.

cardinal number. A number (as 1, 5, 15) that is used in simple counting and that tells how many elements there are in a set but not the order in which they are arranged. Compare with **ordinal number**.

Cartesian plane. A coordinate plane with perpendicular coordinate axes.

causation. The act of causing or inducing. If one action causes another, then the actions are certainly correlated; however, just because two things occur together does not mean that one caused the other (STATS 2013). See also **correlation**.

Cavalieri's principle. If, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, the volumes of the two solids are equal (Kern and Bland 1948).

coefficient. The numerical factor in a product. *Example:* In the term $3ab$, 3 is the coefficient of ab.

commutative property. See table GL-1 in this glossary.

complex fraction. A fraction $\frac{A}{B}$ where A and/or B are fractions (B non-zero).

complex number. A number that can be written in the form $a+bi$, where a and b are real numbers and $i^2=-1$.

complex plane. A Cartesian plane in which the point (a, b) represents the complex number $a+bi$.

compose numbers. To form a new number by "putting together" other numbers, paying special attention to the number 10. *Example:* 1 ten and 6 ones compose the number 16, or 10 + 6 = 16. See also **decompose numbers**.

compose shapes. To join geometric shapes without overlaps and form other shapes.

composite number. A whole number that has more than two distinct positive factors (Harcourt School Publishers 2013).

computation algorithm. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also **algorithm** and **computation strategy**.

computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also **computation algorithm**.

congruent. Two plane or solid figures are congruent if one can be obtained from the other by a rigid motion (i.e., a sequence of rotations, reflections, and translations).

conjugate. The result of writing a sum of two terms as a difference, or vice versa. *Example:* The conjugate of $x-2$ is $x+2$. (*Source:* Mathwords 2013.)

constant of proportionality. The constant k in the equation $y=kx$ that shows that y is directly proportional to x. The unit rate associated with a ratio is an example of a constant of proportionality. See also **proportional relationship**.

coordinate plane. A plane in which points are designated using two coordinates. In the Cartesian or rectangular coordinate plane, the two coordinates correspond to numbers on two perpendicular numbers lines, called *axes*, which intersect at the zero of each axis.

correlation. A measure of the amount of positive or negative relationship existing between two measures. *Example:* If the height and weight of a set of individuals were measured, it could be said that there is a positive correlation between height and weight if the data showed that larger weights tended to be paired with larger heights and smaller weights tended to be paired with smaller heights. The stronger those tendencies, the closer the measure is to –1 or 1. See also **causation**. (*Source:* WDPI 2013.)

cosine. A trigonometric function that for an acute angle of a right triangle is the ratio between a leg adjacent to the angle and the hypotenuse.

counting number. A number used in counting objects (e.g., a number from the set 1, 2, 3, 4, 5, . . .). See figure GL-1 at the end of this glossary.

counting on. A strategy for finding the number of objects in a group without having to count every member of the group. *Example:* If a stack of books is known to have 8 books, and 3 more books are added to the top, it is not necessary to count the stack all over again; one can find the total by counting on—pointing to the top book and saying "Eight," following this with "nine, ten, eleven. There are eleven books now."

decimal expansion. The representation of a real number using base-ten notation (e.g., the decimal expansion of the number $\frac{1}{4}$ is 0.25).

decimal fraction. A fraction (such as $0.25=\frac{25}{100}$ or $0.025=\frac{25}{1000}$) or mixed number (such as $3.025=3\frac{25}{1000}$) in which the denominator is a power of 10. Decimal fractions are usually expressed in base-ten notation with a decimal point.

decimal number. Any real number expressed in base-ten notation, such as 2.673.

decompose numbers. To "break apart" a number and represent it as a sum or difference of two or more other numbers. *Example:* 16 = 10 + 6.

decompose shapes. Given a geometric shape, to identify geometric shapes that meet without overlap to form the given shape.

digit. (**a**) Any of the Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. (**b**) One of the elements that combine to form numbers in a system other than the decimal system.

digital. Having to do with data that are represented in the form of numerical digits; providing a readout in numerical digits (e.g., a *digital* watch).

dilation. A transformation that moves each point along the ray through the point emanating from a fixed center and multiplies distances from the center by a common scale factor (NGA/CCSSO 2010c); a transformation in which a figure is made proportionally larger or smaller.

directrix. A straight line the distance to which from any point of a conic section is in fixed ratio to the distance from the same point to the conic's focus.

discrete mathematics. The branch of mathematics that includes combinatorics, recursion, Boolean algebra, set theory, and graph theory.

dot plot. See **line plot**.

double number line diagram. A diagram in which two number lines subdivided in the same way are set one on top of the other with zeros lined up. Although the number lines are subdivided in the same way, the units in each may be different, which allows for the illustration of ratio relationships. Double number lines can also be constructed vertically.

expanded form. A multi-digit number is expressed in expanded form when it is written as a sum of single-digit multiples of powers of ten. *Example:* $643 = 600 + 40 + 3$.

expected value. For a random variable, the weighted average of its possible values, with weights given by their respective probabilities.

exponent. For positive integer values, the number that indicates how many times the base is used as a factor. *Example:* In $4^3 = 4 \times 4 \times 4 = 64$, the exponent is 3, indicating that 4 is repeated as a factor three times.

exponential function. An exponential function is a function of the form $y(x) = a \bullet b^x$, where $a \neq 0$ and either $0 < b < 1$, or $1 < b$. The variables do not have to be x and y. *Example:* $A = 3.2 \bullet (1.02)^t$ defines A as an exponential function of t.

expression. A mathematical phrase that combines operations, numbers, and/or variables—for example, $3^2 \div a$ (Harcourt School Publishers 2013).

extreme values of a polynomial. The graph of a polynomial of degree n has at most $n-1$ extreme values (local minima and/or maxima). The total number of extreme values could be $n-1$, $n-3$, $n-5$, and so forth. *Example:* A degree 9 polynomial could have 8, 6, 4, 2, or 0 extreme values. A degree 2 (quadratic) polynomial must have 1 extreme value. (*Source:* Mathwords 2013.)

Fibonacci sequence. The sequence of numbers beginning with 1, 1, in which each number that follows is the sum of the previous two numbers (e.g., 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 . . .).

first quartile. For a data set with median M, the first quartile is the median of the data values less than M. *Example:* For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the first quartile is 6.[1] See also **median**, **third quartile**, and **interquartile range**.

fluency.

- ***conceptual fluency.*** Being able to use the relevant ideas or procedures in a wide range of contexts (Smarter Balanced Assessment Consortium [Smarter Balanced] 2012a).
- ***contextual fluency.*** The ability to use certain facts and procedures with enough facility that using them does not slow down or derail the problem solver as he or she works on more complex problems (Smarter Balanced 2012a).
- ***fluency as a special case of assessing individual content standards.*** *Fluent*, according to the Common Core State Standards for Mathematics, means fast and accurate. The word *fluency* was used judiciously in the standards to mark the endpoints of progressions of learning that begin with solid underpinnings and then pass upward through stages of growing maturity. Assessing the full range of the standards means assessing fluency where it is called for in the standards. (*Source:* Smarter Balanced 2012a.)
- ***procedural fluency.*** Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately (NGA/CCSSO 2010c).

focus (**plural *foci***). One of the fixed points from which the distances to any point of a conic curve, such as an ellipse or parabola, are connected by a linear relation (Oxford Dictionaries 2013).

fraction. A number expressible in the form $\frac{a}{b}$ where a is a whole number and b is a positive whole number. (In the context of this publication, the word *fraction* always refers to a non-negative number.) See also **rational number**.

function. (**a**) A mathematical relation for which each element of the domain corresponds to exactly one element of the range. (**b**) A rule that assigns to every element of one set (the domain) exactly one element from another set (the co-domain).

function notation. A notation that describes a function. For a function f when x is a member of the domain, the symbol $f(x)$ denotes the corresponding member of the range (e.g., $f(x) = x + 3$).

fundamental theorem of algebra. A theorem which states that when using complex numbers, all polynomials can be factored into a product of linear terms. An alternative form of the theorem asserts that any polynomial of degree n has exactly n complex roots (counting multiplicity).

geometric sequence (**progression**). An ordered list of numbers that has a common ratio between consecutive terms (e.g., 2, 6, 18, 54, and so on) [Harcourt School Publishers 2013].

histogram. A special type of bar graph used to display the distribution of measurement data across a continuous range.

imaginary number. A complex number of the form bi, where $i = \sqrt{-1}$.

independently combined probability models. Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

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1. Many different methods for computing quartiles are in use. The method defined here is sometimes called the Moore and McCabe method. See Eric Langford, "Quartiles in Elementary Statistics," *Journal of Statistics Education* 14, no. 3 (2006).

initial value (of a function). (**a**) For a function f with domain of the interval $[a, b]$, the *initial value* of f is the value $f(a)$. If the domain of f is discrete, then the initial value of f is $f(n)$, where n is the smallest value in the domain of f (should such a smallest value exist). (**b**) For a function f with domain all real numbers, the initial value is also taken to mean $f(0)$.

integers (set of). The set of numbers that includes whole numbers and their opposites—for example, {. . . –3, –2, –1, 0, 1, 2, 3 . . .}.

interquartile range. A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. *Example:* For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the interquartile range is $15 - 6 = 9$. See also **first quartile**, **third quartile**.

inverse. See **additive inverses**, **multiplicative inverses**.

inverse function. Two functions, $y = h(x)$ and $x = g(y)$, are said to be inverses when $g(h(x)) = x$ and $h(g(y)) = y$. The function inverse to $f(x)$ is denoted $f^{-1}(x)$.

irrational number. A real number that cannot be expressed as a quotient of two integers (e.g., $\sqrt{2}$). A number is irrational only if it cannot be written as a repeating or terminating decimal.

law of cosines. An equation relating the cosine of an interior angle and the lengths of the sides of a triangle (Mathwords 2013).

law of sines. Equations relating the sines of the interior angles of a triangle and the corresponding opposite sides (Mathwords 2013).

linear association. Two variables have a linear association if a scatter plot of the data can be well approximated by a line. See **correlation**.

linear equation. Any equation that can be written in the form $Ax + By + C = 0$, where A and B are not both 0. The graph of such an equation is a line.

linear function. A function, f, which may be brought into the form $f(x) = mx + b$. *Example:* $f(t) = 2(t - 7)$ represents a linear function.

line of symmetry. A line that divides a figure into two congruent parts, so that the reflection of either part across the line maps precisely onto the other part.

line plot. A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. Also known as a *dot plot* (WDPI 2013).

logarithm. The exponent that indicates the power to which a base number is raised to produce a given number. *Example:* The logarithm of 100 to the base 10 is 2 (Merriam-Webster 2013).

logarithmic function. A function $f(x) = \log_b(x)$ which is inverse to the function $g(x) = b^x$.

matrix (plural *matrices*). A rectangular array of numbers or variables.

mean. A measure of central tendency in a set of numerical data, computed by adding the values in a list and then dividing by the number of values in the list.[2] *Example:* For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean is 21.

2. To be more precise, this defines the *arithmetic mean*.

mean absolute deviation. A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. *Example:* For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean absolute deviation is 19.9.

measure of variability. A determination of how much the performance of a group deviates from the mean or median. The most frequently used measure is standard deviation.

median. A measure of central tendency in a set of numerical data. The median of a list of values is the value appearing at the center of a sorted version of the list, or the mean of the two central values if the list contains an even number of values. *Example:* For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 90}, the median is 11.

midline. In the graph of a sine or cosine function, the horizontal line halfway between its maximum and minimum values.

minuend. A number from which another number is to be subtracted (Merriam-Webster 2013).

model. A mathematical representation (e.g., number, graph, matrix, equation[s], geometric figure) for real-world or mathematical objects, properties, actions, or relationships (WDPI 2013).

modulus of a complex number. The distance between a complex number and the origin on the complex plane. The modulus of $a+bi$ is written $|a+bi|$ and computed as $|a+bi|=\sqrt{a^2+b^2}$. For a complex number in polar form, $r(\cos\theta+i\sin\theta)$, the modulus is $|r|$.

multiplication and division within 100. Multiplication or division of two whole numbers with whole-number answers, and with product or dividend in the range 0–100. *Examples:* $4\times21=84$ and $72\div8=9$.

multiplicative inverses. Two numbers whose product is 1 are multiplicative inverses of one another. *Example:* $\frac{3}{4}$ and $\frac{4}{3}$ are multiplicative inverses of one another because $\frac{3}{4}\times\frac{4}{3}=\frac{4}{3}\times\frac{3}{4}=1$.
Source: NGA/CCSSO 2010c.

network. (**a**) A figure consisting of vertices and edges that shows how objects are connected. (**b**) A collection of points (vertices) with certain connections (edges) between them.

non-linear association. The relationship between two variables is non-linear if the value of each variable changes with the value of the other, but the change in one is not simply proportional to the change in the other variable.

number line diagram. A diagram of the number line used to represent numbers and support reasoning about them. In a number line diagram for measurement quantities, the interval from 0 to 1 on the diagram represents the unit of measure for the quantity.

numeral. A symbol or mark used to represent a number. For example, 16, 3, and IV are all numerals.

order of operations. Convention adopted to perform mathematical operations in a consistent order: (1) perform all operations inside parentheses, brackets, and/or above and below a fraction bar in the order specified in steps 3 and 4; (2) find the value of any powers or roots; (3) multiply and divide from left to right; (4) add and subtract from left to right. (*Source:* National Council of Teachers of Mathematics [NCTM] 2013.)

ordinal number. A number designating the place (such as *first*, *second*, or *third*) occupied by an item in an ordered sequence (Merriam-Webster 2013).

partition. The process or result of dividing an object, set of objects, or a number into non-overlapping parts.

partitive division (or ***fair-share division***). A division that determines how many are in each group when some quantity is portioned equally into groups. *Example:* Partitive division is used to determine how many pencils each child gets if a parent divides a dozen pencils equally among three children. The calculation is accomplished with counters by parceling 12 counters into 3 piles ("One for Adam, one for Beth, one for Charlie; two for Adam, two for Beth, two for Charlie," and so on) and checking how many counters are in each pile.

Pascal's triangle. A triangular arrangement of numbers in which each row starts and ends with 1 and each other number is the sum of the two numbers above it (Harcourt School Publishers 2013).

percent rate of change. A rate of change expressed as a percentage. *Example:* If a population grows from 50 to 55 in a year, it grows by $\frac{5}{50}$, or 10% per year.

periodic phenomena. Events that recur at regular, fixed intervals—for example, the solstices.

picture graph. A graph that uses pictures to show and compare information. Also known as a *pictograph*.

piecewise-defined function. A function defined by multiple subfunctions, each of which applies to a certain interval of the main function's domain.

polar form. The polar form of the complex number $a+bi$ is either of the following forms: $r\cos\theta+ri\sin\theta$; or $r(\cos\theta+i\sin\theta)$, which is often simplified to $r\text{cis}\theta$ when (r,θ) are polar coordinates of $a+bi$ on the complex plane. In either of these forms, $|r|$ is called the modulus and θ is called the argument.

polynomial. The sum or difference of terms which have variables raised to non-negative integer powers and which have coefficients that may be real or complex. Each of the following examples is a polynomial: $5x^3-2x^2+x-13$, x^2y^3+xy, x^2y^3 and $(1+i)a^2+ib^2$. (*Source:* Mathwords 2013.)

polynomial function. Any function whose values are determined by evaluating a polynomial.

prime factorization. A number written as the product of all its prime factors (Harcourt School Publishers 2013).

prime number. A positive integer that has only 1 and the number itself as factors. For example, 2, 3, 5, 7, 11, 13 (and so on) are all primes. By convention, the number 1 is not prime. (*Source:* Mathwords 2013.)

probability. A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, or testing for a medical condition).

probability distribution. The set of possible values of a random variable with a probability assigned to each.

probability model. A mathematical representation used to assign probabilities to outcomes of a chance process by examining the nature of the process. The set of all outcomes is called the *sample space*, and their probabilities sum to 1. See also **uniform probability model**.

proof. A method of using deductive reasoning to construct a valid argument.

properties of equality. See table GL-2 in this glossary.

properties of inequality. See table GL-3 in this glossary.

properties of integers. See tables GL-1, GL-2, and GL-3 in this glossary.

properties of operations. See table GL-1 in this glossary.

proportion. (**a**) Another term for a fraction of a whole. *Example:* The "proportion of the population that prefers product A" might be 60 percent. (**b**) An equation that states that two ratios are equivalent. *Example:* $\frac{4}{8}=\frac{1}{2}$ or $4:8=1:2$.

proportional relationship. A collection of pairs of numbers that are in equivalent ratios. A ratio $A:B$ determines a proportional relationship—namely, the collection of pairs (cA, cB) for c positive. A proportional relationship is described by an equation of the form $y=kx$, where k is a positive constant (often called a constant of proportionality). (*Source:* University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2011c.)

Pythagorean Theorem. For any right triangle, the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.

quadratic equation. An equation that includes second-degree (and possibly lower-degree) polynomials. Some examples are $y=3x^2-5x^2+1$, $x^2+5xy+y^2=1$, and $1.6a^2+5.9a-3.14=0$.

quadratic expression. A polynomial expression that contains a term of degree 2, but no term of higher degree.

quadratic function. A function that can be represented by an equation of the form $y=ax^2+bx+c$, where a, b, and c are arbitrary (but fixed) numbers and $a\neq 0$. The graph of this function is a parabola (WDPI 2013).

quadratic polynomial. A polynomial where the highest degree of any of its terms is 2.

quotitive division (**or *measurement division***). A division that determines how many equal-size groups can be formed from a given quantity. For example, quotitive division is used to determine how many pies can be purchased for $12 when each pie costs $3. The calculation can be accomplished with counters by parceling 12 counters into piles of size 3 each ("One, two, three for the first pie; one, two, three for the second pie," and so on) and checking how many piles there are.

radical. The $\sqrt{\ }$ symbol, which is used to indicate square roots or nth roots (Mathwords 2013).

random sampling. A smaller group of people or objects chosen from a larger group or population by a process giving equal chance of selection to all possible people or objects, and all possible subsets of the same size (Harcourt School Publishers 2013).

random variable. An assignment of a numerical value to each outcome in a sample space (Merriam-Webster 2013).

range (**of a set of data**). The numerical difference between the largest and smallest values in a set of data (WDPI 2013).

rate. A *rate* associated with the ratio $A:B$ is $\frac{A}{B}$ units of the first quantity per 1 unit of the second quantity. The two quantities may have different units. (*Source:* UA Progressions Documents 2011c.)

ratio. A multiplicative comparison of two numbers or quantities (e.g., 4 to 7 or 4:7 or $\frac{4}{7}$).

rational expression. A quotient of two polynomials with a non-zero denominator.

rational number. A number expressible in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. See figure GL-1 at the end of this glossary.

real number. A number that corresponds to a point on a number line. See figure GL-1 at the end of this glossary.

rectangular array. An arrangement of mathematical elements into rows and columns.

rectilinear figure. A polygon whose every angle is a right angle.

recursive pattern or sequence. A pattern or sequence wherein each successive term can be computed from some or all of the preceding terms by an algorithmic procedure.

reflection. A type of transformation that flips points about a line or a point.

relative frequency. The empirical counterpart of a probability. If an event occurs N' times in N trials, its relative frequency is $\frac{N'}{N}$ (Merriam-Webster 2013).

remainder theorem. If $f(x)$ is a polynomial in x, then the remainder on dividing $f(x)$ by $x-a$ is $f(a)$ [Merriam-Webster 2013].

repeated reasoning. The reasoning involved in solving one mathematical problem that is used again in a different mathematical problem or problems. Mathematically proficient students notice if calculations are repeated and look for general methods as well as shortcuts. As they work to solve a problem, mathematically proficient students maintain oversight of the process while attending to the details. They continually evaluate the reasonableness of their intermediate results. See Standard for Mathematical Practice 8. (*Source:* NGA/CCSSO 2010c.)

repeating decimal. A decimal expansion of a number in which, after a certain point, a particular digit or sequence of digits repeats itself indefinitely; the decimal form of a rational number (Merriam-Webster 2013). See also **terminating decimal**.

rigid motion. A transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations. Rigid motions preserve distances and angle measures.

rotation. A type of transformation that turns a figure about a fixed point (called the *center of rotation*).

sample space. In a probability model for a random process, a list of the individual outcomes that are to be considered.

SAS congruence (side–angle–side congruence). When two triangles have two pairs of corresponding sides that are congruent, and the included angle formed by those sides is congruent, then the triangles are congruent.

scatter plot. A graph in the coordinate plane representing a set of bivariate data. *Example:* The heights and weights of a group of people could be displayed on a scatter plot (WDPI 2013).

scientific notation. A widely used system in which numbers are expressed as products consisting of a number between 1 and 10 multiplied by an appropriate power of 10 (e.g., $562 = 5.62 \times 10^2$) [Mathwords 2013].

sequence, progression. An ordered set of elements (e.g., 1, 3, 9, 27, 81). In this sequence, 1 is the *first term*, 3 is the *second term*, 9 is the *third term*, and so on.

similarity transformation. A transformation that can be represented as a rigid motion followed by a dilation.

simultaneous equations. Two or more equations containing common variables (Mathwords 2013).

sine. The trigonometric function that, for an acute angle of a right triangle, is the ratio between the leg opposite the angle and the hypotenuse.

SSS congruence (side–side–side congruence). When two triangles have all three pairs of corresponding sides congruent, then the triangles are congruent.

strategy. (**a**) A plan of action designed to achieve a long-term or overall aim. (**b**) An approach to teaching and learning (Oxford Dictionaries 2013).

subitize. To immediately, and without counting, perceive a quantity.

tangent. (**a**) A line passing perpendicular to a radius at the point lying on the circle is said to be tangent to the circle. (**b**) The trigonometric function that, for an acute angle of a right triangle, is the ratio between the leg opposite the angle and the leg adjacent to the angle.

tape diagram. A drawing that looks like a segment of tape and is used to illustrate number relationships. Also known as a *strip diagram*, *bar model*, *fraction strip*, or *length model*.

terminating decimal. A repeating decimal number whose repeating digit is 0. See also **repeating decimal**.

third quartile. For a data set with median M, the third quartile is the median of the data values greater than M. *Example:* For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the third quartile is 15. See also **median**, **first quartile**, and **interquartile range**.

transformation. A prescription, or rule, that sets up a one-to-one correspondence between the points in a geometric object (the *pre-image*) and the points in another geometric object (the *image*). Reflections, rotations, translations, and dilations are examples of transformations.

transitivity principle for indirect measurement. If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other attributes as well, such as time, weight, area, and volume.

translation. A type of transformation that moves every point in a graph or geometric figure by the same distance in the same direction without a change in orientation or size (Mathwords 2013).

trigonometric function. Any of the six functions (sine, cosine, tangent, cotangent, secant, cosecant) that, for an acute angle of a right triangle, may be expressed in terms of ratios of sides of the right triangle.

trigonometry. The study of triangles and trigonometric functions.

uniform probability model. A probability model that assigns equal probability to all outcomes (NGA/CCSSO 2010c). See also **probability model**.

unit fraction. A fraction with a numerator of 1, such as $\frac{1}{3}$ or $\frac{1}{5}$.

unit rate. The numerical part of the rate; the word *unit* in the term *unit rate* is used to highlight the 1 in "per 1 unit" of the second quantity. *Example:* If 3 melons cost $4.50, then the unit rate is simply the number $\frac{4.50}{3} = 1.5$ [without units]. (*Source:* UA Progressions Documents 2011c.)

univariate. Relating to a single variable.

valid. (**a**) Well grounded or justifiable; being at once relevant and meaningful (e.g., a *valid* theory). (**b**) Logically correct. (*Source:* Mathwords 2013.)

variable. A quantity that can change or that may take on different values. Refers to the letter or symbol representing such a quantity in an expression, equation, inequality, or matrix. (*Source:* Mathwords 2013.)

vector. A quantity with magnitude and direction in the plane or in space, defined by an ordered pair or triple of real numbers.

visual fraction model. A tape diagram, number line diagram, or area model.

whole numbers. The numbers 0, 1, 2, 3, . . . See figure GL-1 at the end of this glossary.

Tables and Illustrations of Key Mathematical Properties, Rules, and Number Sets

Table GL-1. The Properties of Operations

In this table, a, b, and c stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

Associative property of addition	$(a+b)+c=a+(b+c)$
Commutative property of addition	$a+b=b+a$
Additive identity property of 0	$a+0=0+a=a$
Existence of additive inverses	For every a there exists $-a$, so that $a+(-a)=(-a)+a=0$.
Associative property of multiplication	$(a\times b)\times c=a\times(b\times c)$
Commutative property of multiplication	$a\times b=b\times a$
Multiplicative identity property of 1	$a\times 1=1\times a=a$
Existence of multiplicative inverses	For every $a\neq 0$ there exists $\frac{1}{a}$, so that $a\times\frac{1}{a}=\frac{1}{a}\times a=1$.
Distributive property of multiplication over addition	$a\times(b+c)=a\times b+a\times c$

Table GL-2. The Properties of Equality

In this table, a, b, and c stand for arbitrary numbers in the rational number system, the real number system, and the complex number system.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a - c = b - c$.
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.

Table GL-3. The Properties of Inequality

In this table, a, b, and c stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$, $a = b$, $a > b$.
If $a > b$ and $b > c$, then $a > c$.
If $a > b$, then $b < a$.
If $a > b$, then $-a < -b$.
If $a > b$, then $a \pm c > b \pm c$.
If $a > b$ and $c > 0$, then $a \times c > b \times c$.
If $a > b$ and $c < 0$, then $a \times c < b \times c$.
If $a > b$ and $c > 0$, then $a \div c > b \div c$.
If $a > b$ and $c < 0$, then $a \div c < b \div c$.

Table GL-4. Common Addition and Subtraction Situations*

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? $2+3=\square$	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were 5 bunnies. How many bunnies hopped over to the first two? $2+\square=5$	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were 5 bunnies. How many bunnies were on the grass before? $\square+3=5$
Take from	Five apples were on the table. I ate 2 apples. How many apples are on the table now? $5-2=\square$	Five apples were on the table. I ate some apples. Then there were 3 apples. How many apples did I eat? $5-\square=3$	Some apples were on the table. I ate 2 apples. Then there were 3 apples. How many apples were on the table before? $\square-2=3$
Put together/ Take apart‡	**Total Unknown** Three red apples and 2 green apples are on the table. How many apples are on the table? $3+2=\square$	**Addend Unknown** Five apples are on the table. Three are red, and the rest are green. How many apples are green? $3+\square=5,\ 5-3=\square$	**Both Addends Unknown†** Grandma has 5 flowers. How many can she put in her red vase and how many in her blue vase? $5=0+5,\ 5=5+0$ $5=1+4,\ 5=4+1$ $5=2+3,\ 5=3+2$
Compare§	**Difference Unknown** ("How many more?" version): Lucy has 2 apples. Julie has 5 apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has 2 apples. Julie has 5 apples. How many fewer apples does Lucy have than Julie? $2+\square=5,\ 5-2=\square$	**Bigger Unknown** (Version with *more*): Julie has 3 more apples than Lucy. Lucy has 2 apples. How many apples does Julie have? (Version with *fewer*): Lucy has 3 fewer apples than Julie. Lucy has 2 apples. How many apples does Julie have? $2+3=\square,\ 3+2=\square$	**Smaller Unknown** (Version with *more*): Julie has 3 more apples than Lucy. Julie has 5 apples. How many apples does Lucy have? (Version with *fewer*): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? $5-3=\square,\ \square+3=5$

*Adapted from Boxes 2–4 of *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity* (National Research Council, Committee on Early Childhood Mathematics 2009, 32–33).

‡ Either addend can be unknown, so there are three variations of these problem situations. "Both Addends Unknown" is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

† These take-apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign (=), help children understand that the equal sign does not always mean *makes* or *results in* but does always mean *is the same number as*.

§ For the "Bigger Unknown" or "Smaller Unknown" situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult.

Table GL-5. Common Multiplication and Division Situations*

	Unknown Product	Group Size Unknown	Number of Groups Unknown
	$3 \times 6 = \square$	$3 \times \square = 18$ and $18 \div 3 = \square$	$\square \times 6 = 18$ and $18 \div 6 = \square$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there altogether? **Measurement example** You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally and packed inside 3 bags, then how many plums will be in each bag? **Measurement example** You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed, with 6 plums to a bag, then how many bags are needed? **Measurement example** You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,† Area‡	There are 3 rows of apples with 6 apples in each row. How many apples are there? **Area example** What is the area of a rectangle that measures 3 centimeters by 6 centimeters?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? **Area example** A rectangle has an area of 18 square centimeters. If one side is 3 centimeters long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? **Area example** A rectangle has an area of 18 square centimeters. If one side is 6 centimeters long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? **Measurement example** A rubber band is 6 centimeters long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18, and that is three times as much as a blue hat costs. How much does a blue hat cost? **Measurement example** A rubber band is stretched to be 18 centimeters long and that is three times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? **Measurement example** A rubber band was 6 centimeters long at first. Now it is stretched to be 18 centimeters long. How many times as long is the rubber band now as it was at first?
General	$a \times b = \square$	$a \times \square = p$ and $p \div a = \square$	$\square \times b = p$ and $p \div b = \square$

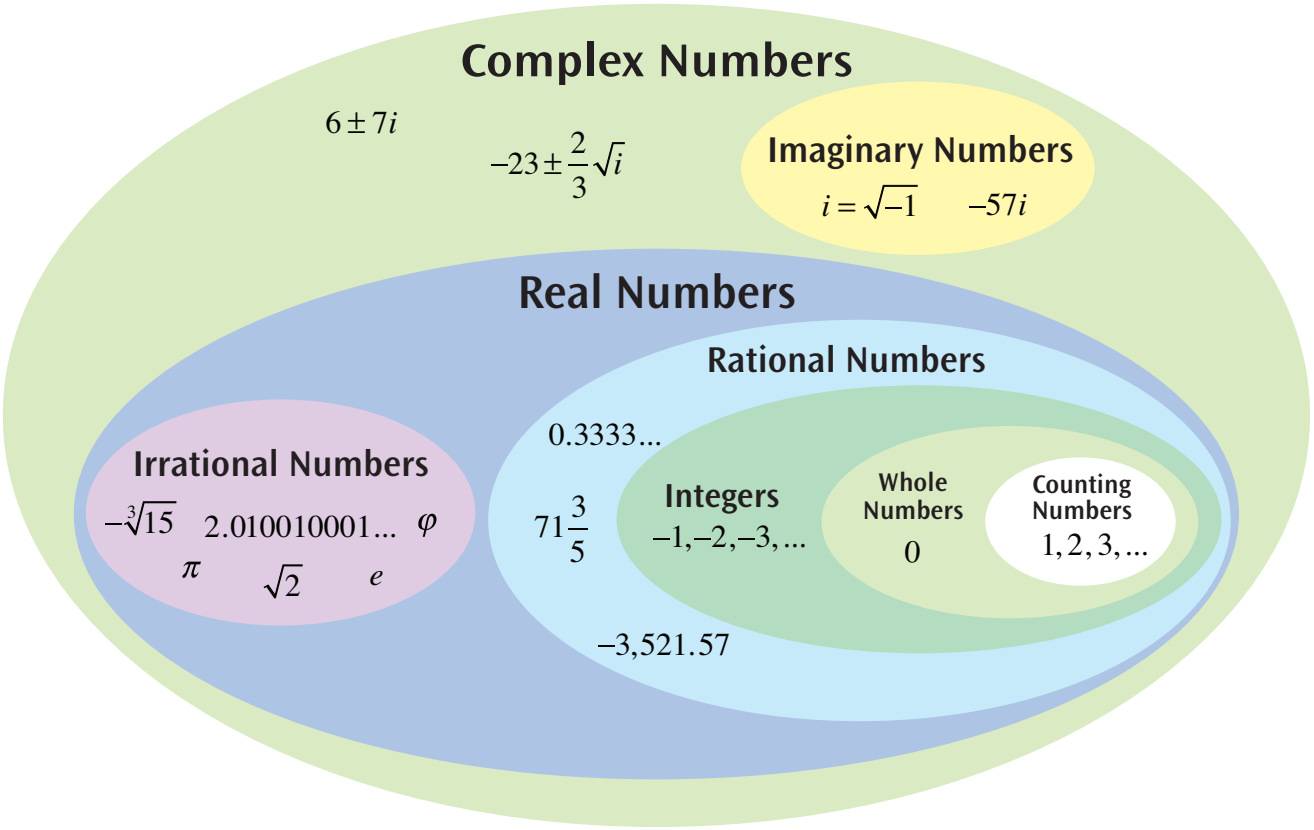
*The first examples in each cell focus on discrete things. These examples are easier for students and should be given before the measurement examples.

† The language in the array examples shows the easiest form of array problems. A more difficult form of these problems uses the terms *rows* and *columns*, as in this example: *"The apples in the grocery window are in 3 rows and 6 columns. How many apples are there?"* Both forms are valuable.

‡ Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps; thus array problems include these especially important measurement situations.

Figure GL-1. The Number System

The number system consists of number sets beginning with *counting numbers* and culminating in the more complete *complex numbers*. The name of each set is written on the boundary of the set, indicating that each increasing oval encompasses the sets contained within. Note that the *real number set* is composed of two parts: rational numbers and irrational numbers.



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Resources for Implementing the California Common Core State Standards for Mathematics

California Department of Education Resources

Common Core State Standards Web Page

The California Department of Education (CDE) Common Core State Standards Web page (http://www.cde.ca.gov/re/cc/) provides a portal to hundreds of resources for teachers, administrators, parents, and families, allowing users to get started and keep current with the California Common Core State Standards. This site is updated frequently and offers a searchable database.

California Standards Documents

California Common Core State Standards: English Language Arts and Literacy in History/Social Studies, Science, and Technical Subjects
http://www.cde.ca.gov/be/st/ss/documents/finalelaccssstandards.pdf

California Common Core State Standards: Mathematics
http://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf

California English Language Development Standards: Kindergarten Through Grade 12
http://www.cde.ca.gov/sp/el/er/eldstandards.asp

All other state-adopted content standards are available online at the CDE Content Standards Web page (http://www.cde.ca.gov/be/st/ss/).

Other CDE Publications

The Alignment of the California Preschool Learning Foundations with Key Early Education Resources is an electronic publication (not available in print form) that displays the alignment of the four domains of infant/toddler learning and development foundations with the nine domains of preschool foundations. The publication clearly defines a full continuum of learning that begins at birth, highlighting the path of school readiness, long-term school success, and child wellness.
http://www.cde.ca.gov/sp/cd/re/documents/psalignment.pdf

A Look at Kindergarten Through Grade Six in California Public Schools: Transitioning to the Common Core State Standards in English Language Arts and Mathematics is an electronic publication (not available in print form) that contains grade-level chapters with short, descriptive narratives and California's K–6 Common Core standards for English language arts and mathematics.
http://www.cde.ca.gov/re/cc/documents/alookatkthrugrade6.pdf

A Look at Grades Seven and Eight in California Public Schools: Transitioning to the California Common Core State Standards in English Language Arts and Mathematics is an electronic publication (not available in print form) that contains grade-level chapters with short, descriptive narratives and California's grade-seven and grade-eight Common Core standards for English language arts and mathematics.
http://www.cde.ca.gov/ci/cr/cf/documents/glcgr78revisedjuly2014.pdf

Improving Education for English Learners: Research-Based Approaches is a CDE publication that offers a comprehensive, user-friendly review and analysis of recent research to inform and improve instructional practices. Ordering information is available from CDE Press at http://www.cde.ca.gov/re/pn/rc/.

The Superintendent's Quality Professional Learning Standards (QPLS) identify essential elements of quality professional learning that cut across specific content knowledge, pedagogical skills, and dispositions. The QPLS serve as a foundation for the content, processes, and conditions essential to all educator professional learning. The document is posted on the CDE Web site at http://www.cde.ca.gov/pd/ps/documents/caqpls.pdf.

CDE Web Pages

The CDE Assistive Technology Checklist Web Page provides a list of examples of assistive technology.
http://www.cde.ca.gov/sp/se/sr/atexmpl.asp

The CDE Common Core Resources for Special Education Web Page offers resources and guidelines on how the California Common Core State Standards and the state's new student assessments will impact students in California's special education community.
http://www.cde.ca.gov/sp/se/cc/

The CDE English Learners Resources Web page includes answers to frequently asked questions and links to Web sites with information about developing programs for linguistically and culturally diverse students.
http://www.cde.ca.gov/sp/el/er/

The CDE Special Education Web Page features information and resources to serve the unique needs of persons with disabilities so that each person will meet or exceed high standards of achievement in academic and nonacademic skills.
http://www.cde.ca.gov/sp/se/

The CDE Special Education Services and Resources Web Page provides a list of programs and services available to students with disabilities, publications, training and technical assistance opportunities, and recruitment resources and materials.
http://www.cde.ca.gov/sp/se/sr/

Additional Mathematics Resources*

The **Achieve the Core** Web site provides free content created by the nonprofit Student Achievement Partners to help educators understand and implement the Common Core State Standards. The site includes practical tools designed to help students and teachers, including annotated lesson plans, videos, and hands-on activities.
http://achievethecore.org/

Appendix A of the Common Core State Standards for Mathematics, from the Council of Chief State School Officers (CCSSO) and the National Governors Association Center for Best Practices (NGA Center), provides guidance on higher mathematics (high school) courses, including compacted courses for acceleration.
http://www.corestandards.org/assets/CCSSI_Mathematics_Appendix_A.pdf

The **California Mathematics Project** (**CMP**), in collaboration with the CDE, the California Mathematics Council (CMC), CCSESA's Mathematics Subcommittee of the Curriculum and Instruction Steering Committee (CISC), and the California Association of Mathematics Teacher Educators (CAMTE), established the CaCCSS-M Task Forces to collect, design, and organize resources that could be used in professional development to strengthen teachers' content knowledge in the California Common Core State Standards for Mathematics: K–2 Number Sense and Place Value, Fractions from a Number Line Approach, Model with Mathematics, Transformational Geometry, and High School Mathematical Modeling. These resources are updated on a regular basis.
http://caccssm.cmpso.org/home

GeoGebra is an online software suite that features interactive geometry, algebra, and calculus applications. It is intended for both teachers and students.
http://www.geogebra.org/cms/

Illustrative Mathematics is a community of educators dedicated to coherent mathematics instruction. The community shares vetted resources for teachers and teacher leaders to give students an understanding of mathematics and skills to use mathematics. The Web site features many sample tasks for use in the classroom.
https://www.illustrativemathematics.org/

Inside Mathematics is a professional learning community focused on improving students' mathematics learning. The Inside Mathematics Web site features examples of innovative teaching methods and insights into student learning, as well as tools for mathematics instruction.
http://www.insidemathematics.org/

Math by Design was produced as part of a national public television collaborative aimed at creating online resources focused on STEM subjects (science, technology, engineering, and mathematics) for middle school students and teachers.
http://mathbydesign.thinkport.org/

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*Inclusion on this list of additional mathematics resources does not imply that the California Department of Education (CDE) endorses the Web sites, the organizations that manage the Web sites, or materials found on the Web sites. The CDE does not monitor or control the Web sites or their content.

The University of Arizona's **Progressions Documents for the Common Core Math Standards** describe the progression of specific mathematics topics across a number of grade levels, informed both by research on children's cognitive development and by the logical structure of mathematics. The documents draw attention to cognitive difficulties and pedagogical solutions and give more detail on particularly complicated areas of mathematics.
http://ime.math.arizona.edu/progressions/

Supporting ELLs in Mathematics is a collection of open-source materials produced by Understanding Language, an initiative of Stanford University. The materials were designed for teachers to support mathematics instruction and language development for English learners. Annotated lessons for classroom use are available through the Web site.
http://ell.stanford.edu/content/supporting-ells-mathematics

TKCalifornia Mathematics Teaching Tools provide examples of teaching strategies with concrete approaches for mathematics instruction in Transitional Kindergarten classrooms. The tools were designed to guide developmentally appropriate instruction.
http://www.tkcalifornia.org/teaching-tools/mathematics/

Note: All URLs referenced in the Resources for Implementing the California Common Core State Standards for Mathematics were accessed on June 19, 2015, and were valid as of that date.