NAME______________________________

My "Laws of Exponents" Cheat Sheet

✓ **Multiplying Powers with the Same Base**

General Rule: $x^a \bullet x^b = x^{a+b}$

Example: $x^5 \bullet x^6 = x^{11}$

✓ **Dividing Powers with the Same Base**

General Rule: $\frac{x^a}{x^b} = x^{a-b}$

Example: $\frac{x^7}{x^4} = x^3$

✓ **Finding a Power of a Power**

General Rule: $(x^a)^b = x^{a \bullet b}$

Example: $(x^3)^6 = x^{18}$

✓ **Negative Exponents**

General Rule: $x^{-a} = \frac{1}{x^a}$

Example: $x^{-7} = \frac{1}{x^7}$

✓ **Zero as an Exponent**

General Rule: $x^0 = 1$

Example: $5^0 = 1$

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Algebra Cheat Sheet

Basic Properties & Facts

Arithmetic Operations

$ab + ac = a(b+c)$ $\qquad a\left(\frac{b}{c}\right) = \frac{ab}{c}$

$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$ $\qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$

$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ $\qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$

$\frac{a-b}{c-d} = \frac{b-a}{d-c}$ $\qquad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

$\frac{ab+ac}{a} = b+c,\ a \neq 0$ $\qquad \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}$

Exponent Properties

$a^n a^m = a^{n+m}$ $\qquad \frac{a^n}{a^m} = a^{n-m} = \frac{1}{a^{m-n}}$

$\left(a^n\right)^m = a^{nm}$ $\qquad a^0 = 1,\quad a \neq 0$

$(ab)^n = a^n b^n$ $\qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$a^{-n} = \frac{1}{a^n}$ $\qquad \frac{1}{a^{-n}} = a^n$

$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ $\qquad a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n = \left(a^n\right)^{\frac{1}{m}}$

Properties of Radicals

$\sqrt[n]{a} = a^{\frac{1}{n}}$ $\qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

$\sqrt[m]{\sqrt[n]{a}} = \sqrt[nm]{a}$ $\qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$\sqrt[n]{a^n} = a$, if n is odd

$\sqrt[n]{a^n} = |a|$, if n is even

Properties of Inequalities

If $a < b$ then $a + c < b + c$ and $a - c < b - c$

If $a < b$ and $c > 0$ then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

If $a < b$ and $c < 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

Properties of Absolute Value

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

$|a| \geq 0$ $\qquad |-a| = |a|$

$|ab| = |a||b|$ $\qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$|a+b| \leq |a| + |b|$ Triangle Inequality

Distance Formula

If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are two points the distance between them is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Complex Numbers

$i = \sqrt{-1}$ $\qquad i^2 = -1$ $\qquad \sqrt{-a} = i\sqrt{a},\quad a \geq 0$

$(a+bi) + (c+di) = a + c + (b+d)i$

$(a+bi) - (c+di) = a - c + (b-d)i$

$(a+bi)(c+di) = ac - bd + (ad+bc)i$

$(a+bi)(a-bi) = a^2 + b^2$

$|a+bi| = \sqrt{a^2+b^2}$ Complex Modulus

$\overline{(a+bi)} = a - bi$ Complex Conjugate

$\overline{(a+bi)}(a+bi) = |a+bi|^2$

Logarithms and Log Properties

Definition

$y = \log_b x$ is equivalent to $x = b^y$

Example

$\log_5 125 = 3$ because $5^3 = 125$

Special Logarithms

$\ln x = \log_e x$ natural log

$\log x = \log_{10} x$ common log

where $e = 2.718281828\ldots$

Logarithm Properties

$\log_b b = 1$ $\qquad \log_b 1 = 0$

$\log_b b^x = x$ $\qquad b^{\log_b x} = x$

$\log_b\left(x^r\right) = r\log_b x$

$\log_b(xy) = \log_b x + \log_b y$

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

The domain of $\log_b x$ is $x > 0$

Factoring and Solving

Factoring Formulas

$x^2 - a^2 = (x+a)(x-a)$

$x^2 + 2ax + a^2 = (x+a)^2$

$x^2 - 2ax + a^2 = (x-a)^2$

$x^2 + (a+b)x + ab = (x+a)(x+b)$

$x^3 + 3ax^2 + 3a^2x + a^3 = (x+a)^3$

$x^3 - 3ax^2 + 3a^2x - a^3 = (x-a)^3$

$x^3 + a^3 = (x+a)\left(x^2 - ax + a^2\right)$

$x^3 - a^3 = (x-a)\left(x^2 + ax + a^2\right)$

$x^{2n} - a^{2n} = \left(x^n - a^n\right)\left(x^n + a^n\right)$

If n is odd then,

$x^n - a^n = (x-a)\left(x^{n-1} + ax^{n-2} + \cdots + a^{n-1}\right)$

$x^n + a^n$

$= (x+a)\left(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \cdots + a^{n-1}\right)$

Quadratic Formula

Solve $ax^2 + bx + c = 0$, $a \neq 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac > 0$ - Two real unequal solns.

If $b^2 - 4ac = 0$ - Repeated real solution.

If $b^2 - 4ac < 0$ - Two complex solutions.

Square Root Property

If $x^2 = p$ then $x = \pm\sqrt{p}$

Absolute Value Equations/Inequalities

If b is a positive number

$|p| = b \quad \Rightarrow \quad p = -b$ or $p = b$

$|p| < b \quad \Rightarrow \quad -b < p < b$

$|p| > b \quad \Rightarrow \quad p < -b$ or $p > b$

Completing the Square

Solve $2x^2 - 6x - 10 = 0$

(1) Divide by the coefficient of the x^2

$$x^2 - 3x - 5 = 0$$

(2) Move the constant to the other side.

$$x^2 - 3x = 5$$

(3) Take half the coefficient of x, square it and add it to both sides

$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 = 5 + \left(-\frac{3}{2}\right)^2 = 5 + \frac{9}{4} = \frac{29}{4}$$

(4) Factor the left side

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$

(5) Use Square Root Property

$$x - \frac{3}{2} = \pm\sqrt{\frac{29}{4}} = \pm\frac{\sqrt{29}}{2}$$

(6) Solve for x

$$x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$$

Functions and Graphs

Constant Function

$y = a$ or $f(x) = a$

Graph is a horizontal line passing through the point $(0, a)$.

Line/Linear Function

$y = mx + b$ or $f(x) = mx + b$

Graph is a line with point $(0, b)$ and slope m.

Slope

Slope of the line containing the two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

Slope – intercept form

The equation of the line with slope m and y-intercept $(0, b)$ is

$$y = mx + b$$

Point – Slope form

The equation of the line with slope m and passing through the point (x_1, y_1) is

$$y = y_1 + m(x - x_1)$$

Parabola/Quadratic Function

$y = a(x-h)^2 + k$ $\quad f(x) = a(x-h)^2 + k$

The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at (h, k).

Parabola/Quadratic Function

$y = ax^2 + bx + c$ $\quad f(x) = ax^2 + bx + c$

The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Parabola/Quadratic Function

$x = ay^2 + by + c$ $\quad g(y) = ay^2 + by + c$

The graph is a parabola that opens right if $a > 0$ or left if $a < 0$ and has a vertex at $\left(g\left(-\frac{b}{2a}\right), -\frac{b}{2a}\right)$.

Circle

$(x-h)^2 + (y-k)^2 = r^2$

Graph is a circle with radius r and center (h, k).

Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Graph is an ellipse with center (h, k) with vertices a units right/left from the center and vertices b units up/down from the center.

Hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Graph is a hyperbola that opens left and right, has a center at (h, k), vertices a units left/right of center and asymptotes that pass through center with slope $\pm\frac{b}{a}$.

Hyperbola

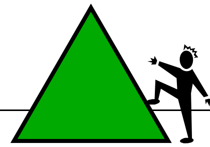
$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

Graph is a hyperbola that opens up and down, has a center at (h, k), vertices b units up/down from the center and asymptotes that pass through center with slope $\pm\frac{b}{a}$.

Common Algebraic Errors

Error	Reason/Correct/Justification/Example
$\frac{2}{0} \neq 0$ and $\frac{2}{0} \neq 2$	Division by zero is undefined!
$-3^2 \neq 9$	$-3^2 = -9$, $(-3)^2 = 9$ Watch parenthesis!
$(x^2)^3 \neq x^5$	$(x^2)^3 = x^2x^2x^2 = x^6$
$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$	$\frac{1}{2} = \frac{1}{1+1} \neq \frac{1}{1} + \frac{1}{1} = 2$
$\frac{1}{x^2 + x^3} \neq x^{-2} + x^{-3}$	A more complex version of the previous error.
$\frac{\not{a} + bx}{\not{a}} \neq 1 + bx$	$\frac{a+bx}{a} = \frac{a}{a} + \frac{bx}{a} = 1 + \frac{bx}{a}$ Beware of incorrect canceling!
$-a(x-1) \neq -ax - a$	$-a(x-1) = -ax + a$ Make sure you distribute the "-"!
$(x+a)^2 \neq x^2 + a^2$	$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2$
$\sqrt{x^2 + a^2} \neq x + a$	$5 = \sqrt{25} = \sqrt{3^2 + 4^2} \neq \sqrt{3^2} + \sqrt{4^2} = 3 + 4 = 7$
$\sqrt{x+a} \neq \sqrt{x} + \sqrt{a}$	See previous error.
$(x+a)^n \neq x^n + a^n$ and $\sqrt[n]{x+a} \neq \sqrt[n]{x} + \sqrt[n]{a}$	More general versions of previous three errors.
$2(x+1)^2 \neq (2x+2)^2$	$2(x+1)^2 = 2(x^2 + 2x + 1) = 2x^2 + 4x + 2$ $(2x+2)^2 = 4x^2 + 8x + 4$ Square first then distribute!
$(2x+2)^2 \neq 2(x+1)^2$	See the previous example. You can not factor out a constant if there is a power on the parethesis!
$\sqrt{-x^2 + a^2} \neq -\sqrt{x^2 + a^2}$	$\sqrt{-x^2 + a^2} = (-x^2 + a^2)^{\frac{1}{2}}$ Now see the previous error.
$\frac{a}{\left(\frac{b}{c}\right)} \neq \frac{ab}{c}$	$\frac{a}{\left(\frac{b}{c}\right)} = \frac{\left(\frac{a}{1}\right)}{\left(\frac{b}{c}\right)} = \left(\frac{a}{1}\right)\left(\frac{c}{b}\right) = \frac{ac}{b}$
$\frac{\left(\frac{a}{b}\right)}{c} \neq \frac{ac}{b}$	$\frac{\left(\frac{a}{b}\right)}{c} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{1}\right)} = \left(\frac{a}{b}\right)\left(\frac{1}{c}\right) = \frac{a}{bc}$

Geometry – Things to Remember!



3-D Figures:

Prism: $V = Bh$

Pyramid: $V = \frac{1}{3}Bh$

Cylinder: $V = \pi r^2 h$; $SA = 2\pi rh + 2\pi r^2$

Cone: $V = \frac{1}{3}\pi r^2 h$; $SA = s\pi r + \pi r^2$

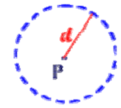
Sphere: $V = \frac{4}{3}\pi r^3$; $SA = 4\pi r^2 = \pi d^2$

Regular Solids:

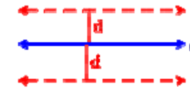
Tetrahedron – 4 faces
Cube – 6 faces
Octahedron – 8 faces
Dodecahedron – 12 faces
Icosahedron – 20 faces

Locus Theorems:

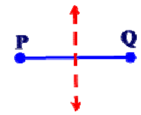
Fixed distance from point.



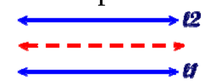
Fixed distance from a line.



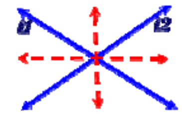
Equidistant from 2 points.



Equidistant 2 parallel lines.



Equidistant from 2 intersecting lines



Polygon Interior/Exterior Angles:

Sum of int. angles = $180(n-2)$

Each int. angle (regular) = $\frac{180(n-2)}{n}$

Sum of ext. angles = 360

Each ext. angle (regular) = $\frac{360}{n}$

Triangles:

By Sides:
Scalene – no congruent sides
Isosceles – 2 congruent sides
Equilateral – 3 congruent sides

By Angles:
Acute – all acute angles
Right – one right angle
Obtuse – one obtuse angle
Equiangular – 3 congruent angles(60º)
Equilateral ↔ Equiangular

Exterior angle of a triangle equals the sum of the 2 non-adjacent interior angles.

Mid-segment of a triangle is parallel to the third side and half the length of the third side.

Congruent Triangles

SSS
SAS
ASA
AAS
HL (right triangles only)

NO donkey theorem
(SSA or ASS)

CPCTC (use after the triangles are congruent)

Related Conditionals:

Converse: switch if and then
Inverse: negate if and then
Contrapositive: inverse of the converse (contrapositive has the same truth value as the original statement)

Inequalities:

--Sum of the lengths of any two sides of a triangle is greater than the length of the third side.
--Longest side of a triangle is opposite the largest angle.
--Exterior angle of a triangle is greater than either of the two non-adjacent interior angles.

Pythagorean Theorem:

$c^2 = a^2 + b^2$

Converse: If the sides of a triangle satisfy $c^2 = a^2 + b^2$ then the triangle is a right triangle.

Similar Triangles:

AA
SSS for similarity
SAS for similarity
Corresponding sides of similar triangles are in proportion.

Mean Proportional in Right Triangle:

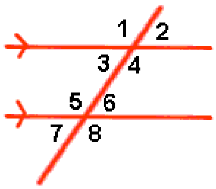
Altitude Rule:

$$\frac{\text{part hyp}}{\text{altitude}} = \frac{\text{altitude}}{\text{other part hyp}}$$

Leg Rule:

$$\frac{\text{hyp}}{\text{leg}} = \frac{\text{leg}}{\text{projection}}$$

Parallels: If lines are parallel …



Corresponding angles are equal.
m<1=*m*<5, *m*<2=*m*<6, *m*<3=*m*<7, *m*<4=*m*<8
Alternate Interior angles are equal.
m<3=*m*<6, *m*<4=*m*<5
Alternate Exterior angles are equal.
m<1=*m*<8, *m*<2=*m*<7
Same side interior angles are supp.
m<3 + *m*<5=180, *m*<4 + *m*<6=180

Quadrilaterals:

Parallelogram:
opp. sides parallel
opp sides =
opp angles =
consec. angles supp
diag bis each other
Rectangle: add 4 rt angles, diag. =
Rhombus: add 4 = sides, diag. perp, diag bisect angles.
Square: All from above.

Trapezoid:
Only one set parallel sides.

Median of trap is parallel to both bases and = ½ sum bases.
Isosceles Trap:
legs =
base angles =
diagonals =
opp angles supp

Transformations:

$r_{x-axis}(x,y)=(x,-y)$
$r_{y-axis}(x,y)=(-x,y)$
$r_{y=x}(x,y)=(y,x)$
$r_{y=-x}(x,y)=(-y,-x)$
$r_{origin}(x,y)=(-x,-y)$
$T_{a,b}(x,y)=(x+a,y+b)$
$D_k(x,y)=(kx,ky)$
$R_{90^\circ}(x,y)=(-y,x)$
$R_{180^\circ}(x,y)=(-x,-y)$
$R_{270^\circ}(x,y)=(y,-x)$

Glide reflection is composition of a reflection and a translation.

Isometry – keeps length.

Orientation – label order

Circle Segments

In a circle, a radius perpendicular to a chord bisects the chord.

Intersecting Chords Rule:
(segment part)•(segment part) = (segment part)•(segment part)

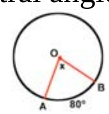
Secant-Secant Rule:
(whole secant)•(external part) = (whole secant)•(external part)

Secant-Tangent Rule:
(whole secant)•(external part) = (tangent)^2

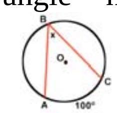
Hat Rule: Two tangents are equal.

Circle Angles:

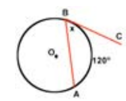
Central angle = arc



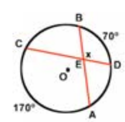
Inscribed angle = half arc



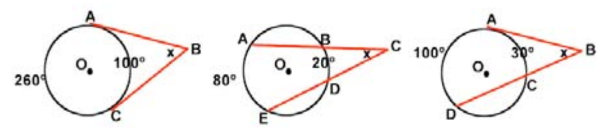
Angle by tangent/chord = half arc



Angle formed by 2 chords = half the sum of arcs



Angle formed by 2 tangents, or 2 secants, or a tangent/secant = half the difference of arcs



Slopes and Equations:

$m=\dfrac{vertical\ change}{horizontal\ change}=\dfrac{y_2-y_1}{x_2-x_1}.$

$y=mx+b$ slope-intercept

$y-y_1=m(x-x_1)$ point-slope

Coordinate Geometry Formulas:

Distance Formula:

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Midpoint Formula:

$(x,y)=\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2}\right)$

Circles:

Equation of circle center at origin:
$x^2+y^2=r^2$ where *r* is the radius.
Equation of circle not at origin:
$(x-h)^2+(y-k)^2=r^2$ where (*h*,*k*) is the center and *r* is the radius.

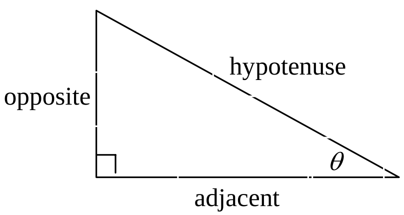
Trig Cheat Sheet

Definition of the Trig Functions

Right triangle definition

For this definition we assume that

$0 < \theta < \frac{\pi}{2}$ or $0° < \theta < 90°$.



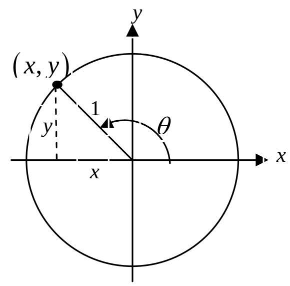
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Unit circle definition

For this definition θ is any angle.



$$\sin\theta = \frac{y}{1} = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{x}{1} = x \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Facts and Properties

Domain

The domain is all the values of θ that can be plugged into the function.

$\sin\theta$, θ can be any angle

$\cos\theta$, θ can be any angle

$\tan\theta$, $\theta \neq \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm1, \pm2, \ldots$

$\csc\theta$, $\theta \neq n\pi, \quad n = 0, \pm1, \pm2, \ldots$

$\sec\theta$, $\theta \neq \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm1, \pm2, \ldots$

$\cot\theta$, $\theta \neq n\pi, \quad n = 0, \pm1, \pm2, \ldots$

Range

The range is all possible values to get out of the function.

$-1 \le \sin\theta \le 1 \qquad \csc\theta \ge 1$ and $\csc\theta \le -1$

$-1 \le \cos\theta \le 1 \qquad \sec\theta \ge 1$ and $\sec\theta \le -1$

$-\infty \le \tan\theta \le \infty \qquad -\infty \le \cot\theta \le \infty$

Period

The period of a function is the number, T, such that $f(\theta + T) = f(\theta)$. So, if ω is a fixed number and θ is any angle we have the following periods.

$$\sin(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \rightarrow T = \frac{\pi}{\omega}$$

$$\csc(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\sec(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\cot(\omega\theta) \rightarrow T = \frac{\pi}{\omega}$$

Formulas and Identities

Tangent and Cotangent Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Even/Odd Formulas

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Periodic Formulas

If n is an integer.

$$\sin(\theta + 2\pi n) = \sin\theta \qquad \csc(\theta + 2\pi n) = \csc\theta$$

$$\cos(\theta + 2\pi n) = \cos\theta \qquad \sec(\theta + 2\pi n) = \sec\theta$$

$$\tan(\theta + \pi n) = \tan\theta \qquad \cot(\theta + \pi n) = \cot\theta$$

Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180} \quad \text{and} \quad x = \frac{180t}{\pi}$$

Half Angle Formulas

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos(2\theta)\right)$$

$$\cos^2\theta = \frac{1}{2}\left(1 + \cos(2\theta)\right)$$

$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Product to Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right]$$

Sum to Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

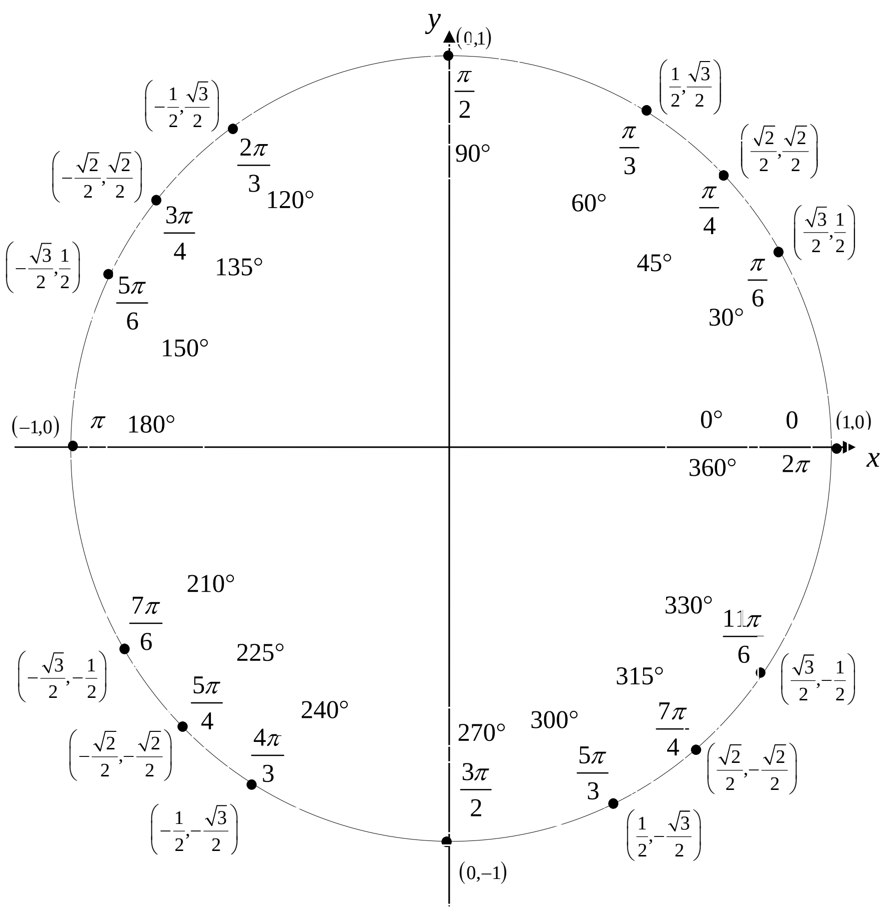
Cofunction Formulas

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

Unit Circle



For any ordered pair on the unit circle (x, y) : $\cos\theta = x$ and $\sin\theta = y$

Example

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \qquad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

Inverse Trig Functions

Definition

$y = \sin^{-1} x$ is equivalent to $x = \sin y$

$y = \cos^{-1} x$ is equivalent to $x = \cos y$

$y = \tan^{-1} x$ is equivalent to $x = \tan y$

Inverse Properties

$\cos\left(\cos^{-1}(x)\right) = x \qquad \cos^{-1}\left(\cos(\theta)\right) = \theta$

$\sin\left(\sin^{-1}(x)\right) = x \qquad \sin^{-1}\left(\sin(\theta)\right) = \theta$

$\tan\left(\tan^{-1}(x)\right) = x \qquad \tan^{-1}\left(\tan(\theta)\right) = \theta$

Domain and Range

Function	Domain	Range
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

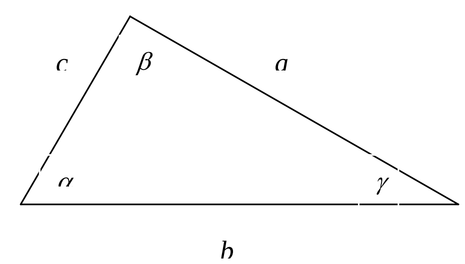
Alternate Notation

$\sin^{-1} x = \arcsin x$

$\cos^{-1} x = \arccos x$

$\tan^{-1} x = \arctan x$

Law of Sines, Cosines and Tangents



Law of Sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Mollweide's Formula

$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}$$

Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$$

$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$$

$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$

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Limits

Definitions

Precise Definition : We say $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

"Working" Definition : We say $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

Right hand limit : $\lim_{x\to a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

Left hand limit : $\lim_{x\to a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

Limit at Infinity : We say $\lim_{x\to\infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim_{x\to-\infty} f(x) = L$ except we require x large and negative.

Infinite Limit : We say $\lim_{x\to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$.

There is a similar definition for $\lim_{x\to a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one-sided limits

$\lim_{x\to a} f(x) = L \Rightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$

$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \Rightarrow \lim_{x\to a} f(x) = L$

$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) \Rightarrow \lim_{x\to a} f(x)$ Does Not Exist

Properties

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and c is any number then,

1. $\lim_{x\to a}\left[c f(x)\right] = c\lim_{x\to a} f(x)$
2. $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x)\ \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$
6. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$

Basic Limit Evaluations at $\pm\infty$

Note : $\operatorname{sgn}(a) = 1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

1. $\lim_{x\to\infty} \mathbf{e}^x = \infty$ & $\lim_{x\to-\infty} \mathbf{e}^x = 0$
2. $\lim_{x\to\infty} \ln(x) = \infty$ & $\lim_{x\to 0^-} \ln(x) = -\infty$
3. If $r > 0$ then $\lim_{x\to\infty} \frac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim_{x\to-\infty} \frac{b}{x^r} = 0$
5. n even : $\lim_{x\to\pm\infty} x^n = \infty$
6. n odd : $\lim_{x\to\infty} x^n = \infty$ & $\lim_{x\to-\infty} x^n = -\infty$
7. n even : $\lim_{x\to\pm\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
8. n odd : $\lim_{x\to\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
9. n odd : $\lim_{x\to-\infty} a\,x^n + \cdots + c\,x + d = -\operatorname{sgn}(a)\infty$

Evaluation Techniques

Continuous Functions

If $f(x)$ is continuous at a then $\lim_{x\to a} f(x) = f(a)$

Continuous Functions and Composition

$f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$ then

$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$

Factor and Cancel

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} = \lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)} = \lim_{x\to 2}\frac{x+6}{x} = \frac{8}{2} = 4$$

Rationalize Numerator/Denominator

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81} = \lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}\,\frac{3+\sqrt{x}}{3+\sqrt{x}} = \lim_{x\to 9}\frac{9-x}{(x^2-81)(3+\sqrt{x})} = \lim_{x\to 9}\frac{-1}{(x+9)(3+\sqrt{x})} = \frac{-1}{(18)(6)} = -\frac{1}{108}$$

Combine Rational Expressions

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

L'Hospital's Rule

If $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then,

$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ a is a number, ∞ or $-\infty$

Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute $\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}$ factor largest power of x out of both $p(x)$ and $q(x)$ and then compute limit.

$$\lim_{x\to-\infty}\frac{3x^2-4}{5x-2x^2} = \lim_{x\to-\infty}\frac{x^2\left(3-\frac{4}{x^2}\right)}{x^2\left(\frac{5}{x}-2\right)} = \lim_{x\to-\infty}\frac{3-\frac{4}{x^2}}{\frac{5}{x}-2} = -\frac{3}{2}$$

Piecewise Function

$\lim_{x\to-2} g(x)$ where $g(x) = \begin{cases} x^2+5 & \text{if } x < -2 \\ 1-3x & \text{if } x \geq -2 \end{cases}$

Compute two one sided limits,

$\lim_{x\to-2^-} g(x) = \lim_{x\to-2^-} x^2+5 = 9$

$\lim_{x\to-2^+} g(x) = \lim_{x\to-2^+} 1-3x = 7$

One sided limits are different so $\lim_{x\to-2} g(x)$ doesn't exist. If the two one sided limits had been equal then $\lim_{x\to-2} g(x)$ would have existed and had the same value.

Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

1. Polynomials for all x.
2. Rational function, except for x's that give division by zero.
3. $\sqrt[n]{x}$ (n odd) for all x.
4. $\sqrt[n]{x}$ (n even) for all $x \geq 0$.
5. $\mathbf{e}^x$ for all x.
6. $\ln x$ for $x > 0$.
7. $\cos(x)$ and $\sin(x)$ for all x.
8. $\tan(x)$ and $\sec(x)$ provided $x \neq \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$
9. $\cot(x)$ and $\csc(x)$ provided $x \neq \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that $a < c < b$ and $f(c) = M$.

Derivatives

Definition and Notation

If $y = f(x)$ then the derivative is defined to be $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

If $y = f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = Df(x)$$

If $y = f(x)$ all of the following are equivalent notations for derivative evaluated at $x = a$.

$$f'(a) = y'\big|_{x=a} = \frac{df}{dx}\bigg|_{x=a} = \frac{dy}{dx}\bigg|_{x=a} = Df(a)$$

Interpretation of the Derivative

If $y = f(x)$ then,

1. $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x - a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$.
3. If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object at $x = a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(cf)' = cf'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(fg)' = f'g + fg'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c) = 0$
6. $\frac{d}{dx}(x^n) = nx^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

   This is the **Chain Rule**

Common Derivatives

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(a^x) = a^x \ln(a)$

$\frac{d}{dx}(\mathbf{e}^x) = \mathbf{e}^x$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x},\ x > 0$

$\frac{d}{dx}(\ln|x|) = \frac{1}{x},\ x \neq 0$

$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a},\ x > 0$

Calculus Cheat Sheet

Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}\left(\left[f(x)\right]^n\right) = n\left[f(x)\right]^{n-1} f'(x)$
2. $\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\mathbf{e}^{f(x)}$
3. $\frac{d}{dx}\left(\ln\left[f(x)\right]\right) = \frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}\left(\sin\left[f(x)\right]\right) = f'(x)\cos\left[f(x)\right]$
5. $\frac{d}{dx}\left(\cos\left[f(x)\right]\right) = -f'(x)\sin\left[f(x)\right]$
6. $\frac{d}{dx}\left(\tan\left[f(x)\right]\right) = f'(x)\sec^2\left[f(x)\right]$
7. $\frac{d}{dx}\left(\sec\left[f(x)\right]\right) = f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\frac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right) = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Higher Order Derivatives

The Second Derivative is denoted as $f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2}$ and is defined as $f''(x) = \left(f'(x)\right)'$, *i.e.* the derivative of the first derivative, $f'(x)$.

The n^th^ Derivative is denoted as $f^{(n)}(x) = \frac{d^n f}{dx^n}$ and is defined as $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$, *i.e.* the derivative of the (n-1)^st^ derivative, $f^{(n-1)}(x)$.

Implicit Differentiation

Find y' if $\mathbf{e}^{2x-9y} + x^3y^2 = \sin(y) + 11x$. Remember $y = y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). After differentiating solve for y'.

$$\mathbf{e}^{2x-9y}(2-9y') + 3x^2y^2 + 2x^3y\,y' = \cos(y)y' + 11$$
$$2\mathbf{e}^{2x-9y} - 9y'\mathbf{e}^{2x-9y} + 3x^2y^2 + 2x^3y\,y' = \cos(y)y' + 11 \quad \Rightarrow \quad y' = \frac{11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2}{2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)}$$
$$\left(2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)\right)y' = 11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2$$

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x = c$ is a critical point of $f(x)$ provided either **1.** $f'(c) = 0$ or **2.** $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x) < 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x = c$ is a inflection point of $f(x)$ if the concavity changes at $x = c$.

Visit http://tutorial.math.lamar.edu for a complete set of Calculus notes.

Extrema

Absolute Extrema
1. $x = c$ is an absolute maximum of $f(x)$ if $f(c) \ge f(x)$ for all x in the domain.
2. $x = c$ is an absolute minimum of $f(x)$ if $f(c) \le f(x)$ for all x in the domain.

Fermat's Theorem
If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$.

Extreme Value Theorem
If $f(x)$ is continuous on the closed interval $[a,b]$ then there exist numbers c and d so that, **1.** $a \le c, d \le b$, **2.** $f(c)$ is the abs. max. in $[a,b]$, **3.** $f(d)$ is the abs. min. in $[a,b]$.

Finding Absolute Extrema
To find the absolute extrema of the continuous function $f(x)$ on the interval $[a,b]$ use the following process.
1. Find all critical points of $f(x)$ in $[a,b]$.
2. Evaluate $f(x)$ at all points found in Step 1.
3. Evaluate $f(a)$ and $f(b)$.
4. Identify the abs. max. (largest function value) and the abs. min.(smallest function value) from the evaluations in Steps 2 & 3.

Relative (local) Extrema
1. $x = c$ is a relative (or local) maximum of $f(x)$ if $f(c) \ge f(x)$ for all x near c.
2. $x = c$ is a relative (or local) minimum of $f(x)$ if $f(c) \le f(x)$ for all x near c.

1st Derivative Test
If $x = c$ is a critical point of $f(x)$ then $x = c$ is
1. a rel. max. of $f(x)$ if $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$.
2. a rel. min. of $f(x)$ if $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$.
3. not a relative extrema of $f(x)$ if $f'(x)$ is the same sign on both sides of $x = c$.

2nd Derivative Test
If $x = c$ is a critical point of $f(x)$ such that $f'(c) = 0$ then $x = c$
1. is a relative maximum of $f(x)$ if $f''(c) < 0$.
2. is a relative minimum of $f(x)$ if $f''(c) > 0$.
3. may be a relative maximum, relative minimum, or neither if $f''(c) = 0$.

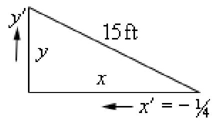
Finding Relative Extrema and/or Classify Critical Points
1. Find all critical points of $f(x)$.
2. Use the 1st derivative test or the 2nd derivative test on each critical point.

Mean Value Theorem
If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) then there is a number $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Newton's Method
If x_n is the n^{th} guess for the root/solution of $f(x) = 0$ then $(n+1)^{st}$ guess is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)$ exists.

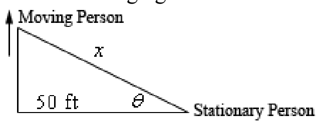
Related Rates
Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to t using implicit differentiation (*i.e.* add on a derivative every time you differentiate a function of t). Plug in known quantities and solve for the unknown quantity.

Ex. A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at $\frac{1}{4}$ ft/sec. How fast is the top moving after 12 sec?



x' is negative because x is decreasing. Using Pythagorean Theorem and differentiating,
$x^2 + y^2 = 15^2 \quad \Rightarrow \quad 2x\,x' + 2y\,y' = 0$
After 12 sec we have $x = 10 - 12\left(\frac{1}{4}\right) = 7$ and so $y = \sqrt{15^2 - 7^2} = \sqrt{176}$. Plug in and solve for y'.
$7\left(-\frac{1}{4}\right) + \sqrt{176}\;y' = 0 \;\Rightarrow\; y' = \dfrac{7}{4\sqrt{176}}$ ft/sec

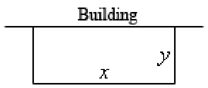
Ex. Two people are 50 ft apart when one starts walking north. The angle θ changes at 0.01 rad/min. At what rate is the distance between them changing when $\theta = 0.5$ rad?



We have $\theta' = 0.01$ rad/min. and want to find x'. We can use various trig fcns but easiest is,
$$\sec\theta = \frac{x}{50} \quad \Rightarrow \quad \sec\theta\tan\theta\;\theta' = \frac{x'}{50}$$
We know $\theta = 0.05$ so plug in θ' and solve.
$$\sec(0.5)\tan(0.5)(0.01) = \frac{x'}{50}$$
$$x' = 0.3112 \text{ ft/sec}$$
Remember to have calculator in radians!

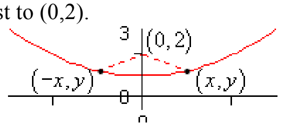
Optimization
Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

Ex. We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.



Maximize $A = xy$ subject to constraint of $x + 2y = 500$. Solve constraint for x and plug into area.
$$x = 500 - 2y \quad \Rightarrow \quad \begin{aligned} A &= y(500 - 2y) \\ &= 500y - 2y^2 \end{aligned}$$
Differentiate and find critical point(s).
$$A' = 500 - 4y \;\Rightarrow\; y = 125$$
By 2nd deriv. test this is a rel. max. and so is the answer we're after. Finally, find x.
$$x = 500 - 2(125) = 250$$
The dimensions are then 250 x 125.

Ex. Determine point(s) on $y = x^2 + 1$ that are closest to (0,2).



Minimize $f = d^2 = (x - 0)^2 + (y - 2)^2$ and the constraint is $y = x^2 + 1$. Solve constraint for x^2 and plug into the function.
$$x^2 = y - 1 \;\Rightarrow\; f = x^2 + (y - 2)^2$$
$$= y - 1 + (y - 2)^2 = y^2 - 3y + 3$$
Differentiate and find critical point(s).
$$f' = 2y - 3 \quad \Rightarrow \quad y = \tfrac{3}{2}$$
By the 2nd derivative test this is a rel. min. and so all we need to do is find x value(s).
$$x^2 = \tfrac{3}{2} - 1 = \tfrac{1}{2} \quad \Rightarrow \quad x = \pm\tfrac{1}{\sqrt{2}}$$
The 2 points are then $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$.

Integrals

Definitions

Definite Integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval.

Then $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{\infty} f(x_i^*)\Delta x$.

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x)=f(x)$.

Indefinite Integral : $\int f(x)dx = F(x)+c$ where $F(x)$ is an anti-derivative of $f(x)$.

Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then $g(x)=\int_a^x f(t)dt$ is also continuous on $[a,b]$ and $g'(x)=\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$ (*i.e.* $F(x)=\int f(x)dx$) then $\int_a^b f(x)dx = F(b)-F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = u'(x)f[u(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)dt = -v'(x)f[v(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = u'(x)f[u(x)]-v'(x)f[v(x)]$$

Properties

$\int f(x)\pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

$\int cf(x)dx = c\int f(x)dx$, c is a constant

$\int_a^b f(x)\pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

$\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant

$\int_a^a f(x)dx = 0$

$\int_a^b f(x)dx = \int_a^b f(t)dt$

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$

If $f(x)\ge g(x)$ on $a\le x\le b$ then $\int_a^b f(x)dx \ge \int_b^a g(x)dx$

If $f(x)\ge 0$ on $a\le x\le b$ then $\int_a^b f(x)dx \ge 0$

If $m\le f(x)\le M$ on $a\le x\le b$ then $m(b-a)\le \int_a^b f(x)dx \le M(b-a)$

Common Integrals

$\int k\,dx = kx+c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+c, n\ne -1$

$\int x^{-1}\,dx = \int \frac{1}{x}dx = \ln|x|+c$

$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|+c$

$\int \ln u\,du = u\ln(u)-u+c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u+c$

$\int \cos u\,du = \sin u+c$

$\int \sin u\,du = -\cos u+c$

$\int \sec^2 u\,du = \tan u+c$

$\int \sec u\tan u\,du = \sec u+c$

$\int \csc u\cot u\,du = -\csc u+c$

$\int \csc^2 u\,du = -\cot u+c$

$\int \tan u\,du = \ln|\sec u|+c$

$\int \sec u\,du = \ln|\sec u+\tan u|+c$

$\int \frac{1}{a^2+u^2}du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c$

$\int \frac{1}{\sqrt{a^2-u^2}}du = \sin^{-1}\left(\frac{u}{a}\right)+c$

Standard Integration Techniques

Note that at many schools all but the Substitution Rule tend to be taught in a Calculus II class.

***u* Substitution :** The substitution $u=g(x)$ will convert $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du$ using $du=g'(x)dx$. For indefinite integrals drop the limits of integration.

Ex. $\int_1^2 5x^2\cos(x^3)dx$

$u=x^3 \Rightarrow du=3x^2dx \Rightarrow x^2dx=\frac{1}{3}du$

$x=1 \Rightarrow u=1^3=1 \;::\; x=2 \Rightarrow u=2^3=8$

$\int_1^2 5x^2\cos(x^3)dx = \int_1^8 \frac{5}{3}\cos(u)du$

$= \frac{5}{3}\sin(u)\Big|_1^8 = \frac{5}{3}(\sin(8)-\sin(1))$

Integration by Parts : $\int u\,dv = uv-\int v\,du$ and $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v=\int dv$.

Ex. $\int x\mathbf{e}^{-x}\,dx$

$u=x \quad dv=\mathbf{e}^{-x} \Rightarrow du=dx \quad v=-\mathbf{e}^{-x}$

$\int x\mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x}+\int \mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x}-\mathbf{e}^{-x}+c$

Ex. $\int_3^5 \ln x\,dx$

$u=\ln x \quad dv=dx \Rightarrow du=\frac{1}{x}dx \quad v=x$

$\int_3^5 \ln x\,dx = x\ln x\Big|_3^5 - \int_3^5 dx = (x\ln(x)-x)\Big|_3^5$

$=5\ln(5)-3\ln(3)-2$

Products and (some) Quotients of Trig Functions

For $\int \sin^n x\cos^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 sine out and convert rest to cosines using $\sin^2 x = 1-\cos^2 x$, then use the substitution $u=\cos x$.
2. ***m* odd.** Strip 1 cosine out and convert rest to sines using $\cos^2 x = 1-\sin^2 x$, then use the substitution $u=\sin x$.
3. ***n* and *m* both odd.** Use either 1. or 2.
4. ***n* and *m* both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n x\sec^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2 x = \sec^2 x-1$, then use the substitution $u=\sec x$.
2. ***m* even.** Strip 2 secants out and convert rest to tangents using $\sec^2 x = 1+\tan^2 x$, then use the substitution $u=\tan x$.
3. ***n* odd and *m* even.** Use either 1. or 2.
4. ***n* even and *m* odd.** Each integral will be dealt with differently.

Trig Formulas : $\sin(2x)=2\sin(x)\cos(x)$, $\cos^2(x)=\frac{1}{2}(1+\cos(2x))$, $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$

Ex. $\int \tan^3 x\sec^5 x\,dx$

$\int \tan^3 x\sec^5 x\,dx = \int \tan^2 x\sec^4 x\tan x\sec x\,dx$

$= \int (\sec^2 x-1)\sec^4 x\tan x\sec x\,dx$

$= \int (u^2-1)u^4\,du \qquad (u=\sec x)$

$= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + c$

Ex. $\int \frac{\sin^5 x}{\cos^3 x}dx$

$\int \frac{\sin^5 x}{\cos^3 x}dx = \int \frac{\sin^4 x\sin x}{\cos^3 x}dx = \int \frac{(\sin^2 x)^2\sin x}{\cos^3 x}dx$

$= \int \frac{(1-\cos^2 x)^2\sin x}{\cos^3 x}dx \qquad (u=\cos x)$

$= -\int \frac{(1-u^2)^2}{u^3}du = -\int \frac{1-2u^2+u^4}{u^3}du$

$= \frac{1}{2}\sec^2 x + 2\ln|\cos x| - \frac{1}{2}\cos^2 x + c$

Trig Substitutions : If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$\sqrt{a^2-b^2x^2} \Rightarrow x=\frac{a}{b}\sin\theta$	$\sqrt{b^2x^2-a^2} \Rightarrow x=\frac{a}{b}\sec\theta$	$\sqrt{a^2+b^2x^2} \Rightarrow x=\frac{a}{b}\tan\theta$
$\cos^2\theta=1-\sin^2\theta$	$\tan^2\theta=\sec^2\theta-1$	$\sec^2\theta=1+\tan^2\theta$

Ex. $\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx$

$x=\frac{2}{3}\sin\theta \Rightarrow dx=\frac{2}{3}\cos\theta\,d\theta$

$\sqrt{4-9x^2}=\sqrt{4-4\sin^2\theta}=\sqrt{4\cos^2\theta}=2\left|\cos\theta\right|$

Recall $\sqrt{x^2}=\left|x\right|$. Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute θ's and remove absolute value bars based on that and,

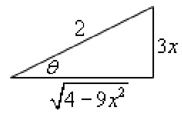
$$\left|x\right|=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0 \end{cases}$$

In this case we have $\sqrt{4-9x^2}=2\cos\theta$.

$$\int \frac{16}{\frac{4}{9}\sin^2\theta(2\cos\theta)}\left(\tfrac{2}{3}\cos\theta\right)d\theta=\int \frac{12}{\sin^2\theta}\,d\theta$$

$$=\int 12\csc^2\,d\theta=-12\cot\theta+c$$

Use Right Triangle Trig to go back to x's. From substitution we have $\sin\theta=\frac{3x}{2}$ so,



From this we see that $\cot\theta=\frac{\sqrt{4-9x^2}}{3x}$. So,

$$\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx=-\frac{4\sqrt{4-9x^2}}{x}+c$$

Partial Fractions : If integrating $\int \frac{P(x)}{Q(x)}\,dx$ where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Ex. $\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx$

$$\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx=\int \frac{4}{x-1}+\frac{3x+16}{x^2+4}\,dx$$

$$=\int \frac{4}{x-1}+\frac{3x}{x^2+4}+\frac{16}{x^2+4}\,dx$$

$$=4\ln\left|x-1\right|+\tfrac{3}{2}\ln\left(x^2+4\right)+8\tan^{-1}\left(\tfrac{x}{2}\right)$$

Here is partial fraction form and recombined.

$$\frac{7x^2+13x}{(x-1)(x^2+4)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}=\frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

Set numerators equal and collect like terms.

$$7x^2+13x=(A+B)x^2+(C-B)x+4A-C$$

Set coefficients equal to get a system and solve to get constants.

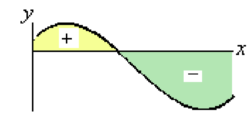
$A+B=7 \quad C-B=13 \quad 4A-C=0$

$A=4 \quad B=3 \quad C=16$

An alternate method that *sometimes* works to find constants. Start with setting numerators equal in previous example : $7x^2+13x=A(x^2+4)+(Bx+C)(x-1)$. Chose *nice* values of x and plug in. For example if $x=1$ we get $20=5A$ which gives $A=4$. This won't always work easily.

Applications of Integrals

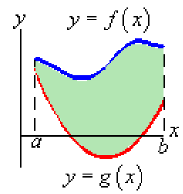
Net Area : $\int_a^b f(x)\,dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.



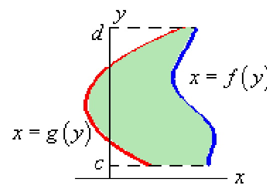
Area Between Curves : The general formulas for the two main cases for each are,

$$y=f(x) \Rightarrow A=\int_a^b\left[\text{upper function}\right]-\left[\text{lower function}\right]dx \quad \& \quad x=f(y) \Rightarrow A=\int_c^d\left[\text{right function}\right]-\left[\text{left function}\right]dy$$

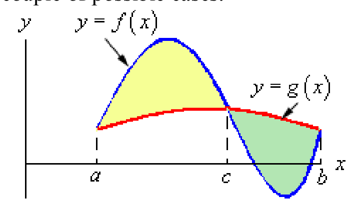
If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



$A=\int_a^b f(x)-g(x)\,dx$



$A=\int_c^d f(y)-g(y)\,dy$

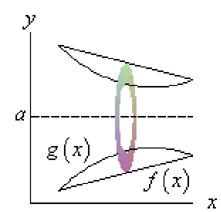


$A=\int_a^c f(x)-g(x)\,dx+\int_c^b g(x)-f(x)\,dx$

Volumes of Revolution : The two main formulas are $V=\int A(x)\,dx$ and $V=\int A(y)\,dy$. Here is some general information about each method of computing and some examples.

Rings		Cylinders	
$A=\pi\left((\text{outer radius})^2-(\text{inner radius})^2\right)$		$A=2\pi\,(\text{radius})\,(\text{width / height})$	
Limits: x/y of right/bot ring to x/y of left/top ring		Limits : x/y of inner cyl. to x/y of outer cyl.	
Horz. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.	Vert. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.	Horz. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.	Vert. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

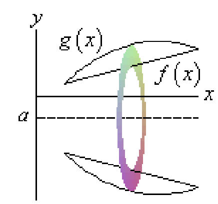
Ex. Axis : $y=a>0$



outer radius : $a-f(x)$

inner radius : $a-g(x)$

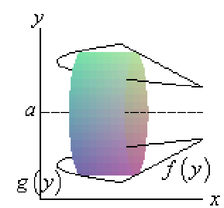
Ex. Axis : $y=a\le 0$



outer radius: $\left|a\right|+g(x)$

inner radius: $\left|a\right|+f(x)$

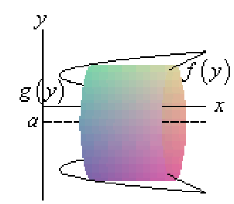
Ex. Axis : $y=a>0$



radius : $a-y$

width : $f(y)-g(y)$

Ex. Axis : $y=a\le 0$



radius : $\left|a\right|+y$

width : $f(y)-g(y)$

These are only a few cases for horizontal axis of rotation. If axis of rotation is the x-axis use the $y=a\le 0$ case with $a=0$. For vertical axis of rotation ($x=a>0$ and $x=a\le 0$) interchange x and y to get appropriate formulas.

Work : If a force of $F(x)$ moves an object in $a \le x \le b$, the work done is $W = \int_a^b F(x)\,dx$

Average Function Value : The average value of $f(x)$ on $a \le x \le b$ is $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Arc Length Surface Area : Note that this is often a Calc II topic. The three basic formulas are,

$L = \int_a^b ds$ $\qquad SA = \int_a^b 2\pi y\,ds$ (rotate about x-axis) $\qquad SA = \int_a^b 2\pi x\,ds$ (rotate about y-axis)

where ds is dependent upon the form of the function being worked with as follows.

$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ if $y = f(x),\ a \le x \le b$ $\qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ if $x = f(t), y = g(t),\ a \le t \le b$

$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ if $x = f(y),\ a \le y \le b$ $\qquad ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ if $r = f(\theta),\ a \le \theta \le b$

With surface area you *may* have to substitute in for the x or y depending on your choice of ds to match the differential in the ds. With parametric and polar you will always need to substitute.

Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called convergent if the limit exists and has a finite value and divergent if the limit doesn't exist or has infinite value. This is typically a Calc II topic.

Infinite Limit

1. $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
2. $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$
3. $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ provided BOTH integrals are convergent.

Discontinuous Integrand

1. Discont. at a: $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$
2. Discont. at b : $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$
3. Discontinuity at $a < c < b$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided both are convergent.

Comparison Test for Improper Integrals : If $f(x) \ge g(x) \ge 0$ on $[a,\infty)$ then,

1. If $\int_a^\infty f(x)\,dx$ conv. then $\int_a^\infty g(x)\,dx$ conv.
2. If $\int_a^\infty g(x)\,dx$ divg. then $\int_a^\infty f(x)\,dx$ divg.

Useful fact : If $a > 0$ then $\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges for $p \le 1$.

Approximating Definite Integrals

For given integral $\int_a^b f(x)\,dx$ and a n (must be even for Simpson's Rule) define $\Delta x = \frac{b-a}{n}$ and divide $[a,b]$ into n subintervals $[x_0,x_1]$, $[x_1,x_2]$, … , $[x_{n-1},x_n]$ with $x_0 = a$ and $x_n = b$ then,

Midpoint Rule : $\int_a^b f(x)\,dx \approx \Delta x\left[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)\right]$, x_i^* is midpoint $[x_{i-1},x_i]$

Trapezoid Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)++2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Simpson's Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$

TABLE OF INFORMATION FOR 2002

CONSTANTS AND CONVERSION FACTORS

1 unified atomic mass unit,	$1\ \text{u} = 1.66 \times 10^{-27}\ \text{kg}$
	$= 931\ \text{MeV}/c^2$
Proton mass,	$m_p = 1.67 \times 10^{-27}\ \text{kg}$
Neutron mass,	$m_n = 1.67 \times 10^{-27}\ \text{kg}$
Electron mass,	$m_e = 9.11 \times 10^{-31}\ \text{kg}$
Magnitude of the electron charge,	$e = 1.60 \times 10^{-19}\ \text{C}$
Avogadro's number,	$N_0 = 6.02 \times 10^{23}\ \text{mol}^{-1}$
Universal gas constant,	$R = 8.31\ \text{J}/(\text{mol} \cdot \text{K})$
Boltzmann's constant,	$k_B = 1.38 \times 10^{-23}\ \text{J/K}$
Speed of light,	$c = 3.00 \times 10^8\ \text{m/s}$
Planck's constant,	$h = 6.63 \times 10^{-34}\ \text{J} \cdot \text{s}$
	$= 4.14 \times 10^{-15}\ \text{eV} \cdot \text{s}$
	$hc = 1.99 \times 10^{-25}\ \text{J} \cdot \text{m}$
	$= 1.24 \times 10^3\ \text{eV} \cdot \text{nm}$
Vacuum permittivity,	$\epsilon_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{N} \cdot \text{m}^2$
Coulomb's law constant,	$k = 1/4\pi\epsilon_0 = 9.0 \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2$
Vacuum permeability,	$\mu_0 = 4\pi \times 10^{-7}\ (\text{T} \cdot \text{m})/\text{A}$
Magnetic constant,	$k' = \mu_0/4\pi = 10^{-7}\ (\text{T} \cdot \text{m})/\text{A}$
Universal gravitational constant,	$G = 6.67 \times 10^{-11}\ \text{m}^3/\text{kg} \cdot \text{s}^2$
Acceleration due to gravity at the Earth's surface,	$g = 9.8\ \text{m/s}^2$
1 atmosphere pressure,	$1\ \text{atm} = 1.0 \times 10^5\ \text{N/m}^2$
	$= 1.0 \times 10^5\ \text{Pa}$
1 electron volt,	$1\ \text{eV} = 1.60 \times 10^{-19}\ \text{J}$

UNITS

Name	Symbol
meter	m
kilogram	kg
second	s
ampere	A
kelvin	K
mole	mol
hertz	Hz
newton	N
pascal	Pa
joule	J
watt	W
coulomb	C
volt	V
ohm	Ω
henry	H
farad	F
tesla	T
degree Celsius	°C
electron-volt	eV

PREFIXES

Factor	Prefix	Symbol
10^9	giga	G
10^6	mega	M
10^3	kilo	k
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p

VALUES OF TRIGONOMETRIC FUNCTIONS FOR COMMON ANGLES

θ	sin θ	cos θ	tan θ
$0°$	0	1	0
$30°$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$
$37°$	3/5	4/5	3/4
$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$53°$	4/5	3/5	4/3
$60°$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
$90°$	1	0	∞

The following conventions are used in this examination.

I. Unless otherwise stated, the frame of reference of any problem is assumed to be inertial.

II. The direction of any electric current is the direction of flow of positive charge (conventional current).

III. For any isolated electric charge, the electric potential is defined as zero at an infinite distance from the charge.

*IV. For mechanics and thermodynamics equations, *W* represents the work done <u>on</u> a system.

*Not on the Table of Information for Physics C, since Thermodynamics is not a Physics C topic.

ADVANCED PLACEMENT PHYSICS B EQUATIONS FOR 2002

NEWTONIAN MECHANICS

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2} at^2$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

$$\sum \mathbf{F} = \mathbf{F}_{net} = m\mathbf{a}$$

$$F_{fric} \le \mu N$$

$$a_c = \frac{v^2}{r}$$

$$\tau = rF \sin\theta$$

$$\mathbf{p} = m\mathbf{v}$$

$$\mathbf{J} = \mathbf{F}\Delta t = \Delta\mathbf{p}$$

$$K = \frac{1}{2} mv^2$$

$$\Delta U_g = mgh$$

$$W = \mathbf{F} \cdot \Delta\mathbf{r} = F\Delta r \cos\theta$$

$$P_{avg} = \frac{W}{\Delta t}$$

$$P = \mathbf{F} \cdot \mathbf{v} = Fv \cos\theta$$

$$\mathbf{F}_s = -k\mathbf{x}$$

$$U_s = \frac{1}{2} kx^2$$

$$T_s = 2\pi\sqrt{\frac{m}{k}}$$

$$T_p = 2\pi\sqrt{\frac{\ell}{g}}$$

$$T = \frac{1}{f}$$

$$F_G = -\frac{Gm_1 m_2}{r^2}$$

$$U_G = -\frac{Gm_1 m_2}{r}$$

a = acceleration
F = force
f = frequency
h = height
J = impulse
K = kinetic energy
k = spring constant
ℓ = length
m = mass
N = normal force
P = power
p = momentum
r = radius or distance
$\mathbf{r}$ = position vector
T = period
t = time
U = potential energy
v = velocity or speed
W = work done on a system
x = position
μ = coefficient of friction
θ = angle
τ = torque

ELECTRICITY AND MAGNETISM

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

$$\mathbf{E} = \frac{\mathbf{F}}{q}$$

$$U_E = qV = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}$$

$$E_{avg} = -\frac{V}{d}$$

$$V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$$

$$C = \frac{Q}{V}$$

$$C = \frac{\epsilon_0 A}{d}$$

$$U_c = \frac{1}{2}QV = \frac{1}{2}CV^2$$

$$I_{avg} = \frac{\Delta Q}{\Delta t}$$

$$R = \frac{\rho\ell}{A}$$

$$V = IR$$

$$P = IV$$

$$C_p = \sum_i C_i$$

$$\frac{1}{C_s} = \sum_i \frac{1}{C_i}$$

$$R_s = \sum_i R_i$$

$$\frac{1}{R_p} = \sum_i \frac{1}{R_i}$$

$$F_B = qvB\sin\theta$$

$$F_B = BI\ell\sin\theta$$

$$B = \frac{\mu_0}{2\pi}\frac{I}{r}$$

$$\phi_m = \mathbf{B} \cdot \mathbf{A} = BA\cos\theta$$

$$\mathcal{E}_{avg} = -\frac{\Delta\phi_m}{\Delta t}$$

$$\mathcal{E} = B\ell v$$

A = area
B = magnetic field
C = capacitance
d = distance
E = electric field
$\mathcal{E}$ = emf
F = force
I = current
ℓ = length
P = power
Q = charge
q = point charge
R = resistance
r = distance
t = time
U = potential (stored) energy
V = electric potential or potential difference
v = velocity or speed
ρ = resistivity
ϕ_m = magnetic flux

ADVANCED PLACEMENT PHYSICS B EQUATIONS FOR 2002

FLUID MECHANICS AND THERMAL PHYSICS

$p = p_0 + \rho g h$

$F_{buoy} = \rho V g$

$A_1 \upsilon_1 = A_2 \upsilon_2$

$p + \rho g y + \frac{1}{2}\rho \upsilon^2 = \text{const.}$

$\Delta \ell = \alpha \ell_0 \Delta T$

$Q = mL$

$Q = mc\Delta T$

$p = \frac{F}{A}$

$pV = nRT$

$K_{avg} = \frac{3}{2} k_B T$

$\upsilon_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3k_B T}{\mu}}$

$W = -p\Delta V$

$Q = nc\Delta T$

$\Delta U = Q + W$

$\Delta U = nc_V \Delta T$

$e = \left|\frac{W}{Q_H}\right|$

$e_c = \frac{T_H - T_C}{T_H}$

A = area
c = specific heat or molar specific heat
e = efficiency
F = force
h = depth
K_{avg} = average molecular kinetic energy
L = heat of transformation
ℓ = length
M = molecular mass
m = mass of sample
n = number of moles
p = pressure
Q = heat transferred to a system
T = temperature
U = internal energy
V = volume
υ = velocity or speed
υ_{rms} = root-mean-square velocity
W = work done on a system
y = height
α = coefficient of linear expansion
μ = mass of molecule
ρ = density

ATOMIC AND NUCLEAR PHYSICS

$E = hf = pc$

$K_{\max} = hf - \phi$

$\lambda = \frac{h}{p}$

$\Delta E = (\Delta m)c^2$

E = energy
f = frequency
K = kinetic energy
m = mass
p = momentum
λ = wavelength
ϕ = work function

WAVES AND OPTICS

$\upsilon = f\lambda$

$n = \frac{c}{\upsilon}$

$n_1 \sin\theta_1 = n_2 \sin\theta_2$

$\sin\theta_c = \frac{n_2}{n_1}$

$\frac{1}{s_i} + \frac{1}{s_0} = \frac{1}{f}$

$M = \frac{h_i}{h_0} = -\frac{s_i}{s_0}$

$f = \frac{R}{2}$

$d \sin\theta = m\lambda$

$x_m \approx \frac{m\lambda L}{d}$

d = separation
f = frequency or focal length
h = height
L = distance
M = magnification
m = an integer
n = index of refraction
R = radius of curvature
s = distance
υ = speed
x = position
λ = wavelength
θ = angle

GEOMETRY AND TRIGONOMETRY

Rectangle
$A = bh$

Triangle
$A = \frac{1}{2}bh$

Circle
$A = \pi r^2$
$C = 2\pi r$

Parallelepiped
$V = \ell wh$

Cylinder
$V = \pi r^2 \ell$
$S = 2\pi r\ell + 2\pi r^2$

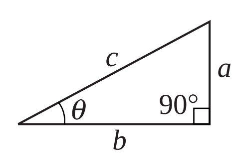
Sphere
$V = \frac{4}{3}\pi r^3$
$S = 4\pi r^2$

Right Triangle
$a^2 + b^2 = c^2$

$\sin\theta = \frac{a}{c}$

$\cos\theta = \frac{b}{c}$

$\tan\theta = \frac{a}{b}$



A = area
C = circumference
V = volume
S = surface area
b = base
h = height
ℓ = length
w = width
r = radius

ADVANCED PLACEMENT PHYSICS C EQUATIONS FOR 2002

MECHANICS

$\upsilon = \upsilon_0 + at$

$x = x_0 + \upsilon_0 t + \frac{1}{2} at^2$

$\upsilon^2 = \upsilon_0^2 + 2a(x - x_0)$

$\sum \mathbf{F} = \mathbf{F}_{net} = m\mathbf{a}$

$\mathbf{F} = \frac{d\mathbf{p}}{dt}$

$\mathbf{J} = \int \mathbf{F}\, dt = \Delta \mathbf{p}$

$\mathbf{p} = m\mathbf{v}$

$F_{fric} \leq \mu N$

$W = \int \mathbf{F} \cdot d\mathbf{r}$

$K = \frac{1}{2} m\upsilon^2$

$P = \frac{dW}{dt}$

$P = \mathbf{F} \cdot \mathbf{v}$

$\Delta U_g = mgh$

$a_c = \frac{\upsilon^2}{r} = \omega^2 r$

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_{net} = I\boldsymbol{\alpha}$

$I = \int r^2 dm = \sum mr^2$

$\mathbf{r}_{cm} = \sum m\mathbf{r} / \sum m$

$\upsilon = r\omega$

$\mathbf{L} = \mathbf{r} \times \mathbf{p} = I\boldsymbol{\omega}$

$K = \frac{1}{2} I\omega^2$

$\omega = \omega_0 + \alpha t$

$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$

$\mathbf{F}_s = -k\mathbf{x}$

$U_s = \frac{1}{2} kx^2$

$T = \frac{2\pi}{\omega} = \frac{1}{f}$

$T_s = 2\pi\sqrt{\frac{m}{k}}$

$T_p = 2\pi\sqrt{\frac{\ell}{g}}$

$\mathbf{F}_G = -\frac{Gm_1m_2}{r^2}\,\hat{\mathbf{r}}$

$U_G = -\frac{Gm_1m_2}{r}$

a = acceleration
F = force
f = frequency
h = height
I = rotational inertia
J = impulse
K = kinetic energy
k = spring constant
ℓ = length
L = angular momentum
m = mass
N = normal force
P = power
p = momentum
r = radius or distance
$\mathbf{r}$ = position vector
T = period
t = time
U = potential energy
υ = velocity or speed
W = work done on a system
x = position
μ = coefficient of friction
θ = angle
τ = torque
ω = angular speed
α = angular acceleration

ELECTRICITY AND MAGNETISM

$F = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}$

$\mathbf{E} = \frac{\mathbf{F}}{q}$

$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0}$

$E = -\frac{dV}{dr}$

$V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$

$U_E = qV = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r}$

$C = \frac{Q}{V}$

$C = \frac{\kappa\epsilon_0 A}{d}$

$C_p = \sum_i C_i$

$\frac{1}{C_s} = \sum_i \frac{1}{C_i}$

$I = \frac{dQ}{dt}$

$U_c = \frac{1}{2}QV = \frac{1}{2}CV^2$

$R = \frac{\rho\ell}{A}$

$V = IR$

$R_s = \sum_i R_i$

$\frac{1}{R_p} = \sum_i \frac{1}{R_i}$

$P = IV$

$\mathbf{F}_M = q\mathbf{v} \times \mathbf{B}$

$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I$

$\mathbf{F} = \int I\, d\boldsymbol{\ell} \times \mathbf{B}$

$B_s = \mu_0 nI$

$\phi_m = \int \mathbf{B} \cdot d\mathbf{A}$

$\mathcal{E} = -\frac{d\phi_m}{dt}$

$\mathcal{E} = -L\frac{dI}{dt}$

$U_L = \frac{1}{2}LI^2$

A = area
B = magnetic field
C = capacitance
d = distance
E = electric field
$\mathcal{E}$ = emf
F = force
I = current
L = inductance
ℓ = length
n = number of loops of wire per unit length
P = power
Q = charge
q = point charge
R = resistance
r = distance
t = time
U = potential or stored energy
V = electric potential
υ = velocity or speed
ρ = resistivity
ϕ_m = magnetic flux
κ = dielectric constant

ADVANCED PLACEMENT PHYSICS C EQUATIONS FOR 2002

GEOMETRY AND TRIGONOMETRY

Rectangle
$A = bh$

Triangle
$A = \frac{1}{2}bh$

Circle
$A = \pi r^2$
$C = 2\pi r$

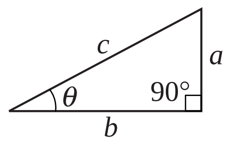
Parallelepiped
$V = \ell wh$

Cylinder
$V = \pi r^2 \ell$
$S = 2\pi r\ell + 2\pi r^2$

Sphere
$V = \frac{4}{3}\pi r^3$
$S = 4\pi r^2$

Right Triangle
$a^2 + b^2 = c^2$
$\sin\theta = \frac{a}{c}$
$\cos\theta = \frac{b}{c}$
$\tan\theta = \frac{a}{b}$

A = area
C = circumference
V = volume
S = surface area
b = base
h = height
ℓ = length
w = width
r = radius



CALCULUS

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1$$

$$\int e^x\,dx = e^x$$

$$\int \frac{dx}{x} = \ln|x|$$

$$\int \cos x\,dx = \sin x$$

$$\int \sin x\,dx = -\cos x$$

TRANSFORMATIONS CHEAT-SHEET!

REFLECTIONS:

- ✓ Reflections are a flip.
- ✓ The flip is performed over the "line of reflection." Lines of symmetry are examples of lines of reflection.
- ✓ Reflections are isometric, but do not preserve orientation.

Coordinate plane rules:

Over the x-axis:	***(x, y) → (x, −y)***
Over the y-axis:	***(x, y) → (−x, y)***
Over the line y = x:	***(x, y) → (y, x)***
Through the origin:	***(x, y) → (−x, −y)***

TRANSLATIONS:

- ✓ Translations are a slide or shift.
- ✓ Translations can be achieved by performing two composite reflections over parallel lines.
- ✓ Translations are isometric, and preserve orientation.

Coordinate plane rules:

(x, y) → (x ± h, y ± k) *where h and k are the horizontal and vertical shifts.*

Note: If movement is left, then h is negative. If movement is down, then k is negative.

DILATIONS:

- ✓ Dilations are an enlargement / shrinking.
- ✓ Dilations multiply the distance from the point of projection (point of dilation) by the scale factor.
- ✓ Dilations are not isometric, and preserve orientation only if the scale factor is positive.

Coordinate plane rules:

From the origin dilated by a factor of "c": ***(x, y) → (cx, cy)***

From non-origin by factor of "c": count slope from point to projection point, multiply by "c," count from projection point.

ROTATIONS:

- ✓ Rotations are a turn.
- ✓ Rotations can be achieved by performing two composite reflections over intersecting lines. The resulting rotation will be double the amount of the angle formed by the intersecting lines.
- ✓ Rotations are isometric, and do not preserve orientation unless the rotation is 360° or exhibit rotational symmetry back onto itself.
- ✓ Rotations of 180° are equivalent to a reflection through the origin.

Coordinate plane rules:

Counter-clockwise:	*Clockwise:*	*Rule:*
90°	*270°*	***(x, y) → (−y, x)***
180°	*180°*	***(x, y) → (−x, −y)***
270°	*90°*	***(x, y) → (y, −x)***