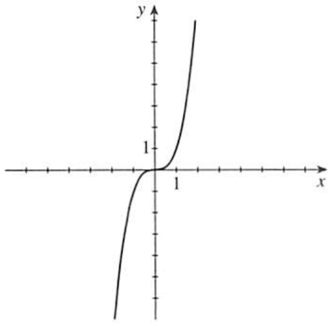
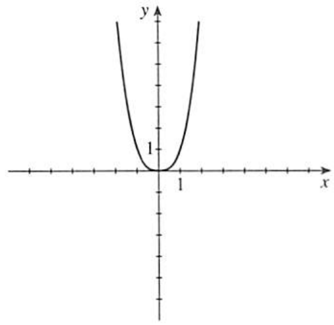
72. (a) $f(x) = x^3$



(b) $g(x) = |x^3|$



73. (a) The graph of $y = t^2$ must be shrunk vertically by a factor of 0.01 and shifted vertically 4 units up to obtain the graph of $y = f(t)$.

(b) The graph of $y = f(t)$ must be shifted horizontally 10 units to the right to obtain the graph of $y = g(t)$. So $g(t) = f(t - 10) = 4 + 0.01(t - 10)^2 = 5 - 0.02t + 0.01t^2$.

74. (a) The graph of $y = t^2$ must be shrunk vertically by a factor of $\frac{1}{2}$ and shifted up 2 units to obtain the graph of $y = C(t)$.

(b) The graph of $y = C(t)$ must be stretched vertically by a factor of $\frac{9}{5}$ and shifted up 32 units to obtain the graph of $y = F(t)$. So $F(t) = \frac{9}{5}C(t) + 32 = \frac{9}{5}\left(\frac{1}{2}t^2 + 2\right) + 32 = \frac{9}{10}t^2 + \frac{178}{5}$.

75. f even implies $f(-x) = f(x)$; g even implies $g(-x) = g(x)$; f odd implies $f(-x) = -f(x)$; and g odd implies $g(-x) = -g(x)$
If f and g are both even, then $(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$ and $f + g$ is even.
If f and g are both odd, then $(f + g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f + g)(x)$ and $f + g$ is odd.
If f odd and g even, then $(f + g)(-x) = f(-x) + g(-x) = -f(x) + g(x)$, which is neither odd nor even.

76. f even implies $f(-x) = f(x)$; g even implies $g(-x) = g(x)$; f odd implies $f(-x) = -f(x)$; and g odd implies $g(-x) = -g(x)$.
If f and g are both even, then $(fg)(-x) = f(-x) \cdot g(-x) = f(x) \cdot g(x) = (fg)(x)$. Thus fg is even.
If f and g are both odd, then $(fg)(-x) = f(-x) \cdot g(-x) = -f(x) \cdot (-g(x)) = f(x) \cdot g(x) = (fg)(x)$. Thus fg is even
If f if odd and g is even, then $(fg)(-x) = f(-x) \cdot g(-x) = f(x) \cdot (-g(x)) = -f(x) \cdot g(x) = -(fg)(x)$. Thus fg is odd.

77. $f(x) = x^n$ is even when n is an even integer and $f(x) = x^n$ is odd when n is an odd integer.
These names were chosen because polynomials with only terms with odd powers are odd functions, and polynomials with only terms with even powers are even functions.

2.5 Quadratic Functions; Maxima and Minima

1. (a) Vertex: $(3, 4)$

(b) Maximum value of f: 4.

2. (a) Vertex: $(-2, 8)$

(b) Maximum value of f: 8.

3. (a) Vertex: $(1, -3)$

(b) Minimum value of f: -3.

4. (a) Vertex: $(-1, -4)$

(b) Minimum value of f: -4.

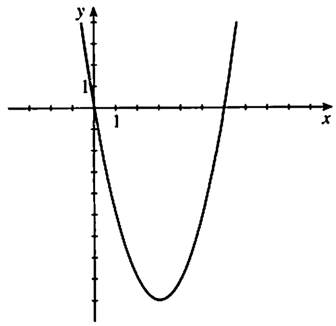
5. (a) $f(x) = x^2 - 6x = (x-3)^2 - 9$

(b) Vertex: $y = x^2 - 6x = x^2 - 6x + 9 - 9 = (x-3)^2 - 9$. So the vertex is at $(3, -9)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad 0 = x^2 - 6x = x(x-6)$. So $x = 0$ or $x = 6$. The x-intercepts are $x = 0$ and $x = 6$.

y-intercept: $x = 0 \quad \Rightarrow \quad y = 0$. The y-intercept is $y = 0$.

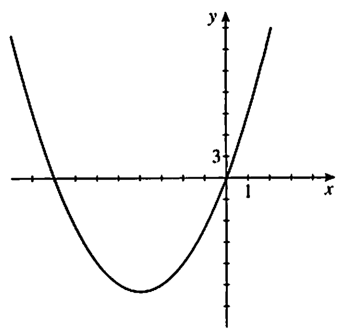
(c)



6. (a) $f(x) = x^2 + 8x = (x+4)^2 - 16$

(b) The vertex is $(-4, -16)$. x-intercepts: $y = 0 \quad \Rightarrow$ $0 = x^2 + 8x = x(x+8)$. So $x = 0$ or $x = -8$. The x-intercepts are $x = 0$ and $x = -8$. y-intercept: $x = 0 \quad \Rightarrow \quad y = 0$. The y-intercept is $y = 0$.

(c)



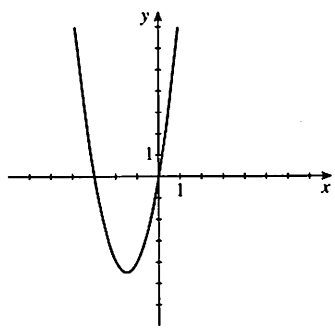
7. (a) $f(x) = 2x^2 + 6x = 2\left(x + \frac{3}{2}\right)^2 - \frac{9}{2}$

(b) The vertex is $\left(-\frac{3}{2}, -\frac{9}{2}\right)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad 0 = 2x^2 + 6x = 2x(x+3) \quad \Rightarrow$ $x = 0$ or $x = -3$. The x-intercepts are $x = 0$ and $x = -3$.

y-intercept: $x = 0 \quad \Rightarrow \quad y = 0$. The y-intercept is $y = 0$.

(c)



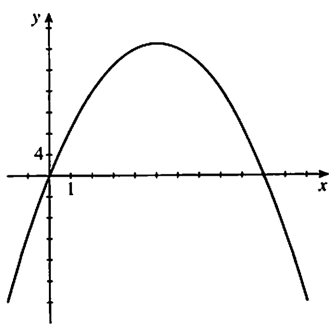
8. (a) $f(x) = -x^2 + 10x = -(x-5)^2 + 25$

(b) The vertex is $(5, 25)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad 0 = -x^2 + 10x = -x(x-10) = 0$ $\Rightarrow x = 0$ or $x = 10$. The x-intercepts are $x = 0$ and $x = 10$.

y-intercept: $x = 0 \quad \Rightarrow \quad y = 0$. The y-intercept is $y = 0$.

(c)



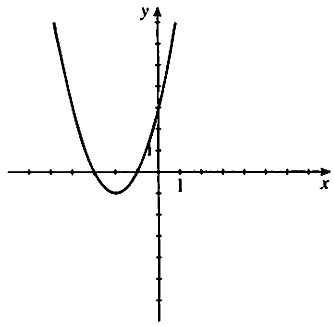
9. **(a)** $f(x) = x^2 + 4x + 3 = (x+2)^2 - 1$

(b) The vertex is $(-2, -1)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad 0 = x^2 + 4x + 3 = (x+1)(x+3)$. So $x = -1$ or $x = -3$. The x-intercepts are $x = -1$ and $x = -3$.

y-intercept: $x = 0 \quad \Rightarrow \quad y = 3$. The y-intercept is $y = 3$.

(c)



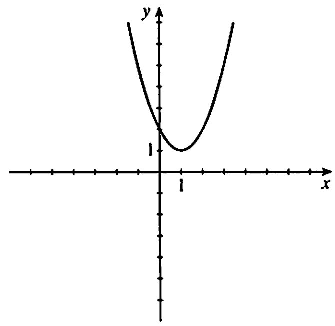
10. **(a)** $f(x) = x^2 - 2x + 2 = (x-1)^2 + 1$

(b) The vertex is $(1, 1)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad (x-1)^2 + 1 = 0 \quad \Leftrightarrow$ $(x-1)^2 = -1$. Since this last equation has no real solution, there is no x-intercept.

y-intercept: $x = 0 \quad \Rightarrow \quad y = 2$. The y-intercept is $y = 2$.

(c)



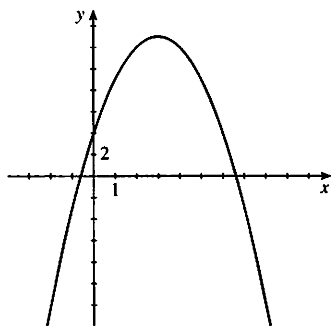
11. **(a)** $f(x) = -x^2 + 6x + 4 = -(x-3)^2 + 13$

(b) The vertex is $(3, 13)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad 0 = -(x-3)^2 + 13 \quad \Leftrightarrow$ $(x-3)^2 = 13 \Rightarrow x - 3 = \pm\sqrt{13} \quad \Leftrightarrow \quad x = 3 \pm \sqrt{13}$. The x-intercepts are $x = 3 - \sqrt{13}$ and $x = 3 + \sqrt{13}$.

y-intercept: $x = 0 \quad \Rightarrow \quad y = 4$. The y-intercept is $y = 4$.

(c)



12. **(a)** $f(x) = -x^2 - 4x + 4 = -(x+2)^2 + 8$

(b) The vertex is $(-2, 8)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad 0 = -x^2 - 4x + 4 \Leftrightarrow 0 = x^2 + 4x - 4$.

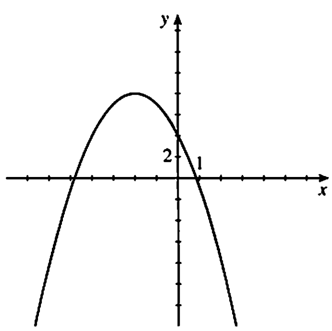
Using the quadratic formula,

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-4)}}{2(1)} = \frac{-4 \pm \sqrt{32}}{2} = \frac{2\left(-2 \pm 2\sqrt{2}\right)}{2} = -2 \pm 2\sqrt{2}.$$

The x-intercepts are $x = -2 + 2\sqrt{2}$ and $x = -2 - 2\sqrt{2}$.

y-intercept: $x = 0 \quad \Rightarrow \quad y = 4$. The y-intercept is $y = 4$.

(c)



13. (a) $f(x) = 2x^2 + 4x + 3 = 2(x+1)^2 + 1$

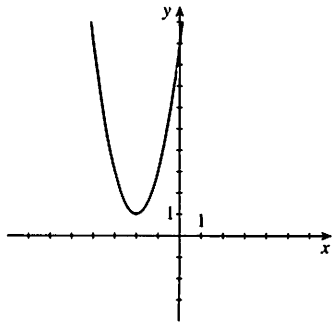
(b) The vertex is $(-1, 1)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad 0 = 2x^2 + 4x + 3 = 2(x+1)^2 + 1$

$\Leftrightarrow \quad 2(x+1)^2 = -1$. Since this last equation has no real solution, there is no x-intercept.

y-intercept: $x = 0 \quad \Rightarrow \quad y = 3$. The y-intercept is $y = 3$.

(c)



14. (a) $f(x) = -3x^2 + 6x - 2 = -3(x-1)^2 + 1$

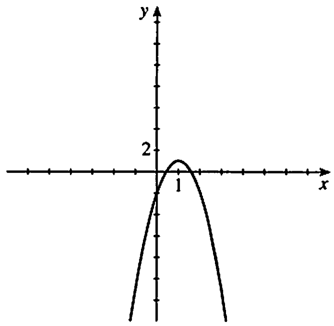
(b) The vertex is $(1, 1)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad 0 = -3(x-1)^2 + 1 = 0$

$\Leftrightarrow (x-1)^2 = \frac{1}{3} \quad \Rightarrow \quad x - 1 = \pm\sqrt{\frac{1}{3}} \quad \Leftrightarrow \quad x = 1 \pm \sqrt{\frac{1}{3}}$. The x-intercepts are $x = 1 + \sqrt{\frac{1}{3}}$ and $x = 1 - \sqrt{\frac{1}{3}}$.

y-intercept: $x = 0 \quad \Rightarrow \quad y = -2$. The y-intercept is $y = -2$.

(c)



15. (a) $f(x) = 2x^2 - 20x + 57 = 2(x-5)^2 + 7$

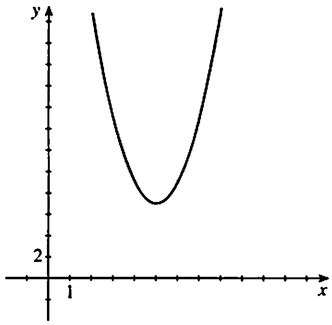
(b) The vertex is $(5, 7)$.

x-intercepts: $y = 0 \quad \Rightarrow$

$0 = 2x^2 - 20x + 57 = 2(x-5)^2 + 7 \Leftrightarrow 2(x-5)^2 = -7$. Since this last equation has no real solution, there is no x-intercept.

y-intercept: $x = 0 \quad \Rightarrow \quad y = 57$. The y-intercept is $y = 57$.

(c)



16. (a) $f(x) = 2x^2 + x - 6 = 2\left(x + \frac{1}{4}\right)^2 - \frac{49}{8}$

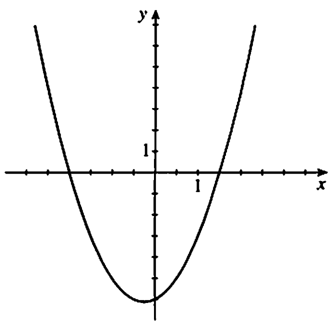
(b) The vertex is $\left(-\frac{1}{4}, -\frac{49}{8}\right)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad 0 = 2x^2 + x - 6 = (2x-3)(x+2)$

$\Rightarrow x = \frac{3}{2}$ or $x = -2$. The x-intercepts are $x = \frac{3}{2}$ and $x = -2$.

y-intercept: $x = 0 \quad \Rightarrow \quad y = -6$. The y-intercept is $y = -6$.

(c)



17. (a) $f(x) = -4x^2 - 16x + 3 = -4(x+2)^2 + 19$

(b) The vertex is $(-2, 19)$.

x-intercepts: $y = 0 \quad \Rightarrow$

$0 = -4x^2 - 16x + 3 = -4(x+2)^2 + 19 \Leftrightarrow 4(x+2)^2 = 19$

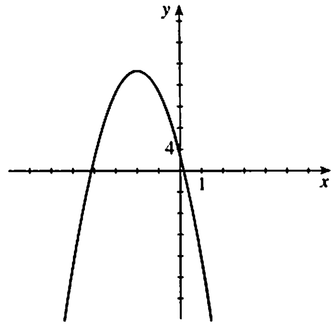
$\Leftrightarrow \quad (x+2)^2 = \frac{19}{4} \Rightarrow x + 2 = \pm\sqrt{\frac{19}{4}} = \pm\frac{\sqrt{19}}{2} \quad \Leftrightarrow$

$x = -2 \pm \frac{\sqrt{19}}{2}$. The x-intercepts are $x = -2 - \frac{\sqrt{19}}{2}$ and

$x = -2 + \frac{\sqrt{19}}{2}$.

y-intercept: $x = 0 \quad \Rightarrow \quad y = 3$. The y-intercept is $y = 3$.

(c)



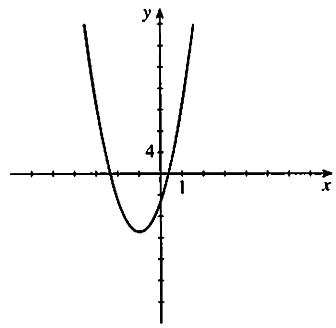
18. (a) $f(x) = 6x^2 + 12x - 5 = 6(x+1)^2 - 11$

(b) The vertex is $(-1, -11)$.

x-intercepts: $y = 0 \quad \Rightarrow \quad 0 = 6x^2 + 12x - 5$. Using the quadratic formula, $x = \frac{-12 \pm \sqrt{(12)^2 - 4(6)(-5)}}{2(6)} = \frac{-12 \pm \sqrt{264}}{12} = \frac{-6 \pm \sqrt{66}}{6}$. The x-intercepts are $x = \frac{-6 \pm \sqrt{66}}{6}$.

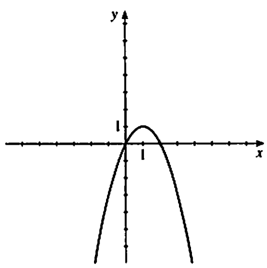
y-intercept: $x = 0 \quad \Rightarrow \quad y = -5$. The y-intercept is $y = -5$.

(c)



19. (a) $f(x) = 2x - x^2 = -(x^2 - 2x)$
$= -(x^2 - 2x + 1) + 1 = -(x-1)^2 + 1$

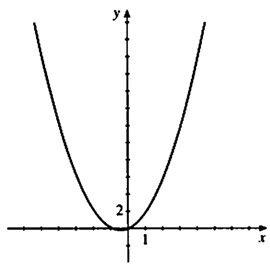
(b)



(c) The maximum value is $f(1) = 1$.

20. (a) $f(x) = x + x^2 = \left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4}$
$= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$

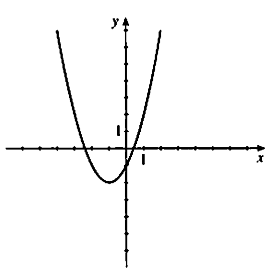
(b)



(c) The minimum value is $f\left(-\frac{1}{2}\right) = -\frac{1}{4}$.

21. (a) $f(x) = x^2 + 2x - 1 = (x^2 + 2x) - 1$
$= (x^2 + 2x + 1) - 1 - 1 = (x+1)^2 - 2$

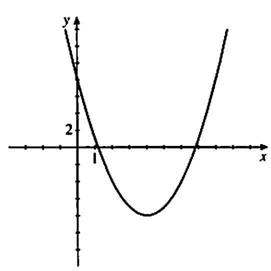
(b)



(c) The minimum value is $f(-1) = -2$.

22. (a) $f(x) = x^2 - 8x + 8 = (x^2 - 8x + 16) + 8 - 16$
$= (x-4)^2 - 8$

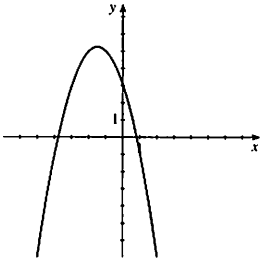
(b)



(c) The minimum value is $f(4) = -8$.

23. (a) $f(x) = -x^2 - 3x + 3 = -\left(x^2 + 3x\right) + 3$
$= -\left(x^2 + 3x + \frac{9}{4}\right) + 3 + \frac{9}{4}$
$= -\left(x + \frac{3}{2}\right)^2 + \frac{21}{4}$

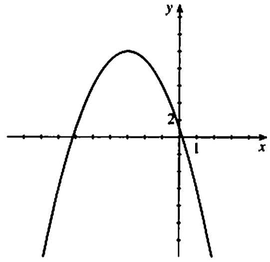
(b)



(c) The maximum value is $f\left(-\frac{3}{2}\right) = \frac{21}{4}$.

24. (a) $f(x) = 1 - 6x - x^2 = -\left(x^2 + 6x\right) + 1$
$= -\left(x^2 + 6x + 9\right) + 1 + 9$
$= -(x+3)^2 + 10$

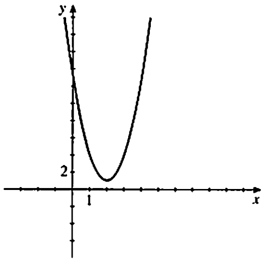
(b)



(c) The maximum value is $f(-3) = 10$.

25. (a) $g(x) = 3x^2 - 12x + 13 = 3\left(x^2 - 4x\right) + 13$
$= 3\left(x^2 - 4x + 4\right) + 13 - 12$
$= 3(x-2)^2 + 1$

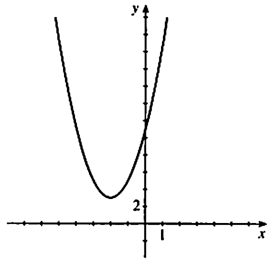
(b)



(c) The minimum value is $g(2) = 1$.

26. (a) $g(x) = 2x^2 + 8x + 11 = 2\left(x^2 + 4x\right) + 11$
$= 2\left(x^2 + 4x + 4\right) + 11 - 8$
$= 2(x+2)^2 + 3$

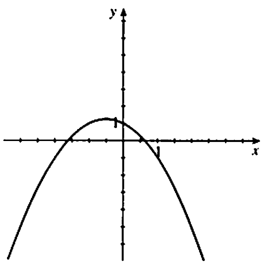
(b)



(c) The minimum value is $g(-2) = 3$.

27. (a) $h(x) = 1 - x - x^2 = -\left(x^2 + x\right) + 1$
$= -\left(x^2 + x + \frac{1}{4}\right) + 1 + \frac{1}{4}$
$= -\left(x + \frac{1}{2}\right)^2 + \frac{5}{4}$

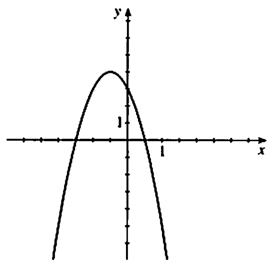
(b)



(c) The maximum value is $h\left(-\frac{1}{2}\right) = \frac{5}{4}$.

28. (a) $h(x) = 3 - 4x - 4x^2 = -4\left(x^2 + x\right) + 3$
$= -4\left(x^2 + x + \frac{1}{4}\right) + 3 + 1$
$= -4\left(x + \frac{1}{2}\right)^2 + 4$

(b)



(c) The maximum value is $h\left(-\frac{1}{2}\right) = 4$.

29. $f(x) = x^2 + x + 1 = \left(x^2 + x\right) + 1 = \left(x^2 + x + \frac{1}{4}\right) + 1 + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$.
Therefore, the minimum value is $f\left(-\frac{1}{2}\right) = \frac{3}{4}$.

30. $f(x) = 1 + 3x - x^2 = -\left(x^2 - 3x\right) + 1 = -\left(x^2 - 3x + \frac{9}{4}\right) + 1 + \frac{9}{4} = -\left(x - \frac{3}{2}\right)^2 + \frac{13}{4}$. Therefore, the maximum value is $f\left(\frac{3}{2}\right) = \frac{13}{4}$.

31. $f(t) = 100 - 49t - 7t^2 = -7\left(t^2 + 7t\right) + 100 = -7\left(t^2 + 7t + \frac{49}{4}\right) + 100 + \frac{343}{4} = -7\left(t + \frac{7}{2}\right)^2 + \frac{743}{4}$.
Therefore, the maximum value is $f\left(-\frac{7}{2}\right) = \frac{743}{4} = 185.75$.

32. $f(t) = 10t^2 + 40t + 113 = 10(t^2 + 4t) + 113 = 10(t^2 + 4t + 4) + 113 - 40 = 10(t + 2)^2 + 73$. Therefore, the minimum value is $f(-2) = 73$.

33. $f(s) = s^2 - 1.2s + 16 = (s^2 - 1.2s) + 16 = (s^2 - 1.2s + 0.36) + 16 - 0.36 = (s - 0.6)^2 + 15.64$. Therefore, the minimum value is $f(0.6) = 15.64$.

34. $g(x) = 100x^2 - 1500x = 100(x^2 - 15x) = 100(x^2 - 15x + \frac{225}{4}) - 5625 = 10(x - \frac{15}{2})^2 - 5625$. Therefore, the minimum value is $g(\frac{15}{2}) = -5625$.

35. $h(x) = \frac{1}{2}x^2 + 2x - 6 = \frac{1}{2}(x^2 + 4x) - 6 = \frac{1}{2}(x^2 + 4x + 4) - 6 - 2 = \frac{1}{2}(x + 2)^2 - 8$. Therefore, the minimum value is $h(-2) = -8$.

36. $f(x) = -\dfrac{x^2}{3} + 2x + 7 = -\frac{1}{3}(x^2 + 6x) + 7 = -\frac{1}{3}(x^2 + 6x + 9) + 7 + 3 = -\frac{1}{3}(x + 3)^2 + 10$. Therefore, the maximum value is $f(-3) = 10$.

37. $f(x) = 3 - x - \frac{1}{2}x^2 = -\frac{1}{2}(x^2 + 2x) + 3 = -\frac{1}{2}(x^2 + 2x + 1) + 3 + \frac{1}{2} = -\frac{1}{2}(x + 1) + \frac{7}{2}$. Therefore, the maximum value is $f(-1) = \frac{7}{2}$.

38. $g(x) = 2x(x - 4) + 7 = 2x^2 - 8x + 7 = 2(x^2 - 4x) + 7 = 2(x^2 - 4x + 4) + 7 - 8 = 2(x - 2)^2 - 1$. Therefore, the minimum value is $g(2) = -1$.

39. Since the vertex is at $(1, -2)$, the function is of the form $f(x) = a(x - 1)^2 - 2$. Substituting the point $(4, 16)$, we get $16 = a(4 - 1)^2 - 2 \Leftrightarrow 16 = 9a - 2 \Leftrightarrow 9a = 18 \Leftrightarrow a = 2$. So the function is $f(x) = 2(x - 1)^2 - 2 = 2x^2 - 4x$.

40. Since the vertex is $(3, 4)$, the function is of the form $y = a(x - 3)^2 + 4$. Since the parabola passes through the point $(1, -8)$, it must satisfy $-8 = a(1 - 3)^2 + 4 \Leftrightarrow -8 = 4a + 4 \Leftrightarrow 4a = -12 \Leftrightarrow a = -3$. So the function is $y = -3(x - 3)^2 + 4 = -3x^2 + 18x - 23$.

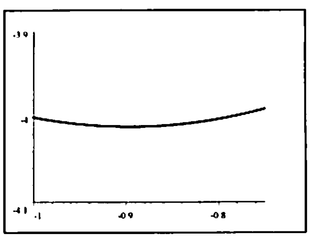
41. $f(x) = -x^2 + 4x - 3 = -(x^2 - 4x) - 3 = -(x^2 - 4x + 4) - 3 + 4 = -(x - 2)^2 + 1$. So the domain of $f(x)$ is $(-\infty, \infty)$. Since $f(x)$ has a maximum value of 1, the range is $(-\infty, 1]$.

42. $f(x) = x^2 - 2x - 3 = (x^2 - 2x + 1) - 3 - 1 = (x - 1)^2 - 4$. Then the domain of the function is all real numbers, and since the minimum value of the function is $f(1) = -4$, the range of the function is $[-4, \infty)$.

43. $f(x) = 2x^2 + 6x - 7 = 2(x + 3)^2 - 7 - 9 = 2(x + 3)^2 - 16$. The domain of the function is all real numbers, and since the minimum value of the function is $f(-3) = -16$, the range of the function is $[-16, \infty)$.

44. $f(x) = -3x^2 + 6x + 4 = -3(x - 1)^2 + 4 + 3 = -3(x - 1)^2 + 7$. The domain of the function is all real numbers, and since the maximum value of the function is $f(1) = 7$, the range of the function is $(-\infty, 7]$.

45. **(a)** The graph of $f(x) = x^2 + 1.79x - 3.21$ is shown. The minimum value is $f(x) \approx -4.01$.



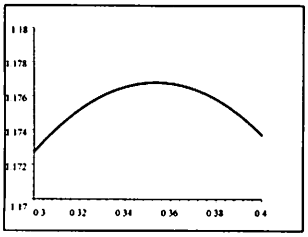
(b) $f(x) = x^2 + 1.79x - 3.21$

$$= \left[x^2 + 1.79x + \left(\tfrac{1.79}{2}\right)^2\right] - 3.21 - \left(\tfrac{1.79}{2}\right)^2$$

$$= (x + 0.895)^2 - 4.011025$$

Therefore, the exact minimum of $f(x)$ is -4.011025.

46. **(a)** The graph of $f(x) = 1 + x - \sqrt{2}x^2$ is shown. The maximum value is $f(x) \approx 1.18$.



(b) $f(x) = 1 + x - \sqrt{2}x^2 = -\sqrt{2}\left(x^2 - \frac{\sqrt{2}}{2}x\right) + 1$

$$= -\sqrt{2}\left[x^2 - \frac{\sqrt{2}}{2}x + \left(\frac{\sqrt{2}}{4}\right)^2\right] + 1 + \frac{\sqrt{2}}{8}$$

$$= -\sqrt{2}\left(x - \frac{\sqrt{2}}{4}\right)^2 + \frac{8+\sqrt{2}}{8}$$

Therefore, the exact maximum of $f(x)$ is $\frac{8+\sqrt{2}}{8}$.

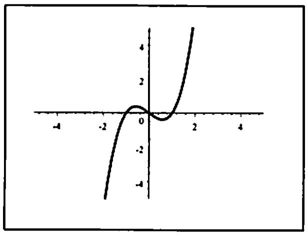
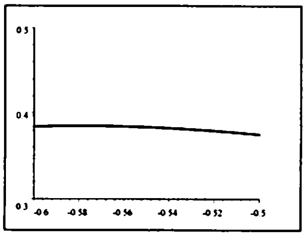
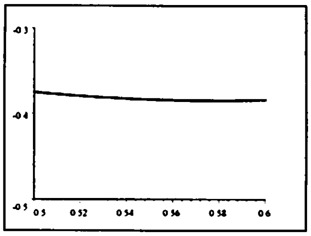
47. Local maximum: 2 at $x = 0$. Local minimum: -1 at $x = -2$ and 0 at $x = 2$.

48. Local maximum: 2 at $x = -2$ and $x = 1$ at $x = 2$. Local minimum: -1 at $x = 0$.

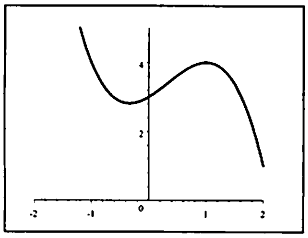
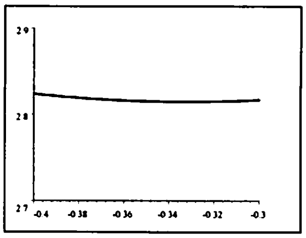
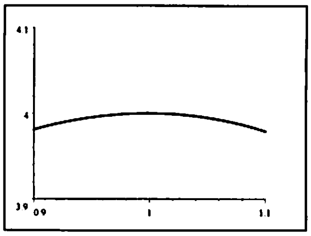
49. Local maximum: 0 at $x = 0$ and 1 at $x = 3$. Local minimum: -2 at $x = -2$ and -1 at $x = 1$.

50. Local maximum: 3 at $x = -2$ and 2 at $x = 1$. Local minimum: 0 at $x = -1$ and -1 at $x = 2$.

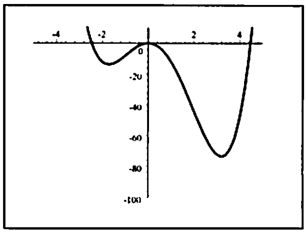
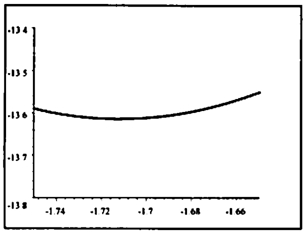
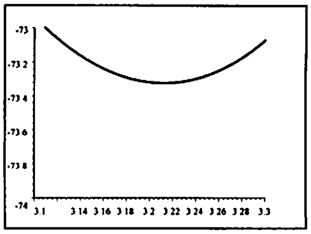
51. In the first graph, we see that $f(x) = x^3 - x$ has a local minimum and a local maximum. Smaller x- and y-ranges show that $f(x)$ has a local maximum of about 0.38 when $x \approx -0.58$ and a local minimum of about -0.38 when $x \approx 0.58$.

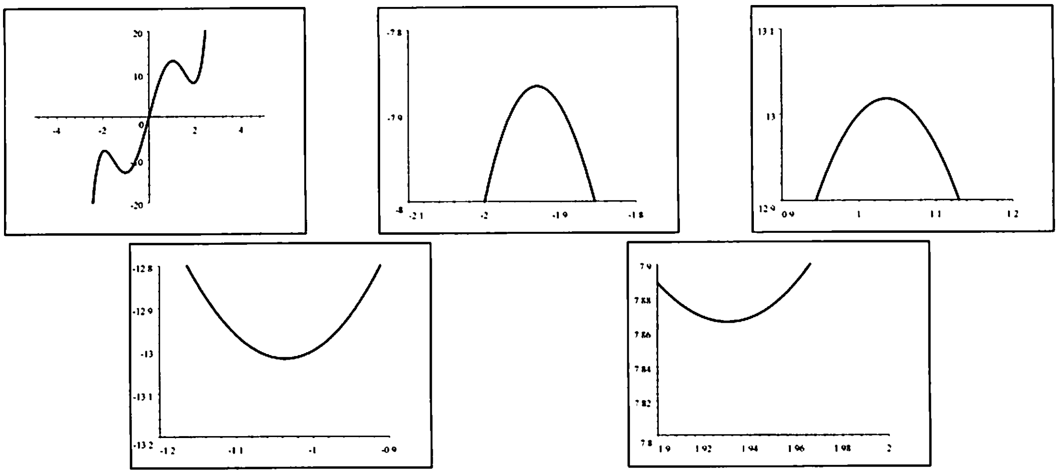
52. In the first graph, we see that $f(x) = 3 + x + x^2 - x^3$ has a local minimum and a local maximum. Smaller x- and y-ranges show that $f(x)$ has a local maximum of about 4.00 when $x \approx 1.00$ and a local minimum of about 2.81 when $x \approx -0.33$.

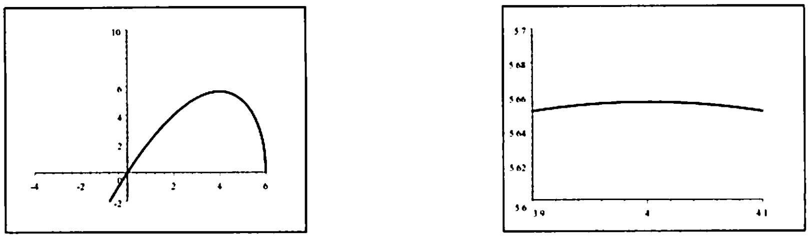
53. In the first graph, we see that $g(x) = x^4 - 2x^3 - 11x^2$ has two local minimums and a local maximum. The local maximum is $g(x) = 0$ when $x = 0$. Smaller x- and y-ranges show that local minima are $g(x) \approx -13.61$ when $x \approx -1.71$ and $g(x) \approx -73.32$ when $x \approx 3.21$.

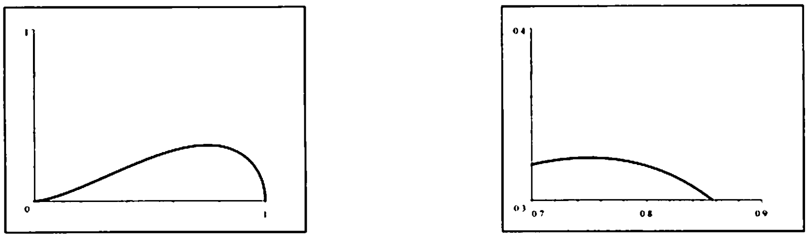
54. In the first graph, we see that $g(x) = x^5 - 8x^3 + 20x$ has two local minimums and two local maximums. The local maximums are $g(x) \approx -7.87$ when $x \approx -1.93$ and $g(x) \approx 13.02$ when $x = 1.04$. Smaller x- and y-ranges show that local minimums are $g(x) \approx -13.02$ when $x = -1.04$ and $g(x) \approx 7.87$ when $x \approx 1.93$. Notice that since $g(x)$ is odd, the local maxima and minima are related.



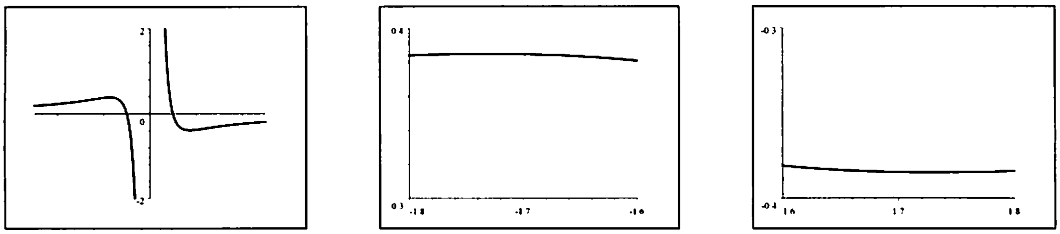
55. In the first graph, we see that $U(x) = x\sqrt{6-x}$ only has a local maximum. Smaller x- and y-ranges show that $U(x)$ has a local maximum of about 5.66 when $x \approx 4.00$.



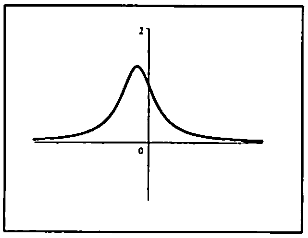
56. In the first viewing rectangle below, we see that $U(x) = x\sqrt{x - x^2}$ has only a local maximum. Smaller x- and y-ranges show that $U(x)$ has a local maximum of about 0.32 when $x \approx 0.75$.

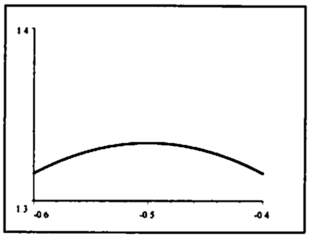


57. In the first graph, we see that $V(x) = \dfrac{1-x^2}{x^3}$ has a local minimum and a local maximum. Smaller x- and y-ranges show that $V(x)$ has a local maximum of about 0.38 when $x \approx -1.73$ and a local minimum of about -0.38 when $x \approx 1.73$.



58. In the first viewing rectangle below, we see that $V(x) = \dfrac{1}{x^2 + x + 1}$ only has a local maximum. Smaller x- and y-ranges show that $V(x)$ has a local maximum of about 1.33 when $x \approx -0.50$.





59. $y = f(t) = 40t - 16t^2 = -16\left(t^2 - \frac{5}{2}\right) = -16\left[t^2 - \frac{5}{2}t + \left(\frac{5}{4}\right)^2\right] + 16\left(\frac{5}{4}\right)^2 = -16\left(t - \frac{5}{4}\right)^2 + 25$. Thus the maximum height attained by the ball is $f\left(\frac{5}{4}\right) = 25$ feet.

60. **(a)** $y = -0.005x^2 + x + 5 = -0.005\left(x^2 - 200x\right) + 5 = -0.005\left(x^2 - 200x + 10,000\right) + 5 + 50 = -0.005\left(x - 100\right)^2 + 55$. Thus the maximum height attained by the football is 55 ft.

(b) We solve for $y = 0$: $0 = -0.005x^2 + x + 5$. Using the quadratic formula, we have $x = \frac{-1 \pm \sqrt{1^2 - (4)(-0.005)(5)}}{2(-0.005)} = \frac{-1 \pm \sqrt{1.1}}{-0.01} = 100 \pm 100\sqrt{1.1}$. Since he throws the football downfield, we take the positive root, so $x = 100 + 100\sqrt{1.1} \approx 204.9$ feet.

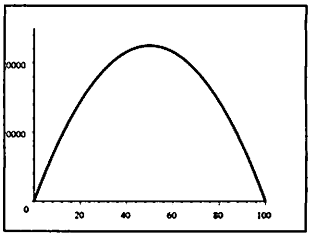
61. $R(x) = 80x - 0.4x^2 = -0.4\left(x^2 - 200x\right) = -0.4\left(x^2 - 200x + 10{,}000\right) + 4{,}000 = -0.4\left(x - 100\right)^2 + 4{,}000$. So revenue is maximized at \$4,000 when 100 units are sold.

62. $P(x) = -0.001x^2 + 3x - 1800 = -0.001\left(x^2 - 3000x\right) - 1800 = -0.001\left(x^2 - 3000x + 2,250,000\right) - 1800 + 2250 = -0.001\left(x - 1500\right)^2 + 450$. The vendor's maximum profit occurs when he sells 1500 cans and the profit is \$450.

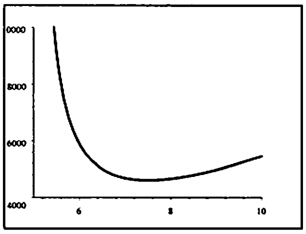
63. $E(n) = \frac{2}{3}n - \frac{1}{90}n^2 = -\frac{1}{90}\left(n^2 - 60n\right) = -\frac{1}{90}\left(n^2 - 60n + 900\right) + 10 = -\dfrac{1}{90}\left(n - 30\right)^2 + 10$. Since the maximum of the function occurs when $n = 30$, the viewer should watch the commercial 30 times for maximum effectiveness.

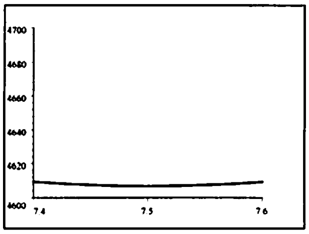
64. $C(t) = 0.06t - 0.0002t^2 = -0.0002\left(t^2 - 300t\right) = -0.0002\left(t^2 - 300t + 22,500\right) + 4.5 = -0.0002\left(t - 150\right)^2 + 4.5$. The maximum concentration of 4.5 mg/L occurs after 150 minutes.

65. Graphing $A(n) = n(900 - 9n)$ in the viewing rectangle $[0, 100]$ by $[0, 25000]$, we see that maximum yield of apples occurs when there are 50 trees per acre.

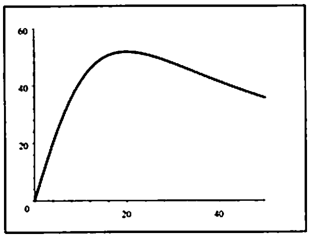


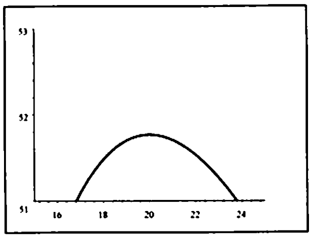
66. In the first graph, we see the general location of the minimum of $E(v) = 2.73v^3 \dfrac{10}{v - 5}$. In the second graph, we isolate the minimum, and from this graph, we see that energy is minimized when $v \approx 7.5$ mi/h.



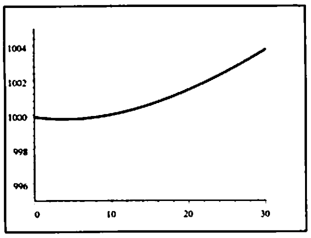


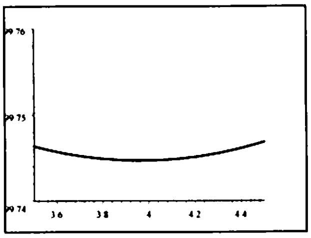
67. In the first graph, we see the general location of the maximum of $N(s) = \dfrac{88s}{17 + 17\left(\frac{s}{20}\right)^2}$. In the second graph we isolate the maximum, and from this graph we see that at the speed of 20 mi/h the largest number of cars that can use the highway safely is 52.



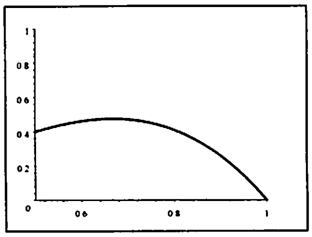
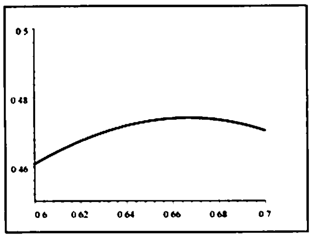


68. In the first graph, we see the general location of the minimum of $V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$ is around $T = 4$. In the second graph, we isolate the minimum, and from this graph, we see that the minimum volume of 1 kg of water occurs at $T \approx 3.96°$ C.





69. In the first graph, we see the general location of the maximum of $v(r) = 3.2(1 - r)r^2$ is around $r = 0.7$ cm. In the second graph, we isolate the maximum, and from this graph we see that at the maximum velocity is 0.47 when $r \approx 0.67$ cm.

70. Numerous answers are possible.

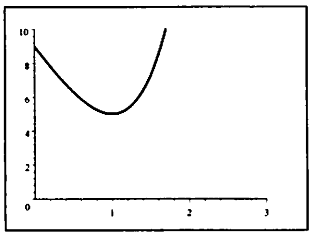
71. (a) If $x = a$ is a local maximum of $f(x)$ then $f(a) \geq f(x) \geq 0$ for all x around $x = a$. So $[g(a)]^2 \geq [g(x)]^2$ and thus $g(a) \geq g(x)$. Similarly, if $x = b$ is a local minimum of $f(x)$, then $f(x) \geq f(b) \geq 0$ for all x around $x = b$. So $[g(x)]^2 \geq [g(b)]^2$ and thus $g(x) \geq g(b)$.

(b) Using the distance formula,

$$g(x) = \sqrt{(x-3)^2 + (x^2 - 0)^2} = \sqrt{x^4 + x^2 - 6x + 9}$$

(c) Let $f(x) = x^4 + x^2 - 6x + 9$. From the graph, we see that $f(x)$ has a minimum at $x = 1$. Thus $g(x)$ also has a minimum at $x = 1$ and this minimum value is

$$g(1) = \sqrt{1^4 + 1^2 - 6(1) + 9} = \sqrt{5}.$$



72. $f(x) = 3 + 4x^2 - x^4$. Substituting $t = x^2$, we have $f(\sqrt{t}) = 3 + 4t - t^2 = -(t^2 - 4t) + 3 = -(t^2 - 4t + 4) + 3 + 4 = -(t-2)^2 + 7$. Therefore, the function $f(\sqrt{t}) = f(x)$ has a maximum value of 7.

2.6 Modeling with Functions

1. Let w be the width of the building lot. Then the length of the lot is $3w$. So the area of the building lot is $A(w) = 3w^2$, $w > 0$.

2. Let w be the width of the poster. Then the length of the poster is $w + 10$. So the area of the poster is $A(w) = w(w+10) = w^2 + 10w$.

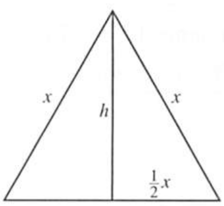
3. Let w be the width of the base of the rectangle. Then the height of the rectangle is $\frac{1}{2}w$. Thus the volume of the box is given by the function $V(w) = \frac{1}{2}w^3$, $w > 0$.

4. Let r be the radius of the cylinder. Then the height of the cylinder is $4r$. Since for a cylinder $V = \pi r^2 h$, the volume of the cylinder is given by the function $V(r) = \pi r^2 (4r) = 4\pi r^3$.

5. Let P be the perimeter of the rectangle and y be the length of the other side. Since $P = 2x + 2y$ and the perimeter is 20, we have $2x + 2y = 20 \quad \Leftrightarrow \quad x + y = 10 \quad \Leftrightarrow \quad y = 10 - x$. Since area is $A = xy$, substituting gives $A(x) = x(10 - x) = 10x - x^2$, and since A must be positive, the domain is $0 < x < 10$.

6. Let A be the area and y be the length of the other side. Then $A = xy = 16 \quad \Leftrightarrow \quad y = \dfrac{16}{x}$. Substituting into $P = 2x + 2y$ gives $P = 2x + 2 \cdot \dfrac{16}{x} = 2x + \dfrac{32}{x}$, where $x > 0$.

7.



Let h be the height of an altitude of the equilateral triangle whose side has length x, as shown in the diagram. Thus the area is given by $A = \frac{1}{2}xh$. By the Pythagorean Theorem, $h^2 + \left(\frac{1}{2}x\right)^2 = x^2 \quad \Leftrightarrow \quad h^2 + \frac{1}{4}x^2 = x^2 \quad \Leftrightarrow$ $h^2 = \frac{3}{4}x^2 \Leftrightarrow h = \frac{\sqrt{3}}{2}x$. Substituting into the area of a triangle, we get $A(x) = \frac{1}{2}xh = \frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, $x > 0$.

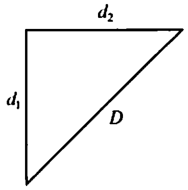
8. Let d represent the length of any side of a cube. Then the surface area is $S = 6d^2$, and the volume is $V = d^3 \quad \Leftrightarrow$ $d = \sqrt[3]{V}$. Substituting for d gives $S(V) = 6\left(\sqrt[3]{V}\right)^2 = 6V^{2/3}$, $V > 0$.

9. We solve for r in the formula for the area of a circle. This gives $A = \pi r^2 \quad \Leftrightarrow \quad r^2 = \dfrac{A}{\pi} \quad \Rightarrow \quad r = \sqrt{\dfrac{A}{\pi}}$, so the model is $r(A) = \sqrt{\dfrac{A}{\pi}}$, $A > 0$.

10. Let r be the radius of a circle. Then the area is $A = \pi r^2$, and the circumference is $C = 2\pi r \quad \Leftrightarrow \quad r = \dfrac{C}{2\pi}$. Substituting for r gives $A(C) = \pi\left(\dfrac{C}{2\pi}\right)^2 = \dfrac{C^2}{4\pi}$, $C > 0$.

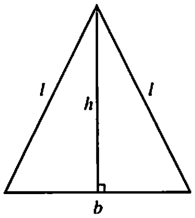
11. Let h be the height of the box in feet. The volume of the box is $V = 60$. Then $x^2 h = 60 \quad \Leftrightarrow \quad h = \dfrac{60}{x^2}$. The surface area, S, of the box is the sum of the area of the 4 sides and the area of the base and top. Thus $S = 4xh + 2x^2 = 4x\left(\dfrac{60}{x^2}\right) + 2x^2 = \dfrac{240}{x} + 2x^2$, so the model is $S(x) = \dfrac{240}{x} + 2x^2$, $x > 0$.

12. By similar triangles, $\dfrac{5}{L} = \dfrac{12}{L+d} \quad \Leftrightarrow \quad 5(L+d) = 12L \quad \Leftrightarrow \quad 5d = 7L \quad \Leftrightarrow \quad L = \dfrac{5d}{7}$. The model is $L(d) = \frac{5}{7}d$.

13. 

Let d_1 be the distance traveled south by the first ship and d_2 be the distance traveled east by the second ship. The first ship travels south for t hours at 5 mi/h, so $d_1 = 15t$ and, similarly, $d_2 = 20t$. Since the ships are traveling at right angles to each other, we can apply the Pythagorean Theorem to get $D^2 = d_1^2 + d_2^2 = (15t)^2 + (20t)^2 = 225t^2 + 400t^2 = 625t^2$.

14. Let n be one of the numbers. Then the other number is $60 - n$, so the product is given by the function $P(n) = n(60 - n) = 60n - n^2$.

15. 

Let b be the length of the base, l be the length of the equal sides, and h be the height in centimeters. Since the perimeter is 8, $2l + b = 8 \Leftrightarrow 2l = 8 - b \Leftrightarrow l = \frac{1}{2}(8 - b)$. By the Pythagorean Theorem, $h^2 + \left(\frac{1}{2}b\right)^2 = l^2 \Leftrightarrow h = \sqrt{l^2 - \frac{1}{4}b^2}$. Therefore the area of the triangle is

$$A = \tfrac{1}{2} \cdot b \cdot h = \tfrac{1}{2} \cdot b\sqrt{l^2 - \tfrac{1}{4}b^2} = \frac{b}{2}\sqrt{\tfrac{1}{4}(8-b)^2 - \tfrac{1}{4}b^2}$$
$$= \frac{b}{4}\sqrt{64 - 16b + b^2 - b^2} = \frac{b}{4}\sqrt{64 - 16b} = \frac{b}{4} \cdot 4\sqrt{4 - b} = b\sqrt{4 - b}$$

so the model is $A(b) = b\sqrt{4 - b}$, $0 < b < 4$.

16. Let x be the length of the shorter leg of the right triangle. Then the length of the other triangle is $2x$. Since it is a right triangle, the length of the hypotenuse is $\sqrt{x^2 + (2x)^2} = \sqrt{5x^2} = \sqrt{5}\,x$ (since $x \geq 0$). Thus the perimeter of the triangle is $P(x) = x + 2x + \sqrt{5}\,x = \left(3 + \sqrt{5}\right)x$.

17. Let w be the length of the rectangle. By the Pythagorean Theorem, $\left(\frac{1}{2}w\right)^2 + h^2 = 10^2 \Leftrightarrow \dfrac{w^2}{4} + h^2 = 10^2 \Leftrightarrow w^2 = 4\left(100 - h^2\right) \Leftrightarrow w = 2\sqrt{100 - h^2}$ (since $w > 0$). Therefore, the area of the rectangle is $A = wh = 2h\sqrt{100 - h^2}$, so the model is $A(h) = 2h\sqrt{100 - h^2}$, $0 < h < 10$.

18. Using the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, we substitute $V = 100$ and solve for h. Thus $100 = \frac{1}{3}\pi r^2 h \Leftrightarrow h(r) = \dfrac{300}{\pi r^2}$.

19. (a) We complete the table.

First number	Second number	Product
1	18	18
2	17	34
3	16	48
4	15	60
5	14	70
6	13	78
7	12	84
8	11	88
9	10	90
10	9	90
11	8	88

From the table we conclude that the numbers is still increasing, the numbers whose product is a maximum should both be 9.5.

(b) Let x be one number: then $19 - x$ is the other number, and so the product, p, is

$p(x) = x(19 - x) = 19x - x^2$.

(c)
$$p(x) = 19x - x^2 = -\left(x^2 - 19x\right)$$
$$= -\left[x^2 - 19x + \left(\tfrac{19}{2}\right)^2\right] + \left(\tfrac{19}{2}\right)^2$$
$$= -(x - 9.5)^2 + 90.25$$

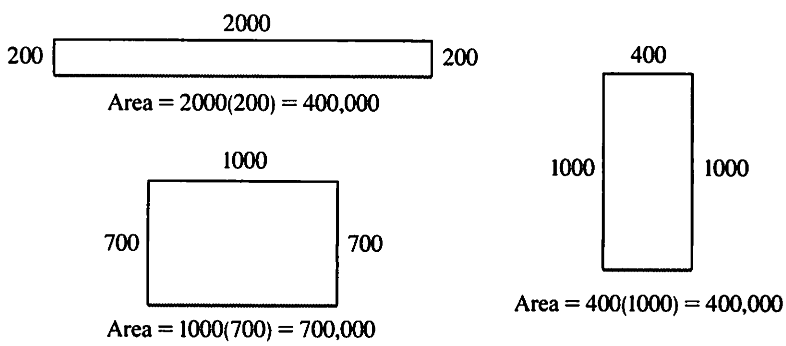
So the product is maximized when the numbers are both 9.5.

20. Let the positive numbers be x and y. Since their sum is 100, we have $x+y=100 \Leftrightarrow y=100-x$. We wish to minimize the sum of squares, which is $S=x^2+y^2=x^2+(100-x)^2$. So $S(x)=x^2+(100-x)^2=x^2+10{,}000-200x+x^2=2x^2-200x+10{,}000=2(x^2-100x)+10{,}000=2(x^2-100x+2500)+10{,}000-5000=2(x-50)^2+5000$. Thus the minimum sum of squares occurs when $x=50$. Then $y=100-50=50$. Therefore both numbers are 50.

21. Let x and y be the two numbers. Since their sum is -24, we have $x+y=-24 \Leftrightarrow y=-x-24$. The product of the two numbers is $P=xy=x(-x-24)=-x^2-24x$, which we wish to maximize. So $P=-x^2-24x=-(x^2+24x)=-(x^2+24x+144)+144=-(x+12)^2+144$. Thus the maximum product is 144, and it occurs when $x=-12$ and $y=-(-12)-24=-12$. Thus the two numbers are -12 and -12.

22. Let w and l be the width and the length of the rectangle in feet. We want all rectangles with perimeter equal to 20, so we have $2w+2l=20 \Leftrightarrow l=10-w$. The area of a rectangle is given by $A(w)=l\cdot w=(10-w)w=10w-w^2=-(w^2-10w)=-(w^2-10w+25)+25=-(w-5)^2+25$. So the area is maximized when $w=5$, and hence the largest rectangle is a square where the dimension of each side is 5 feet.

23. (a) Let x be the width of the field (in feet) and l be the length of the field (in feet). Since the farmer has 2400 ft of fencing we must have $2x+l=2400$.



Width	Length	Area
200	2000	400,000
300	1800	540,000
400	1600	640,000
500	1400	700,000
600	1200	720,000
700	1000	700,000
800	800	640,000

It appears that the field of largest area is about 600 ft × 1200 ft.

(b) Let x be the width of the field (in feet) and l be the length of the field (in feet). Since the farmer has 2400 ft of fencing we must have $2x+l=2400 \Leftrightarrow l=2400-2x$. The area of the fenced-in field is given by $A(x)=l\cdot x=(2400-2x)x=-2x^2+2400x=-2(x^2-1200x)$.

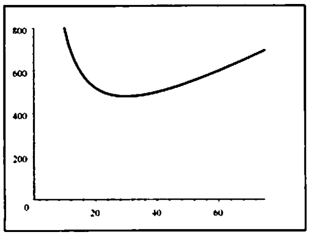
(c) The area is $A(x)=-2(x^2-1200x+600^2)+2(600^2)=-2(x-600)^2+720000$. So the maximum area occurs when $x=600$ feet and $l=2400-2(600)=1200$ feet.

24. (a) Let w be the width of the rectangular area (in feet) and l be the length of the field (in feet). Since the farmer has 750 feet of fencing, we must have $5w+2l=750 \Leftrightarrow 2l=750-5w \Leftrightarrow l=\frac{5}{2}(150-w)$. Thus the total area of the four pens is $A(w)=l\cdot w=\frac{5}{2}w(150-w)=-\frac{5}{2}(w^2-150w)$.

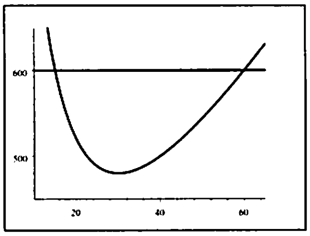
(b) We complete the square to get $A(w)=-\frac{5}{2}(w^2-150w)=-\frac{5}{2}(w^2-150w+75^2)+\left(\frac{5}{2}\right)\cdot 75^2=-\frac{5}{2}(w-75)^2+14062.5$. Therefore, the largest possible total area of the four pens is 14,062.5 square feet.

25. **(a)** Let x be the length of the fence along the road. If the area is 1200, we have $1200 = x\cdot$ width, so the width of the garden is $\dfrac{1200}{x}$. Then the cost of the fence is given by the function $C(x) = 5(x) + 3\left[x + 2\cdot\dfrac{1200}{x}\right] = 8x + \dfrac{7200}{x}$.

(b) We graph the function $y = C(x)$ in the viewing rectangle $[0, 75] \times [0, 800]$. From this we get the cost is minimized when $x = 30$ ft. Then the width is $\frac{1200}{30} = 40$ ft. So the length is 30 ft and the width is 40 ft.



(c) We graph the function $y = C(x)$ and $y = 600$ in the viewing rectangle $[10, 65] \times [450, 650]$. From this we get that the cost is at most \$600 when $15 \le x \le 60$. So the range of lengths he can fence along the road is 15 feet to 60 feet.



26. **(a)** Let x be the length of wire in cm that is bent into a square. So $10 - x$ is the length of wire in cm that is bent into the second square. The width of each square is $\dfrac{x}{4}$ and $\dfrac{10-x}{4}$, and the area of each square is $\left(\dfrac{x}{4}\right)^2 = \dfrac{x^2}{16}$ and $\left(\dfrac{10-x}{4}\right)^2 = \dfrac{100-20x+x^2}{16}$. Thus the sum of the areas is $A(x) = \dfrac{x^2}{16} + \dfrac{100-20x+x^2}{16} = \dfrac{100-20x+2x^2}{16} = \frac{1}{8}x^2 - \frac{5}{4}x + \frac{25}{4}$.

(b) We complete the square. $A(x) = \frac{1}{8}x^2 - \frac{5}{4}x + \frac{25}{4} = \frac{1}{8}\left(x^2 - 10x\right) + \frac{25}{4} = \frac{1}{8}\left(x^2 - 10x + 25\right) + \dfrac{25}{4} - \frac{25}{8} = \frac{1}{8}(x-5)^2 + \frac{25}{8}$ So the minimum area is $\frac{25}{8}$ cm^2 when each piece is 5 cm long.

27. **(a)** Let p be the price of the ticket. So $10 - p$ is the difference in ticket price and therefore the number of tickets sold is $27{,}000 + 3000(10 - p) = 57{,}000 - 3000p$. Thus the revenue is $R(p) = p(57{,}000 - 3000p) = 57{,}000p - 3000p^2$.

(b) $R(p) = 0 = p(57{,}000 - 3000p)$. So $p = 0$ *or* $= \dfrac{57{,}000}{3000} = 19$. So at \$19 no one will come.

(c) We complete the square:

$$\begin{aligned} R(p) &= 57{,}000p - 3000p^2 = -3000\left(p^2 - 19p\right) = -3000\left(p^2 - 19p + \tfrac{19^2}{4}\right) + 270{,}750 \\ &= -3000\left(p - \tfrac{19}{2}\right)^2 + 270{,}750 \end{aligned}$$

The revenue is maximized when $p = \frac{19}{2}$, and so the price should be set at \$9.50.

28. **(a)** Let x be the number of one dollar increases in the price of a bird feeder. So the selling price will be $10+x$ dollars, and the number of bird feeders sold will be $20 - 2x$. The revenue from the sales will be $(10 + x)(20 - 2x)$, and the cost will be $6(20 - 2x)$. The profits will be $P(x) = (10 + x)(20 - 2x) - 6(20 - 2x) = (4 + x)(20 - 2x) = 80 + 12x - 2x^2$.

(b) Completing the square we get $P(x) = 80 + 12x - 2x^2 = -2\left(x^2 - 6x\right) + 80 = -2\left(x^2 - 6x + 9\right) + 80 + 18 = -2(x-3)^2 + 98$. Thus the profit would be maximized at \$98 when $x = 3$. So the bird society should set the selling price at $10 + x = \$13$.

29. (a) Let h be the height in feet of the straight portion of the window. The circumference of the semicircle is $C = \frac{1}{2}\pi x$. Since the perimeter of the window is 30 feet, we have $x + 2h + \frac{1}{2}\pi x = 30$. Solving for h, we get $2h = 30 - x - \frac{1}{2}\pi x \quad \Leftrightarrow \quad h = 15 - \frac{1}{2}x - \frac{1}{4}\pi x$. The area of the window is $A(x) = xh + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = x\left(15 - \frac{1}{2}x - \frac{1}{4}\pi x\right) + \frac{1}{8}\pi x^2 = 15x - \frac{1}{2}x^2 - \frac{1}{8}\pi x^2$.

(b) $A(x) = 15x - \frac{1}{2}x^2 - \frac{1}{8}\pi x^2 = 15x - \frac{1}{8}(\pi + 4)x^2 = -\frac{1}{8}(\pi + 4)\left[x^2 - \frac{120}{\pi + 4}x\right]$

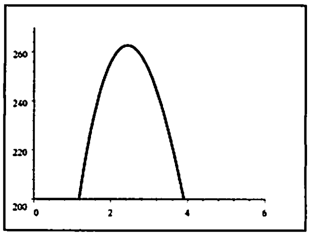
$$= -\frac{1}{8}(\pi + 4)\left[x^2 - \frac{120}{\pi + 4}x + \left(\frac{60}{\pi + 4}\right)^2\right] + \frac{450}{\pi + 4} = -\frac{1}{8}(\pi + 4)\left(x - \frac{60}{\pi + 4}\right)^2 + \frac{450}{\pi + 4}$$

The area is maximized when $x = \frac{60}{\pi + 4} \approx 8.40$, and hence $h \approx 15 - \frac{1}{2}(8.40) - \frac{1}{4}\pi(8.40) \approx 4.20$.

30. (a) The height of the box is x, the width of the box is $12 - 2x$, and the length of the box is $20 - 2x$. Therefore, the volume of the box is

$$V(x) = x(12 - 2x)(20 - 2x) = 4x^3 - 64x^2 + 240x, 0 < x < 6$$

(b) We graph the function $y = V(x)$ in the viewing rectangle $[0, 6] \times [200, 270]$.

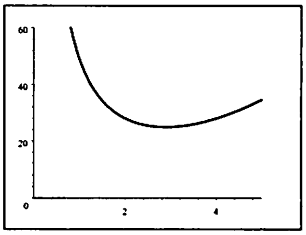


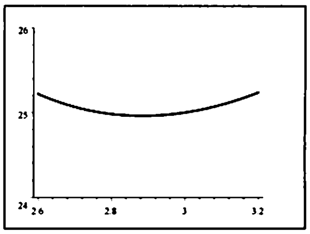
From the calculator we get that the volume of the box is greater than 200 in^3 for $1.174 \leq x \leq 3.898$ (accurate to 3 decimal places).

(c) From the graph, the volume of the box with the largest volume is 262.682 in^3 when $x \approx 2.427$.

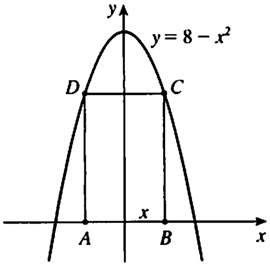
31. (a) Let x be the length of one side of the base and let h be the height of the box in feet. Since the volume of the box is $V = x^2 h = 12$, we have $x^2 h = 12 \quad \Leftrightarrow \quad h = \frac{12}{x^2}$. The surface area, A, of the box is sum of the area of the four sides and the area of the base. Thus the surface area of the box is given by the formula $A(x) = 4xh + x^2 = 4x\left(\frac{12}{x^2}\right) + x^2 = \frac{48}{x} + x^2, x > 0$.

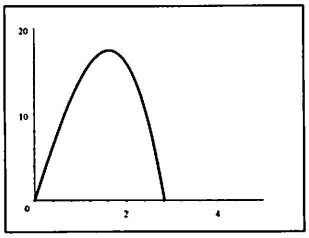
(b) The function $y = A(x)$ is shown in the first viewing rectangle below. In the second viewing rectangle, we isolate the minimum, and we see that the amount of material is minimized when x (the length and width) is 2.88 ft. Then the height is $h = \frac{12}{x^2} \approx 1.44$ ft.

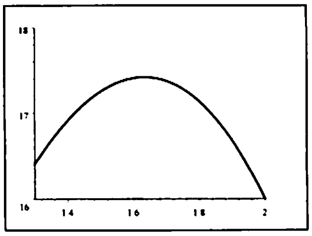




32. Let A, B, C, and D be the vertices of a rectangle with base AB on the x-axis and its other two vertices C and D above the x-axis and lying on the parabola $y = 8 - x^2$. Let C have the coordinates (x, y), $x > 0$. By symmetry, the coordinates of D must be $(-x, y)$. So the width of the rectangle is $2x$, and the length is $y = 8 - x^2$. Thus the area of the rectangle is $A(x) = \text{length} \cdot \text{width} = 2x(8 - x^2) = 16x - 2x^3$. The graphs of $A(x)$ below show that the area is maximized when $x \approx 1.63$. Hence the maximum area occurs when the width is 3.26 and the length is 5.33.

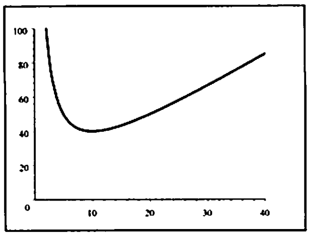


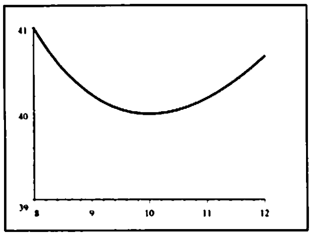




33. **(a)** Let w be the width of the pen and l be the length in meters. We use the area to establish a relationship between w and l. Since the area is 100 m^2, we have $l \cdot w = 100 \quad \Leftrightarrow \quad l = \dfrac{100}{w}$. So the amount of fencing used is $F = 2l + 2w = 2\left(\dfrac{100}{w}\right) + 2w = \dfrac{200 + 2w^2}{w}$.

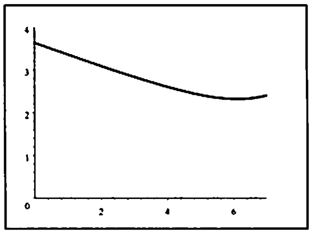
(b) Using a graphing device, we first graph F in the viewing rectangle $[0, 40]$ by $[0, 100]$, and locate the approximate location of the minimum value. In the second viewing rectangle, $[8, 12]$ by $[39, 41]$, we see that the minimum value of F occurs when $w = 10$. Therefore the pen should be a square with side 10 m.





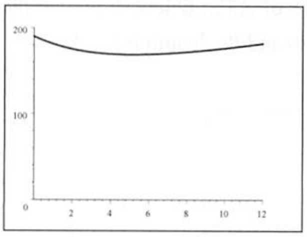
34. **(a)** Let t_1 represent the time, in hours, spent walking, and let t_2 represent the time spent rowing. Since the distance walked is x and the walking speed is 5 mi/h, the time spent walking is $t_1 = \frac{1}{5}x$. By the Pythagorean Theorem, the distance rowed is $d = \sqrt{2^2 + (7 - x)^2} = \sqrt{x^2 - 14x + 53}$, and so the time spent rowing is $t_2 = \frac{1}{2} \cdot \sqrt{x^2 - 14x + 53}$. Thus the total time is $T(x) = \frac{1}{2}\sqrt{x^2 - 14x + 53} + \frac{1}{5}x$.

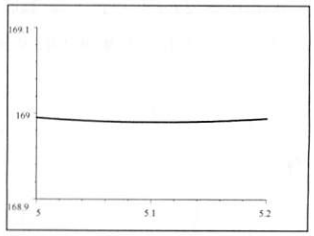
(b) We graph $y = T(x)$. Using the zoom function, we see that T is minimized when $x \approx 6.13$. He should land at a point 6.13 miles from point B.



35. (a) Let x be the distance from point B to C, in miles. Then the distance from A to C is $\sqrt{x^2+25}$, and the energy used in flying from A to C then C to D is $f(x)=14\sqrt{x^2+25}+10(12-x)$.

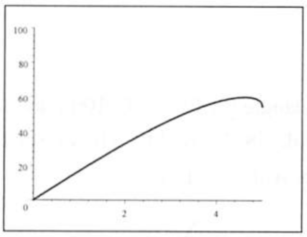
(b) By using a graphing device, the energy expenditure is minimized when the distance from B to C is about 5.1 miles.

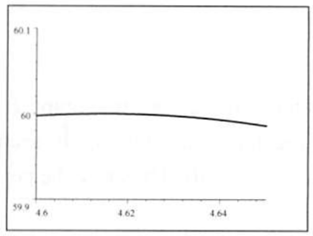




36. (a) Using the Pythagorean Theorem, we have that the height of the upper triangles is $\sqrt{25-x^2}$ and the height of the lower triangles is $\sqrt{144-x^2}$. So the area of the each of the upper triangles is $\frac{1}{2}x\sqrt{25-x^2}$, and the area of the each of the lower triangles is $\frac{1}{2}x\sqrt{144-x^2}$. Since there are two upper triangles and two lower triangles, we get that the total area is $A(x)=2\cdot\left[\frac{1}{2}x\sqrt{25-x^2}\right]+2\cdot\left[\frac{1}{2}x\sqrt{144-x^2}\right]=x\left(\sqrt{25-x^2}+\sqrt{144-x^2}\right)$.

(b) The function $y=A(x)=x\left(\sqrt{25-x^2}+\sqrt{144-x^2}\right)$ is shown in the first viewing rectangle below. In the second viewing rectangle, we isolate the maximum, and we see that the area of the kite is maximized when $x\approx 4.615$. So the length of the horizontal crosspiece must be $2\cdot 4.615=9.23$. The length of the vertical crosspiece is $\sqrt{5^2-(4.615)^2}+\sqrt{12^2-(4.615)^2}\approx 13.00$.





2.7 Combining Functions

1. $f(x)=x-3$ has domain $(-\infty,\infty)$. $g(x)=x^2$ has domain $(-\infty,\infty)$. The intersection of the domains of f and g is $(-\infty,\infty)$.

$(f+g)(x)=(x-3)+(x^2)=x^2+x-3$, and the domain is $(-\infty,\infty)$.

$(f-g)(x)=(x-3)-(x^2)=-x^2+x-3$, and the domain is $(-\infty,\infty)$.

$(fg)(x)=(x-3)(x^2)=x^3-3x^2$, and the domain is $(-\infty,\infty)$.

$\left(\dfrac{f}{g}\right)(x)=\dfrac{x-3}{x^2}$, and the domain is $\{x\mid x\neq 0\}$.

2. $f(x)=x^2+2x$ has domain $(-\infty,\infty)$. $g(x)=3x^2-1$ has domain $(-\infty,\infty)$. The intersection of the domains of f and g is $(-\infty,\infty)$.

$(f+g)(x)=x^2+2x+(3x^2-1)=4x^2+2x-1$, and the domain is $(-\infty,\infty)$.

$(f-g)(x)=x^2+2x-(3x^2-1)=-2x^2+2x+1$, and the domain is $(-\infty,\infty)$.

$(fg)(x)=(x^2+2x)(3x^2-1)=3x^4+6x^3-x^2-2x$, and the domain is $(-\infty,\infty)$.

$\left(\dfrac{f}{g}\right)(x)=\dfrac{x^2+2x}{3x^2-1}$, $3x^2-1\neq 0 \quad\Rightarrow\quad x\neq\pm\frac{\sqrt{3}}{3}$, and the domain is $\left\{x\mid x\neq\pm\frac{\sqrt{3}}{3}\right\}$.

3. $f(x) = \sqrt{4 - x^2}$, has domain $[-2, 2]$. $g(x) = \sqrt{1 + x}$, has domain $[-1, \infty)$. The intersection of the domains of f and g is $[-1, 2]$.

$(f + g)(x) = \sqrt{4 - x^2} + \sqrt{1 + x}$, and the domain is $[-1, 2]$.

$(f - g)(x) = \sqrt{4 - x^2} - \sqrt{1 + x}$, and the domain is $[-1, 2]$.

$(fg)(x) = \sqrt{4 - x^2}\sqrt{1 + x} = \sqrt{-x^3 - x^2 + 4x + 4}$, and the domain is $[-1, 2]$.

$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{4 - x^2}}{\sqrt{1 + x}} = \sqrt{\frac{4 - x^2}{1 + x}}$, and the domain is $(-1, 2]$.

4. $f(x) = \sqrt{9 - x^2}$ has domain $[-3, 3]$. $g(x) = \sqrt{x^2 - 4}$ has domain $(-\infty, -2] \cup [2, \infty)$. The intersection of the domains of f and g is $[-3, -2] \cup [2, 3]$.

$(f + g)(x) = \sqrt{9 - x^2} + \sqrt{x^2 - 4}$, and the domain is $[-3, -2] \cup [2, 3]$.

$(f - g)(x) = \sqrt{9 - x^2} - \sqrt{x^2 - 4}$, and the domain is $[-3, -2] \cup [2, 3]$.

$(fg)(x) = \sqrt{9 - x^2} \cdot \sqrt{x^2 - 4} = \sqrt{-x^4 + 13x^2 - 36}$, and the domain is $[-3, -2] \cup [2, 3]$.

$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{9 - x^2}}{\sqrt{x^2 - 4}} = \sqrt{\frac{9 - x^2}{x^2 - 4}}$, and the domain is $[-3, -2) \cup (2, 3]$.

5. $f(x) = \frac{2}{x}$ has domain $x \neq 0$. $g(x) = \frac{4}{x + 4}$, has domain $x \neq -4$. The intersection of the domains of f and g is $\{x \mid x \neq 0, -4\}$; in interval notation, this is $(-\infty, -4) \cup (-4, 0) \cup (0, \infty)$.

$(f + g)(x) = \frac{2}{x} + \frac{4}{x + 4} = \frac{2}{x} + \frac{4}{x + 4} = \frac{2(3x + 4)}{x(x + 4)}$, and the domain is $(-\infty, -4) \cup (-4, 0) \cup (0, \infty)$.

$(f - g)(x) = \frac{2}{x} - \frac{4}{x + 4} = -\frac{2(x - 4)}{x(x + 4)}$, and the domain is $(-\infty, -4) \cup (-4, 0) \cup (0, \infty)$.

$(fg)(x) = \frac{2}{x} \cdot \frac{4}{x + 4} = \frac{8}{x(x + 4)}$, and the domain is $(-\infty, -4) \cup (-4, 0) \cup (0, \infty)$.

$\left(\frac{f}{g}\right)(x) = \frac{\frac{2}{x}}{\frac{4}{x + 4}} = \frac{x + 4}{2x}$, and the domain is $(-\infty, -4) \cup (-4, 0) \cup (0, \infty)$.

6. $f(x) = \frac{2}{x + 1}$ has domain $x \neq -1$. $g(x) = \frac{x}{x + 1}$ has domain $x \neq -1$. The intersection of the domains of f and g is $\{x \mid x \neq -1\}$; in interval notation, this is $(-\infty, -1) \cup (-1, \infty)$.

$(f + g)(x) = \frac{2}{x + 1} + \frac{x}{x + 1} = \frac{x + 2}{x + 1}$, and the domain is $(-\infty, -1) \cup (-1, \infty)$.

$(f - g)(x) = \frac{2}{x + 1} - \frac{x}{x + 1} = \frac{2 - x}{x + 1}$, and the domain is $(-\infty, -1) \cup (-1, \infty)$.

$(fg)(x) = \frac{2}{x + 1} \cdot \frac{x}{x + 1} = \frac{2x}{(x + 1)^2}$, and the domain is $(-\infty, -1) \cup (-1, \infty)$.

$\left(\frac{f}{g}\right)(x) = \frac{\frac{2}{x + 1}}{\frac{x}{x + 1}} = \frac{2}{x}$, so $x \neq 0$ as well. Thus the domain is $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$.

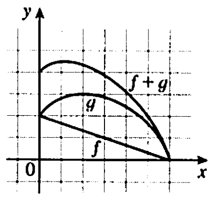
7. $f(x) = \sqrt{x} + \sqrt{1 - x}$. The domain of $\sqrt{x}$ is $[0, \infty)$, and the domain of $\sqrt{1 - x}$ is $(-\infty, 1]$. Thus the domain is $(-\infty, 1] \cap [0, \infty) = [0, 1]$.

8. $g(x) = \sqrt{x + 1} + \frac{1}{x}$. The domain of $\sqrt{x + 1}$ is $[-1, \infty)$, and the domain of $\frac{1}{x}$ is $x \neq 0$. Since $x \neq 0$ is $(-\infty, 0) \cup (0, \infty)$, the domain is $[-1, \infty) \cap \{(-\infty, 0) \cup (0, \infty)\} = [-1, 0) \cup (0, \infty)$.

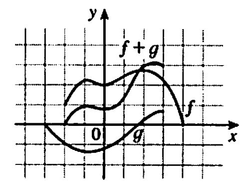
9. $h(x) = (x - 3)^{-1/4} = \frac{1}{(x - 3)^{1/4}}$. Since $1/4$ is an even root and the denominator can not equal 0, $x - 3 > 0 \Leftrightarrow x > 3$. So the domain is $(3, \infty)$.

10. $k(x) = \dfrac{\sqrt{x+3}}{x-1}$. The domain of $\sqrt{x+3}$ is $[-3,\infty)$, and the domain of $\dfrac{1}{x-1}$ is $x \neq 1$. Since $x \neq 1$ is $(-\infty,1)\cup(1,\infty)$, the domain is $[-3,\infty)\cap\{(-\infty,1)\cup(1,\infty)\} = [-3,1)\cup(1,\infty)$.

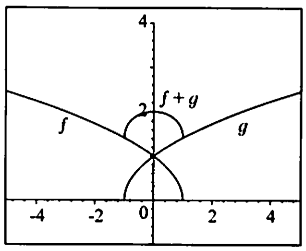
11.



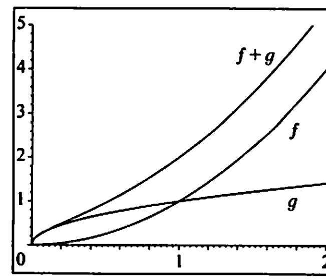
12.



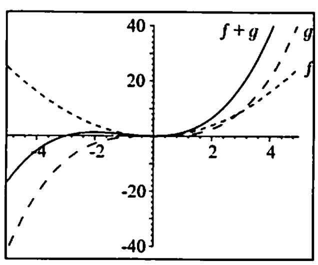
13.



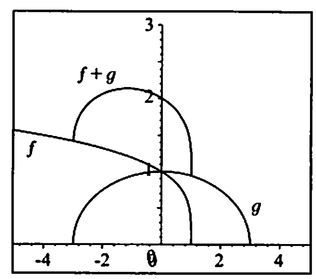
14.



15.



16.



17. (a) $f(g(0)) = f\left(2-(0)^2\right) = f(2) = 3(2)-5 = 1$

(b) $g(f(0)) = g(3(0)-5) = g(-5) = 2-(-5)^2 = -23$

18. (a) $f(f(4)) = f(3(4)-5) = f(7) = 3(7)-5 = 16$

(b) $g(g(3)) = g\left(2-(3)^2\right) = g(-7) = 2-(-7)^2 = -47$

19. (a) $(f\circ g)(-2) = f(g(-2)) = f\left(2-(-2)^2\right) = f(-2) = 3(-2)-5 = -11$

(b) $(g\circ f)(-2) = g(f(-2)) = g(3(-2)-5) = g(-11) = 2-(-11)^2 = -119$

20. (a) $(f\circ f)(-1) = f(f(-1)) = f(3(-1)-5) = f(-8) = 3(-8)-5 = -29$

(b) $(g\circ g)(2) = g(g(2)) = g\left(2-(2)^2\right) = g(-2) = 2-(-2)^2 = -2$

21. (a) $(f\circ g)(x) = f(g(x)) = f\left(2-x^2\right) = 3\left(2-x^2\right)-5 = 6-3x^2-5 = 1-3x^2$

(b) $(g\circ f)(x) = g(f(x)) = g(3x-5) = 2-(3x-5)^2 = 2-\left(9x^2-30x+25\right) = -9x^2+30x-23$

22. (a) $(f\circ f)(x) = f(f(x)) = f(3x-5) = 3(3x-5)-5 = 9x-15-5 = 9x-20$

(b) $(g\circ g)(x) = g(g(x)) = g\left(2-x^2\right) = 2-\left(2-x^2\right)^2 = 2-\left(4-4x^2+x^4\right) = -x^4+4x^2-2$

23. $f(g(2)) = f(5) = 4$

24. $f(0) = 0$, so $g(f(0)) = g(0) = 3$.

25. $(g\circ f)(4) = g(f(4)) = g(2) = 5$

26. $g(0) = 3$, so $(f\circ g)(0) = f(3) = 0$.

27. $(g\circ g)(-2) = g(g(-2)) = g(1) = 4$

28. $f(4) = 2$, so $(f\circ f)(4) = f(2) = -2$.

29. $f(x) = 2x + 3$, has domain $(-\infty, \infty)$; $g(x) = 4x - 1$, has domain $(-\infty, \infty)$.
$(f \circ g)(x) = f(4x - 1) = 2(4x - 1) + 3 = 8x + 1$, and the domain is $(-\infty, \infty)$.
$(g \circ f)(x) = g(2x + 3) = 4(2x + 3) - 1 = 8x + 11$, and the domain is $(-\infty, \infty)$.
$(f \circ f)(x) = f(2x + 3) = 2(2x + 3) + 3 = 4x + 9$, and the domain is $(-\infty, \infty)$.
$(g \circ g)(x) = g(4x - 1) = 4(4x - 1) - 1 = 16x - 5$, and the domain is $(-\infty, \infty)$.

30. $f(x) = 6x - 5$ has domain $(-\infty, \infty)$. $g(x) = \frac{x}{2}$ has domain $(-\infty, \infty)$.
$(f \circ g)(x) = f\left(\frac{x}{2}\right) = 6\left(\frac{x}{2}\right) - 5 = 3x - 5$, and the domain is $(-\infty, \infty)$.
$(g \circ f)(x) = g(6x - 5) = \dfrac{6x - 5}{2} = 3x - \frac{5}{2}$, and the domain is $(-\infty, \infty)$.
$(f \circ f)(x) = f(6x - 5) = 6(6x - 5) - 5 = 36x - 35$, and the domain is $(-\infty, \infty)$.
$(g \circ g)(x) = g\left(\frac{x}{2}\right) = \dfrac{\frac{x}{2}}{2} = \frac{x}{4}$, and the domain is $(-\infty, \infty)$.

31. $f(x) = x^2$, has domain $(-\infty, \infty)$; $g(x) = x + 1$, has domain $(-\infty, \infty)$.
$(f \circ g)(x) = f(x + 1) = (x + 1)^2 = x^2 + 2x + 1$, and the domain is $(-\infty, \infty)$.
$(g \circ f)(x) = g(x^2) = (x^2) + 1 = x^2 + 1$, and the domain is $(-\infty, \infty)$.
$(f \circ f)(x) = f(x^2) = (x^2)^2 = x^4$, and the domain is $(-\infty, \infty)$.
$(g \circ g)(x) = g(x + 1) = (x + 1) + 1 = x + 2$, and the domain is $(-\infty, \infty)$.

32. $f(x) = x^3 + 2$ has domain $(-\infty, \infty)$. $g(x) = \sqrt[3]{x}$ has domain $(-\infty, \infty)$.
$(f \circ g)(x) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 + 2 = x + 2$, and the domain is $(-\infty, \infty)$.
$(g \circ f)(x) = g(x^3 + 2) = \sqrt[3]{x^3 + 2}$ and the domain is $(-\infty, \infty)$.
$(f \circ f)(x) = f(x^3 + 2) = (x^3 + 2)^3 + 2 = x^9 + 6x^6 + 12x^3 + 8 + 2 = x^9 + 6x^6 + 12x^3 + 10$, and the domain is $(-\infty, \infty)$.
$(g \circ g)(x) = g(\sqrt[3]{x}) = \sqrt[3]{\sqrt[3]{x}} = \left(x^{1/3}\right)^{1/3} = x^{1/9}$, and the domain is $(-\infty, \infty)$.

33. $f(x) = \frac{1}{x}$, has domain $\{x \mid x \neq 0\}$; $g(x) = 2x + 4$, has domain $(-\infty, \infty)$.
$(f \circ g)(x) = f(2x + 4) = \dfrac{1}{2x + 4}$. $(f \circ g)(x)$ is defined for $2x + 4 \neq 0 \quad \Leftrightarrow \quad x \neq -2$. So the domain is $\{x \mid x \neq -2\} = (-\infty, -2) \cup (-2, \infty)$.
$(g \circ f)(x) = g\left(\dfrac{1}{x}\right) = 2\left(\dfrac{1}{x}\right) + 4 = \dfrac{2}{x} + 4$, the domain is $\{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.
$(f \circ f)(x) = f\left(\dfrac{1}{x}\right) = \dfrac{1}{\left(\dfrac{1}{x}\right)} = x$. $(f \circ f)(x)$ is defined whenever both $f(x)$ and $f(f(x))$ are defined; that is, whenever $\{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.
$(g \circ g)(x) = g(2x + 4) = 2(2x + 4) + 4 = 4x + 8 + 4 = 4x + 12$, and the domain is $(-\infty, \infty)$.

34. $f(x) = x^2$ has domain $(-\infty, \infty)$. $g(x) = \sqrt{x - 3}$ has domain $[3, \infty)$.
$(f \circ g)(x) = f(\sqrt{x - 3}) = (\sqrt{x - 3})^2 = x - 3$, and the domain is $[3, \infty)$.
$(g \circ f)(x) = g(x^2) = \sqrt{x^2 - 3}$. For the domain we must have $x^2 \geq 3 \quad \Rightarrow \quad x \leq -\sqrt{3}$ or $x \geq \sqrt{3}$. Thus the domain is $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$.
$(f \circ f)(x) = f(x^2) = (x^2)^2 = x^4$, and the domain is $(-\infty, \infty)$.
$(g \circ g)(x) = g(\sqrt{x - 3}) = \sqrt{\sqrt{x - 3} - 3}$. For the domain we must have $\sqrt{x - 3} \geq 3 \quad \Rightarrow \quad x - 3 \geq 9 \quad \Rightarrow$ $x \geq 12$, so the domain is $[12, \infty)$.

35. $f(x) = |x|$, has domain $(-\infty, \infty)$; $g(x) = 2x + 3$, has domain $(-\infty, \infty)$

$(f \circ g)(x) = f(2x + 4) = |2x + 3|$, and the domain is $(-\infty, \infty)$.

$(g \circ f)(x) = g(|x|) = 2|x| + 3$, and the domain is $(-\infty, \infty)$.

$(f \circ f)(x) = f(|x|) = ||x|| = |x|$, and the domain is $(-\infty, \infty)$.

$(g \circ g)(x) = g(2x + 3) = 2(2x + 3) + 3 = 4x + 6 + 3 = 4x + 9$. Domain is $(-\infty, \infty)$.

36. $f(x) = x - 4$ has domain $(-\infty, \infty)$. $g(x) = |x + 4|$ has domain $(-\infty, \infty)$.

$(f \circ g)(x) = f(|x + 4|) = |x + 4| - 4$, and the domain is $(-\infty, \infty)$.

$(g \circ f)(x) = g(x - 4) = |(x - 4) + 4| = |x|$, and the domain is $(-\infty, \infty)$.

$(f \circ f)(x) = f(x - 4) = (x - 4) - 4 = x - 8$, and the domain is $(-\infty, \infty)$.

$(g \circ g)(x) = g(|x + 4|) = ||x + 4| + 4| = |x + 4| + 4$ ($|x + 4| + 4$ is always positive). The domain is $(-\infty, \infty)$.

37. $f(x) = \dfrac{x}{x + 1}$, has domain $\{x \mid x \neq -1\}$; $g(x) = 2x - 1$, has domain $(-\infty, \infty)$

$(f \circ g)(x) = f(2x - 1) = \dfrac{2x - 1}{(2x - 1) + 1} = \dfrac{2x - 1}{2x}$, and the domain is $\{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.

$(g \circ f)(x) = g\left(\dfrac{x}{x + 1}\right) = 2\left(\dfrac{x}{x + 1}\right) - 1 = \dfrac{2x}{x + 1} - 1$, and the domain is $\{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$

$(f \circ f)(x) = f\left(\dfrac{x}{x + 1}\right) = \dfrac{\dfrac{x}{x + 1}}{\dfrac{x}{x + 1} + 1} \cdot \dfrac{x + 1}{x + 1} = \dfrac{x}{x + x + 1} = \dfrac{x}{2x + 1}$. $(f \circ f)(x)$ is defined whenever both $f(x)$ and $f(f(x))$ are defined; that is, whenever $x \neq -1$ and $2x + 1 \neq 0 \quad \Rightarrow \quad x \neq -\frac{1}{2}$, which is $(-\infty, -1) \cup \left(-1, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$.

$(g \circ g)(x) = g(2x - 1) = 2(2x - 1) - 1 = 4x - 2 - 1 = 4x - 3$, and the domain is $(-\infty, \infty)$.

38. $f(x) = \dfrac{1}{\sqrt{x}}$ has domain $\{x \mid x > 0\}$; $g(x) = x^2 - 4x$ has domain $(-\infty, \infty)$.

$(f \circ g)(x) = f\left(x^2 - 4x\right) = \dfrac{1}{\sqrt{x^2 - 4x}}$. $(f \circ g)(x)$ is defined whenever $0 < x^2 - 4x = x(x - 4)$. The product of two numbers is positive either when both numbers are negative or when both numbers are positive. So the domain of $f \circ g$ is $\{x \mid x < 0 \text{ and } x < 4\} \cup \{x \mid x > 0 \text{ and } x > 4\}$ which is $(-\infty, 0) \cup (4, \infty)$.

$(g \circ f)(x) = g\left(\dfrac{1}{\sqrt{x}}\right) = \left(\dfrac{1}{\sqrt{x}}\right)^2 - 4\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{1}{x} - \dfrac{4}{\sqrt{x}}$. $(g \circ f)(x)$ is defined whenever both $f(x)$ and $g(f(x))$ are defined, that is, whenever $x > 0$. So the domain of $g \circ f$ is $(0, \infty)$.

$(f \circ f)(x) = f\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{1}{\sqrt{\dfrac{1}{\sqrt{x}}}} = x^{1/4}$. $(f \circ f)(x)$ is defined whenever both $f(x)$ and $f(f(x))$ are defined, that is, whenever $x > 0$. So the domain of $f \circ f$ is $(0, \infty)$.

$(g \circ g)(x) = g\left(x^2 - 4x\right) = \left(x^2 - 4x\right)^2 - 4\left(x^2 - 4x\right) = x^4 - 8x^3 + 16x^2 - 4x^2 + 16x = x^4 - 8x^3 + 12x^2 + 16x$, and the domain is $(-\infty, \infty)$.

39. $f(x) = \sqrt[3]{x}$, has domain $(-\infty, \infty)$; $g(x) = \sqrt[4]{x}$, has domain $[0, \infty)$.

$(f \circ g)(x) = f(\sqrt[4]{x}) = \sqrt[3]{\sqrt[4]{x}} = \sqrt[12]{x}$. $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. Since $f(x)$ has no restriction, the domain is $[0, \infty)$.

$(g \circ f)(x) = g(\sqrt[3]{x}) = \sqrt[4]{\sqrt[3]{x}} = \sqrt[12]{x}$. $(g \circ f)(x)$ is defined whenever both $f(x)$ and $g(f(x))$ are defined; that is, whenever $x \geq 0$. So the domain is $[0, \infty)$.

$(f \circ f)(x) = f(\sqrt[3]{x}) = \sqrt[3]{\sqrt[3]{x}} = \sqrt[9]{x}$. $(f \circ f)(x)$ is defined whenever both $f(x)$ and $f(f(x))$ are defined. Since $f(x)$ is defined everywhere, the domain is $(-\infty, \infty)$.

$(g \circ g)(x) = g(\sqrt[4]{x}) = \sqrt[4]{\sqrt[4]{x}} = \sqrt[16]{x}$. $(g \circ g)(x)$ is defined whenever both $g(x)$ and $g(g(x))$ are defined; that is, whenever $x \geq 0$. So the domain is $[0, \infty)$.

40. $f(x) = \frac{2}{x}$ has domain $\{x \mid x \neq 0\}$ $g(x) = \frac{x}{x+2}$ has domain $\{x \mid x \neq -2\}$.

$(f \circ g)(x) = f\left(\frac{x}{x+2}\right) = \frac{2}{\frac{x}{x+2}} = \frac{2x+4}{x}$. $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined; that is, whenever $x \neq 0$ and $x \neq -2$. So the domain is $\{x \mid x \neq 0, -2\}$.

$(g \circ f)(x) = g\left(\frac{2}{x}\right) = \frac{\frac{2}{x}}{\frac{2}{x}+2} = \frac{2}{2+2x} = \frac{1}{1+x}$. $(g \circ f)(x)$ is defined whenever both $f(x)$ and $g(f(x))$ are defined; that is, whenever $x \neq 0$ and $x \neq -1$. So the domain is $\{x \mid x \neq 0, -1\}$.

$(f \circ f)(x) = f\left(\frac{2}{x}\right) = \frac{2}{\frac{2}{x}} = x$. $(f \circ f)(x)$ is defined whenever both $f(x)$ and $f(f(x))$ are defined; that is, whenever $x \neq 0$. So the domain is $\{x \mid x \neq 0\}$.

$(g \circ g)(x) = g\left(\frac{x}{x+2}\right) = \frac{\frac{x}{x+2}}{\frac{x}{x+2}+2} = \frac{x}{x+2(x+2)} = \frac{x}{3x+4}$. $(g \circ g)(x)$ is defined whenever both $g(x)$ and $g(g(x))$ are defined; that is whenever $x \neq -2$ and $x \neq -\frac{4}{3}$. So the domain is $\left\{x \mid x \neq -2, -\frac{4}{3}\right\}$.

41. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x-1)) = f\left(\sqrt{x-1}\right) = \sqrt{x-1} - 1$

42. $(g \circ h)(x) = g(x^2+2) = (x^2+2)^3 = x^6 + 6x^4 + 12x^2 + 8$.

$(f \circ g \circ h)(x) = f\left(x^6 + 6x^4 + 12x^2 + 8\right) = \frac{1}{x^6 + 6x^4 + 12x^2 + 8}$.

43. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(\sqrt{x} - 5) = (\sqrt{x} - 5)^4 + 1$

44. $(g \circ h)(x) = g(\sqrt[3]{x}) = \frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}$. $(f \circ g \circ h)(x) = f\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right) = \sqrt{\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}}$.

45. $F(x) = (x-9)^5$. Let $f(x) = x^5$ and $g(x) = x - 9$, then $F(x) = (f \circ g)(x)$.

46. $F(x) = \sqrt{x} + 1$. If $f(x) = x + 1$ and $g(x) = \sqrt{x}$, then $F(x) = (f \circ g)(x)$.

47. $G(x) = \frac{x^2}{x^2+4}$. Let $f(x) = \frac{x}{x+4}$ and $g(x) = x^2$, then $G(x) = (f \circ g)(x)$.

48. $G(x) = \frac{1}{x+3}$. If $f(x) = \frac{1}{x}$ and $g(x) = x + 3$, then $G(x) = (f \circ g)(x)$.

49. $H(x) = \left|1 - x^3\right|$. Let $f(x) = |x|$ and $g(x) = 1 - x^3$, then $H(x) = (f \circ g)(x)$. For Exercises 51 and 53 there are several possible solutions, only one of which is shown.

50. $H(x) = \sqrt{1 + \sqrt{x}}$. If $f(x) = \sqrt{1+x}$ and $g(x) = \sqrt{x}$, then $H(x) = (f \circ g)(x)$. For Exercises 52 and 54 there are several possible solutions only one of which is shown.

51. $F(x) = \frac{1}{x^2+1}$. Let $f(x) = \frac{1}{x}$, $g(x) = x + 1$, and $h(x) = x^2$, then $F(x) = (f \circ g \circ h)(x)$.

52. $F(x) = \sqrt[3]{\sqrt{x} - 1}$. If $g(x) = x - 1$ and $h(x) = \sqrt{x}$, then $(g \circ h)(x) = \sqrt{x} - 1$, and if $f(x) = \sqrt[3]{x}$, then $F(x) = (f \circ g \circ h)(x)$.

53. $G(x) = (4 + \sqrt[3]{x})^9$. Let $f(x) = x^9$, $g(x) = 4 + x$, and $h(x) = \sqrt[3]{x}$, then $G(x) = (f \circ g \circ h)(x)$.

54. $G(x) = \frac{2}{(3+\sqrt{x})^2}$. If $g(x) = 3 + x$ and $h(x) = \sqrt{x}$, then $(g \circ h)(x) = 3 + \sqrt{x}$, and if $f(x) = \frac{2}{x^2}$, then $G(x) = (f \circ g \circ h)(x)$.

55. The price per sticker is $0.15 - 0.000002x$ and the number sold is x, so the revenue is $R(x) = (0.15 - 0.000002x)\,x = 0.15x - 0.000002x^2$.

56. As found in Exercise 55, the revenue is $R(x) = 0.15x - 0.000002x^2$, and the cost is $0.095x - 0.0000005x^2$, so the profit is $P(x) = 0.15x - 0.000002x^2 - \left(0.095x - 0.0000005x^2\right) = 0.055x - 0.0000015x^2$.

57. **(a)** Since the ripple travels at a speed of 60 cm/s, the distance traveled in t seconds is the radius, so $g(t) = 60t$.

(b) The area of a circle is πr^2, so $f(r) = \pi r^2$.

(c) $f \circ g = \pi (g(t))^2 = \pi (60t)^2 = 3600\pi t^2$ cm^2. This function represents the area of the ripple as a function of time.

58. **(a)** Let $f(t)$ be the radius of the spherical balloon in centimeters. Since the radius is increasing at a rate of 1 cm/s, the radius is $f(t) = t$ after t seconds.

(b) The volume of the balloon can be written as $g(r) = \frac{4}{3}\pi r^3$.

(c) $g \circ f = \frac{4}{3}\pi (t)^3 = \frac{4}{3}\pi t^3$. $g \circ f$ represents the volume as a function of time.

59. Let r be the radius of the spherical balloon in centimeters. Since the radius is increasing at a rate of 2 cm/s, the radius is $r = 2t$ after t seconds. Therefore, the surface area of the balloon can be written as $S = 4\pi r^2 = 4\pi (2t)^2 = 4\pi \left(4t^2\right) = 16\pi t^2$.

60. **(a)** $f(x) = 0.80x$

(b) $g(x) = x - 50$

(c) $f \circ g = f(x - 50) = 0.80(x - 50) = 0.80x - 40$. $f \circ g$ represents applying the \$50 coupon, then the 20% discount. $g \circ f = g(0.80x) = 0.80x - 50$. $g \circ f$ represents applying the 20% discount, then the \$50 coupon. So applying the 20% discount, then the \$50 coupon gives the lower price.

61. **(a)** $f(x) = 0.90x$

(b) $g(x) = x - 100$

(c) $f \circ g = f(x - 100) = 0.90(x - 100) = 0.90x - 90$. $f \circ g$ represents applying the \$100 coupon, then the 10% discount. $g \circ f = g(0.90x) = 0.90x - 100$. $g \circ f$ represents applying the 10% discount, then the \$100 coupon. So applying the 10% discount, then the \$100 coupon gives the lower price.

62. Let t be the time since the plane flew over the radar station.

(a) Let s be the distance in miles between the plane and the radar station, and let d be the horizontal distance that the plane has flown. Using the Pythagorean theorem, $s = f(d) = \sqrt{1 + d^2}$.

(b) Since distance=rate× time we have $d = g(t) = 350t$.

(c) $s(t) = (f \circ g)(t) = f(350t) = \sqrt{1 + (350t)^2} = \sqrt{1 + 122{,}500t^2}$.

63. $A(x) = 1.05x$. $(A \circ A)(x) = A(A(x)) = A(1.05x) = 1.05(1.05x) = (1.05)^2 x$.
$(A \circ A \circ A)(x) = A(A \circ A(x)) = A\left((1.05)^2 x\right) = 1.05\left[(1.05)^2 x\right] = (1.05)^3 x$.
$(A \circ A \circ A \circ A)(x) = A(A \circ A \circ A(x)) = A\left((1.05)^3 x\right) = 1.05\left[(1.05)^3 x\right] = (1.05)^4 x$. A represents the amount in the account after 1 year; $A \circ A$ represents the amount in the account after 2 years; $A \circ A \circ A$ represents the amount in the account after 3 years; and $A \circ A \circ A \circ A$ represents the amount in the account after 4 years. We can see that if we compose n copies of A, we get $(1.05)^n x$.

64. Yes. If $f(x) = m_1 x + b_1$ and $g(x) = m_2 x + b_2$, then
$(f \circ g)(x) = f(m_2 x + b_2) = m_1(m_2 x + b_2) + b_1 = m_1 m_2 x + m_1 b_2 + b_1$, which is a linear function, because it is of the form $y = mx + b$. The slope is $m_1 m_2$.

65. $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$.

Method 1: Notice that $(2x+1)^2 = 4x^2 + 4x + 1$. We see that adding 6 to this quantity gives $(2x+1)^2 + 6 = 4x^2 + 4x + 1 + 6 = 4x^2 + 4x + 7$, which is $h(x)$. So let $f(x) = x^2 + 6$, and we have $(f \circ g)(x) = (2x+1)^2 + 6 = h(x)$.

Method 2: Since $g(x)$ is linear and $h(x)$ is a second degree polynomial, $f(x)$ must be a second degree polynomial, that is, $f(x) = ax^2 + bx + c$ for some a, b, and c. Thus $f(g(x)) = f(2x+1) = a(2x+1)^2 + b(2x+1) + c \Leftrightarrow 4ax^2 + 4ax + a + 2bx + b + c = 4ax^2 + (4a+2b)x + (a+b+c) = 4x^2 + 4x + 7$. Comparing this with $f(g(x))$, we have $4a = 4$ (the x^2 coefficients), $4a + 2b = 4$ (the x coefficients), and $a + b + c = 7$ (the constant terms) $\Leftrightarrow a = 1$ and $2a + b = 2$ and $a + b + c = 7 \Leftrightarrow a = 1, b = 0, c = 6$. Thus $f(x) = x^2 + 6$.

$f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$.

Note since $f(x)$ is linear and $h(x)$ is quadratic, $g(x)$ must also be quadratic. We can then use trial and error to find $g(x)$. Another method is the following: We wish to find g so that $(f \circ g)(x) = h(x)$. Thus $f(g(x)) = 3x^2 + 3x + 2 \Leftrightarrow 3(g(x)) + 5 = 3x^2 + 3x + 2 \Leftrightarrow 3(g(x)) = 3x^2 + 3x - 3 \Leftrightarrow g(x) = x^2 + x - 1$.

66. If $g(x)$ is even, then $h(-x) = f(g(-x)) = f(g(x)) = h(x)$. So yes, h is always an even function.

If $g(x)$ is odd, then h is not necessarily an odd function. For example, if we let $f(x) = x - 1$ and $g(x) = x^3$, g is an odd function, but $h(x) = (f \circ g)(x) = f(x^3) = x^3 - 1$ is not an odd function.

If $g(x)$ is odd and f is also odd, then $h(-x) = (f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x) = -h(x)$. So in this case, h is also an odd function.

If $g(x)$ is odd and f is even, then $h(-x) = (f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x) = h(x)$, so in this case, h is an even function.

2.8 One-to-One Functions and Their Inverses

1. By the Horizontal Line Test, f is not one-to-one.

2. By the Horizontal Line Test, f is one-to-one.

3. By the Horizontal Line Test, f is one-to-one.

4. By the Horizontal Line Test, f is not one-to-one.

5. By the Horizontal Line Test, f is not one-to-one.

6. By the Horizontal Line Test, f is one-to-one.

7. $f(x) = -2x + 4$. If $x_1 \neq x_2$, then $-2x_1 \neq -2x_2$ and $-2x_1 + 4 \neq -2x_2 + 4$. So f is a one-to-one function.

8. $f(x) = 3x - 2$. If $x_1 \neq x_2$, then $3x_1 \neq 3x_2$ and $3x_1 - 2 \neq 3x_2 - 2$. So f is a one-to-one function.

9. $g(x) = \sqrt{x}$. If $x_1 \neq x_2$, then $\sqrt{x_1} \neq \sqrt{x_2}$ because two different numbers cannot have the same square root. Therefore, g is a one-to-one function.

10. $g(x) = |x|$. Since every number and its negative have the same absolute value, that is, $|-1| = 1 = |1|$, g is not a one-to-one function.

11. $h(x) = x^2 - 2x$. Since $h(0) = 0$ and $h(2) = (2) - 2(2) = 0$ we have $h(0) = h(2)$. So f is not a one-to-one function.

12. $h(x) = x^3 + 8$. If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ and $x_1^3 + 8 \neq x_2^3 + 8$. So f is a one-to-one function.

13. $f(x) = x^4 + 5$. Every nonzero number and its negative have the same fourth power. For example, $(-1)^4 = 1 = (1)^4$, so $f(-1) = f(1)$. Thus f is not a one-to-one function.

14. $f(x) = x^4 + 5, 0 \le x \le 2$. If $x_1 \neq x_2$, then $x_1^4 \neq x_2^4$ because two different positive numbers cannot have the same fourth power. Thus, $x_1^4 + 5 \neq x_2^4 + 5$. So f is a one-to-one function.

15. $f(x) = \dfrac{1}{x^2}$. Every nonzero number and its negative have the same square. For example, $\dfrac{1}{(-1)^2} = 1 = \dfrac{1}{(1)^2}$, so $f(-1) = f(1)$. Thus f is not a one-to-one function.

16. $f(x) = \dfrac{1}{x}$. If $x_1 \neq x_2$, then $\dfrac{1}{x_1} \neq \dfrac{1}{x_2}$. So f is a one-to-one function.

17. (a) $f(2) = 7$. Since f is one-to-one, $f^{-1}(7) = 2$.

(b) $f^{-1}(3) = -1$. Since f is one-to-one, $f(-1) = 3$.

18. (a) $f(5) = 18$. Since f is one-to-one, $f^{-1}(18) = 5$.

(b) $f^{-1}(4) = 2$. Since f is one-to-one, $f(2) = 4$.

19. $f(x) = 5 - 2x$. Since f is one-to-one and $f(1) = 5 - 2(1) = 3$, then $f^{-1}(3) = 1$. (Find 1 by solving the equation $5 - 2x = 3$.)

20. To find $g^{-1}(5)$, we find the x value such that $g(x) = 5$; that is, we solve the equation $g(x) = x^2 + 4x = 5$. Now $x^2 + 4x = 5 \quad \Leftrightarrow \quad x^2 + 4x - 5 = 0 \quad \Leftrightarrow \quad (x-1)(x+5) = 0 \quad \Leftrightarrow \quad x = 1$ or $x = -5$. Since the domain of g is $[-2, \infty)$, $x = 1$ is the only value where $g(x) = 5$. Therefore, $g^{-1}(5) = 1$.

21. $f(g(x)) = f(x+6) = (x+6) - 6 = x$ for all x.
$g(f(x)) = g(x-6) = (x-6) + 6 = x$ for all x. Thus f and g are inverses of each other.

22. $f(g(x)) = f\left(\dfrac{x}{3}\right) = 3\left(\dfrac{x}{3}\right) = x$ for all x.
$g(f(x)) = g(3x) = \dfrac{3x}{3} = x$ for all x. Thus f and g are inverses of each other.

23. $f(g(x)) = f\left(\dfrac{x+5}{2}\right) = 2\left(\dfrac{x+5}{2}\right) - 5 = x + 5 - 5 = x$ for all x.
$g(f(x)) = g(2x-5) = \dfrac{(2x-5)+5}{2} = x$ for all x. Thus f and g are inverses of each other.

24. $f(g(x)) = f(3-4x) = \dfrac{3-(3-4x)}{4} = \dfrac{3-3+4x}{4} = x$ for all x.
$g(f(x)) = g\left(\dfrac{3-x}{4}\right) = 3 - 4\left(\dfrac{3-x}{4}\right) = 3 - 3 + x = x$ for all x. Thus f and g are inverses of each other.

25. $f(g(x)) = f\left(\dfrac{1}{x}\right) = \dfrac{1}{1/x} = x$ for all x. Since $f(x) = g(x)$, we also have $g(f(x)) = x$. Thus f and g are inverses of each other.

26. $f(g(x)) = f(\sqrt[5]{x}) = (\sqrt[5]{x})^5 = x$ for all x.
$g(f(x)) = g(x^5) = \sqrt[5]{x^5} = x$ for all x. Thus f and g are inverses of each other.

27. $f(g(x)) = f(\sqrt{x+4}) = (\sqrt{x+4})^2 - 4 = x + 4 - 4 = x$ for all $x \geq -4$.
$g(f(x)) = g(x^2 - 4) = \sqrt{(x^2-4)+4} = \sqrt{x^2} = x$ for all $x \geq 0$. Thus f and g are inverses of each other.

28. $f(g(x)) = f\left((x-1)^{1/3}\right) = \left((x-1)^{1/3}\right)^3 + 1 = x - 1 + 1 = x$ for all x.
$g(f(x)) = g(x^3+1) = \left[(x-1)^{1/3}\right]^3 + 1 = x - 1 + 1 = x$ for all x. Thus f and g are inverses of each other.

29. $f(g(x)) = f\left(\dfrac{1}{x} + 1\right) = \dfrac{1}{\left(\dfrac{1}{x}+1\right) - 1} = x$ for all $x \neq 0$.
$g(f(x)) = g\left(\dfrac{1}{x-1}\right) = \dfrac{1}{\left(\dfrac{1}{x-1}\right)} + 1 = (x-1) + 1 = x$ for all $x \neq 1$. Thus f and g are inverses of each other.

30. $f(g(x)) = f\left(\sqrt{4-x^2}\right) = \sqrt{4 - \left(\sqrt{4-x^2}\right)^2} = \sqrt{4-4+x^2} = \sqrt{x^2} = x$, for all $0 \leq x \leq 2$. (Note that the last equality is possible since $x \geq 0$.)
$g(f(x)) = g\left(\sqrt{4-x^2}\right) = \sqrt{4 - \left(\sqrt{4-x^2}\right)^2} = \sqrt{4-4+x^2} = \sqrt{x^2} = x$, for all $0 \leq x \leq 2$. (Again, the last equality is possible since $x \geq 0$.) Thus f and g are inverses of each other.

31. $f(x) = 2x + 1$. $y = 2x + 1 \quad \Leftrightarrow \quad 2x = y - 1 \quad \Leftrightarrow \quad x = \frac{1}{2}(y - 1)$. So $f^{-1}(x) = \frac{1}{2}(x - 1)$.

32. $f(x) = 6 - x$. $y = 6 - x \quad \Leftrightarrow \quad x = 6 - y$. So $f^{-1}(x) = 6 - x$.

33. $f(x) = 4x + 7$. $y = 4x + 7 \quad \Leftrightarrow \quad 4x = y - 7 \quad \Leftrightarrow \quad x = \frac{1}{4}(y - 7)$. So $f^{-1}(x) = \frac{1}{4}(x - 7)$.

34. $f(x) = 3 - 5x$. $y = 3 - 5x \quad \Leftrightarrow \quad -5x = y - 3 \quad \Leftrightarrow \quad x = -\frac{1}{5}(y - 3) = \dfrac{1}{5}(3 - y)$. So $f^{-1}(x) = \frac{1}{5}(3 - x)$.

35. $f(x) = \dfrac{x}{2}$. $y = \dfrac{x}{2} \quad \Leftrightarrow \quad x = 2y$. So $f^{-1}(x) = 2x$.

36. $f(x) = \dfrac{1}{x^2}$, $x > 0$. Since the function f is restricted to positive values, f is one-to-one. So $y = \dfrac{1}{x^2} \quad \Leftrightarrow \quad \sqrt{y} = \dfrac{1}{x}$ $\Leftrightarrow \quad x = \dfrac{1}{\sqrt{y}}$. Thus $f^{-1}(x) = \dfrac{1}{\sqrt{x}}$, $x > 0$.

37. $f(x) = \dfrac{1}{x+2}$. $y = \dfrac{1}{x+2} \quad \Leftrightarrow \quad x + 2 = \dfrac{1}{y} \quad \Leftrightarrow \quad x = \dfrac{1}{y} - 2$. So $f^{-1}(x) = \dfrac{1}{x} - 2$.

38. $f(x) = \dfrac{x-2}{x+2}$. $y = \dfrac{x-2}{x+2} \quad \Leftrightarrow \quad y(x+2) = x - 2 \quad \Leftrightarrow \quad xy + 2y = x - 2 \quad \Leftrightarrow \quad xy - x = -2 - 2y \quad \Leftrightarrow$ $x(y-1) = -2(y+1) \quad \Leftrightarrow \quad x = \dfrac{-2(y+1)}{y-1}$. So $f^{-1}(x) = \dfrac{-2(x+1)}{x-1}$.

39. $f(x) = \dfrac{1+3x}{5-2x}$. $y = \dfrac{1+3x}{5-2x} \quad \Leftrightarrow \quad y(5-2x) = 1 + 3x \quad \Leftrightarrow \quad 5y - 2xy = 1 + 3x \quad \Leftrightarrow \quad 3x + 2xy = 5y - 1$ $\Leftrightarrow \quad x(3 + 2y) = 5y - 1 \quad \Leftrightarrow \quad x = \dfrac{5y-1}{2y+3}$. So $f^{-1}(x) = \dfrac{5x-1}{2x+3}$.

40. $f(x) = 5 - 4x^3$. $y = 5 - 4x^3 \quad \Leftrightarrow \quad 4x^3 = 5 - y \quad \Leftrightarrow \quad x^3 = \frac{1}{4}(5 - y) \quad \Leftrightarrow \quad x = \sqrt[3]{\frac{1}{4}(5-y)}$. So $f^{-1}(x) = \sqrt[3]{\frac{1}{4}(5-x)}$.

41. $f(x) = \sqrt{2 + 5x}$, $x \ge -\frac{2}{5}$. $y = \sqrt{2+5x}$, $y \ge 0 \quad \Leftrightarrow \quad y^2 = 2 + 5x \quad \Leftrightarrow \quad 5x = y^2 - 2 \quad \Leftrightarrow \quad x = \frac{1}{5}(y^2 - 2)$ and $y \ge 0$. So $f^{-1}(x) = \frac{1}{5}(x^2 - 2)$, $x \ge 0$.

42. $f(x) = x^2 + x = \left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$, $x \ge -\dfrac{1}{2}$. $y = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} \quad \Leftrightarrow \quad \left(x + \frac{1}{2}\right)^2 = y + \frac{1}{4} \quad \Leftrightarrow$ $x + \frac{1}{2} = \sqrt{y + \frac{1}{4}} \quad \Leftrightarrow \quad x = \sqrt{y + \frac{1}{4}} - \frac{1}{2}$, $y \ge -\frac{1}{4}$. So $f^{-1}(x) = \sqrt{x + \frac{1}{4}} - \frac{1}{2}$, $x \ge -\frac{1}{4}$. (Note that $x \ge -\frac{1}{2}$, so that $x + \frac{1}{2} \ge 0$, and hence $\left(x + \frac{1}{2}\right)^2 = y + \frac{1}{4} \quad \Leftrightarrow \quad x + \dfrac{1}{2} = \sqrt{y + \frac{1}{4}}$. Also, since $x \ge -\frac{1}{2}$, $y = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} \ge -\frac{1}{4}$ so that $y + \frac{1}{4} \ge 0$, and hence $\sqrt{y + \frac{1}{4}}$ is defined.)

43. $f(x) = 4 - x^2$, $x \ge 0$. $y = 4 - x^2 \quad \Leftrightarrow \quad x^2 = 4 - y \quad \Leftrightarrow \quad x = \sqrt{4 - y}$. So $f^{-1}(x) = \sqrt{4 - x}$. Note: $x \ge 0 \quad \Rightarrow$ $f(x) \le 4$.

44. $f(x) = \sqrt{2x - 1}$. $y = \sqrt{2x-1} \quad \Leftrightarrow \quad 2x - 1 = y^2 \quad \Leftrightarrow \quad x = \frac{1}{2}\left(y^2 + 1\right)$. Since the range of f is $f(x) \ge 0$ so $f^{-1}(x) = \frac{1}{2}\left(x^2 + 1\right)$ for $x \ge 0$.

45. $f(x) = 4 + \sqrt[3]{x}$. $y = 4 + \sqrt[3]{x} \quad \Leftrightarrow \quad \sqrt[3]{x} = y - 4 \quad \Leftrightarrow \quad x = (y - 4)^3$. So $f^{-1}(x) = (x - 4)^3$.

46. $f(x) = \left(2 - x^3\right)^5$. $y = \left(2 - x^3\right)^5 \quad \Leftrightarrow \quad 2 - x^3 = \sqrt[5]{y} \quad \Leftrightarrow \quad x^3 = 2 - \sqrt[5]{y} \quad \Leftrightarrow \quad x = \sqrt[3]{2 - \sqrt[5]{y}}$. So $f^{-1}(x) = \sqrt[3]{2 - \sqrt[5]{x}}$.

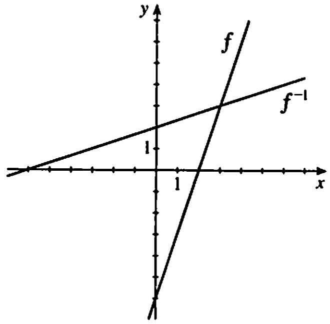
47. $f(x) = 1 + \sqrt{1 + x}$. $y = 1 + \sqrt{1+x}$, $y \ge 1 \quad \Leftrightarrow \quad \sqrt{1+x} = y - 1 \quad \Leftrightarrow \quad 1 + x = (y-1)^2$ $\Leftrightarrow x = (y-1)^2 - 1 = y^2 - 2y$. So $f^{-1}(x) = x^2 - 2x$, $x \ge 1$.

48. $f(x) = \sqrt{9 - x^2}$, $0 \le x \le 3$. $y = \sqrt{9 - x^2} \quad \Leftrightarrow \quad y^2 = 9 - x^2 \quad \Leftrightarrow \quad x^2 = 9 - y^2 \quad \Rightarrow \quad x = \sqrt{9 - y^2}$ (since we must have $x \ge 0$). So $f^{-1}(x) = \sqrt{9 - x^2}$, $0 \le x \le 3$.

49. $f(x) = x^4$, $x \ge 0$. $y = x^4$, $y \ge 0 \quad \Leftrightarrow \quad x = \sqrt[4]{y}$. So $f^{-1}(x) = \sqrt[4]{x}$, $x \ge 0$.

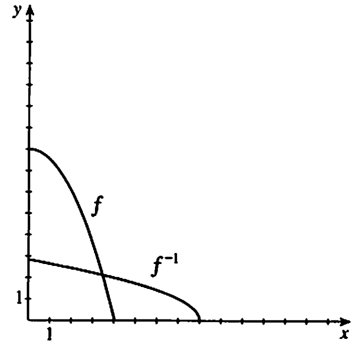
50. $f(x) = 1 - x^3$. $y = 1 - x^3 \quad \Leftrightarrow \quad x^3 = 1 - y \quad \Leftrightarrow \quad x = \sqrt[3]{1 - y}$. So $f^{-1}(x) = \sqrt[3]{1 - x}$.

51. **(a), (b)** $f(x) = 3x - 6$



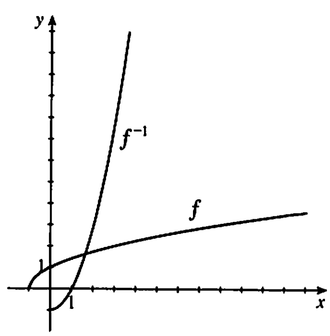
(c) $f(x) = 3x - 6$. $y = 3x - 6 \quad \Leftrightarrow$
$3x = y + 6 \quad \Leftrightarrow \quad x = \frac{1}{3}(y + 6)$. So
$f^{-1}(x) = \frac{1}{3}(x + 6)$.

52. **(a), (b)** $f(x) = 16 - x^2, x \geq 0$



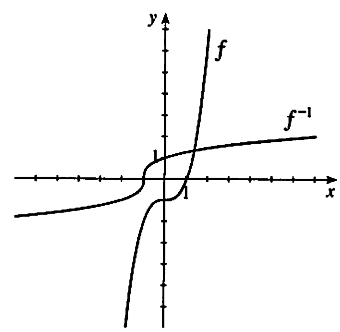
(c) $f(x) = 16 - x^2, x \geq 0$. $y = 16 - x^2 \quad \Leftrightarrow$
$x^2 = 16 - y \quad \Leftrightarrow \quad x = \sqrt{16 - y}$. So
$f^{-1}(x) = \sqrt{16 - x}, x \leq 16$. (*Note:* $x \geq 0$
$\Rightarrow \quad f(x) = 16 - x^2 \leq 16$.)

53. **(a), (b)** $f(x) = \sqrt{x + 1}$



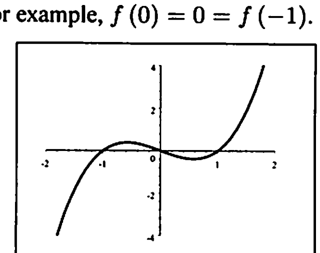
(c) $f(x) = \sqrt{x + 1}, x \geq -1$.
$y = \sqrt{x + 1}, y \geq 0 \quad \Leftrightarrow \quad y^2 = x + 1 \quad \Leftrightarrow$
$x = y^2 - 1$ and $y \geq 0$. So $f^{-1}(x) = x^2 - 1$,
$x \geq 0$.

54. **(a), (b)** $f(x) = x^3 - 1$

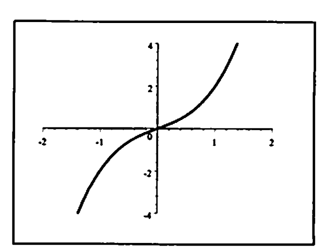


(c) $f(x) = x^3 - 1 \quad \Leftrightarrow \quad y = x^3 - 1 \quad \Leftrightarrow$
$x^3 = y + 1 \quad \Leftrightarrow \quad x = \sqrt[3]{y + 1}$. So
$f^{-1}(x) = \sqrt[3]{x + 1}$.

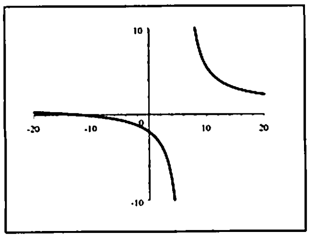
55. $f(x) = x^3 - x$. Using a graphing device and the Horizontal Line Test, we see that f is not a one-to-one function. For example, $f(0) = 0 = f(-1)$.



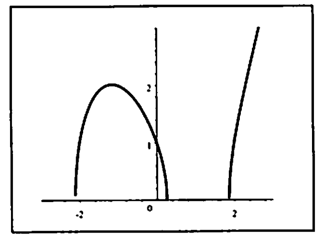
56. $f(x) = x^3 + x$. Using a graphing device and the Horizontal Line Test, we see that f is a one-to-one function.



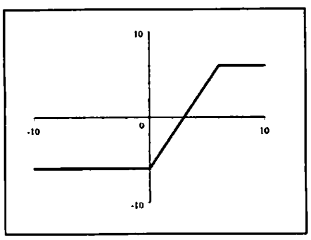
57. $f(x) = \dfrac{x+12}{x-6}$. Using a graphing device and the Horizontal Line Test, we see that f is a one-to-one function.



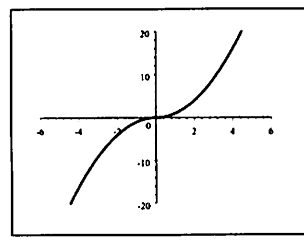
58. $f(x) = \sqrt{x^3 - 4x + 1}$. Using a graphing device and the Horizontal Line Test, we see that f is not a one-to-one function. For example, $f(0) = 1 = f(2)$.



59. $f(x) = |x| - |x-6|$. Using a graphing device and the Horizontal Line Test, we see that f is not a one-to-one function. For example $f(0) = -6 = f(-2)$.

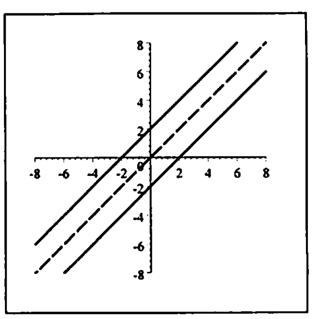


60. $f(x) = x \cdot |x|$. Using a graphing device and the Horizontal Line Test, we see that f is a one-to-one function.



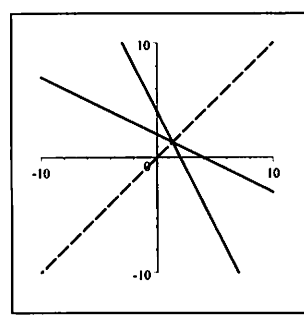
61. **(a)** $y = f(x) = 2 + x \quad \Leftrightarrow \quad x = y - 2$. So $f^{-1}(x) = x - 2$.

(b)



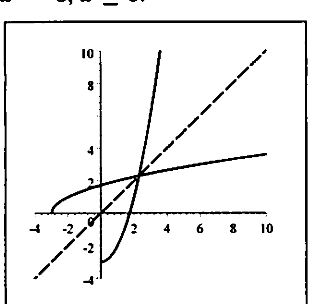
62. **(a)** $y = f(x) = 2 - \frac{1}{2}x \quad \Leftrightarrow \quad \frac{1}{2}x = 2 - y \quad \Leftrightarrow \quad x = 4 - 2y$. So $f^{-1}(x) = 4 - 2x$.

(b)



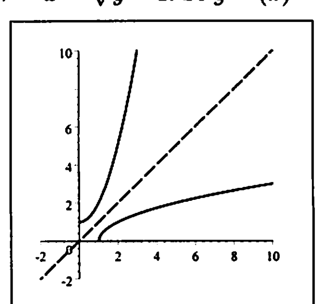
63. **(a)** $y = g(x) = \sqrt{x+3}, y \geq 0 \quad \Leftrightarrow \quad x + 3 = y^2, y \geq 0 \quad \Leftrightarrow \quad x = y^2 - 3, y \geq 0$. So $g^{-1}(x) = x^2 - 3, x \geq 0$.

(b)



64. **(a)** $y = g(x) = x^2 + 1, x \geq 0 \quad \Leftrightarrow \quad x^2 = y - 1, x \geq 0 \quad \Leftrightarrow \quad x = \sqrt{y-1}$. So $g^{-1}(x) = \sqrt{x-1}$.

(b)



65. If we restrict the domain of $f(x)$ to $[0,\infty)$, then $y = 4 - x^2 \Leftrightarrow x^2 = 4 - y \Rightarrow x = \sqrt{4-y}$ (since $x \geq 0$, we take the positive square root). So $f^{-1}(x) = \sqrt{4-x}$.
If we restrict the domain of $f(x)$ to $(-\infty, 0]$, then $y = 4 - x^2 \Leftrightarrow x^2 = 4 - y \Rightarrow x = -\sqrt{4-y}$ (since $x \leq 0$, we take the negative square root). So $f^{-1}(x) = -\sqrt{4-x}$.

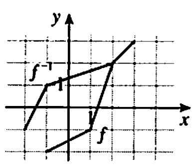
66. If we restrict the domain of $g(x)$ to $[1,\infty)$, then $y = (x-1)^2 \Rightarrow x - 1 = \sqrt{y}$ (since $x \geq 1$ we take the positive square root) $\Leftrightarrow x = 1 + \sqrt{y}$. So $g^{-1}(x) = 1 + \sqrt{x}$.
If we restrict the domain of $g(x)$ to $(-\infty, 1]$, then $y = (x-1)^2 \Rightarrow x - 1 = -\sqrt{y}$ (since $x \leq 1$ we take the negative square root) $\Leftrightarrow x = 1 - \sqrt{y}$. So $g^{-1}(x) = 1 - \sqrt{x}$.

67. If we restrict the domain of $h(x)$ to $[-2,\infty)$, then $y = (x+2)^2 \Rightarrow x + 2 = \sqrt{y}$ (since $x \geq -2$, we take the positive square root) $\Leftrightarrow x = -2 + \sqrt{y}$. So $h^{-1}(x) = -2 + \sqrt{x}$.
If we restrict the domain of $h(x)$ to $(-\infty, -2]$, then $y = (x+2)^2 \Rightarrow x + 2 = -\sqrt{y}$ (since $x \leq -2$, we take the negative square root) $\Leftrightarrow x = -2 - \sqrt{y}$. So $h^{-1}(x) = -2 - \sqrt{x}$.

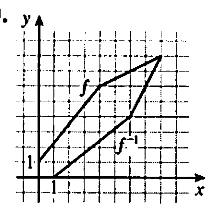
68. $k(x) = |x-3| = \begin{cases} -(x-3) & \text{if } x - 3 < 0 \quad \Leftrightarrow \quad x < 3 \\ x - 3 & \text{if } x - 3 \geq 0 \quad \Leftrightarrow \quad x \geq 3 \end{cases}$

If we restrict the domain of $k(x)$ to $[3,\infty)$, then $y = x - 3 \Leftrightarrow x = 3 + y$. So $k^{-1}(x) = 3 + x$.
If we restrict the domain of $k(x)$ to $(-\infty, 3]$, then $y = -(x-3) \Leftrightarrow y = -x + 3 \Leftrightarrow x = 3 - y$. So $k^{-1}(x) = 3 - x$.

69.



70.



71. **(a)** $f(x) = 500 + 80x$.

(b) $f(x) = 500 + 80x$. $y = 500 + 80x \Leftrightarrow 80x = y - 500 \Leftrightarrow x = \dfrac{y-500}{80}$. So $f^{-1}(x) = \dfrac{x-500}{80}$. f^{-1} represents the number of hours of investigation the investigate spends on a case for x dollars.

(c) $f^{-1}(1220) = \dfrac{1220-500}{80} = \dfrac{720}{80} = 9$. The investigator spent 9 hours investigating this case.

72. **(a)** $V(t) = 100\left(1 - \dfrac{t}{40}\right)^2$, $0 \leq t \leq 40$. $y = 100\left(1 - \dfrac{t}{40}\right)^2 \Leftrightarrow \dfrac{y}{100} = \left(1 - \dfrac{t}{40}\right)^2 \Rightarrow$
$1 - \dfrac{t}{40} = \pm\sqrt{\dfrac{y}{100}} \Leftrightarrow \dfrac{t}{40} = 1 \pm \dfrac{\sqrt{y}}{10} \Leftrightarrow t = 40 \pm 4\sqrt{y}$. Since $t \leq 40$ we must have $V^{-1}(t) = 40 - 4\sqrt{t}$.
V^{-1} represents time that has elapsed since the tank started to leak.

(b) $V^{-1}(15) = 40 - 4\sqrt{15} \approx 24.5$ minutes. In 24.5 minutes the tank has drained to just 15 gallons of water.

73. **(a)** $v(r) = 18{,}500(0.25 - r^2)$. $t = 18{,}500(0.25 - r^2) \Leftrightarrow t = 4625 - 18{,}500r^2 \Leftrightarrow 18500r^2 = 4625 - t$
$\Leftrightarrow r^2 = \dfrac{4625-t}{18{,}500} \Rightarrow r = \pm\sqrt{\dfrac{4625-t}{18{,}500}}$. Since r represents a distance, $r \geq 0$, so $v^{-1}(t) = \sqrt{\dfrac{4625-t}{18{,}500}}$.
v^{-1} represents the radius in the vein that has the velocity v.

(b) $v^{-1}(30) = \sqrt{\dfrac{4625-30}{18{,}500}} \approx 0.498$ cm. The velocity is 30 at 0.498 cm from the center of the artery or vein.

74. (a) $D(p) = -3p + 150$. $y = -3p + 150 \Leftrightarrow 3p = 150 - y \Leftrightarrow p = 50 - \frac{1}{3}y$. So $D^{-1}(p) = 50 - \frac{1}{3}p$. D^{-1} represents the price that is associated with demand D.

(b) $D^{-1}(30) = 50 - \frac{1}{3}(30) = 40$. So when the demand is 30 units the price per unit is \$40.

75. (a) $F(x) = \frac{9}{5}x + 32$. $y = \frac{9}{5}x + 32 \Leftrightarrow \frac{9}{5}x = y - 32 \Leftrightarrow x = \frac{5}{9}(y - 32)$. So $F^{-1}(x) = \frac{5}{9}(x - 32)$. F^{-1} represents the Celsius temperature that corresponds to the Fahrenheit temperature of F.

(b) $F^{-1}(86) = \frac{5}{9}(86 - 32) = \frac{5}{9}(54) = 30$. So 86° Fahrenheit is the same as 30° Celsius.

76. (a) $f(x) = 0.8159x$.

(b) $f(x) = 0.8159x$. $y = 0.8159x \Leftrightarrow x = 1.2256y$. So $f^{-1}(x) = 1.2256x$. f^{-1} represents the exchange rate from US dollars to Canadian dollars.

(c) $f^{-1}(12{,}250) = 1.2256(12{,}250) = 15{,}013.60$. So \$12,250 in US currency is worth \$15,013.60 in Canadian currency.

77. (a) $f(x) = \begin{cases} 0.1x, & \text{if } 0 \le x \le 20{,}000 \\ 2000 + 0.2(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$

(b) We will find the inverse of each piece of the function f.

$f_1(x) = 0.1x$. $y = 0.1x \Leftrightarrow x = 10y$. So $f_1^{-1}(x) = 10x$.

$f_2(x) = 2000 + 0.2(x - 20{,}000) = 0.2x - 2000$. $y = 0.2x - 2000 \Leftrightarrow 0.2x = y + 2000 \Leftrightarrow x = 5y + 10{,}000$. So $f_2^{-1}(x) = 5x + 10{,}000$.

Since $f(0) = 0$ and $f(20{,}000) = 2000$ we have $f^{-1}(x) = \begin{cases} 10x, & \text{if } 0 \le x \le 2000 \\ 5x + 10{,}000 & \text{if } x > 2000 \end{cases}$ It represents the taxpayer's income.

(b) $f^{-1}(10{,}000) = 5(10{,}000) + 10{,}000 = 60{,}000$. The required income is \$60,000.

78. (a) $f(x) = 0.85x$.

(b) $g(x) = x - 1000$.

(c) $H = f \circ g = f(x - 1000) = 0.85(x - 1000) = 0.85x - 850$.

(d) $H(x) = 0.85x - 850$. $y = 0.85x - 850 \Leftrightarrow 0.85x = y + 850 \Leftrightarrow x = 1.176y + 1000$. So $H^{-1}(x) = 1.176x + 1000$. The function H^{-1} represents the original sticker price for a given discounted price.

(e) $H^{-1}(13{,}000) = 1.176(13{,}000) + 1000 = 16{,}266$. So the original price of the car is \$16,288 when the discounted price (\$1000 rebate, then 15% off) is \$13,000.

79. $f(x) = 7 + 2x$. $y = 7 + 2x \Leftrightarrow 2x = y - 7 \Leftrightarrow x = \dfrac{y - 7}{2}$. So $f^{-1}(x) = \dfrac{x - 7}{2}$. f^{-1} is the number of toppings on a pizza that costs x dollars.

80. $f(x) = mx + b$. Notice that $f(x_1) = f(x_2) \Leftrightarrow mx_1 + b = mx_2 + b \Leftrightarrow mx_1 = mx_2$. We can conclude that $x_1 = x_2$ if and only if $m \neq 0$. Therefore f is one-to-one if and only if $m \neq 0$. If $m \neq 0$, $f(x) = mx + b \Leftrightarrow y = mx + b \Leftrightarrow mx = y - b \Leftrightarrow x = \dfrac{y - b}{m}$. So, $f^{-1}(x) = \dfrac{x - b}{m}$.

81. (a) $f(x) = \dfrac{2x + 1}{5}$ is "multiply by 2, add 1, and then divide by 5 ". So the reverse is "multiply by 5, subtract 1, and then divide by 2 " or $f^{-1}(x) = \dfrac{5x - 1}{2}$. Check: $f \circ f^{-1}(x) = f\left(\dfrac{5x - 1}{2}\right) = \dfrac{2\left(\dfrac{5x - 1}{2}\right) + 1}{5} = \dfrac{5x - 1 + 1}{5} = \dfrac{5x}{5} = x$ and $f^{-1} \circ f(x) = f^{-1}\left(\dfrac{2x + 1}{5}\right) = \dfrac{5\left(\dfrac{2x + 1}{5}\right) - 1}{2} = \dfrac{2x + 1 - 1}{2} = \dfrac{2x}{2} = x$.

(b) $f(x) = 3 - \dfrac{1}{x} = \dfrac{-1}{x} + 3$ is "take the negative reciprocal and add 3 ". Since the reverse of "take the negative reciprocal" is "take the negative reciprocal ", $f^{-1}(x)$ is "subtract 3 and take the negative reciprocal ", that is, $f^{-1}(x) = \dfrac{-1}{x-3}$. Check: $f \circ f^{-1}(x) = f\left(\dfrac{-1}{x-3}\right) = 3 - \dfrac{1}{\dfrac{-1}{x-3}} = 3 - \left(1 \cdot \dfrac{x-3}{-1}\right) = 3 + x - 3 = x$ and

$$f^{-1} \circ f(x) = f^{-1}\left(3 - \frac{1}{x}\right) = \frac{-1}{\left(3 - \frac{1}{x}\right) - 3} = \frac{-1}{-\frac{1}{x}} = -1 \cdot \frac{x}{-1} = x.$$

(c) $f(x) = \sqrt{x^3 + 2}$ is "cube, add 2, and then take the square root". So the reverse is "square, subtract 2, then take the cube root " or $f^{-1}(x) = \sqrt[3]{x^2 - 2}$. Domain for $f(x)$ is $\left[-\sqrt[3]{2}, \infty\right)$; domain for $f^{-1}(x)$ is $[0, \infty)$. Check:

$f \circ f^{-1}(x) = f\left(\sqrt[3]{x^2 - 2}\right) = \sqrt{\left(\sqrt[3]{x^2 - 2}\right)^3 + 2} = \sqrt{x^2 - 2 + 2} = \sqrt{x^2} = x$ (on the appropriate domain) and

$f^{-1} \circ f(x) = f^{-1}\left(\sqrt{x^3 + 2}\right) = \sqrt[3]{\left(\sqrt{x^3 + 2}\right)^2 - 2} = \sqrt[3]{x^3 + 2 - 2} = \sqrt[3]{x^3} = x$ (on the appropriate domain).

(d) $f(x) = (2x - 5)^3$ is "double, subtract 5, and then cube". So the reverse is "take the cube root, add 5, and divide by 2" or $f^{-1}(x) = \dfrac{\sqrt[3]{x} + 5}{2}$ Domain for both $f(x)$ and $f^{-1}(x)$ is $(-\infty, \infty)$. Check:

$$f \circ f^{-1}(x) = f\left(\frac{\sqrt[3]{x} + 5}{2}\right) = \left[2\left(\frac{\sqrt[3]{x} + 5}{2}\right) - 5\right]^3 = \left(\sqrt[3]{x} + 5 - 5\right)^3 = \left(\sqrt[3]{x}\right)^3 = \sqrt[3]{x^3} = x \text{ and}$$

$$f^{-1} \circ f(x) = f^{-1}\left((2x - 5)^3\right) = \frac{\sqrt[x]{(2x - 5)^3} + 5}{2} = \frac{(2x - 5) + 5}{2} = \frac{2x}{2} = x.$$

In a function like $f(x) = 3x - 2$, the variable occurs only once and it easy to see how to reverse the operations step by step. But in $f(x) = x^3 + 2x + 6$, you apply two different operations to the variable x (cubing and multiplying by 2) and then add 6, so it is not possible to reverse the operations step by step.

82. $f(I(x)) = f(x)$; therefore $f \circ I = f$. $I(f(x)) = f(x)$; therefore $I \circ f = f$.
By definition, $f \circ f^{-1}(x) = x = I(x)$; therefore $f \circ f^{-1} = I$. Similarly, $f^{-1} \circ f(x) = x = I(x)$; therefore $f^{-1} \circ f = I$.

83. (a) We find $g^{-1}(x)$: $y = 2x + 1 \Leftrightarrow 2x = y - 1 \Leftrightarrow x = \frac{1}{2}(y - 1)$. So $g^{-1}(x) = \frac{1}{2}(x - 1)$. Thus $f(x) = h \circ g^{-1}(x) = h\left(\frac{1}{2}(x - 1)\right) = 4\left[\frac{1}{2}(x - 1)\right]^2 + 4\left[\frac{1}{2}(x - 1)\right] + 7 = x^2 - 2x + 1 + 2x - 2 + 7 = x^2 + 6$.

(b) $f \circ g = h \Leftrightarrow f^{-1} \circ f \circ g = f^{-1} \circ h \Leftrightarrow I \circ g = f^{-1} \circ h \Leftrightarrow g = f^{-1} \circ h$.
Note that we compose with f^{-1} on the left on each side of the equation. We find f^{-1}:
$y = 3x + 5 \Leftrightarrow 3x = y - 5 \Leftrightarrow x = \dfrac{1}{3}(y - 5)$. So $f^{-1}(x) = \frac{1}{3}(x - 5)$. Thus
$g(x) = f^{-1} \circ h(x) = f^{-1}\left(3x^2 + 3x + 2\right) = \frac{1}{3}\left[\left(3x^2 + 3x + 2\right) - 5\right] = \frac{1}{3}\left[3x^2 + 3x - 3\right] = x^2 + x - 1$.

Chapter 2 Review

1. $f(x) = x^2 - 4x + 6$; $f(0) = (0)^2 - 4(0) + 6 = 6$; $f(2) = (2)^2 - 4(2) + 6 = 2$;
$f(-2) = (-2)^2 - 4(-2) + 6 = 18$; $f(a) = (a)^2 - 4(a) + 6 = a^2 - 4a + 6$; $f(-a) = (-a)^2 - 4(-a) + 6 = a^2 + 4a + 6$;
$f(x + 1) = (x + 1)^2 - 4(x + 1) + 6 = x^2 + 2x + 1 - 4x - 4 + 6 = x^2 - 2x + 3$; $f(2x) = (2x)^2 - 4(2x) + 6 = 4x^2 - 8x + 6$;
$2f(x) - 2 = 2\left(x^2 - 4x + 6\right) - 2 = 2x^2 - 8x + 12 - 2 = 2x^2 - 8x + 10$.

2. $f(x) = 4 - \sqrt{3x - 6}$; $f(5) = 4 - \sqrt{15 - 6} = 3$; $f(9) = 4 - \sqrt{27 - 6} = 4 - \sqrt{21}$;
$f(a + 2) = 4 - \sqrt{3a + 6 - 6} = 4 - \sqrt{3a}$; $f(-x) = 4 - \sqrt{3(-x) - 6} = 4 - \sqrt{-3x - 6}$; $f\left(x^2\right) = 4 - \sqrt{3x^2 - 6}$.
$[f(x)]^2 = \left(4 - \sqrt{3x - 6}\right)^2 = 4 - 8\sqrt{3x - 6} + 3x - 6 = -2 - 8\sqrt{3x - 6} + 3x$.