HL2 Name________________________________ ID: 1

Summer Skills Review

Date________________ Period____

Evaluate each limit using L'Hôpital's Rule.

1) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

2) $\lim_{x \to \infty} \frac{\ln (x + 5)^5}{\ln x^5}$

3) $\lim_{x \to 0} \frac{e^{2x} - 1}{\sin (5x)}$

Solve each optimization problem.

4) Two vertical poles, one 5 ft high and the other 10 ft high, stand 36 feet apart on a flat field. A worker wants to support both poles by running rope from the ground to the top of each post. If the worker wants to stake both ropes in the ground at the same point, where should the stake be placed to use the least amount of rope?

For each problem, find the equation of the line tangent to the function at the given point. Your answer should be in slope-intercept form.

5) $y = \frac{x^2}{2x + 4}$ at $\left(1, \frac{1}{6}\right)$

6) $y = -e^{x + 1}$ at $(0, -e)$

For each problem, find the equation of the line normal to the function at the given point. If the normal line is a vertical line, indicate so. Otherwise, your answer should be in slope-intecept form.

7) $y = x^3 - 4x^2 + 3$ at $(1, 0)$

8) $y = -(x + 2)^{\frac{1}{2}}$ at $(-1, -1)$

For each problem, find the indicated derivative with respect to *x*.

9) $f(x) = 3x^3$ Find f'''

10) $f(x) = 2x^5 + x^3 + 3x$ Find f'''

11) $f(x) = 2x^4 + x$ Find f'''

Differentiate each function with respect to *x*.

12) $f(x) = (-x^3 + 4)^{\frac{1}{2}}$

13) $f(x) = (-4x^5 + 1)^2$

14) $f(x) = \sqrt[3]{4x^4 - 3}$

15) $y = \left(\frac{3x^3 - 1}{4x^2 + 5}\right)^4$

16) $y = (-2x^3 + 5)^{-5}(-3x^5 - 2)$

17) $y = e^{e^{x^2}}$

18) $y = e^{\ln 2x^4 + 5}$

19) $y = (\ln 4x^2)^5$

20) $y = \sqrt{3x^4 + 2} \ln x^3$

21) $y = \ln (2x^5 + 3)^2$

22) $y = \ln 5x^5 \cdot \sqrt{3x^4 - 2}$

23) $y = \log_5 4x^5$

24) $y = 5^{4x^3}$

25) $y = (x^2 + 4) \cdot 5^{4x^3}$

26) $y = (x^4 + 3)\log_2 4x^2$

27) $y = 5^{2x^3}(4x^4 + 3)$

28) $y = 4^{4x^5}(2x^4 - 5)$

29) $y = \frac{3x^2 + 5}{\ln 5x^5}$

30) $y = \ln x^5$

31) $y = 2^{3x^5}(5x^4 + 3)$

32) $y = \frac{2x^5 + 1}{\log_5 x^3}$

33) $y = \cos(\sec x^3)$

34) $y = \sin\left(\cot 2x^4\right)$

35) $y = \cot 3x^2$

36) $y = \tan 4x^3$

37) $y = \csc 5x^4$

38) $y = \csc x^3$

39) $y = \sec\left(\cot 4x^5\right)$

40) $y = \tan 2x^3$

41) $y = \sec\left(\sec 4x^5\right)$

42) $y = \cos^{-1} 2x^4$

43) $y = \cos^{-1} 5x^5$

44) $y = \cos^{-1}\left(2x^4 + 5\right)^5$

45) $y = (\tan^{-1} 4x^2)^5$

Solve each related rate problem.

46) A 10 ft ladder is leaning against a wall and sliding towards the floor. The foot of the ladder is sliding away from the base of the wall at a rate of 6 ft/sec. How fast is the top of the ladder sliding down the wall when the top of the ladder is 8 ft from the ground?

47) A hypothetical square shrinks at a rate of 36 m^2/min. At what rate are the sides of the square changing when the sides are 11 m each?

48) A spherical balloon is deflated at a rate of $\frac{256\pi}{3}$ cm³/sec. At what rate is the radius of the balloon changing when the radius is 6 cm?

49) A hypothetical cube shrinks at a rate of 64 m³/min. At what rate are the sides of the cube changing when the sides are 4 m each?

50) A hypothetical square shrinks at a rate of $\frac{2}{A}$ m²/min, where A is the area of the square. At what rate are the diagonals of the square changing when the diagonals are 4 m each?

Answers to Summer Skills Review (ID: 1)

1) $\frac{1}{2}$

2) 1

3) $\frac{2}{5}$

4) 12 ft from the short pole (or 24 ft from the long pole)

5) $y = \frac{5}{18}x - \frac{1}{9}$

6) $y = -ex - e$

7) $y = \frac{1}{5}x - \frac{1}{5}$

8) $y = 2x + 1$

9) $f'''(x) = 18$

10) $f'''(x) = 120x^2 + 6$

11) $f'''(x) = 48x$

12) $f'(x) = \frac{1}{2}(-x^3 + 4)^{-\frac{1}{2}} \cdot -3x^2$
$= -\frac{3x^2}{2(-x^3 + 4)^{\frac{1}{2}}}$

13) $f'(x) = 2(-4x^5 + 1) \cdot -20x^4$
$= -40x^4(-4x^5 + 1)$

14) $f'(x) = \frac{1}{3}(4x^4 - 3)^{-\frac{2}{3}} \cdot 16x^3$
$= \frac{16x^3}{3(4x^4 - 3)^{\frac{2}{3}}}$

15) $\frac{dy}{dx} = 4 \cdot \left(\frac{3x^3 - 1}{4x^2 + 5}\right)^3 \cdot \frac{(4x^2 + 5) \cdot 9x^2 - (3x^3 - 1) \cdot 8x}{(4x^2 + 5)^2}$
$= \frac{4x(3x^3 - 1)^3(12x^3 + 45x + 8)}{(4x^2 + 5)^5}$

16) $\frac{dy}{dx} = (-2x^3 + 5)^{-5} \cdot -15x^4 + (-3x^5 - 2) \cdot -5(-2x^3 + 5)^{-6} \cdot -6x^2$
$= \frac{15x^2(-4x^5 - 5x^2 - 4)}{(-2x^3 + 5)^6}$

17) $\frac{dy}{dx} = e^{e^{x^2}} \cdot e^{x^2} \cdot 2x$
$= 2xe^{e^{x^2} + x^2}$

18) $\frac{dy}{dx} = e^{\ln 2x^4 + 5} \cdot \frac{1}{2x^4} \cdot 8x^3$
$= \frac{4e^{\ln 2x^4 + 5}}{x}$

19) $\frac{dy}{dx} = 5 \cdot (\ln 4x^2)^4 \cdot \frac{1}{4x^2} \cdot 8x$
$= \frac{10 \cdot (\ln 4x^2)^4}{x}$

20) $\frac{dy}{dx} = (3x^4 + 2)^{\frac{1}{2}} \cdot \frac{1}{x^3} \cdot 3x^2 + \ln x^3 \cdot \frac{1}{2}(3x^4 + 2)^{-\frac{1}{2}} \cdot 12x^3$
$= \frac{3(3x^4 + 2 + 2x^4 \ln x^3)}{x\sqrt{3x^4 + 2}}$

21) $\frac{dy}{dx} = \frac{1}{(2x^5 + 3)^2} \cdot 2(2x^5 + 3) \cdot 10x^4$
$= \frac{20x^4}{2x^5 + 3}$

22) $\frac{dy}{dx} = \ln 5x^5 \cdot \frac{1}{2}(3x^4 - 2)^{-\frac{1}{2}} \cdot 12x^3 + (3x^4 - 2)^{\frac{1}{2}} \cdot \frac{1}{5x^5} \cdot 25x^4$
$= \frac{6x^4 \ln 5x^5 + 15x^4 - 10}{x\sqrt{3x^4 - 2}}$

23) $\frac{dy}{dx} = \frac{1}{4x^5 \ln 5} \cdot 20x^4$
$= \frac{5}{x \ln 5}$

24) $\frac{dy}{dx} = 5^{4x^3} \ln 5 \cdot 12x^2$

25) $\frac{dy}{dx} = (x^2 + 4) \cdot 5^{4x^3} \ln 5 \cdot 12x^2 + 5^{4x^3} \cdot 2x$
$= 2x \cdot 5^{4x^3}(6x^3 \ln 5 + 24x \ln 5 + 1)$

26) $\frac{dy}{dx} = (x^4 + 3) \cdot \frac{1}{4x^2 \ln 2} \cdot 8x + \log_2 4x^2 \cdot 4x^3$
$= \frac{2(2x^4 \log_2 4x^2 \cdot \ln 2 + x^4 + 3)}{x \ln 2}$

27) $\frac{dy}{dx} = 5^{2x^3} \cdot 16x^3 + (4x^4 + 3) \cdot 5^{2x^3} \ln 5 \cdot 6x^2$
$= 2x^2 \cdot 5^{2x^3}(8x + 12x^4 \ln 5 + 9 \ln 5)$

28) $\frac{dy}{dx} = 4^{4x^5} \cdot 8x^3 + (2x^4 - 5) \cdot 4^{4x^5} \ln 4 \cdot 20x^4$
$= x^3 \cdot 4^{4x^5 + 1}(2 + 10x^5 \ln 4 - 25x \ln 4)$

29) $\frac{dy}{dx} = \frac{\ln 5x^5 \cdot 6x - (3x^2 + 5) \cdot \frac{1}{5x^5} \cdot 25x^4}{(\ln 5x^5)^2}$
$= \frac{6x^2 \ln 5x^5 - 15x^2 - 25}{x \cdot (\ln 5x^5)^2}$

30) $\frac{dy}{dx} = \frac{1}{x^5} \cdot 5x^4$
$= \frac{5}{x}$

31) $\frac{dy}{dx} = 2^{3x^5} \cdot 20x^3 + (5x^4 + 3) \cdot 2^{3x^5} \ln 2 \cdot 15x^4$
$= 5x^3 \cdot 2^{3x^5}(4 + 15x^5 \ln 2 + 9x \ln 2)$

32) $\frac{dy}{dx} = \frac{\log_5 x^3 \cdot 10x^4 - (2x^5 + 1) \cdot \frac{1}{x^3 \ln 5} \cdot 3x^2}{(\log_5 x^3)^2}$
$= \frac{10x^5 \log_5 x^3 \cdot \ln 5 - 6x^5 - 3}{x \ln 5 \cdot (\log_5 x^3)^2}$

33) $\frac{dy}{dx} = -\sin(\sec x^3) \cdot \sec x^3 \tan x^3 \cdot 3x^2$
$= -3x^2 \sin(\sec x^3) \sec x^3 \tan x^3$

34) $\frac{dy}{dx} = \cos(\cot 2x^4) \cdot -\csc^2 2x^4 \cdot 8x^3$
$= -8x^3 \cos(\cot 2x^4) \csc^2 2x^4$

35) $\frac{dy}{dx} = -\csc^2 3x^2 \cdot 6x$
$= -6x \csc^2 3x^2$

36) $\frac{dy}{dx} = \sec^2 4x^3 \cdot 12x^2$
$= 12x^2 \sec^2 4x^3$

37) $\frac{dy}{dx} = -\csc 5x^4 \cot 5x^4 \cdot 20x^3$
$= -20x^3 \csc 5x^4 \cot 5x^4$

38) $\frac{dy}{dx} = -\csc x^3 \cot x^3 \cdot 3x^2$
$= -3x^2 \csc x^3 \cot x^3$

39) $\frac{dy}{dx} = \sec(\cot 4x^5) \tan(\cot 4x^5) \cdot -\csc^2 4x^5 \cdot 20x^4$
$= -20x^4 \sec(\cot 4x^5) \tan(\cot 4x^5) \csc^2 4x^5$

40) $\frac{dy}{dx} = \sec^2 2x^3 \cdot 6x^2$
$= 6x^2 \sec^2 2x^3$

41) $\frac{dy}{dx} = \sec(\sec 4x^5) \tan(\sec 4x^5) \cdot \sec 4x^5 \tan 4x^5 \cdot 20x^4$
$= 20x^4 \sec(\sec 4x^5) \tan(\sec 4x^5) \sec 4x^5 \tan 4x^5$

42) $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - (2x^4)^2}} \cdot 8x^3$
$= -\frac{8x^3}{\sqrt{1 - 4x^8}}$

43) $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - (5x^5)^2}} \cdot 25x^4$
$= -\frac{25x^4}{\sqrt{1 - 25x^{10}}}$

44) $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - ((2x^4 + 5)^5)^2}} \cdot 5(2x^4 + 5)^4 \cdot 8x^3$
$= -\frac{40x^3(2x^4 + 5)^4}{\sqrt{1 - (2x^4 + 5)^{10}}}$

45) $\frac{dy}{dx} = 5(\tan^{-1} 4x^2)^4 \cdot \frac{1}{(4x^2)^2 + 1} \cdot 8x$
$= \frac{40x(\tan^{-1} 4x^2)^4}{16x^4 + 1}$

46) x = horizontal distance from base of ladder to wall y = vertical distance from top of ladder to floor t = time

Equation: $x^2 + y^2 = 10^2$ Given rate: $\frac{dx}{dt} = 6$ Find: $\left.\frac{dy}{dt}\right|_{y=8}$

$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{x}{y} \cdot \frac{dx}{dt} = -\frac{9}{2}$ ft/sec, therefore: $\frac{9}{2}$ ft/sec down the wall

47) A = area of square s = length of sides t = time

Equation: $A = s^2$ Given rate: $\frac{dA}{dt} = -36$ Find: $\left.\frac{ds}{dt}\right|_{s=11}$

$\left.\frac{ds}{dt}\right|_{s=11} = \frac{1}{2s} \cdot \frac{dA}{dt} = -\frac{18}{11}$ m/min

48) V = volume of sphere r = radius t = time

Equation: $V = \frac{4}{3}\pi r^3$ Given rate: $\frac{dV}{dt} = -\frac{256\pi}{3}$ Find: $\left.\frac{dr}{dt}\right|_{r=6}$

$\left.\frac{dr}{dt}\right|_{r=6} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt} = -\frac{16}{27}$ cm/sec

49) V = volume of cube s = length of sides t = time

Equation: $V = s^3$ Given rate: $\frac{dV}{dt} = -64$ Find: $\left.\frac{ds}{dt}\right|_{s=4}$

$\left.\frac{ds}{dt}\right|_{s=4} = \frac{1}{3s^2} \cdot \frac{dV}{dt} = -\frac{4}{3}$ m/min

50) A = area of square x = length of diagonals t = time

Equation: $A = \frac{x^2}{2}$ Given rate: $\frac{dA}{dt} = -\frac{2}{A}$ Find: $\left.\frac{dx}{dt}\right|_{x=4}$

$\left.\frac{dx}{dt}\right|_{x=4} = \frac{1}{x} \cdot \frac{dA}{dt} = -\frac{1}{16}$ m/min

HL2 Summer Questions *[151 marks]*

1a. Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *[5 marks]*

Let the roots of the equation $z^3 = -3 + \sqrt{3}\mathrm{i}$ be u, v and w.

1b. Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *[5 marks]*

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

1c. Find the area of triangle UVW. *[4 marks]*

1d. By considering the sum of the roots u, v and w, show that *[4 marks]*

$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0.$

2. The plane *Π* has the Cartesian equation $2x + y + 2z = 3$ *[7 marks]*

The line *L* has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line *L* and the plane *Π* is 30°.

Find the possible values of p.

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3a. Show that $\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$. *[1 mark]*

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3b. Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)\,x - 1 = 0$. *[7 marks]*

3c. Hence, or otherwise, show that the exact value of $\tan \frac{\pi}{12} = 2 - \sqrt{3}$. *[5 marks]*

3d. Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$. *[6 marks]*

Give your answer in the form $a + b\sqrt{3}$ where a, $b \in \mathbb{Z}$.

4. Two ships, A and B , are observed from an origin O. Relative to O, their position vectors at time t hours after midday are given by *[5 marks]*

$$\boldsymbol{r}_{\mathrm{A}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{B}} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

where distances are measured in kilometres.

Find the minimum distance between the two ships.

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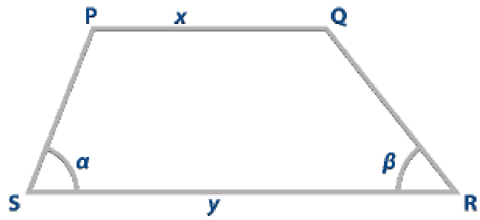
5. Consider the graphs of $y = \frac{x^2}{x-3}$ and $y = m\left(x+3\right)$, $m \in \mathbb{R}$. *[5 marks]*

Find the set of values for m such that the two graphs have no intersection points.

6. Consider quadrilateral PQRS where $[\mathrm{PQ}]$ is parallel to $[\mathrm{SR}]$. *[5 marks]*



In PQRS, $\mathrm{PQ} = x$, $\mathrm{SR} = y$, $\mathrm{R\hat{S}P} = \alpha$ and $\mathrm{Q\hat{R}S} = \beta$.

Find an expression for PS in terms of $x,\ y,\ \sin\beta$ and $\sin(\alpha+\beta)$.

The lines l_1 and l_2 have the following vector equations where $\lambda,\ \mu \in \mathbb{R}$ and $m \in \mathbb{R}$.

$$l_1 : r_1 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} \quad l_2 : r_2 = \begin{pmatrix} -1 \\ -4 \\ -2m \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ -m \end{pmatrix}$$

7a. Show that l_1 and l_2 are never perpendicular to each other. *[3 marks]*

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The plane Π has Cartesian equation $x + 4y - z = p$ where $p \in \mathbb{R}$.

Given that l_1 and Π have no points in common, find

7b. the value of m. *[2 marks]*

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7c. the condition on the value of p. *[2 marks]*

The function f is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of f at $x = 4$.

8a. Write down the value of $f'(4)$. *[1 mark]*

8b. Find $f(4)$. *[1 mark]*

The function g is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

8c. Find $h(4)$. *[2 marks]*

8d. Hence find the equation of the tangent to the graph of h at $x = 4$. *[3 marks]*

Consider the function $f(x) = \frac{x^2 - x - 12}{2x - 15},\ x \in \mathbb{R},\ x \neq \frac{15}{2}$.

Find the coordinates where the graph of f crosses the

9a. x-axis. *[2 marks]*

9b. y-axis. *[1 mark]*

9c. Write down the equation of the vertical asymptote of the graph of f. *[1 mark]*

9d. The oblique asymptote of the graph of f can be written as $y = ax + b$ where $a,\ b \in \mathbb{Q}$. *[4 marks]*

Find the value of a and the value of b.

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9e. Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. *[3 marks]*

9f. Express $\frac{1}{f(x)}$ in partial fractions. *[3 marks]*

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9g. *[4 marks]*

Hence find the exact value of $\int_0^3 \frac{1}{f(x)} \mathrm{d}x$, expressing your answer as a single logarithm.

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Three points $\mathrm{A}(3,\ 0,\ 0),\ \mathrm{B}(0,\ -2,\ 0)$ and $\mathrm{C}(1,\ 1,\ -7)$ lie on the plane Π_1.

10a. *[2 marks]*

Find the vector $\overrightarrow{\mathrm{AB}}$ and the vector $\overrightarrow{\mathrm{AC}}$.

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10b. Hence find the equation of Π_1, expressing your answer in the form $ax + by + cz = d$, where $a,\ b,\ c,\ d \in \mathbb{Z}$. *[5 marks]*

Plane Π_2 has equation $3x - y + 2z = 2$.

10c. The line L is the intersection of Π_1 and Π_2. Verify that the vector equation of L can be written as $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. *[2 marks]*

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The plane Π_3 is given by $2x - 2z = 3$. The line L and the plane Π_3 intersect at the point P.

10d. Show that at the point P, $\lambda = \frac{3}{4}$. *[2 marks]*

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10e. Hence find the coordinates of P. *[1 mark]*

The point $\mathrm{B}(0,\ -2,\ 0)$ lies on L.

10f. Find the reflection of the point B in the plane Π_3. *[7 marks]*

10g. Hence find the vector equation of the line formed when L is reflected in the plane Π_3. *[2 marks]*

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11. The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots α, β and $\alpha + \beta$. *[5 marks]*

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k.

The points A and B have position vectors $\begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$ respectively.

Point C has position vector $\begin{pmatrix} -1 \\ k \\ 0 \end{pmatrix}$. Let O be the origin.

Find, in terms of k,

12a. $\overrightarrow{\mathrm{OA}} \bullet \overrightarrow{\mathrm{OC}}$. *[2 marks]*

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12b. $\overrightarrow{\mathrm{OB}} \bullet \overrightarrow{\mathrm{OC}}$. *[1 mark]*

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12c. Given that $\mathrm{A\widehat{O}C} = \mathrm{B\widehat{O}C}$, show that $k = 7$. *[8 marks]*

12d. Calculate the area of triangle AOC. *[6 marks]*

Consider the equation $x^5 - 3x^4 + mx^3 + nx^2 + px + q = 0$, where $m, n, p, q \in \mathbb{R}$.

The equation has three distinct real roots which can be written as $\log_2 a$, $\log_2 b$ and $\log_2 c$.

The equation also has two imaginary roots, one of which is $d\mathrm{i}$ where $d \in \mathbb{R}$.

13a. Show that $abc = 8$. *[5 marks]*

The values a, b, and c are consecutive terms in a geometric sequence.

13b. Show that one of the real roots is equal to 1. *[3 marks]*

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13c. Given that $q = 8d^2$, find the other two real roots. *[9 marks]*

HL2 Summer Questions *[151 marks]*

1a. Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *[5 marks]*

Markscheme

attempt to find modulus ***(M1)***

$r=2\sqrt{3}\left(=\sqrt{12}\right)$ ***A1***

attempt to find argument in the correct quadrant ***(M1)***

$\theta=\pi+\arctan\left(-\frac{\sqrt{3}}{3}\right)$ ***A1***

$=\frac{5\pi}{6}$ ***A1***

$-3+\sqrt{3}\mathrm{i}=\sqrt{12}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\left(=2\sqrt{3}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\right)$

[5 marks]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be u, v and w.

1b. Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *[5 marks]*

Markscheme

attempt to find a root using de Moivre's theorem ***M1***

$12^{\frac{1}{6}}e^{\frac{5\pi i}{18}}$ ***A1***

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument ***M1***

$12^{\frac{1}{6}}e^{-\frac{7\pi i}{18}}$ ***A1***

$12^{\frac{1}{6}}e^{\frac{17\pi i}{18}}$ ***A1***

Note: Ignore labels for u, v and w at this stage.

[5 marks]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

1c. Find the area of triangle UVW. *[4 marks]*

Markscheme

METHOD 1

attempting to find the total area of (congruent) triangles UOV, VOW and UOW ***M1***

Area $= 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$

= $\frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

METHOD 2

UV$^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$ (or equivalent) ***A1***

UV $= \sqrt{3}\left(12^{\frac{1}{6}}\right)$ (or equivalent) ***A1***

attempting to find the area of UVW using Area = $\frac{1}{2}$ × UV × VW × sin α for example ***M1***

Area $= \frac{1}{2}\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$

= $\frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

1d. By considering the sum of the roots u, v and w, show that $\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$. *[4 marks]*

Markscheme

$u + v + w = 0$ ***R1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \mathrm{i}\sin\frac{5\pi}{18} + \cos\frac{17\pi}{18} + \mathrm{i}\sin\frac{17\pi}{18}\right) = 0$ ***A1***

consideration of real parts ***M1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \cos\frac{17\pi}{18}\right) = 0$

$\cos\left(-\frac{7\pi}{18}\right) = \cos\frac{17\pi}{18}$ explicitly stated ***A1***

$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$ ***AG***

[4 marks]

2. The plane Π has the Cartesian equation $2x + y + 2z = 3$ *[7 marks]*

The line L has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line L and the plane Π is 30°.

Find the possible values of p.

Markscheme

recognition that the angle between the normal and the line is 60° (seen anywhere) ***R1***

attempt to use the formula for the scalar product ***M1***

$\cos 60° = \frac{\left|\begin{pmatrix}2\\1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\p\end{pmatrix}\right|}{\sqrt{9}\times\sqrt{1+4+p^2}}$ ***A1***

$\frac{1}{2} = \frac{|2p|}{3\sqrt{5+p^2}}$ ***A1***

$3\sqrt{5+p^2} = 4\,|p|$

attempt to square both sides ***M1***

$9\left(5+p^2\right) = 16p^2 \Rightarrow 7p^2 = 45$

$p = \pm 3\sqrt{\frac{5}{7}}$ (or equivalent) ***A1A1***

[7 marks]

3a. Show that $\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$. *[1 mark]*

Markscheme

stating the relationship between $\cot$ and $\tan$ and stating the identity for $\tan 2\theta$ ***M1***

$\cot 2\theta = \frac{1}{\tan 2\theta}$ and $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$ ***AG***

[1 mark]

3b. Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)\,x - 1 = 0$. *[7 marks]*

Markscheme

METHOD 1

attempting to substitute $\tan\theta$ for x and using the result from (a) ***M1***

LHS = $\tan^2\theta + 2\tan\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$\tan^2\theta + 1 - \tan^2\theta - 1 = 0$(= RHS) ***A1***

so $x = \tan\theta$ satisfies the equation ***AG***

attempting to substitute $-\cot\theta$ for x and using the result from (a) ***M1***

LHS = $\cot^2\theta - 2\cot\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$= \frac{1}{\tan^2\theta} - \left(\frac{1-\tan^2\theta}{\tan^2\theta}\right) - 1$ ***A1***

$\frac{1}{\tan^2\theta} - \frac{1}{\tan^2\theta} + 1 - 1 = 0$(= RHS) ***A1***

so $x = -\cot\theta$ satisfies the equation ***AG***

METHOD 2

let $\alpha = \tan\theta$ and $\beta = -\cot\theta$

attempting to find the sum of roots ***M1***

$\alpha + \beta = \tan\theta - \frac{1}{\tan\theta}$

$= \frac{\tan^2\theta - 1}{\tan\theta}$ ***A1***

$= -2\cot 2\theta$ (from part (a)) ***A1***

attempting to find the product of roots ***M1***

$\alpha\beta = \tan\theta \times (-\cot\theta)$ ***A1***

= −1 ***A1***

the coefficient of x and the constant term in the quadratic are $2\cot 2\theta$ and −1 respectively ***R1***

hence the two roots are $\alpha = \tan\theta$ and $\beta = -\cot\theta$ ***AG***

[7 marks]

3c. Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. *[5 marks]*

Markscheme

METHOD 1

$x = \tan\frac{\pi}{12}$ and $x = -\cot\frac{\pi}{12}$ are roots of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$ ***R1***

Note: Award ***R1*** if only $x = \tan\frac{\pi}{12}$ is stated as a root of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$.

$x^2 + 2\sqrt{3}x - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$x = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ $\left(-\cot\frac{\pi}{12} < 0\right)$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

METHOD 2

attempting to substitute $\theta = \frac{\pi}{12}$ into the identity for $\tan 2\theta$ ***M1***

$\tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$

$\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$\tan\frac{\pi}{12} = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

[5 marks]

3d. Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$. *[6 marks]*

Give your answer in the form $a + b\sqrt{3}$ where a, $b \in \mathbb{Z}$.

Markscheme

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$ is the sum of the roots of $x^2 + \left(2\cot\frac{\pi}{12}\right)x - 1 = 0$ ***R1***

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24} = -2\cot\frac{\pi}{12}$ ***A1***

$= \frac{-2}{2-\sqrt{3}}$ ***A1***

attempting to rationalise **their** denominator ***(M1)***

$= -4 - 2\sqrt{3}$ ***A1A1***

[6 marks]

4. Two ships, A and B , are observed from an origin O. Relative to O, their position vectors at time t hours after midday are given by *[5 marks]*

$\boldsymbol{r}_A = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 8 \end{pmatrix}$

$\boldsymbol{r}_B = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 12 \end{pmatrix}$

where distances are measured in kilometres.

Find the minimum distance between the two ships.

Markscheme

attempting to find $\boldsymbol{r}_B - \boldsymbol{r}_A$ for example ***(M1)***

$\boldsymbol{r}_B - \boldsymbol{r}_A = \begin{pmatrix} 3 \\ -6 \end{pmatrix} + t\begin{pmatrix} -5 \\ 4 \end{pmatrix}$

attempting to find $|\boldsymbol{r}_B - \boldsymbol{r}_A|$ ***M1***

distance $d(t) = \sqrt{(3-5t)^2 + (4t-6)^2}\left(= \sqrt{41t^2 - 78t + 45}\right)$ ***A1***

using a graph to find the d − coordinate of the local minimum ***M1***

the minimum distance between the ships is 2.81 (km) $\left(= \frac{11\sqrt{41}}{41}\,(\text{km})\right)$ ***A1***

[5 marks]

5. Consider the graphs of $y = \frac{x^2}{x-3}$ and $y = m\left(x+3\right)$, $m \in \mathbb{R}$. *[5 marks]*

Find the set of values for m such that the two graphs have no intersection points.

Markscheme

METHOD 1

sketching the graph of $y = \frac{x^2}{x-3}$ $\left(y = x + 3 + \frac{9}{x-3}\right)$ ***M1***

the (oblique) asymptote has a gradient equal to 1

and so the maximum value of m is 1 ***R1***

consideration of a straight line steeper than the horizontal line joining (−3, 0) and (0, 0) ***M1***

so m > 0 ***R1***

hence 0 < m ≤ 1 ***A1***

METHOD 2

attempting to eliminate y to form a quadratic equation in x ***M1***

$x^2 = m\left(x^2 - 9\right)$

$\Rightarrow \left(m-1\right)x^2 - 9m = 0$ ***A1***

EITHER

attempting to solve $-4\left(m-1\right)\left(-9m\right) < 0$ for m ***M1***

OR

attempting to solve x^2 < 0 *ie* $\frac{9m}{m-1} < 0$ $\left(m \neq 1\right)$ for m ***M1***

THEN

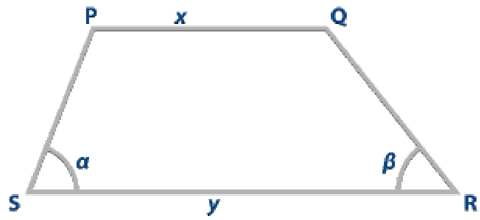
$\Rightarrow 0 < m < 1$ ***A1***

a valid reason to explain why $m = 1$ gives no solutions *eg* if $m = 1$,

$\left(m-1\right)x^2 - 9m = 0 \Rightarrow -9 = 0$ and so 0 < m ≤ 1 ***R1***

[5 marks]

6. Consider quadrilateral PQRS where $[\mathrm{PQ}]$ is parallel to $[\mathrm{SR}]$. *[5 marks]*



In PQRS, $\mathrm{PQ} = x$, $\mathrm{SR} = y$, $\mathrm{R\hat{S}P} = \alpha$ and $\mathrm{Q\hat{R}S} = \beta$.

Find an expression for PS in terms of $x,\ y,\ \sin\beta$ and $\sin(\alpha+\beta)$.

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

METHOD 1

from vertex P, draws a line parallel to $[\mathrm{QR}]$ that meets $[\mathrm{SR}]$ at a point X **(M1)**

uses the sine rule in $\Delta\mathrm{PSX}$ **M1**

$\frac{\mathrm{PS}}{\sin\beta} = \frac{y-x}{\sin(180^\circ-\alpha-\beta)}$ **A1**

$\sin(180^\circ-\alpha-\beta) = \sin(\alpha+\beta)$ **(A1)**

$\mathrm{PS} = \frac{(y-x)\sin\beta}{\sin(\alpha+\beta)}$ **A1**

METHOD 2

let the height of quadrilateral PQRS be h

$h = \mathrm{PS}\sin\alpha$ **A1**

attempts to find a second expression for h **M1**

$h = (y-x-\mathrm{PS}\cos\alpha)\tan\beta$

$\mathrm{PS}\sin\alpha = (y-x-\mathrm{PS}\cos\alpha)\tan\beta$

writes $\tan\beta$ as $\frac{\sin\beta}{\cos\beta}$, multiplies through by $\cos\beta$ and expands the RHS **M1**

$\mathrm{PS}\sin\alpha\cos\beta = (y-x)\sin\beta - \mathrm{PS}\cos\alpha\sin\beta$

$\mathrm{PS} = \frac{(y-x)\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$ **A1**

$\mathrm{PS} = \frac{(y-x)\sin\beta}{\sin(\alpha+\beta)}$ **A1**

[5 marks]

The lines l_1 and l_2 have the following vector equations where $\lambda,\ \mu \in \mathbb{R}$ and $m \in \mathbb{R}$.

$$l_1 : r_1 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} l_2 : r_2 = \begin{pmatrix} -1 \\ -4 \\ -2m \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ -m \end{pmatrix}$$

7a. Show that l_1 and l_2 are never perpendicular to each other. *[3 marks]*

Markscheme

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attempts to calculate $\begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ -m \end{pmatrix}$ **(M1)**

$= -1 - m^2$ **A1**

since $m^2 \geq 0,\ -1 - m^2 < 0$ for $m \in \mathbb{R}$ **R1**

so l_1 and l_2 are never perpendicular to each other **AG**

[3 marks]

The plane Π has Cartesian equation $x + 4y - z = p$ where $p \in \mathbb{R}$.

Given that l_1 and Π have no points in common, find

7b. the value of m. *[2 marks]*

Markscheme

(since l_1 is parallel to Π, l_1 is perpendicular to the normal of Π and so)

$\begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = 0$ **R1**

$2 + 4 - m = 0$

$m = 6$ **A1**

[2 marks]

7c. the condition on the value of p. *[2 marks]*

Markscheme

since there are no points in common, $(3,\ -2,\ 0)$ does not lie in Π

EITHER

substitutes $(3,\ -2,\ 0)$ into $x + 4y - z (\neq p)$ **(M1)**

OR

$\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} (\neq p)$ **(M1)**

THEN

$p \neq -5$ **A1**

[2 marks]

The function f is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of f at $x = 4$.

8a. Write down the value of $f'(4)$. *[1 mark]*

Markscheme

$f'(4) = 6$ ***A1***

[1 mark]

8b. Find $f(4)$. *[1 mark]*

Markscheme

$f(4) = 6 \times 4 - 1 = 23$ ***A1***

[1 mark]

The function g is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

8c. Find $h(4)$. *[2 marks]*

Markscheme

$h(4) = f(g(4))$ ***(M1)***

$h(4) = f\left(4^2 - 3 \times 4\right) = f(4)$

$h(4) = 23$ ***A1***

[2 marks]

8d. Hence find the equation of the tangent to the graph of h at $x = 4$. *[3 marks]*

Markscheme

attempt to use chain rule to find h' ***(M1)***

$f'(g(x))\times g'(x)$ OR $\left(x^2-3x\right)'\times f'\left(x^2-3x\right)$

$h'(4)=(2\times 4-3)f'\left(4^2-3\times 4\right)$ ***A1***

$=30$

$y-23=30(x-4)$ OR $y=30x-97$ ***A1***

[3 marks]

Consider the function $f(x)=\frac{x^2-x-12}{2x-15},\ x\in\mathbb{R},\ x\neq\frac{15}{2}$.

Find the coordinates where the graph of f crosses the

9a. x-axis. *[2 marks]*

Markscheme

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2-x-12=0$ ***(M1)***

$(-3,0)$ and $(4,0)$ ***A1***

[2 marks]

9b. y-axis. *[1 mark]*

Markscheme

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

$\left(0, \frac{4}{5}\right)$ ***A1***

[1 mark]

9c. Write down the equation of the vertical asymptote of the graph of f. *[1 mark]*

Markscheme

$x = \frac{15}{2}$ ***A1***

Note: Award ***A0*** for $x \neq \frac{15}{2}$.
Award ***A1*** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark]

9d. The oblique asymptote of the graph of f can be written as $y = ax + b$ where $a,\ b \in \mathbb{Q}$. *[4 marks]*

Find the value of a and the value of b.

Markscheme

METHOD 1

$(ax + b)(2x - 15) \equiv x^2 - x - 12$

attempts to expand $(ax + b)(2x - 15)$ ***(M1)***

$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$

$a = \frac{1}{2}$ ***A1***

equates coefficients of x ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 2

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x}{2} + \frac{13}{4} + \ldots$ ***M1***

$a = \frac{1}{2}$ ***A1***

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 3

$a = \frac{1}{2}$ ***A1***

$\frac{x^2-x-12}{2x-15} \equiv \frac{x}{2} + b + \frac{c}{2x-15}$ ***M1***

$x^2 - x - 12 \equiv \frac{(2x-15)x}{2} + (2x-15)b + c$

equates coefficients of x : ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 4

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x^2-x-12}{2x-15} = \frac{x}{2} + \frac{\frac{13x}{2}-12}{2x-15}$

$a = \frac{1}{2}$ ***A1***

$\frac{\frac{13x}{2}-12}{2x-15} = \frac{13}{4} + \ldots$ ***M1***

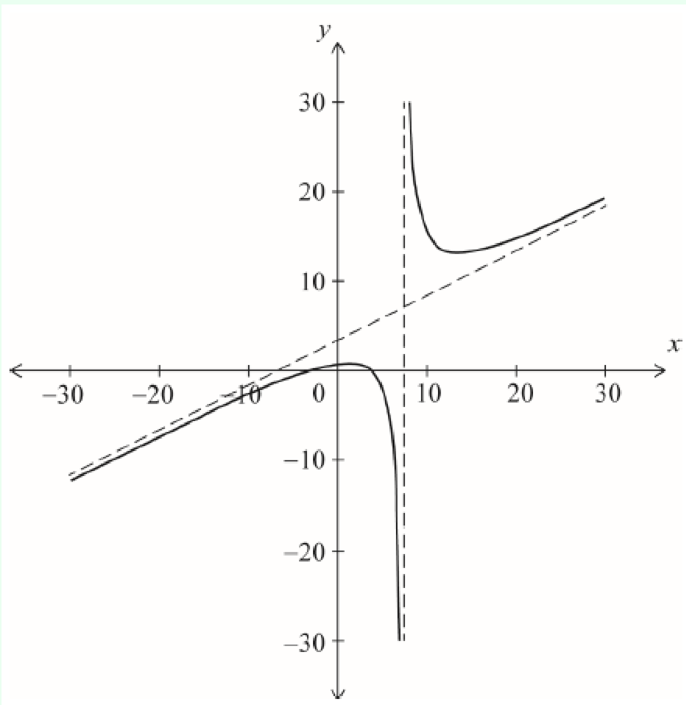
$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

[4 marks]

9e. Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. *[3 marks]*

Markscheme



two branches with approximately correct shape (for $-30 \leq x \leq 30$) ***A1***

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1***

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

9f. Express $\frac{1}{f(x)}$ in partial fractions. *[3 marks]*

Markscheme

attempts to split into partial fractions: ***(M1)***

$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3} + \frac{B}{x-4}$

$2x - 15 \equiv A(x-4) + B(x+3)$

$A = 3$ ***A1***

$B = -1$ ***A1***

$\left(\frac{3}{x+3} - \frac{1}{x-4}\right)$

[3 marks]

9g. *[4 marks]*

Hence find the exact value of $\int_0^3 \frac{1}{f(x)}\,\mathrm{d}x$, expressing your answer as a single logarithm.

Markscheme

$\int_0^3 \left(\frac{3}{x+3} - \frac{1}{x-4}\right)\mathrm{d}x$

attempts to integrate and obtains two terms involving 'ln' ***(M1)***

$= [3\ln|x+3| - \ln|x-4|]_0^3$ ***A1***

$= 3\ln 6 - \ln 1 - 3\ln 3 + \ln 4$ ***A1***

$= 3\ln 2 + \ln 4 \ (= \ln 8 + \ln 4)$

$= \ln 32 \ (= 5\ln 2)$ ***A1***

Note: The final ***A1*** is dependent on the previous two ***A*** marks.

[4 marks]

Three points $\mathrm{A}(3,\ 0,\ 0),\ \mathrm{B}(0,\ -2,\ 0)$ and $\mathrm{C}(1,\ 1,\ -7)$ lie on the plane Π_1.

10a. Find the vector $\overrightarrow{\mathrm{AB}}$ and the vector $\overrightarrow{\mathrm{AC}}$. *[2 marks]*

Markscheme

attempts to find either $\overrightarrow{\mathrm{AB}}$ or $\overrightarrow{\mathrm{AC}}$ ***(M1)***

$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$ and $\overrightarrow{\mathrm{AC}} = \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ ***A1***

[2 marks]

10b. Hence find the equation of Π_1, expressing your answer in the form $ax + by + cz = d$, where $a,\ b,\ c,\ d \in \mathbb{Z}$. *[5 marks]*

Markscheme

METHOD 1

attempts to find $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}$ ***(M1)***

$\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix}$ ***A1***

EITHER

equation of plane is of the form $14x - 21y - 7z = d \quad (2x - 3y - z = d)$ ***(A1)***

substitutes a valid point e.g $(3,\ 0,\ 0)$ to obtain a value of d ***M1***

$d = 42 \ \ (d = 6)$

OR

attempts to use $r \cdot n = a \cdot n$ ***(M1)***

$r \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} \ \left(r \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = 42 \right)$ ***A1***

$$r\cdot\begin{pmatrix}2\\-3\\-1\end{pmatrix}=\begin{pmatrix}3\\0\\0\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\-1\end{pmatrix}\ \left(r\cdot\begin{pmatrix}2\\-3\\-1\end{pmatrix}=6\right)$$

THEN

$14x-21y-7z=42\ \ (2x-3y-z=6)$ ***A1***

METHOD 2

equation of plane is of the form $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\0\\0\end{pmatrix}+s\begin{pmatrix}-3\\-2\\0\end{pmatrix}+t\begin{pmatrix}-2\\1\\-7\end{pmatrix}$ ***A1***

attempts to form equations for $x,\ y,\ z$ in terms of their parameters ***(M1)***

$x=3-3s-2t\ ,\ y=-2s+t\ ,\ z=-7t$ ***A1***

eliminates at least one of their parameters ***(M1)***

for example, $2x-3y=6-7t(\Rightarrow 2x-3y=6+z)$

$2x-3y-z=6$ ***A1***

[5 marks]

Plane Π_2 has equation $3x-y+2z=2$.

10c. The line L is the intersection of Π_1 and Π_2. Verify that the vector equation of L can be written as $r=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$. *[2 marks]*

Markscheme

METHOD 1

substitutes $r=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ into their Π_1 and Π_2 (given) ***(M1)***

$\Pi_1:\ 2\lambda-3(-2+\lambda)-(-\lambda)=6$ and $\Pi_2:\ 3\lambda-3(-2+\lambda)+2(-\lambda)=2$ ***A1***

Note: Award ***(M1)A0*** for correct verification using a specific value of λ.

so the vector equation of L can be written as $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

AG

METHOD 2

EITHER

attempts to find $\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ ***M1***

$= \begin{pmatrix} -7 \\ -7 \\ 7 \end{pmatrix}$

OR

$\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (2 - 3 + 1) = 0$ and $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (3 - 1 - 2) = 0$

M1

THEN

substitutes $(0, -2, 0)$ into Π_1 and Π_2

$\Pi_1 :\ 2(0) - 3(-2) - (0) = 6$ and $\Pi_2 :\ 3(0) - (-2) + 2(0) = 2$ ***A1***

so the vector equation of L can be written as $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

AG

METHOD 3

attempts to solve $2x - 3y - z = 6$ and $3x - y + 2z = 2$ ***(M1)***

for example, $x = -\lambda,\ y = -2 - \lambda,\ z = \lambda$ ***A1***

Note: Award **A1** for substituting $x = 0$ (or $y = -2$ or $z = 0$) into Π_1 and Π_2 and solving simultaneously. For example, solving $-3y - z = 6$ and $-y + 2z = 2$ to obtain $y = -2$ and $z = 0$.

so the vector equation of L can be written as $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

AG

[2 marks]

The plane Π_3 is given by $2x - 2z = 3$. The line L and the plane Π_3 intersect at the point P.

10d. Show that at the point P, $\lambda = \frac{3}{4}$. *[2 marks]*

Markscheme

substitutes the equation of L into the equation of Π_3 ***(M1)***

$2\lambda + 2\lambda = 3 \Rightarrow 4\lambda = 3$ ***A1***

$\lambda = \frac{3}{4}$ ***AG***

[2 marks]

10e. Hence find the coordinates of P. *[1 mark]*

Markscheme

P has coordinates $\left(\frac{3}{4}, -\frac{5}{4}, -\frac{3}{4}\right)$ ***A1***

[1 mark]

The point $\mathrm{B}(0,\ -2,\ 0)$ lies on L.

10f. Find the reflection of the point B in the plane Π_3. *[7 marks]*

Markscheme

normal to Π_3 is $n = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ ***(A1)***

Note: May be seen or used anywhere.

considers the line normal to Π_3 passing through $\mathrm{B}(0, -2, 0)$ ***(M1)***

$r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ ***A1***

EITHER

finding the point on the normal line that intersects Π_3
attempts to solve simultaneously with plane $2x - 2z = 3$ ***(M1)***

$4\mu + 4\mu = 3$

$\mu = \frac{3}{8}$ ***A1***

point is $\left(\frac{3}{4},\ -2, -\frac{3}{4}\right)$

OR

$$\left(\begin{pmatrix} 2\mu \\ -2 \\ -2\mu \end{pmatrix} - \begin{pmatrix} \frac{3}{4} \\ -\frac{5}{4} \\ -\frac{3}{4} \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = 0$$ ***(M1)***

$4\mu - \frac{3}{2} + 4\mu - \frac{3}{2} = 0$

$\mu = \frac{3}{8}$ ***A1***

OR

attempts to find the equation of the plane parallel to Π_3 containing
B' $(x - z = 3)$ and solve simultaneously with L ***(M1)***

$2\mu' + 2\mu' = 3$

$\mu' = \frac{3}{4}$ ***A1***

THEN

so, another point on the reflected line is given by

$$r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$ ***(A1)***

$\Rightarrow$ B'$\left(\frac{3}{2}, -2, -\frac{3}{2}\right)$ ***A1***

[7 marks]

10g. Hence find the vector equation of the line formed when L is reflected in the plane Π_3. *[2 marks]*

Markscheme

EITHER

attempts to find the direction vector of the reflected line using their P and $\mathrm{B'}$ ***(M1)***

$$\overrightarrow{\mathrm{PB'}} = \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$

OR

attempts to find their direction vector of the reflected line using a vector approach ***(M1)***

$$\overrightarrow{\mathrm{PB'}} = \overrightarrow{\mathrm{PB}} + \overrightarrow{\mathrm{BB'}} = -\frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

THEN

$r = \begin{pmatrix} \frac{3}{2} \\ -2 \\ -\frac{3}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** for either '$r =$' or '$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$' not stated. Award ***A0*** for '$L' =$'

[2 marks]

11. The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots α, β and $\alpha + \beta$. *[5 marks]*

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k.

Markscheme

$\alpha + \beta + \alpha + \beta = k$ ***(A1)***

$\alpha + \beta = \frac{k}{2}$

$\alpha\beta(\alpha + \beta) = -3k$ ***(A1)***

$\left(-\frac{k^2}{4}\right)\left(\frac{k}{2}\right) = -3k\left(-\frac{k^3}{8} = -3k\right)$ ***M1***

attempting to solve $-\frac{k^3}{8} + 3k = 0$ (or equivalent) for k ***(M1)***

$k = 2\sqrt{6}\left(= \sqrt{24}\right)(k > 0)$ ***A1***

Note: Award ***A0*** for $k = \pm 2\sqrt{6}\left(\pm\sqrt{24}\right)$.

[5 marks]

The points A and B have position vectors $\begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$ respectively.

Point C has position vector $\begin{pmatrix} -1 \\ k \\ 0 \end{pmatrix}$. Let O be the origin.

Find, in terms of k,

12a. $\overrightarrow{\mathrm{OA}} \bullet \overrightarrow{\mathrm{OC}}$. *[2 marks]*

Markscheme

correct substitution into either $\overrightarrow{\mathrm{OA}} \bullet \overrightarrow{\mathrm{OC}}$ or into $\overrightarrow{\mathrm{OB}} \bullet \overrightarrow{\mathrm{OC}}$ (in (ii)) ***(A1)***

eg $-2 \times (-1) + 4 \times k,\ 6 \times (-1) + 8 \times k$

correct expression ***A1 N1***

eg $2 + 4k,\ 4k + 2$

[2 marks]

12b. $\overrightarrow{OB} \bullet \overrightarrow{OC}$. *[1 mark]*

Markscheme

correct expression ***A1 N1***

eg $8k - 6,\ -6 + 8k$

[1 mark]

12c. Given that $A\hat{O}C = B\hat{O}C$, show that $k = 7$. *[8 marks]*

Markscheme

finding magnitudes (seen anywhere) ***A1A1***

eg $\sqrt{(-2)^2 + (4)^2 + (-4)^2}\ (= 6),\ \sqrt{(6)^2 + (8)^2 + 0^2}\ (= 10)$

correct substitution of their values into formula for angle AOC ***(A1)***

eg $\cos\theta = \frac{2+4k}{\sqrt{(-2)^2+(4)^2+(-4)^2}\left|\overrightarrow{OC}\right|}$

correct substitution of their values into formula for angle BOC ***(A1)***

eg $\cos\theta = \frac{8k-6}{\sqrt{(6)^2+(8)^2+0^2}\left|\overrightarrow{OC}\right|}$

recognizing that $\cos A\hat{O}C = \cos B\hat{O}C$ (seen anywhere) ***(M1)***

eg $\frac{2+4k}{\left|\overrightarrow{OC}\right|\sqrt{(-2)^2+(4)^2+(-4)^2}} = \frac{8k-6}{\left|\overrightarrow{OC}\right|\sqrt{6^2+(8)^2+0^2}},\ \frac{2+4k}{6\sqrt{1+k^2}} = \frac{8k-6}{10\sqrt{1+k^2}}$

correct working (without radicals) ***(A2)***

eg $10(2 + 4k) = 6(8k - 6),\ 11k^2 - 79k + 14 = 0$

correct working clearly leading to the required answer ***A1***

eg $20 + 36 = 48k - 40k,\ 56 = 8k,\ k = 7$ and $k = \frac{2}{11}$,
$(k - 7)(11k - 2) = 0$

$k = 7$ ***AG N0***

[8 marks]

12d. Calculate the area of triangle AOC. *[6 marks]*

Markscheme

finding magnitude of $\overrightarrow{\mathrm{OC}}$ (seen anywhere) ***A1***

eg $\sqrt{(-1)^2 + 7^2 + 0^2}$, $\sqrt{50}$

valid attempt to find $\cos\theta$ ***(M1)***

eg $\cos\theta = \frac{2+28}{6\sqrt{(-1)^2+7^2+0^2}}$, $\cos\theta = \frac{56-6}{10\sqrt{(-1)^2+7^2+0^2}}$,

$\left(\sqrt{26}\right)^2 = 6^2 + \left(\sqrt{50}\right)^2 - 2(6)\sqrt{50}\cos\theta$

finding $\cos\theta$ ***A1***

eg $\cos\theta = \frac{5}{\sqrt{50}} \left(= \frac{1}{\sqrt{2}}\right)$

valid approach to find $\sin\theta$ (seen anywhere) ***(M1)***

eg $\theta = \frac{\pi}{4}$, $\sin\theta = \cos\theta$, $\sin\theta = \sqrt{1 - \frac{25}{50}}$, $\sin\theta = \sqrt{1 - \cos^2\theta}$,
$\sin\theta = \frac{\sqrt{2}}{2}$

correct substitution of **their** values into $\frac{1}{2}ab\sin\mathrm{C}$ ***(A1)***

eg $\frac{1}{2} \times 6 \times \sqrt{50} \times \sqrt{1 - \frac{25}{50}}$, $\frac{1}{2} \times 6 \times \sqrt{50} \times \frac{5}{\sqrt{50}}$

area is 15 ***A1 N3***

[6 marks]

Consider the equation $x^5 - 3x^4 + mx^3 + nx^2 + px + q = 0$, where $m, n, p, q \in \mathbb{R}$
.

The equation has three distinct real roots which can be written as $\log_2 a$, $\log_2 b$ and $\log_2 c$.

The equation also has two imaginary roots, one of which is $d\mathrm{i}$ where $d \in \mathbb{R}$.

13a. Show that $abc = 8$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

recognition of the other root $= -d\text{i}$ ***(A1)***

$\log_2 a + \log_2 b + \log_2 c + d\text{i} - d\text{i} = 3$ ***M1A1***

Note: Award ***M1*** for sum of the roots, ***A1*** for 3. Award ***A0M1A0*** for just $\log_2 a + \log_2 b + \log_2 c = 3$.

$\log_2 abc = 3$ ***(M1)***

$\Rightarrow abc = 2^3$ ***A1***

$abc = 8$ ***AG***

[5 marks]

The values a, b, and c are consecutive terms in a geometric sequence.

13b. Show that one of the real roots is equal to 1. *[3 marks]*

Markscheme

METHOD 1

let the geometric series be u_1, $u_1 r$, $u_1 r^2$

$(u_1 r)^3 = 8$ ***M1***

$u_1 r = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

METHOD 2

$\frac{b}{a} = \frac{c}{b}$

$b^2 = ac \Rightarrow b^3 = abc = 8$ ***M1***

$b = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

[3 marks]

13c. Given that $q = 8d^2$, find the other two real roots. *[9 marks]*

Markscheme

METHOD 1

product of the roots is $r_1 \times r_2 \times 1 \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$ ***(M1)(A1)***

$r_1 \times r_2 = -8$ ***A1***

sum of the roots is $r_1 + r_2 + 1 + d\mathrm{i} + -d\mathrm{i} = 3$ ***(M1)(A1)***

$r_1 + r_2 = 2$ ***A1***

solving simultaneously ***(M1)***

$r_1 = -2,\ r_2 = 4$ ***A1A1***

METHOD 2

product of the roots $\log_2 a \times \log_2 b \times \log_2 c \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$ ***M1A1***

$\log_2 a \times \log_2 b \times \log_2 c = -8$ ***A1***

EITHER

$a,\ b,\ c$ can be written as $\frac{2}{r},\ 2,\ 2r$ ***M1***

$\left(\log_2 \frac{2}{r}\right)(\log_2 2)(\log_2 2r) = -8$

attempt to solve ***M1***

$(1 - \log_2 r)(1 + \log_2 r) = -8$

$\log_2 r = \pm 3$

$r = \frac{1}{8},\ 8$ ***A1A1***

OR

$a,\ b,\ c$ can be written as $a,\ 2,\ \frac{4}{a}$ ***M1***

$(\log_2 a)(\log_2 2)\left(\log_2 \frac{4}{a}\right) = -8$

attempt to solve ***M1***

$a = \frac{1}{4},\ 16$ ***A1A1***

THEN

a and c are $\frac{1}{4},\ 16$ ***(A1)***

roots are −2, 4 ***A1***

[9 marks]