Notre Dame High School

Established 1957

To: Discrete Math Students

I hope that this letter finds you relaxing in the warmth of the summer sun! In getting ready for the school year, I want to make sure that you have everything you need in order to start out the semester strong. Discrete Mathematics is going to be somewhat different from what you are used too. We are going to take a deeper look into additional topics of mathematics that will help prepare you for the college classroom. Please read and complete all of the problems in the packet that follows. You will be quizzed on this material during the first week of class. Make sure the packet is done on time, to the best of your ability, and have all work shown for each problem. Any calculator will do for this class. Cellphone calculators are NOT permitted at any time during the semester.

Please have your completed packet on the first day of class so that we can have a strong review before the quiz. You will find that this course will give you a whole new outlook on Mathematics and provide you with applications that will be useful in your lives!

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1.1 Solving Problems by Inductive Reasoning

The development of mathematics can be traced to the Egyptian and Babylonian cultures (3000 B.C.–A.D. 260) as a necessity for problem solving. Their approach was an example of the "do thus and so" method: in order to solve a problem or perform an operation, a cookbook-like recipe was given, and it was performed over and over to solve similar problems. The classical Greek period (600 B.C.–A.D. 450) gave rise to a more formal type of mathematics, in which general concepts were applied to specific problems, resulting in a structured, logical development of mathematics.

By observing that a specific method worked for a certain type of problem, the Babylonians and the Egyptians concluded that the same method would work for any similar type of problem. Such a conclusion is called a *conjecture.* A **conjecture** is an educated guess based upon repeated observations of a particular process or pattern. The method of reasoning we have just described is called *inductive reasoning.*

Inductive Reasoning

Inductive reasoning is characterized by drawing a general conclusion (making a conjecture) from repeated observations of specific examples. The conjecture may or may not be true.

In testing a conjecture obtained by inductive reasoning, it takes only one example that does not work in order to prove the conjecture false. Such an example is called a **counterexample.** Inductive reasoning provides a powerful method of drawing conclusions, but it is also important to realize that there is no assurance that the observed conjecture will always be true. For this reason, mathematicians are reluctant to accept a conjecture as an absolute truth until it is formally proved using methods of *deductive reasoning.* Deductive reasoning characterized the development and approach of Greek mathematics, as seen in the works of Euclid, Pythagoras, Archimedes, and others.

Deductive Reasoning

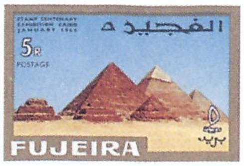
Deductive reasoning is characterized by applying general principles to specific examples.

Let us now look at examples of these two types of reasoning. In this chapter we will often refer to the **natural** or **counting numbers:**

$$1, 2, 3, \ldots.$$

The three dots indicate that the numbers continue indefinitely in the pattern that has been established. The most probable rule for continuing this pattern is "add 1 to the previous number," and this is indeed the rule that we follow. Now consider the following list of natural numbers:

$$2, 9, 16, 23, 30.$$



The Moscow papyrus, which dates back to about 1850 B.C., provides an example of inductive reasoning by the early Egyptian mathematicians. Problem 14 in the document reads:

> *You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4, result 16. You are to double 4, result 8. You are to square 2, result 4. You are to add the 16, the 8, and the 4, result 28. You are to take one-third of 6, result 2. You are to take 28 twice, result 56. See, it is 56. You will find it right.*

What does all this mean? A *frustum* of a pyramid is that part of the pyramid remaining after its top has been cut off by a plane parallel to the base of the pyramid. The actual formula for finding the volume of the frustum of a pyramid with a square base is

$$V = \frac{1}{3}h(b^2 + bB + B^2),$$

where b is the area of the upper base, B is the area of the lower base, and h is the height (or altitude). The writer of the problem is giving a method of determining the volume of the frustum of a pyramid with square bases on the top and bottom, with bottom base side of length 4, top base side of length 2, and height equal to 6.

A truncated pyramid, or frustum of a pyramid

As indicated earlier, not until a general relationship is proved can one be sure about a conjecture since one counterexample is always sufficient to make the conjecture false.

1.1 EXERCISES

In Exercises 1–12, determine whether the reasoning is an example of deductive or inductive reasoning.

1. If the mechanic says that it will take two days to repair your car, then it will actually take four days. The mechanic says, "I figure it'll take a couple of days to fix it, ma'am." Then you can expect it to be ready four days from now.

2. If you take your medicine, you'll feel a lot better. You take your medicine. Therefore, you'll feel a lot better.

3. It has rained every day for the past five days, and it is raining today as well. So it will also rain tomorrow.

4. Natalie's first three children were boys. If she has another baby, it will be a boy.

5. Josh had 95 Pokémon trading cards. Margaret gave him 20 more for his birthday. Therefore, he now has 115 of them.

6. If the same number is subtracted from both sides of a true equation, the new equation is also true. I know that $9 + 18 = 27$. Therefore, $(9 + 18) - 12 = 27 - 12$.

7. If you build it, they will come. You build it. So, they will come.

8. All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

9. It is a fact that every student who ever attended Brainchild University was accepted into medical school. Since I am attending Brainchild, I can expect to be accepted to medical school, too.

10. For the past 25 years, a rare plant has bloomed in Columbia each summer, alternating between yellow and green flowers. Last summer, it bloomed with green flowers, so this summer it will bloom with yellow flowers.

11. In the sequence $5, 10, 15, 20, \ldots$, the most probable next number is 25.

12. Britney Spears' last four single releases have reached the Top Ten list, so her current release will also reach the Top Ten.

✎**13.** Discuss the differences between inductive and deductive reasoning. Give an example of each.

✎**14.** Give an example of faulty inductive reasoning.

Determine the most probable next term in each list of numbers.

15. 6, 9, 12, 15, 18

16. 13, 18, 23, 28, 33

17. 3, 12, 48, 192, 768

18. 32, 16, 8, 4, 2

19. 3, 6, 9, 15, 24, 39

20. 1/3, 3/5, 5/7, 7/9, 9/11

21. 1/2, 3/4, 5/6, 7/8, 9/10

22. 1, 4, 9, 16, 25

23. 1, 8, 27, 64, 125

24. 2, 6, 12, 20, 30, 42

25. 4, 7, 12, 19, 28, 39

26. $-1, 2, -3, 4, -5, 6$

27. 5, 3, 5, 5, 3, 5, 5, 5, 3, 5, 5, 5, 5, 3, 5, 5, 5, 5

28. 8, 2, 8, 2, 2, 8, 2, 2, 2, 8, 2, 2, 2, 2, 8, 2, 2, 2, 2

29. Construct a list of numbers similar to those in Exercise 15 such that the most probable next number in the list is 60.

30. Construct a list of numbers similar to those in Exercise 26 such that the most probable next number in the list is 9.

61. Explain how a toddler might use inductive reasoning to decide on something that will be of benefit to him or her.

62. Discuss one example of inductive reasoning that you have used recently in your life. Test your premises and your conjecture. Did your conclusion ultimately prove to be true or false?

1.2 An Application of Inductive Reasoning: Number Patterns

In the previous section we introduced inductive reasoning, and we showed how it can be applied in predicting "what comes next" in a list of numbers or equations. In this section we will continue our investigation of number patterns.

An ordered list of numbers, such as

$$3, 9, 15, 21, 27, \ldots,$$

is called a *sequence*. A **number sequence** is a list of numbers having a first number, a second number, a third number, and so on, called the **terms** of the sequence. To indicate that the terms of a sequence continue past the last term written, we use three dots (ellipsis points). The sequences in Examples 2(a) and 2(c) in the previous section are called *arithmetic* and *geometric sequences,* respectively. An arithmetic sequence has a common *difference* between successive terms, while a geometric sequence has a common *ratio* between successive terms. The Fibonacci sequence in Example 2(b) is covered in a later chapter.

Successive Differences The sequences seen in the previous section were usually simple enough for us to make an obvious conjecture about the next term. However, some sequences may provide more difficulty in making such a conjecture, and often the **method of successive differences** may be applied to determine the next term if it is not obvious at first glance. For example, consider the sequence

$$2, 6, 22, 56, 114, \ldots.$$

Since the next term is not obvious, subtract the first term from the second term, the second from the third, the third from the fourth, and so on.

2   6   22   56   114

$6 - 2 = 4$ $22 - 6 = 16$ $56 - 22 = 34$ $114 - 56 = 58$

Now repeat the process with the sequence 4, 16, 34, 58 and continue repeating until the difference is a constant value, as shown in line (4):

2		6		22		56		114		**(1)**
	4		16		34		58			**(2)**
		12		18		24				**(3)**
			6		6					**(4)**

Other such relationships among figurate numbers are examined in the exercises of this section.

The method of successive differences, introduced at the beginning of this section, can be used to predict the next figurate number in a sequence of figurate numbers.

EXAMPLE 5 The first five pentagonal numbers are

1, 5, 12, 22, 35.

Use the method of successive differences to predict the sixth pentagonal number.

1	5	12	22	35	**51**
	4	7	10	13	**16**
		3	3	3	**3**

After the second line of successive differences, we can work backward to find that the sixth pentagonal number is 51, which was also found in Example 3(c).

FOR FURTHER THOUGHT

Take any three-digit number whose digits are not all the same. Arrange the digits in decreasing order, and then arrange them in increasing order. Now subtract. Repeat the process, using a 0 if necessary in the event that the difference consists of only two digits. For example, suppose that we choose the digits 1, 4, and 8.

841	963	954
−148	−369	−459
693	594	495

Notice that we have obtained the number 495, and the process will lead to 495 again. The number 495 is called a Kaprekar number, and it will eventually always be generated if this process is followed for such a three-digit number.

For Group Discussion

1. Have each student in a group of students apply the process of Kaprekar to a different two-digit number, in which the digits are not the same. (Interpret 9 as 09 if necessary.) Compare the results. What seems to be true?
2. Repeat the process for four digits, with each student in the group comparing results after several steps. What conjecture can the group make for this situation?

1.2 EXERCISES

Use the method of successive differences to determine the next number in each sequence.

1. 1, 4, 11, 22, 37, 56, . . .

2. 3, 14, 31, 54, 83, 118, . . .

3. 6, 20, 50, 102, 182, 296, . . .

4. 1, 11, 35, 79, 149, 251, . . .

5. 0, 12, 72, 240, 600, 1260, 2352, . . .

6. 2, 57, 220, 575, 1230, 2317, . . .

7. 5, 34, 243, 1022, 3121, 7770, 16799, . . .

8. 3, 19, 165, 771, 2503, 6483, 14409, . . .

1.3 Strategies for Problem Solving

George Polya, author of the classic *How to Solve it,* died at the age of 97 on September 7, 1985. A native of Budapest, Hungary, he was once asked why there were so many good mathematicians to come out of Hungary at the turn of the century. He theorized that it was because mathematics is the cheapest science. It does not require any expensive equipment, only pencil and paper. He authored or coauthored more than 250 papers in many languages, wrote a number of books, and was a brilliant lecturer and teacher. Yet, interestingly enough, he never learned to drive a car.

In the first two sections of this chapter we stressed the importance of pattern recognition and the use of inductive reasoning in solving problems. There are other useful approaches. These ideas will be used throughout the text. (Problem solving is not a topic to be covered one day and then forgotten the next!)

Probably the most famous study of problem-solving techniques was developed by George Polya (1888–1985), among whose many publications was the modern classic *How to Solve It*. In this book, Polya proposed a four-step process for problem solving.

Polya's Four-Step Process for Problem Solving

1. **Understand the problem.** You cannot solve a problem if you do not understand what you are asked to find. The problem must be read and analyzed carefully. You will probably need to read it several times. After you have done so, ask yourself, "What must I find?"
2. **Devise a plan.** There are many ways to attack a problem and decide what plan is appropriate for the particular problem you are solving. See the list headed "Problem-Solving Strategies" for a number of possible approaches.
3. **Carry out the plan.** Once you know how to approach the problem, carry out your plan. You may run into "dead ends" and unforeseen roadblocks, but be persistent. If you are able to solve a problem without a struggle, it isn't much of a problem, is it?
4. **Look back and check.** Check your answer to see that it is reasonable. Does it satisfy the conditions of the problem? Have you answered all the questions the problem asks? Can you solve the problem a different way and come up with the same answer?

In Step 2 of Polya's problem-solving process, we are told to devise a plan. Here are some strategies that may prove useful.

Problem-Solving Strategies

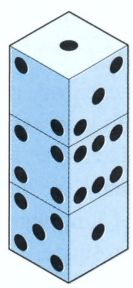
Make a table or a chart.
Look for a pattern.
Solve a similar simpler problem.
Draw a sketch.
Use inductive reasoning.
Write an equation and solve it.
If a formula applies, use it.
Work backward.
Guess and check.
Use trial and error.
Use common sense.
Look for a "catch" if an answer seems too obvious or impossible.

A particular problem solution may involve one or more of the strategies listed here, and you should try to be creative in your problem-solving techniques. The examples that follow illustrate some of these strategies. As you read through them, keep in mind that it is one thing to read a problem solution or watch a teacher solve a problem, but it is another to be able to do it yourself. Have you ever watched your teacher solve a problem and said to yourself, "It looks easy when she does it, but I

2. *Vertical Symmetry in States' Names* (If a vertical line is drawn through the center of a figure and the left and right sides are reflections of each other across this line, the figure is said to have vertical symmetry.) When spelled with all capital letters, each letter in HAWAII has vertical symmetry. Find the name of a state whose letters all have vertical and horizontal symmetry. (September 11, 2001)

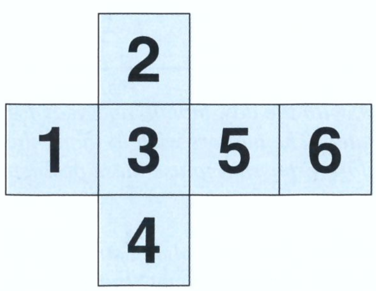
3. *Sum of Hidden Dots on Dice* Three dice with faces numbered 1 through 6 are stacked as shown. Seven of the eighteen faces are visible, leaving eleven faces hidden on the back, on the bottom, and between faces. The total number of dots not visible in this view is ________.

   A. 21
   B. 22
   C. 31
   D. 41
   E. 53
   (September 17, 2001)

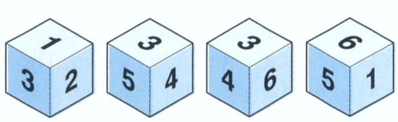


4. *Mr. Green's Age* At his birthday party, Mr. Green would not directly tell how old he was. He said, "If you add the year of my birth to this year, subtract the year of my tenth birthday and the year of my fiftieth birthday, and then add my present age, the result is eighty." How old was Mr Green? (December 14, 1997)

5. *Unfolding and Folding a Box* An unfolded box is shown below.



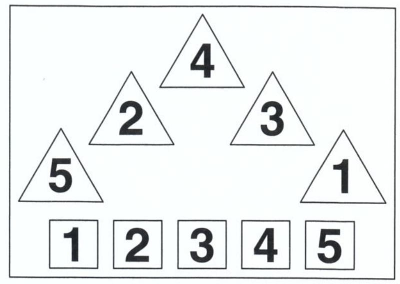
Which figure shows the box folded up? (November 7, 2001)



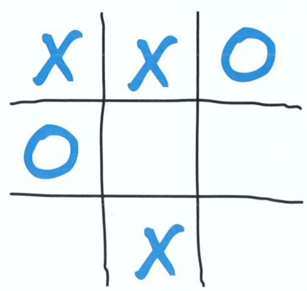
A B C D

6. *Age of the Bus Driver* Today is your first day driving a city bus. When you leave downtown, you have twenty-three passengers. At the first stop, three people exit and five people get on the bus. At the second stop, eleven people exit and eight people get on the bus. At the third stop, five people exit and ten people get on. How old is the bus driver? (April 1, 2002)

7. *Matching Triangles and Squares* How can you connect each square with the triangle that has the same number? Lines cannot cross, enter a square or triangle, or go outside the diagram. (October 15, 1999)



8. *Ticktacktoe Strategy* You and a friend are playing ticktacktoe, where three in a row loses. You are O. If you want to win, what must your next move be? (October 21, 2001)



9. *Forming Perfect Square Sums* How must one place the integers from 1 to 15 in each of the spaces below in such a way that no number is repeated and the sum of the numbers in any two consecutive spaces is a perfect square? (November 11, 2001)

10. *How Old?* Pat and Chris have the same birthday. Pat is twice as old as Chris was when Pat was as old as Chris is now. If Pat is now 24 years old, how old is Chris? (December 3, 2001)

11. *Difference Triangle* Balls numbered 1 through 6 are arranged in a *difference triangle* on the next page. Note that in any row, the difference between the larger and the smaller of two successive balls is the number of the ball that appears below them.

"E13" following 5.627062301 means that this number is multiplied by 10^{13}. This answer is still only an approximation, because the product $6{,}265{,}804 \times 8{,}980{,}591$ contains more digits than the calculator can display.)

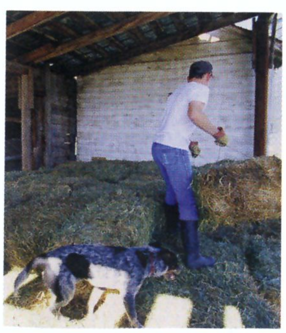
Estimation While calculators can make life easier when it comes to computations, many times we need only estimate an answer to a problem, and in these cases a calculator may not be necessary or appropriate.

EXAMPLE 1 A birdhouse for swallows can accommodate up to 8 nests. How many birdhouses would be necessary to accommodate 58 nests?

If we divide 58 by 8 either by hand or with a calculator, we get 7.25. Can this possibly be the desired number? Of course not, since we cannot consider fractions of birdhouses. Do we need 7 or 8 birdhouses? To provide nesting space for the nests left over after the 7 birdhouses (as indicated by the decimal fraction), we should plan to use 8 birdhouses. In this problem, we must round our answer *up* to the next counting number. ■

EXAMPLE 2 In 2001, David Boston of the Arizona Cardinals caught 98 passes for a total of 1598 yards. What was his approximate average number of yards per catch?

Since we are asked only to find David's approximate average, we can say that he caught about 100 passes for about 1600 yards, and his average was approximately $1600/100 = 16$ yards per catch. (A calculator shows that his average to the nearest tenth was 16.3 yards. Verify this.) ■



EXAMPLE 3 In a recent year there were approximately 127,000 males in the 25–29-year age bracket working on farms. This represented part of the total of 238,000 farm workers in that age bracket. Of the 331,000 farm workers in the 40–44-year age bracket, 160,000 were males. Without using a calculator, determine which age bracket had a larger proportion of males.

Here, it is best to think in terms of thousands instead of dealing with all the zeros. First, let us analyze the age bracket 25–29 years. Since there were a total of 238 thousand workers, of which 127 thousand were males, there were $238 - 127 = 111$ thousand female workers. Here, more than half of the workers were males. In the 40–44-year age bracket, of the 331 thousand workers, there were 160 thousand males, giving $331 - 160 = 171$ thousand females, meaning fewer than half were males. A comparison, then, shows that the 25–29-year age bracket had the larger proportion of males. ■

Reading Graphs Using graphs has become an efficient means of transmitting information in a concise way. Any issue of the newspaper *USA Today* will verify this. There are many ways to represent information using graphs and charts. Pie charts, bar graphs, and line graphs are the most common.

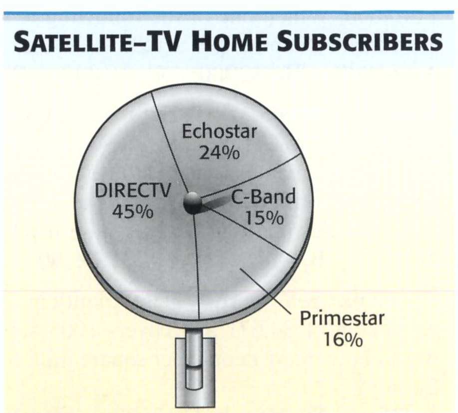
48. *Revolutions of Mercury* The planet Mercury takes 88.0 Earth days to revolve around the sun once. Pluto takes 90,824.2 days to do the same. When Pluto has revolved around the sun once, about how many times will Mercury have revolved around the sun?
A. 100,000 **B.** 10,000 **C.** 1000 **D.** 100

49. *Rushing Average* In 1998, Terrell Davis of the Denver Broncos rushed for 2008 yards in 392 attempts. His approximate number of yards gained per attempt was ______ .
A. 1/5 **B.** 50 **C.** 4 **D.** 5

50. *Area of the Sistine Chapel* The Sistine Chapel in Vatican City measures 40.5 meters by 13.5 meters.

Which one of the following is the closest approximation to its area?
A. 110 meters **B.** 55 meters
C. 110 square meters **D.** 600 square meters

Satellite Television Market Share *The 1999 market share for satellite television home subscribers is shown in the chart.*



Source: Skyreport.com; *USA Today.*

The total number of users reached 12,000,000 *in August 1999. Use this information and the pie chart to work Exercises 51–54.*

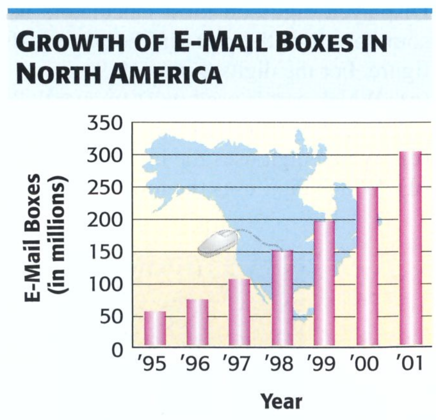
51. Which provider had the largest share of the home subscriber market in August 1999? What was that share?

52. Determine the number of home subscribers to Primestar in August 1999.

53. C-Band is associated with "large dishes" while all other subscribers have "small dishes." How many subscribers had small dishes?

54. How many more subscribers did Primestar have than C-Band?

Growth of E-mail *The latter half of the 1990s was characterized by incredible growth in a new method of communication: electronic mail ("e-mail"). The accompanying bar graph shows the number of e-mail boxes in North America for the years 1995–2001. Use the graph to answer the questions in Exercises 55–58.*



Source: IDC research.

55. How many e-mail boxes were there in 1998?

56. By how many did the number of e-mail boxes grow between 1995 and 2001?

57. In what year was the number of e-mail boxes 150 million?

58. Suppose that the number of boxes in 2002 increased the same amount from the previous year as it did in 2001. What would have been the number of boxes in 2002?

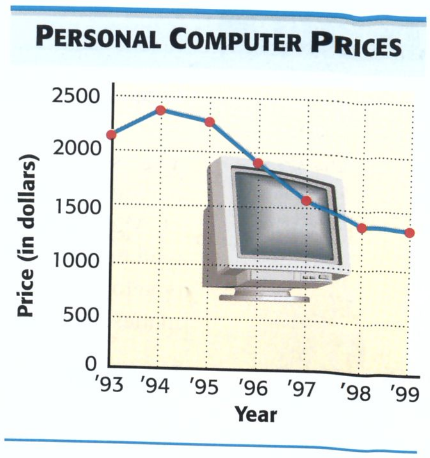
Prices for Personal Computers *The accompanying line graph at the right shows average prices for personal computers (PCs) for the years 1993 through 1999. Use this information to answer the questions in Exercises 59–62.*

59. Between which years did the average price of a PC increase?

60. What has been the general trend in average PC prices since 1994?

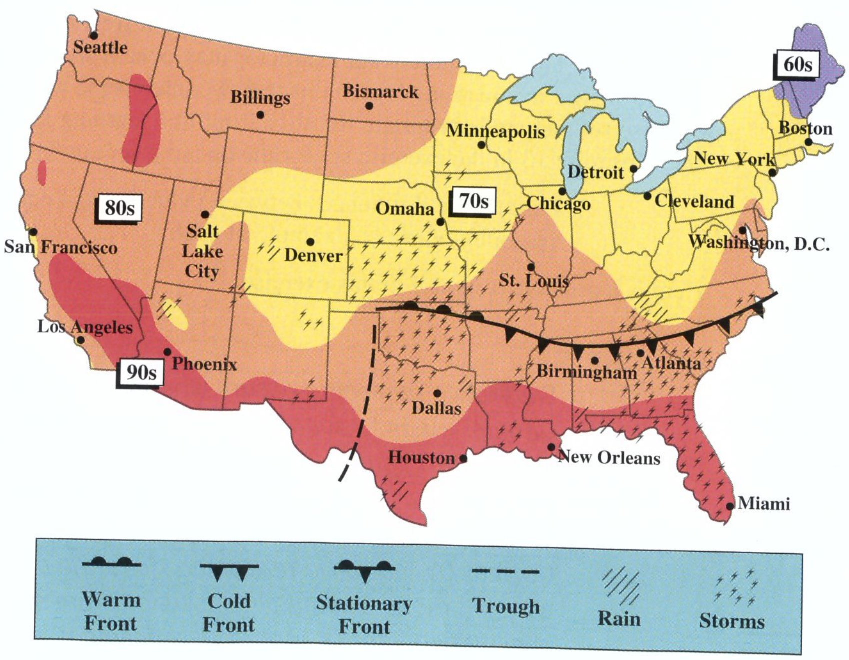
61. What were the average PC prices in 1996 and in 1999?

62. About how much did PC prices decline between 1994 and 1999?



Source: CNW Marketing/Research; *USA Today.*

Weather Prediction *The map shows the predicted weather information for a summer day in July. Use the map and the accompanying legend to answer the questions in Exercises 63–68.*



63. Which temperature range (that is, 60s, 70s, 80s, or 90s) would we expect for Detroit?

64. What type of front is moving toward Atlanta?

65. Assuming that you think anything over 80° is hot, how would you describe the weather for Miami? Use at least two descriptive words.

66. Augusta is the capital of a northeastern state. In what temperature range will Augusta be?

67. In what state is a trough located?

68. Is there a good chance that a baseball game between the Yankees and the Indians, to be played at Jacobs Field in Cleveland, will be rained out?