What Is Calculus?

We begin to answer this question by saying that calculus is the reformulation of elementary mathematics through the use of a limit process. If limit processes are unfamiliar to you, then this answer is, at least for now, somewhat less than illuminating. From an elementary point of view, we may think of calculus as a "limit machine" that generates new formulas from old. Actually, the study of calculus involves three distinct stages of mathematics: *precalculus mathematics* (the length of a line segment, the area of a recangle, and so forth), the *limit process*, and new *calculus* formulations (derivatives, integrals, and so forth).

PRECALCULUS MATHEMATICS → LIMIT PROCESS → CALCULUS

Some students try to learn calculus as if it were simply a collection of new formulas. This is unfortunate. When students reduce calculus to the memorization of differentiation and integration formulas, they miss a great deal of understanding, self-confidence, and satisfaction.

On the following two pages we have listed some familiar precalculus concepts coupled with their more powerful calculus versions. Throughout this text, our goal is to show you how precalculus formulas and techniques are used as building blocks to produce the more general calculus formulas and techniques. Don't worry if you are unfamiliar with some of the "old formulas" listed on the following two pages—we will be reviewing all of them.

As you proceed through this text, we suggest that you come back to this discussion repeatedly. Try to keep track of where you are relative to the three stages involved in the study of calculus. For example, the first three chapters break down as follows: precalculus (Chapter 1), the limit process (Chapter 2), and new calculus formulas (Chapter 3). This cycle is repeated many times on a smaller scale throughout the text. We wish you well in your venture into calculus.

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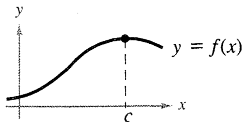
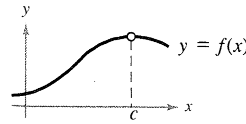
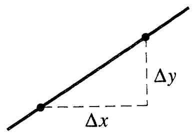
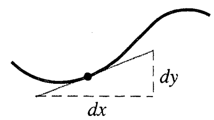
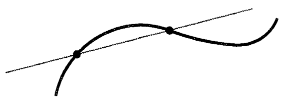
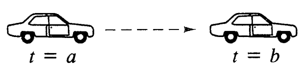
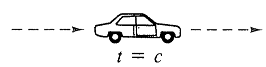
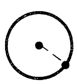
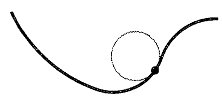
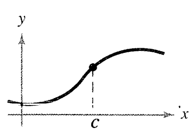
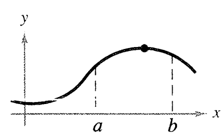
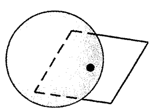
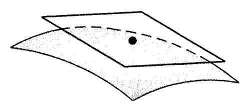
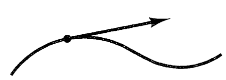
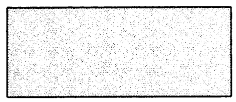
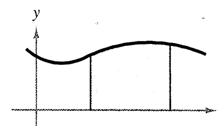
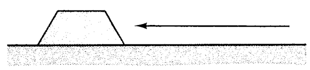
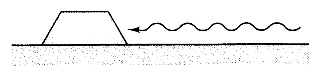
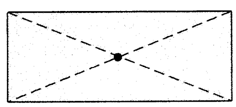
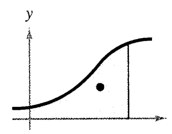
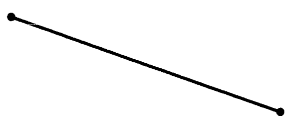
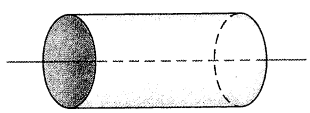
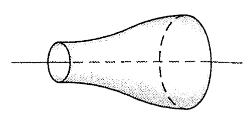
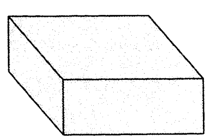
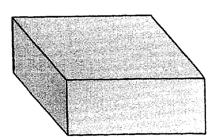
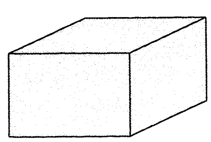
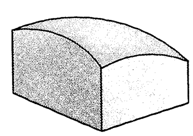
value of $f(x)$ when $x = c$		limit of $f(x)$ as x approaches c	
slope of a line		slope of a curve	
secant line to a curve		tangent line to a curve	
average rate of change between $t = a$ and $t = b$		instantaneous rate of change at $t = c$	
curvature of a circle		curvature of a curve	
height of a curve when $x = c$		maximum height of a curve on an interval	
tangent plane to a sphere		tangent plane to a surface	
direction of motion along a straight line		direction of motion along a curved line	

WITHOUT CALCULUS		WITH INTEGRAL CALCULUS	
area of a rectangle		area under a curve	
work done by a constant force		work done by a variable force	
center of a rectangle		centroid of a region	
length of a line segment		length of an arc	
surface area of a cylinder		surface area of a solid of revolution	
mass of a solid of constant density		mass of a solid of variable density	
volume of a rectangular solid		volume of a region under a surface	
sum of a finite number of terms	$a_1 + a_2 + \cdots + a_n = S$	sum of an infinite number of terms	$a_1 + a_2 + a_3 \cdots = S$

Chapter 1 Application

***The Ferris wheel* was designed by the American mechanical engineer George Ferris (1859–1896). The first and largest Ferris wheel was built for the World's Columbian Exposition in Chicago in 1893, and later used at the World's Fair in St. Louis in 1904. It had a diameter of 250 feet, and each of its 36 cars could hold 60 passengers.**

Height of a Ferris Wheel Car

A Ferris wheel with a radius of 50 feet rotates at a constant rate of 4 revolutions per minute. If the center of the Ferris wheel is considered to be the origin, then each car travels around the circle given by

$$x^2 + y^2 = 50^2$$

where x and y are measured in feet. The height of a car located at the point (x, y) is given by

$$h = 50 + y$$

where y is related to the angle θ by the equation

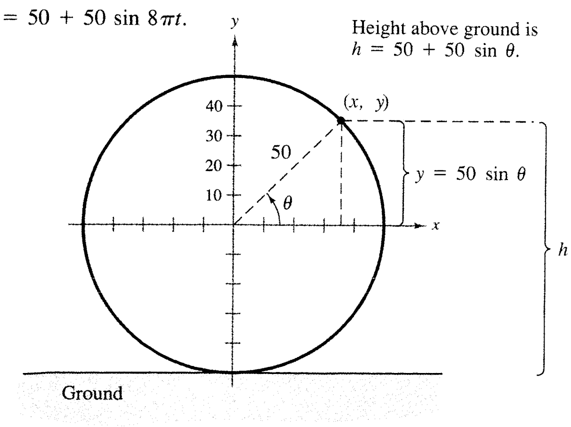
$$y = 50 \sin \theta$$

as shown in the accompanying figure. Since the wheel makes 4 revolutions per minute (with one revolution corresponding to 2π radians), it follows that

$$\theta = 4(2\pi)t = 8\pi t$$

where t is measured in minutes. Thus, as a function of time, the height of a car on the Ferris wheel is given by

$$h = 50 + 50 \sin 8\pi t.$$



See Exercise 76, Section 1.6.

Chapter Overview

This first chapter contains a review of basic algebra, analytic geometry, and trigonometry. The more familiar you are with the material in this chapter, the more successful you will be in calculus.

Section 1.1 reviews the properties of the real numbers and the real number line. The next two sections review the fundamental concepts of plane analytic geometry, the Cartesian plane, and graphs of equations in two variables.

Section 1.4 discusses the slope of a line—this concept is critical in calculus. This section begins by showing how the slope of a line is related to the *average rate of change* of one variable with respect to another.

The concept of a **function** is also critical in calculus, and we review several fundamental ideas related to functions in Section 1.5. For instance, this section reviews the graphs of such basic functions as

$$f(x) = x \qquad f(x) = x^2$$
$$f(x) = x^3 \qquad f(x) = \sqrt{x}$$
$$f(x) = |x| \qquad f(x) = \frac{1}{x}.$$

Familiarity with the graphs of these functions will help you in later chapters.

Finally, Section 1.6 contains a brief review of trigonometry.

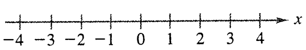
1 The Cartesian Plane and Functions

1.1 Real Numbers and the Real Line

Real numbers ▪ The real line ▪ Order and inequalities ▪ Absolute value ▪ Distance on the real line ▪ Intervals on the real line

In this first chapter we will lay the foundation for studying calculus. We assume that you have a good working knowledge of basic algebra. This is essential for the study of calculus.

The real line

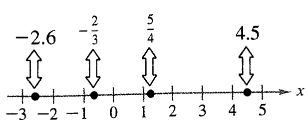


The Real Line

FIGURE 1.1

To represent the set of real numbers we use a coordinate system called the **real line** or x-axis (Figure 1.1). The real number corresponding to a particular point on the real line is called the **coordinate** of the point. As Figure 1.1 shows, it is customary to identify those points whose coordinates are integers.

The point on the real line corresponding to zero is called the **origin** and is denoted by 0. The **positive direction** (to the right) is denoted by an arrowhead and indicates the direction of increasing values of x. Numbers to the right of the origin are **positive;** numbers to the left of the origin are **negative.** We use the term **nonnegative** to describe a number that is either positive or zero. Similarly, the term **nonpositive** is used to describe a number that is either negative or zero.

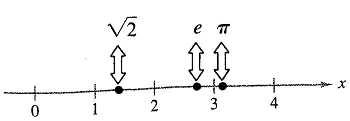


One-to-one correspondence between real numbers and points on the real line.

FIGURE 1.2

Each point on the real line corresponds to one and only one real number, and each real number corresponds to one and only one point on the real line. This type of relationship is called a **one-to-one correspondence.**

Each of the four points in Figure 1.2 corresponds to a real number that can be expressed as the ratio of two integers. (Note that $4.5 = \frac{9}{2}$ and $-2.6 = -\frac{13}{5}$.) We call such numbers **rational.** Rational numbers can be represented either by *terminating decimals* such as $\frac{2}{5} = 0.4$, or by *repeating decimals* such as $\frac{1}{3} = 0.333\ldots = 0.\overline{3}$.



Irrational Numbers

FIGURE 1.3

Real numbers that are not rational are called **irrational.** They cannot be represented as terminating or repeating decimals. To represent an irrational number, we usually resort to a decimal approximation. For example, $\sqrt{2} \approx 1.4142135623$, $\pi \approx 3.1415926535$, and $e \approx 2.7182818284$. (See Figure 1.3.)

Order and inequalities

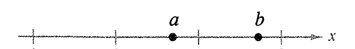
One important property of real numbers is that they are **ordered.**

DEFINITION OF ORDER ON THE REAL LINE

If a and b are real numbers, then a is **less than** b if $b - a$ is positive. We denote this order by the **inequality**

$$a < b.$$

The symbol $a \leq b$ means that a is **less than or equal to** b. The statement b is **greater than** a is equivalent to saying a is less than b.



$a < b$ if and only if a lies to the left of b.

FIGURE 1.4

Geometrically, $a < b$ if and only if a lies to the *left* of b on the real line. (See Figure 1.4.) For example, $1 < 2$ because 1 lies to the left of 2 on the real line.

The following properties are often used to work with inequalities. Similar properties are obtained if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

THEOREM 1.1 PROPERTIES OF INEQUALITIES

1. If $a < b$ and $b < c$, then $a < c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and k is any real number, then $a + k < b + k$.
4. If $a < b$ and $k > 0$, then $ak < bk$.
5. If $a < b$ and $k < 0$, then $ak > bk$.

REMARK Note that we *reverse the inequality* when we multiply by a negative number. For example, if $x < 3$, then $-4x > -12$. This principle also applies to division by a negative number. Thus, if $-2x > 4$, then $x < -2$.

When three real numbers a, b, and c are ordered such that $a < b$ and $b < c$, we say that b is **between** a and c and we write $a < b < c$.

Occasionally it is convenient to use set notation to describe collections of real numbers. A **set** is a collection of elements. For example, the two major sets we have been discussing are the set of real numbers and the set of points on the real line. Often, we will restrict our interest to a **subset** of one of these two sets, in which case it is convenient to use **set notation** of the form

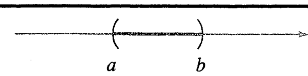
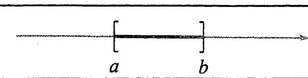
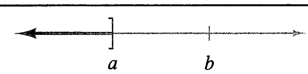
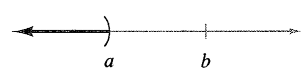
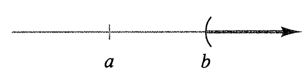
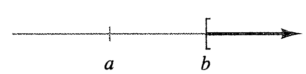
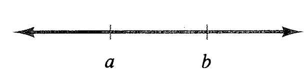
$$\{x\text{: condition on } x\}.$$

The set of all x such that a certain condition is true

For example, we can describe the set of positive real numbers as $\{x: 0 < x\}$. The **union** of two sets A and B is the set of elements that are members of A *or* B *or both*. This union is denoted by $A \cup B$. The **intersection** of two sets A and B is the set of elements that are members of A *and* B. This intersection is denoted by $A \cap B$. Two sets are called **disjoint** if they have no elements in common.

The most common sets we work with are subsets of the real line called **intervals.** For example, the **open** interval $(a, b) = \{x: a < x < b\}$ is the set of all real numbers greater than a and less than b, where a and b are called the **endpoints** of the interval. Note that the endpoints are not included in an open interval. Intervals that include their endpoints are called **closed** and are denoted by $[a, b] = \{x: a \leq x \leq b\}$. The nine basic types of intervals on the real line are shown in Table 1.1. The first four are called **bounded intervals** and the remaining five are called **unbounded intervals.**

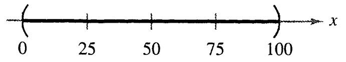
TABLE 1.1 Intervals on the Real Line

	Interval notation	*Set notation*	*Graph*
Open interval	(a, b)	$\{x: a < x < b\}$	
Closed interval	$[a, b]$	$\{x: a \leq x \leq b\}$	
Half-open intervals	$[a, b)$	$\{x: a \leq x < b\}$	
	$(a, b]$	$\{x: a < x \leq b\}$	
Infinite intervals	$(-\infty, a]$	$\{x: x \leq a\}$	
	$(-\infty, a)$	$\{x: x < a\}$	
	(b, ∞)	$\{x: b < x\}$	
	$[b, \infty)$	$\{x: b \leq x\}$	
	$(-\infty, \infty)$	$\{x: x \text{ is a real number}\}$	

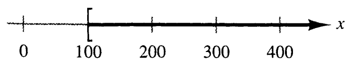
REMARK We use the symbols ∞ and $-\infty$ to refer to positive and negative infinity. These symbols do not denote real numbers; they merely enable us to describe unbounded conditions more concisely. For instance, the interval $[b, \infty)$ is unbounded to the right since it includes *all* real numbers that are greater than or equal to b.

EXAMPLE 1 Intervals on the real line

Describe the intervals on the real line that correspond to the temperature ranges (in degrees Celsius) for water in the following two states.
(a) liquid (b) gas



(a) Temperature range of water (in degrees Celsius)



(b) Temperature range of steam (in degrees Celsius)

FIGURE 1.5

SOLUTION

(a) Since water is in a liquid state at temperatures greater than 0° and less than 100°, we have the interval

$$(0, 100) = \{x: 0 < x < 100\}$$

as shown in Figure 1.5(a).

(b) Since water is in a gaseous state (steam) at temperatures greater than or equal to 100°, we have the interval

$$[100, \infty) = \{x: 100 \le x\}$$

as shown in Figure 1.5(b). ▭

In calculus we are frequently asked to solve inequalities involving variable expressions such as $2x - 5 < 7$. We say that a is a **solution** of this inequality if the inequality is true when a is substituted for x. The set of all values of x that satisfy the inequality is called the **solution set** of the inequality.

EXAMPLE 2 Solving an inequality

Find the solution set of the inequality $2x - 5 < 7$.

SOLUTION

Using the properties in Theorem 1.1, we have

$$2x - 5 < 7$$

$$2x - 5 + 5 < 7 + 5 \qquad \text{Add 5 to both sides}$$

$$2x < 12$$

$$\frac{1}{2}(2x) < \frac{1}{2}(12) \qquad \text{Multiply both sides by } \tfrac{1}{2}$$

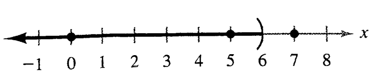
$$x < 6.$$

Thus, the interval representing the solution is $(-\infty, 6)$. ▭

REMARK In Example 2, all five inequalities listed as steps in the solution have the same solution set and are called **equivalent.**

If $x = 0$, $2(0) - 5 = -5 < 7$.

If $x = 5$, $2(5) - 5 = 5 < 7$.



If $x = 7$, $2(7) - 5 = 9 > 7$.

FIGURE 1.6

Once you have solved an inequality, check some x-values in your solution interval to see whether they satisfy the original inequality. You also might check some values outside your solution interval to verify that they do not satisfy the inequality. For example, Figure 1.6 shows that when $x = 0$ or $x = 5$ the inequality is satisfied, but when $x = 7$ the inequality is not satisfied.

EXAMPLE 3 Finding the intersection of two solution sets

Find the intersection of the solution sets of the inequalities

$$-3 \le 2 - 5x \qquad \text{and} \qquad 2 - 5x \le 12.$$

SOLUTION

We could solve both inequalities and then find the intersection of the resulting solution sets. However, since the expression $2 - 5x$ occurs on the left side of one inequality and the right side of the other, it is convenient to work with both inequalities at the same time.

$$\begin{aligned} -3 \le\ & 2 - 5x \le 12 & \\ -3 - 2 \le\ & 2 - 5x - 2 \le 12 - 2 & \text{Subtract 2} \\ -5 \le\ & -5x \le 10 & \\ \frac{-5}{-5} \ge\ & \frac{-5x}{-5} \ge \frac{10}{-5} & \text{Divide by } -5 \text{ and reverse the inequality} \\ 1 \ge\ & x \ge -2 & \end{aligned}$$

[−2, 1]

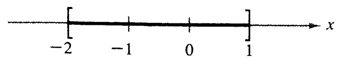


FIGURE 1.7

Thus, the interval representing the solution is $[-2, 1]$, as shown in Figure 1.7. ▭

The inequalities in Examples 2 and 3 involve first-degree polynomials. For inequalities involving polynomials of higher degree we use the fact that a polynomial can change signs *only* at its real zeros (a **zero** of a polynomial is a number at which the value of the polynomial is zero). Between two consecutive real zeros a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real line into **test intervals** in which the polynomial has no sign changes. That is, if a polynomial has the factored form

$$(x - r_1)(x - r_2) \cdots (x - r_n), \qquad r_1 < r_2 < r_3 < \cdots < r_n$$

then the test intervals are

$$(-\infty, r_1),\ (r_1, r_2),\ \ldots,\ (r_{n-1}, r_n), \text{ and } (r_n, \infty).$$

For example, the polynomial

$$x^2 - x - 6 = (x - 3)(x + 2)$$

can change signs only at $x = -2$ and $x = 3$.

EXAMPLE 4 Solving an inequality involving a quadratic

Find the solution set of the inequality $x^2 < x + 6$.

SOLUTION

$$\begin{aligned} x^2 &< x + 6 & \text{Given} \\ x^2 - x - 6 &< 0 & \text{Polynomial form} \\ (x - 3)(x + 2) &< 0 & \text{Factor} \end{aligned}$$

Thus, the polynomial $x^2 - x - 6$ has $x = -2$ and $x = 3$ as its zeros, and we can solve the inequality by testing the sign of $x^2 - x - 6$ in each of the following open intervals.

$$(-\infty, -2), \qquad (-2, 3), \qquad (3, \infty)$$

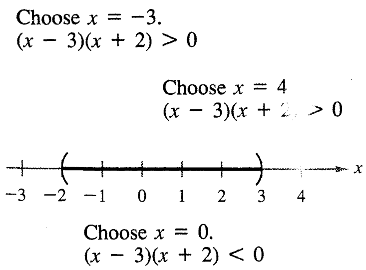


FIGURE 1.8

To test an interval, we choose an arbitrary number in the interval and compute the sign of each factor of $x^2 - x - 6$. For example, for any x in the open interval $(-\infty, -2)$, the factors $(x - 3)$ and $(x + 2)$ are both negative. Consequently, the product (of two negative numbers) is positive and the inequality is *not* satisfied in the interval $(-\infty, -2)$. We suggest that you use the testing format shown in Figure 1.8. Since the inequality $(x - 3)(x + 2) < 0$ is satisfied only for values of x in the center interval, we conclude that the solution set is the open interval $(-2, 3)$. ▭

Absolute value and distance

The **absolute value** of a real number a is denoted by $|a|$ and is defined as follows.

DEFINITION OF ABSOLUTE VALUE

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

The absolute value of a number can never be negative. For example, let $a = -4$. Then, since $-4 < 0$, we have

$$|a| = |-4| = -(-4) = 4.$$

Remember that the symbol $-a$ does not necessarily mean that $-a$ is negative.

Theorems 1.2 and 1.3 contain some useful properties of absolute value.

THEOREM 1.2
OPERATIONS WITH ABSOLUTE VALUE

If a and b are real numbers and n is a positive integer, then the following properties are true.

1. $|ab| = |a||b|$
2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \; b \neq 0$
3. $|a| = \sqrt{a^2}$
4. $|a^n| = |a|^n$

REMARK You are asked to prove these properties in Exercises 67–71.

THEOREM 1.3
INEQUALITIES AND ABSOLUTE VALUE

If a and b are real numbers and k is positive, then the following properties are true.

1. $-|a| \leq a \leq |a|$
2. $|a| \leq k$ if and only if $-k \leq a \leq k$.
3. $k \leq |a|$ if and only if $k \leq a$ or $a \leq -k$.
4. Triangle Inequality: $|a + b| \leq |a| + |b|$

Properties 2 and 3 are also true if $\leq$ is replaced by $<$.

PROOF We give a proof of Property 4 and leave the proofs of the first three properties as exercises (see Exercises 72–74). Using Property 1, we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Adding these two inequalities produces

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

Now, by Property 2 (using $k = |a| + |b|$), we can conclude that

$$|a + b| \leq |a| + |b|.$$

EXAMPLE 5 Solving an inequality involving an absolute value

Sketch the solution set of $|x - 3| \leq 2$.

SOLUTION

Using Property 2 of Theorem 1.3, we have

$$\begin{aligned} -2 \leq\ & x - 3 \leq 2 \\ -2 + 3 \leq\ & x - 3 + 3 \leq 2 + 3 \\ 1 \leq\ & x \leq 5. \end{aligned}$$

Thus, the solution set is the closed interval [1, 5], as shown in Figure 1.9.

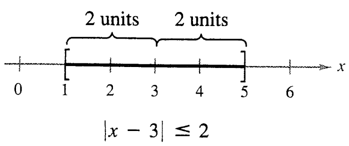


FIGURE 1.9

EXAMPLE 6 A two-interval solution set

Find the solution set of $3 < |x + 2|$.

SOLUTION

Using Property 3 of Theorem 1.3, we have

$$\begin{aligned} 3 < x + 2 \quad &\text{or} \quad x + 2 < -3 \\ 1 < x \quad &\text{or} \quad x < -5. \end{aligned}$$

Thus, the solution set consists of the union of the disjoint intervals $(-\infty, -5)$ and $(1, \infty)$, as shown in Figure 1.10.

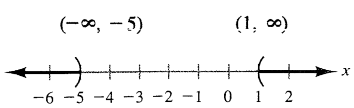


FIGURE 1.10

Examples 5 and 6 illustrate the general results shown in Figure 1.11. Note that if $d > 0$, the solution set for the inequality $|x - a| \leq d$ consists of a *single* interval, while the solution set for the inequality $|x - a| \geq d$ consists of *two* disjoint intervals.

FIGURE 1.11

The **distance between two points** a and b on the real line is given by

$$d = |a - b|.$$

The **directed distance from** a **to** b is $b - a$ and the **directed distance from** b **to** a is $a - b$, as shown in Figure 1.12.

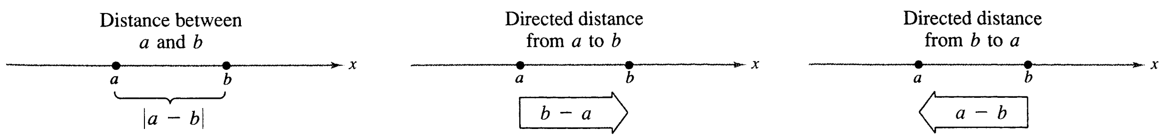


FIGURE 1.12

EXAMPLE 7 Distance on the real line

(a) The distance between -3 and 4 is given by

$$|4 - (-3)| = |7| = 7 \quad \text{or} \quad |-3 - 4| = |-7| = 7.$$

(See Figure 1.13.)

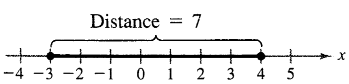


FIGURE 1.13

(b) The directed distance from -3 to 4 is $4 - (-3) = 7$.

(c) The directed distance from 4 to -3 is $-3 - 4 = -7$. ▭

To find the **midpoint** of an interval with endpoints a and b, we simply find the average value of a and b. That is,

$$\text{midpoint of interval } (a, b) = \frac{a + b}{2}.$$

To show that this is the midpoint, you need only show that $(a + b)/2$ is equidistant from a and b.

EXERCISES for Section 1.1

In Exercises 1–10, determine whether the real number is rational or irrational.

1. 0.7

2. -3678

3. $\frac{3\pi}{2}$

4. $3\sqrt{2} - 1$

5. $4.3451\overline{451}$

6. $\frac{22}{7}$

***7.** $\sqrt[3]{64}$

8. $0.8177\overline{8177}$

9. $4\frac{5}{8}$

10. $(\sqrt{2})^3$

*A blue number indicates that a detailed solution can be found in the *Study and Solutions Guide*.

In Exercises 11–14, express the repeating decimal as a ratio of integers using the following procedure. Let $x = 0.6363\ldots$. Then $100x = 63.6363\ldots$. Subtracting the first equation from the second produces $99x = 63$ or $x = \frac{63}{99} = \frac{7}{11}$.

11. $0.36\overline{36}$

12. $0.318\overline{18}$

13. $0.297\overline{297}$

14. $0.9900\overline{9900}$

15. Given $a < b$, determine which of the following are true.

(a) $a + 2 < b + 2$

(b) $5b < 5a$

(c) $5 - a > 5 - b$

(d) $\frac{1}{a} < \frac{1}{b}$

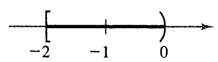
(e) $(a - b)(b - a) > 0$

(f) $a^2 < b^2$

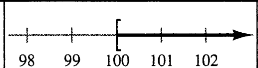
16. For $A = \{x: 0 < x\}$, $B = \{x: -2 \le x \le 2\}$, and $C = \{x: x < 1\}$, find the indicated interval.
(a) $A \cup B$ (b) $A \cap B$
(c) $B \cap C$ (d) $A \cup C$
(e) $A \cap B \cap C$

In Exercises 17 and 18, complete the table by filling in the appropriate interval notation, set notation, and graph on the real line.

17.

Interval notation	*Set notation*	*Graph*
		
$(-\infty, -4]$		
	$\{x: 3 \le x \le \frac{11}{2}\}$	
$(-1, 7)$		

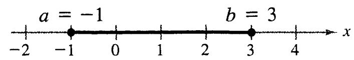
18.

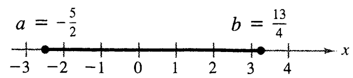
Interval notation	*Set notation*	*Graph*
		
	$\{x: 10 < x\}$	
$(\sqrt{2}, 8]$		
	$\{x: \frac{1}{3} < x \le \frac{22}{7}\}$	

In Exercises 19–44, solve the inequality and graph the solution on the real line.

19. $x - 5 \ge 7$
20. $2x > 3$
21. $4x + 1 < 2x$
22. $2x + 7 < 3$
23. $2x - 1 \ge 0$
24. $3x + 1 \ge 2x + 2$
25. $-4 < 2x - 3 < 4$
26. $0 \le x + 3 < 5$
27. $\frac{3}{4}x > x + 1$
28. $-1 < -\frac{x}{3} < 1$
29. $\frac{x}{2} + \frac{x}{3} > 5$
30. $x > \frac{1}{x}$
31. $|x| < 1$
32. $\frac{x}{2} - \frac{x}{3} > 5$
33. $\left|\frac{x - 3}{2}\right| \ge 5$
34. $\left|\frac{x}{2}\right| > 3$
35. $|x - a| < b$
36. $|x + 2| < 5$
37. $|2x + 1| < 5$
38. $|3x + 1| \ge 4$
39. $\left|1 - \frac{2}{3}x\right| < 1$
40. $|9 - 2x| < 1$
41. $x^2 \le 3 - 2x$
42. $x^4 - x \le 0$
43. $x^2 + x - 1 \le 5$
44. $2x^2 + 1 < 9x - 3$

In Exercises 45–48, find the directed distance from a to b, the directed distance from b to a, and the distance between a and b.

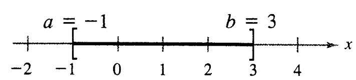
45. 

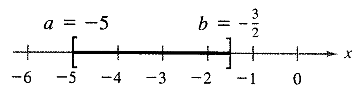
46. 

47. (a) $a = 126$, $b = 75$
(b) $a = -126$, $b = -75$

48. (a) $a = 9.34$, $b = -5.65$
(b) $a = \frac{16}{5}$, $b = \frac{112}{75}$

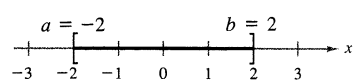
In Exercises 49–52, find the midpoint of the given interval.

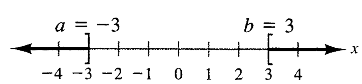
49. 

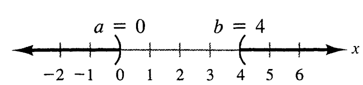
50. 

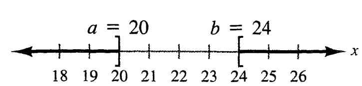
51. (a) $[7, 21]$ (b) $[8.6, 11.4]$
52. (a) $[-6.85, 9.35]$ (b) $[-4.6, -1.3]$

In Exercises 53–58, use absolute values to define each interval (or pair of intervals) on the real line.

53. 

54. 

55. 

56. 

57. (a) All numbers that are at most 10 units from 12.
(b) All numbers that are at least 10 units from 12.

58. (a) y is at most 2 units from a.
(b) y is less than δ units from c.

59. The balance in an account after t years is given by

$$A = P + Prt$$

where P dollars is the initial investment and r is the simple interest rate (in decimal form). In order for an investment of \$1000 to attain a balance that is greater than \$1250 in two years, what should the interest rate be?

60. In the manufacture and sale of a certain product, the revenue for selling x units is

$$R = 115.95x$$

and the cost of producing x units is

$$C = 95x + 750.$$

In order for a profit to be realized, R must be greater than C. For what values of x will this product return a profit?

61. A utility company has a fleet of vans. The annual operating cost of each van is estimated to be

$$C = 0.32m + 2300$$

where C is measured in dollars and m is measured in miles. If the company wants the annual operating cost of each van to be less than \$10,000, then m must be less than what value?

62. The heights, h, of two-thirds of the members of a certain population satisfy the inequality

$$\left|\frac{h - 68.5}{2.7}\right| \leq 1$$

where h is measured in inches. Determine the interval on the real line in which these heights lie.

63. To determine if a coin is fair (has an equal probability of landing tails up or heads up), an experimenter tosses it 100 times and records the number of heads, x. Through statistic ' theory, the coin is declared unfair if

$$\left|\frac{x - 50}{5}\right| \geq 1.645.$$

For what values of x will the coin be declared unfair?

64. The estimated daily production, p, at a refinery is given by

$$|p - 2{,}250{,}000| < 125{,}000$$

where p is measured in barrels of oil. Determine the high and low production levels.

In Exercises 65 and 66, determine which of the two given real numbers is greater.

65. (a) π or $\dfrac{355}{113}$ (b) π or $\dfrac{22}{7}$

66. (a) $\dfrac{224}{151}$ or $\dfrac{144}{97}$ (b) $\dfrac{73}{81}$ or $\dfrac{6427}{7132}$

In Exercises 67–74, prove the given property.

67. $|ab| = |a||b|$

68. $|a - b| = |b - a|$ [Hint: Use Exercise 67 and the fact that $(a - b) = (-1)(b - a)$.]

69. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, $b \neq 0$

70. $|a| = \sqrt{a^2}$

71. $|a^n| = |a|^n$, $n = 1, 2, 3, \ldots$

72. $-|a| \leq a \leq |a|$

73. $|a| \leq k$ if and only if $-k \leq a \leq k$, $k > 0$.

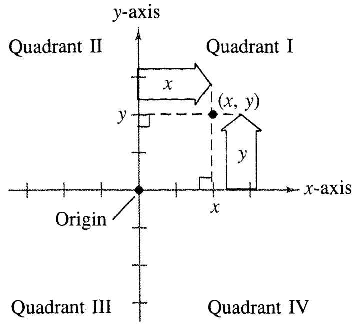
74. $k \leq |a|$ if and only if $k \leq a$ or $a \leq -k$, $k > 0$.

1.2 The Cartesian Plane

The Cartesian plane ▪ The Distance Formula ▪ The Midpoint Formula ▪ Equations of circles ▪ Completing the square

Just as real numbers can be represented by points on the real line, we can represent ordered pairs of real numbers by points in a plane. An **ordered pair** (x, y) of real numbers has x as its *first* member and y as its *second* member. The model for representing ordered pairs is called the **rectangular coordinate system,** or the **Cartesian plane.** It is developed by considering two real lines intersecting at right angles (Figure 1.14).

The horizontal real line is usually called the **x-axis,** and the vertical real line is usually called the **y-axis.** Their point of intersection is called the **origin.** The two axes divide the plane into four parts called **quadrants.**



The Cartesian Plane

FIGURE 1.14

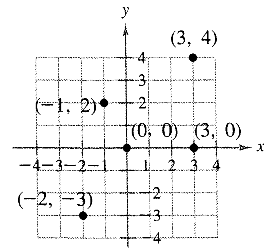
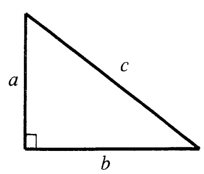


FIGURE 1.15

We identify each point in the plane by an ordered pair (x, y) of real numbers x and y, called **coordinates** of the point. The number x represents the directed distance from the y-axis to the point, and y represents the directed distance from the x-axis to the point (Figure 1.14). For the point (x, y), the first coordinate is called the x-coordinate or **abscissa,** and the second coordinate is called the y-coordinate or **ordinate.** For example, Figure 1.15 shows the location of the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$ in the Cartesian plane.

REMARK Note that we use an ordered pair (a, b) to denote either a point in the plane *or* an open interval on the real line. As the nature of the problem clarifies whether a point in the plane or an open interval is being discussed, there should be no confusion.



Pythagorean Theorem:
$a^2 + b^2 = c^2$

FIGURE 1.16

The Distance and Midpoint Formulas

Section 1.1 defined the distance between two points x_1 and x_2 on the real line. We will now find the distance between two points in the plane. Recall from the Pythagorean Theorem that, for a right triangle with hypotenuse c and sides a and b, we have the relationship $a^2 + b^2 = c^2$. Conversely, if $a^2 + b^2 = c^2$, then the triangle is a right triangle (Figure 1.16).

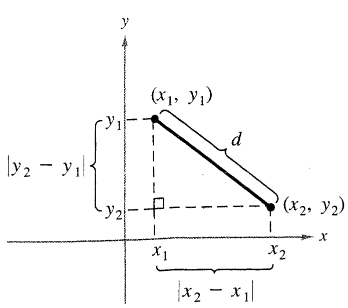
Suppose we want to determine the distance d between the two points (x_1, y_1) and (x_2, y_2) in the plane. With these two points, a right triangle can be formed, as shown in Figure 1.17. The length of the vertical side of the triangle is $|y_2 - y_1|$. Similarly, the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem, it follows that

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}.$$

Replacing $|x_2 - x_1|^2$ and $|y_2 - y_1|^2$ by the equivalent expressions $(x_2 - x_1)^2$ and $(y_2 - y_1)^2$, we obtain

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We choose the positive square root for d because the distance *between* two points is not a directed distance. We have therefore established the following theorem.



Distance Between Two Points

FIGURE 1.17

THEOREM 1.4
DISTANCE FORMULA

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 1 Finding the distance between two points

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

SOLUTION

Applying the Distance Formula, we have

$$\begin{aligned} d &= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} \\ &= \sqrt{(5)^2 + (3)^2} \\ &= \sqrt{25 + 9} \\ &= \sqrt{34} \approx 5.83. \end{aligned}$$

EXAMPLE 2 Verifying a right triangle

Plot the points $(2, 1)$, $(4, 0)$, and $(5, 7)$ and use the Distance Formula to show that the three points form the vertices of a right triangle.

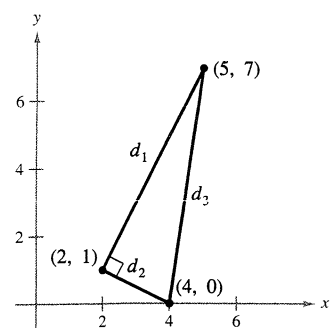


FIGURE 1.18

SOLUTION

Figure 1.18 shows the triangle formed by the three points. Moreover, the three sides of the triangle have the following lengths.

$$\begin{aligned} d_1 &= \sqrt{(5 - 2)^2 + (7 - 1)^2} = \sqrt{9 + 36} = \sqrt{45} \\ d_2 &= \sqrt{(4 - 2)^2 + (0 - 1)^2} = \sqrt{4 + 1} = \sqrt{5} \\ d_3 &= \sqrt{(5 - 4)^2 + (7 - 0)^2} = \sqrt{1 + 49} = \sqrt{50} \end{aligned}$$

Since $d_1^2 + d_2^2 = 45 + 5 = 50 = d_3^2$, we can apply the Pythagorean Theorem to conclude that the triangle must be a right triangle.

The formula for the midpoint of a line segment in the plane is similar to that for an interval on the real line. The proof is left as an exercise (see Exercise 63).

THEOREM 1.5
MIDPOINT FORMULA

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

EXAMPLE 3 Finding the midpoint of a line segment

Find the midpoint of the line segment joining the points $(-5, -3)$ and $(9, 3)$.

SOLUTION

By the Midpoint Formula, the midpoint is

$$\left(\frac{-5 + 9}{2}, \frac{-3 + 3}{2}\right) = (2, 0).$$

(See Figure 1.19.)

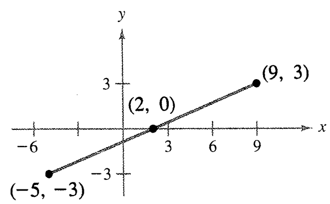


FIGURE 1.19

EXAMPLE 4 Finding points at a specified distance from a given point

Find x so that the distance between $(x, 3)$ and $(2, -1)$ is 5.

SOLUTION

Using the Distance Formula, we have

$$\begin{aligned} d = 5 &= \sqrt{(x - 2)^2 + (3 + 1)^2} \\ 25 &= (x^2 - 4x + 4) + 16 \\ 0 &= x^2 - 4x - 5 \\ 0 &= (x - 5)(x + 1). \end{aligned}$$

Therefore, $x = 5$ or $x = -1$, and we conclude that there are two solutions. That is, both of the points $(5, 3)$ and $(-1, 3)$ lie 5 units from the point $(2, -1)$, as shown in Figure 1.20.

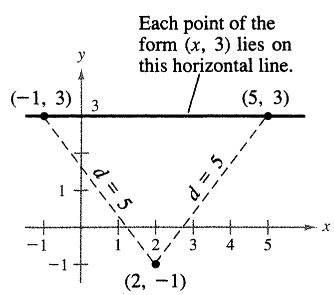


FIGURE 1.20

Circles

One straightforward application of the Distance Formula is in developing an equation for a circle in the plane.

DEFINITION OF A CIRCLE IN THE PLANE

Let (h, k) be a point in the plane and let $r > 0$. The set of all points (x, y) such that r is the distance between (h, k) and (x, y) is called a **circle.** The point (h, k) is the **center** of the circle, and r is the **radius** (see Figure 1.21).

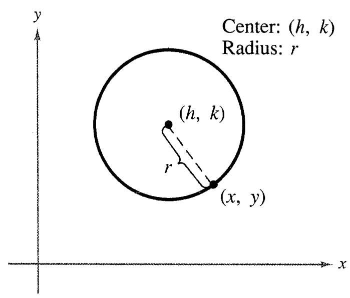


FIGURE 1.21

We can use the Distance Formula to write an equation for the circle with center (h, k) and radius r as

$$[\text{distance between } (h, k) \text{ and } (x, y)] = r$$
$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

By squaring both sides of this equation, we obtain the **standard form of the equation of a circle,** as indicated in the following theorem.

THEOREM 1.6 STANDARD FORM OF THE EQUATION OF A CIRCLE

The point (x, y) lies on the circle of radius r and center (h, k) if and only if

$$(x-h)^2 + (y-k)^2 = r^2.$$

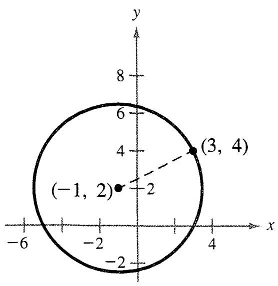
It follows from Theorem 1.6 that the standard form of the equation of a circle with center at the origin, $(h, k) = (0, 0)$, is

$$x^2 + y^2 = r^2.$$

If $r = 1$, then the graph of this equation is called the **unit circle.**

EXAMPLE 5 Finding the equation of a circle

The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure 1.22. Find an equation for the circle.



$(x + 1)^2 + (y - 2)^2 = 20$

FIGURE 1.22

SOLUTION

The radius of the circle is the distance between $(-1, 2)$ and $(3, 4)$. Thus,

$$r = \sqrt{[3 - (-1)]^2 + (4 - 2)^2} = \sqrt{16 + 4} = \sqrt{20}.$$

Therefore, the standard form of the equation of this circle is

$$[x - (-1)]^2 + (y - 2)^2 = (\sqrt{20})^2$$
$$(x + 1)^2 + (y - 2)^2 = 20.$$

By squaring and simplifying, the equation $(x - h)^2 + (y - k)^2 = r^2$ can be written in the following **general form of the equation of a circle.**

$$Ax^2 + Ay^2 + Cx + Dy + F = 0, \quad A \neq 0$$

To convert such an equation to the standard form

$$(x - h)^2 + (y - k)^2 = p$$

we use a process called **completing the square.** If $p > 0$, then the graph of the equation is a circle. If $p = 0$, then the graph is the single point (h, k). Finally, if $p < 0$, then the equation has no graph.

EXAMPLE 6 Completing the square

Sketch the graph of the circle whose general equation is

$$4x^2 + 4y^2 + 20x - 16y + 37 = 0.$$

SOLUTION

To complete the square, first divide by 4 so that the coefficients of x^2 and y^2 are both 1.

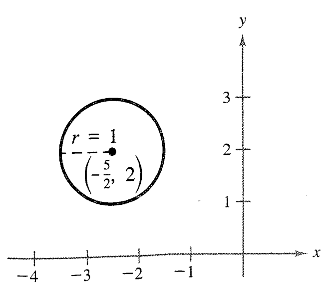
$$4x^2 + 4y^2 + 20x - 16y + 37 = 0 \qquad \text{General form}$$

$$x^2 + y^2 + 5x - 4y + \frac{37}{4} = 0 \qquad \text{Divide by 4}$$

$$(x^2 + 5x + \quad) + (y^2 - 4y + \quad) = -\frac{37}{4} \qquad \text{Group terms}$$

$$\left(x^2 + 5x + \underbrace{\frac{25}{4}}_{(\text{half})^2}\right) + (y^2 - 4y + \underbrace{4}_{(\text{half})^2}) = -\frac{37}{4} + \frac{25}{4} + 4 \qquad \text{Complete the square by adding } \tfrac{25}{4} \text{ and 4 to both sides}$$

$$\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = 1 \qquad \text{Standard form}$$



$\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = 1$

FIGURE 1.23

Note that we complete the square by adding the square of half the coefficient of x *and* the square of half the coefficient of y to both sides of the equation. Therefore, the circle is centered at $\left(-\frac{5}{2}, 2\right)$, and its radius is 1, as shown in Figure 1.23.

Pierre de Fermat

René Descartes

We have now introduced some fundamental concepts of *analytic geometry*. Because these concepts are in common use today, it is easy to overlook their revolutionary character. At the time analytic geometry was being developed by Pierre de Fermat (1601–1655) and René Descartes (1596–1650), the two major branches of mathematics—geometry and algebra—were largely independent of each other. Circles belonged to geometry and equations belonged to algebra. The coordination of the points on a circle and the solutions of an equation belongs to what is now called analytic geometry. It is important to become skilled in analytic geometry so that you may move easily between geometry and algebra. For instance, in Example 5, we are given a geometric description of a circle and are asked to find an algebraic equation for the circle. Thus, we are moving from geometry to algebra. Similarly, in Example 6 we are given an algebraic equation and asked to sketch a geometric picture. In this case, we are moving from algebra to geometry. These two examples illustrate the two most common problems in analytic geometry.

1. Given a graph, find its equation.

   Geometry → Algebra

2. Given an equation, find its graph.

   Algebra → Geometry

In the next two sections, we will look more closely at these two types of problems.

EXERCISES for Section 1.2

In Exercises 1–6, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

1. $(2, 1), (4, 5)$ **2.** $(-3, 2), (3, -2)$

3. $\left(\frac{1}{2}, 1\right), \left(-\frac{3}{2}, -5\right)$ **4.** $\left(\frac{2}{3}, -\frac{1}{3}\right), \left(\frac{5}{6}, 1\right)$

5. $(1, \sqrt{3}), (-1, 1)$ **6.** $(-2, 0), (0, \sqrt{2})$

In Exercises 7–10, show that the given points form the vertices of the indicated polygon. (A rhombus is a quadrilateral whose sides are all of the same length.)

	Vertices	*Figure*
7.	$(4, 0), (2, 1), (-1, -5)$	Right triangle
8.	$(1, -3), (3, 2), (-2, 4)$	Isosceles triangle
9.	$(0, 0), (1, 2), (2, 1), (3, 3)$	Rhombus
10.	$(0, 1), (3, 7), (4, 4), (1, -2)$	Parallelogram

In Exercises 11–14, use the Distance Formula to determine whether the given points are collinear (lie on the same line).

11. $(0, -4), (2, 0), (3, 2)$
12. $(0, 4), (7, -6), (-5, 11)$
13. $(-2, 1), (-1, 0), (2, -2)$
14. $(-1, 1), (3, 3), (5, 5)$

In Exercises 15 and 16, find x so that the distance between the points is 5.

15. $(0, 0), (x, -4)$ **16.** $(2, -1), (x, 2)$

In Exercises 17 and 18, find y so that the distance between the points is 8.

17. $(0, 0), (3, y)$ **18.** $(5, 1), (5, y)$

In Exercises 19 and 20, find the relationship between x and y so that (x, y) is equidistant from the two given points.

19. $(4, -1), (-2, 3)$ **20.** $\left(3, \frac{5}{2}\right), (-7, -1)$

21. Use the Midpoint Formula to find the three points that divide the line segment joining (x_1, y_1) and (x_2, y_2) into four equal parts.

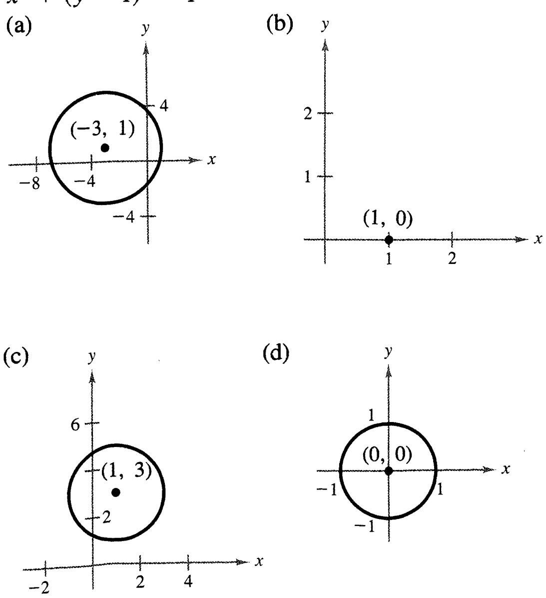
22. Use the result of Exercise 21 to find the points that divide the line segment joining the given points into four equal parts.
(a) $(1, -2), (4, -1)$ (b) $(-2, -3), (0, 0)$

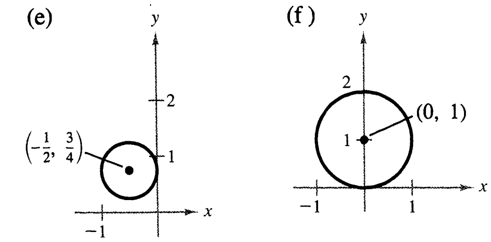
In Exercises 23 and 24, complete the square for each expression.

23. (a) $x^2 + 5x$ (b) $x^2 + 8x + 7$
24. (a) $4x^2 - 4x - 39$ (b) $5x^2 + x$

In Exercises 25–30, match the given equation with its graph. [Graphs are labeled (a)–(f).]

25. $x^2 + y^2 = 1$
26. $(x - 1)^2 + (y - 3)^2 = 4$
27. $(x - 1)^2 + y^2 = 0$
28. $\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{3}{4}\right)^2 = \frac{1}{4}$
29. $(x + 3)^2 + (y - 1)^2 = 16$
30. $x^2 + (y - 1)^2 = 1$





In Exercises 31–40, write the equation of the specified circle in general form.

31. Center: $(0, 0)$; radius: 3
32. Center: $(0, 0)$; radius: 5
33. Center: $(2, -1)$; radius: 4
34. Center: $(-4, 3)$; radius: $\frac{5}{8}$
35. Center: $(-1, 2)$; point on circle: $(0, 0)$
36. Center: $(3, -2)$; point on circle: $(-1, 1)$
37. Endpoints of diameter: $(2, 5), (4, -1)$
38. Endpoints of diameter: $(1, 1), (-1, -1)$
39. Points on circle: $(0, 0), (0, 8), (6, 0)$
40. Points on circle: $(1, -1), (2, -2), (0, -2)$

In Exercises 41–48, write the given equation (of a circle) in standard form and sketch its graph.

41. $x^2 + y^2 - 2x + 6y + 6 = 0$
42. $x^2 + y^2 - 2x + 6y - 15 = 0$
43. $x^2 + y^2 - 2x + 6y + 10 = 0$
44. $3x^2 + 3y^2 - 6y - 1 = 0$
45. $2x^2 + 2y^2 - 2x - 2y - 3 = 0$
46. $4x^2 + 4y^2 - 4x + 2y - 1 = 0$
47. $16x^2 + 16y^2 + 16x + 40y - 7 = 0$
48. $x^2 + y^2 - 4x + 2y + 3 = 0$

49. Find an equation for the path of a communications satellite in a circular orbit 22,000 miles above the earth. (Assume that the radius of the earth is 4000 miles.)

50. Find the equation of the circle passing through the points $(1, 2), (-1, 2)$, and $(2, 1)$.

51. Find the equation of the circle passing through the points $(4, 3), (-2, -5)$, and $(5, 2)$.

52. Find the equations of the circles passing through the points $(4, 1)$ and $(6, 3)$ and having radius $\sqrt{10}$.

In Exercises 53–56, sketch the set of all points satisfying the given inequality.

53. $x^2 + y^2 - 4x + 2y + 1 \le 0$
54. $x^2 + y^2 - 4x + 2y + 1 > 0$
55. $(x + 3)^2 + (y - 1)^2 < 9$
56. $(x - 1)^2 + \left(y - \frac{1}{2}\right)^2 > 1$

57. Prove that

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

is one of the points of trisection of the line segment joining (x_1, y_1) and (x_2, y_2). Also, find the midpoint of the line segment joining

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

and (x_2, y_2) to find the second point of trisection.

58. Use the results of Exercise 57 to find the points of trisection of the line segment joining the following points.
(a) $(1, -2)$ and $(4, 1)$ (b) $(-2, -3)$ and $(0, 0)$

59. Prove that the line segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.

60. Prove that the midpoint of the hypotenuse of a right triangle is equidistant from each of the three vertices.

61. Prove that an angle inscribed in a semicircle is a right angle.

62. Prove that the perpendicular bisector of a chord of a circle passes through the center of the circle.

63. Prove the Midpoint Formula (Theorem 1.5).

1.3 Graphs of Equations

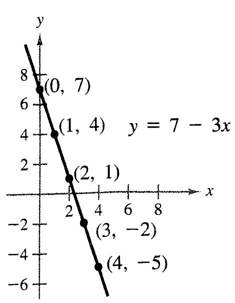
The graph of an equation ▪ Point-plotting method ▪ Intercepts of a graph ▪ Symmetry of a graph ▪ Points of intersection ▪ Mathematical models

Using a graph to show how two quantities are related is common. News magazines frequently show graphs that compare the gross national product or the unemployment rate to the time of year. Industries and businesses use graphs to report their monthly production and sales statistics. The value of such graphs is that they provide a geometric picture of the way one quantity changes with respect to another.

Frequently a relationship between two quantities is expressed as an equation. For instance, degrees on the Fahrenheit scale are related to degrees on the Celsius scale by the equation $F = \frac{9}{5}C + 32$. In this section we introduce a basic procedure for sketching the graph of such an equation.

TABLE 1.2

x	0	1	2	3	4
y	7	4	1	-2	-5



Solution points of $y = 7 - 3x$

FIGURE 1.24

The graph of an equation

Consider the equation $3x + y = 7$. If $x = 2$ and $y = 1$, the equation is satisfied and we call the point $(2, 1)$ a **solution point** of the equation. Of course, there are other solution points, such as $(1, 4)$ and $(0, 7)$. We can construct a **table of values** for x and y by choosing arbitrary values for x and determining the corresponding values for y. To determine the values for y, it is convenient to write the equation in the form

$$y = 7 - 3x.$$

Thus, $(0, 7)$, $(1, 4)$, $(2, 1)$, $(3, -2)$, and $(4, -5)$ are all solution points of the equation $3x + y = 7$, as shown in Table 1.2. Actually, there are infinitely many solution points of this equation, and the set of all such points is called the **graph** of the equation, as shown in Figure 1.24.

REMARK Even though we refer to the sketch shown in Figure 1.24 as the graph of $y = 7 - 3x$, it really represents only a *portion* of the graph. The entire graph would extend beyond the page.

DEFINITION OF THE GRAPH OF AN EQUATION IN TWO VARIABLES

The **graph of an equation** involving two variables x and y is the set of all points in the plane that are solution points of the equation.

EXAMPLE 1 The point-plotting method

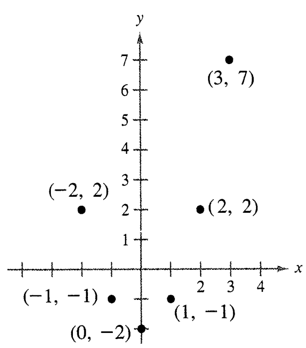
Sketch the graph of the equation $y = x^2 - 2$.

SOLUTION

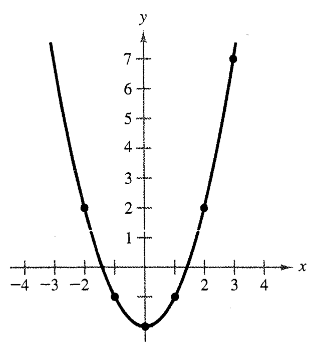
First, we make a table of values (Table 1.3) by choosing several convenient values of x and calculating the corresponding values of $y = x^2 - 2$. Next, we locate these points in the plane, as in Figure 1.25(a). Finally, we connect the points by a *smooth curve*, as shown in Figure 1.25(b). This particular graph is called a **parabola.** It is one of the conic sections we will study in Chapter 10.

TABLE 1.3

x	-2	-1	0	1	2	3
y	2	-1	-2	-1	2	7



(a) Plot several points.



(b) Connect points with a smooth curve.

FIGURE 1.25

We call this method of sketching a graph the *point-plotting method*. It consists of three basic steps.

1. Make up a table of several solution points of the equation.
2. Plot these points in the plane.
3. Connect the points with a smooth curve.

In later chapters, we will discuss more sophisticated graphing techniques. In the meantime, when using the point-plotting method, we must plot a sufficient number of points to reveal the basic shape of the graph. With too few solution points, we can grossly misrepresent the graph of a given equation. For instance, how would you connect the four points shown in Figure 1.26? Without additional points or more information about the equation, any one of the three graphs shown in Figure 1.27 would be reasonable.

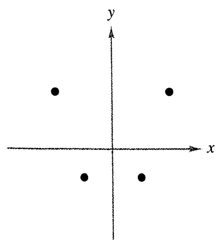


FIGURE 1.26

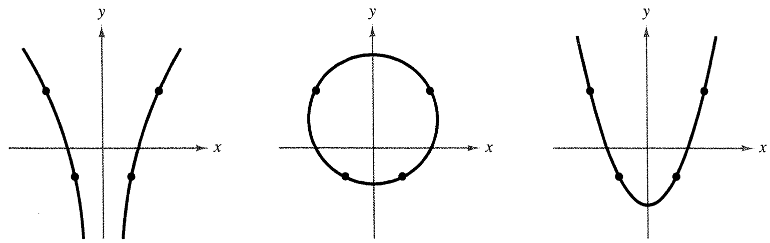
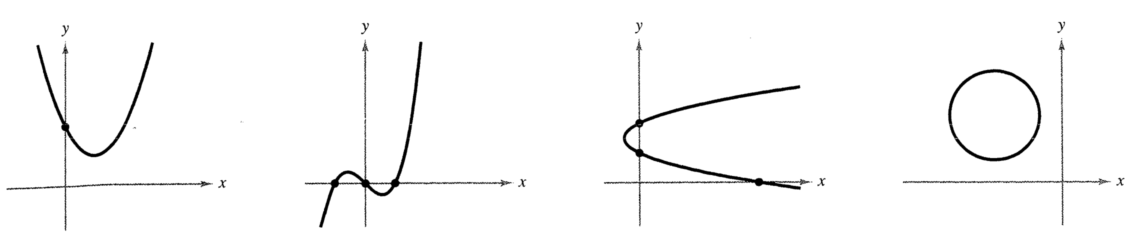


FIGURE 1.27

Two types of solution points that are especially useful are those having zero as either their x- or y-coordinate. Such points are called **intercepts,** because they are the points at which the graph intersects the x- or y-axis. Specifically, the point $(a, 0)$ is called an ***x*-intercept** of the graph of an equation if it is a solution point of the equation. Such points can be found by letting y be zero and solving the equation for x. Similarly, the point $(0, b)$ is called a ***y*-intercept** of the graph of an equation if it is a solution point of the equation. Such points can be found by letting x be zero and solving the equation for y.

REMARK Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, we will use the term *intercept* to mean either the point or the coordinate.

It is possible for a graph to have no intercepts, or it might have several. For instance, consider the four graphs shown in Figure 1.28.



No x-intercept
One y-intercept

Three x-intercepts
One y-intercept

One x-intercept
Two y-intercepts

No intercepts

FIGURE 1.28

EXAMPLE 2 Finding x- and y-intercepts

Find the x- and y-intercepts for the graphs of the following equations.

(a) $y = x^3 - 4x$ (b) $y^2 - 3 = x$

SOLUTION

(a) Let $y = 0$. Then, $0 = x(x^2 - 4)$ has solutions $x = 0$ and $x = \pm 2$.

x-intercepts: $(0, 0)$, $(2, 0)$, $(-2, 0)$

Let $x = 0$. Then $y = 0$.

y-intercept: $(0, 0)$

(See Figure 1.29.)

(b) Let $y = 0$. Then $-3 = x$.

x-intercept: $(-3, 0)$

Let $x = 0$. Then $y^2 - 3 = 0$ has solutions $y = \pm\sqrt{3}$.

y-intercepts: $(0, \sqrt{3})$, $(0, -\sqrt{3})$

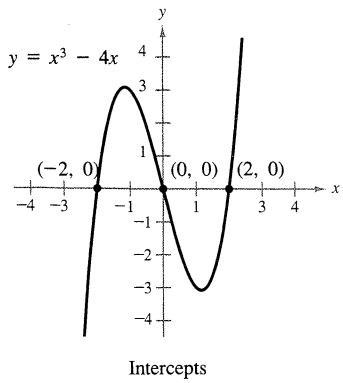


FIGURE 1.29

EXAMPLE 3 Sketching the graph of an equation

Sketch the graph of the equation $x^2 + 4y^2 = 16$.

SOLUTION

Sometimes it helps to rewrite an equation before calculating solution points. For example, if we rewrite the equation $x^2 + 4y^2 = 16$ as

$$x = \pm\sqrt{16 - 4y^2} = \pm 2\sqrt{4 - y^2}$$

then we can easily determine several solution points by choosing values for y and calculating the corresponding values for x. (Note that $x = \pm 2\sqrt{4 - y^2}$ is defined only when $|y| \leq 2$.) By plotting these points and connecting them with a smooth curve, we create the graph shown in Figure 1.30. This particular graph is called an **ellipse.** It is one of the conic sections we will study in Chapter 10.

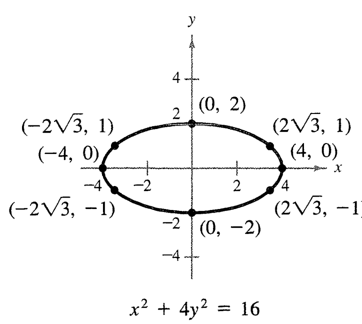


FIGURE 1.30

Symmetry of a graph

The graphs shown in Figures 1.25(b) and 1.30 are said to be **symmetric** with respect to the y-axis. This means that if the Cartesian plane were folded along the y-axis, the portion of the graph to the left of the y-axis would coincide with the portion to the right of the y-axis. Another way to describe this symmetry is to say that the graph is a reflection of itself with respect to the y-axis. Symmetry with respect to the x-axis can be described similarly.

Knowing that a graph has symmetry *before* attempting to sketch it is helpful because then we need only half as many solution points as we would otherwise. We define three basic types of symmetry, as shown in Figure 1.31.

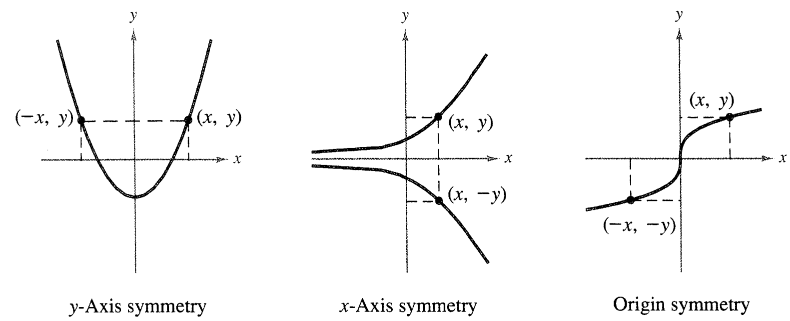


FIGURE 1.31

DEFINITION OF SYMMETRY

A graph is said to be **symmetric with respect to the y-axis** if, whenever (x, y) is a point on the graph, $(-x, y)$ is also a point on the graph.

A graph is said to be **symmetric with respect to the x-axis** if, whenever (x, y) is a point on the graph, $(x, -y)$ is also a point on the graph.

A graph is said to be **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph.

REMARK Note that a graph is symmetric with respect to the origin if a rotation of 180° (about the origin) leaves the graph unchanged.

Suppose we apply the definition of symmetry to the graph of the equation shown in Figure 1.25(b).

$y = x^2 - 2$ Given equation

$y = (-x)^2 - 2$ Replace x by $-x$

$y = x^2 - 2$ Equivalent equation

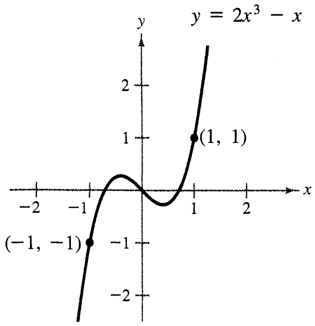
Since substituting $-x$ for x produces an equivalent equation, it follows that if (x, y) is a solution point of the given equation, then $(-x, y)$ must also be a solution point. Therefore, the graph of $y = x^2 - 2$ is symmetric with respect to the y-axis.

A similar test can be made for symmetry with respect to the x-axis or the origin. These three tests are summarized as follows.

TESTS FOR SYMMETRY

1. The graph of an equation in x and y is symmetric with respect to the y-axis if replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis if replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin if replacing x by $-x$ *and* y by $-y$ yields an equivalent equation.

EXAMPLE 4 Testing for origin symmetry



Origin Symmetry

FIGURE 1.32

Show that the graph of $y = 2x^3 - x$ is symmetric with respect to the origin.

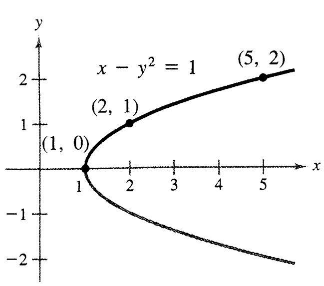
SOLUTION

We apply the test for origin symmetry as follows.

$y = 2x^3 - x$	Given equation
$-y = 2(-x)^3 - (-x)$	Replace x by $-x$ and y by $-y$
$-y = -2x^3 + x$	
$y = 2x^3 - x$	Equivalent equation

Since the replacement produces an equivalent equation, we conclude that the graph of $y = 2x^3 - x$ is symmetric with respect to the origin, as shown in Figure 1.32.

EXAMPLE 5 Using symmetry to sketch a graph



First, plot the points above the x-axis, then use symmetry to complete the graph.

FIGURE 1.33

Sketch the graph of $x - y^2 = 1$.

SOLUTION

The graph is symmetric with respect to the x-axis since replacing y by $-y$ yields

$$x - (-y)^2 = 1$$
$$x - y^2 = 1.$$

This means that the graph below the x-axis is a mirror image of the graph above the x-axis. Hence, we first sketch the graph above the x-axis and then reflect it to obtain the entire graph, as shown in Figure 1.33.

Points of intersection

Since each point of a graph is a solution point of its corresponding equation, a **point of intersection** of two graphs is simply a solution point that satisfies both equations. Moreover, the points of intersection of two graphs can be found by solving the equations simultaneously.

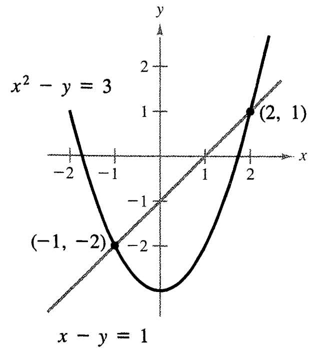
EXAMPLE 6 Finding points of intersection

Find all points of intersection of the graphs of

$$x^2 - y = 3 \quad \text{and} \quad x - y = 1.$$

SOLUTION

It is helpful to begin by making a sketch of each equation on the *same* coordinate plane, as shown in Figure 1.34. Having done this, it appears that the two graphs have two points of intersection. To find these two points, we proceed as follows.



Two Points of Intersection

FIGURE 1.34

$$y = x^2 - 3 \qquad \text{Solve first equation for } y$$
$$y = x - 1 \qquad \text{Solve second equation for } y$$
$$x^2 - 3 = x - 1 \qquad \text{Equate } y\text{-values}$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0 \qquad \text{Solve for } x$$

The corresponding values of y are obtained by substituting $x = 2$ and $x = -1$ into either of the original equations. For instance, if we choose the equation $y = x - 1$, then the values of y are 1 and -2, respectively. Therefore, the two points of intersection are $(2, 1)$ and $(-1, -2)$. ▭

Mathematical models

In applications we frequently use equations to form **mathematical models** of real-world phenomena. In developing a mathematical model to represent actual data, we strive for two (often conflicting) goals: accuracy and simplicity. That is, we want the model to be simple enough to be workable, yet accurate enough to produce meaningful results. Our next example describes a typical mathematical model.

EXAMPLE 7 A mathematical model

The median income (between 1955 and 1985) for married couples in the United States is given in Table 1.4. A mathematical model* for these data is given by

$$y = 0.033286t^2 - 0.130718t + 5.05716$$

where y represents the median income in thousands of dollars and t represents the year, with $t = 0$ corresponding to 1955. Using a graph, compare the data with the model and use the model to predict the median income for 1990.

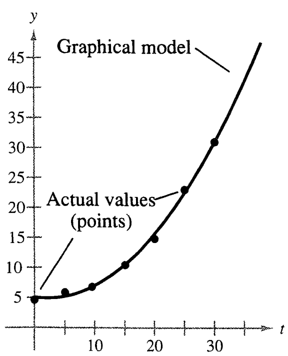
*This model was developed using a procedure called the method of least squares. For a discussion of this method, see Section 14.9, Exercise 21.

TABLE 1.4

Year	1955	1960	1965	1970	1975	1980	1985
Income (in 1000s)	4.6	5.9	7.3	10.5	14.9	23.1	31.1

SOLUTION

Table 1.5 and Figure 1.35 compare the values given by the model with the actual values.



Model:
$y = 0.033286t^2 - 0.130718t + 5.05716$

FIGURE 1.35

TABLE 1.5

t	0	5	10	15	20	25	30
y	5.1	5.2	7.1	10.6	15.8	22.6	31.1
Actual (income)	4.6	5.9	7.3	10.5	14.9	23.1	31.1

To predict the median income for 1990, we let $t = 35$ and calculate y as follows.

$$y = 0.033286(35^2) - 0.130718(35) + 5.05716 \approx 41.3$$

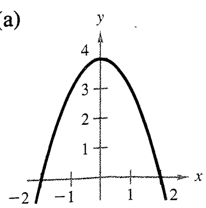
Thus, we estimate that the 1990 median income will be \$41,300.

EXERCISES for Section 1.3

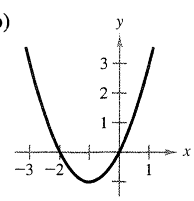
In Exercises 1–6, match the given equation with its graph. [Graphs are labeled (a)–(f).]

1. $y = x - 2$
2. $y = -\frac{1}{2}x + 2$
3. $y = x^2 + 2x$
4. $y = \sqrt{9 - x^2}$
5. $y = 4 - x^2$
6. $y = x^3 - x$

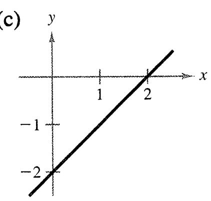
(a)



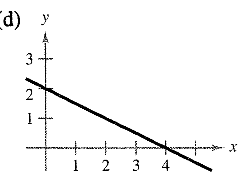
(b)



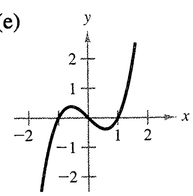
(c)



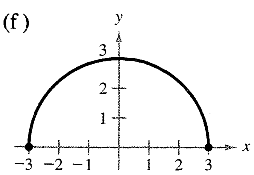
(d)



(e)



(f)



In Exercises 7–16, find the intercepts.

7. $y = 2x - 3$

8. $y = (x - 1)(x - 3)$

9. $y = x^2 + x - 2$

10. $y^2 = x^3 - 4x$

11. $y = x^2\sqrt{9 - x^2}$

12. $xy = 4$

13. $y = \dfrac{x - 1}{x - 2}$

14. $y = \dfrac{x^2 + 3x}{(3x + 1)^2}$

15. $x^2y - x^2 + 4y = 0$

16. $y = 2x - \sqrt{x^2 + 1}$

In Exercises 17–26, check for symmetry with respect to both axes and to the origin.

17. $y = x^2 - 2$

18. $y = x^4 - x^2 + 3$

19. $x^2y - x^2 + 4y = 0$

20. $x^2y - x^2 - 4y = 0$

21. $y^2 = x^3 - 4x$

22. $xy^2 = -10$

23. $y = x^3 + x$

24. $xy = 1$

25. $y = \dfrac{x}{x^2 + 1}$

26. $y = x^3 + x - 3$

In Exercises 27–46, sketch the graph of each equation. Identify the intercepts and test for symmetry.

27. $y = x$

28. $y = x - 2$

29. $y = x + 3$

30. $y = 2x - 3$

31. $y = -3x + 2$

32. $y = -\frac{1}{2}x + 2$

33. $y = \frac{1}{2}x - 4$

34. $y = x^2 + 3$

35. $y = 1 - x^2$

36. $y = 2x^2 + x$

37. $y = -2x^2 + x + 1$

38. $y = x^3 - 1$

39. $y = x^3 + 2$

40. $y = \sqrt{9 - x^2}$

41. $x^2 + 4y^2 = 4$

42. $9x^2 + y^2 = 9$

43. $y = (x + 2)^2$

44. $x = y^2 - 4$

45. $y = \dfrac{1}{x}$

46. $y = 2x^4$

In Exercises 47–56, find the points of intersection of the graphs of the equations; check your results.

47. $x + y = 2,\ 2x - y = 1$

48. $2x - 3y = 13,\ 5x + 3y = 1$

49. $x + y = 7,\ 3x - 2y = 11$

50. $x^2 + y^2 = 25,\ 2x + y = 10$

51. $x^2 + y^2 = 5,\ x - y = 1$

52. $x^2 + y = 4,\ 2x - y = 1$

53. $y = x^3,\ y = x$

54. $y = x^4 - 2x^2 + 1,\ y = 1 - x^2$

55. $y = x^3 - 2x^2 + x - 1,\ y = -x^2 + 3x - 1$

56. $x = 3 - y^2,\ y = x - 1$

In Exercises 57 and 58, find the sales necessary to break even ($R = C$) for the given cost C of x units and the given revenue R obtained by selling x units.

57. $C = 8650x + 250{,}000$
$R = 9950x$

58. $C = 5.5\sqrt{x} + 10{,}000$
$R = 3.29x$

In Exercises 59–62, determine whether the points lie on the graph of the given equation.

59. Equation: $2x - y - 3 = 0$
Points: $(1, 2), (1, -1), (4, 5)$

60. Equation: $x + y^2 = 4$
Points: $(1, -\sqrt{3}), \left(\frac{1}{2}, -1\right), \left(\frac{3}{2}, \frac{7}{2}\right)$

61. Equation: $x^2y - x^2 + 4y = 0$
Points: $\left(1, \frac{1}{5}\right), \left(2, \frac{1}{2}\right), (-1, -2)$

62. Equation: $x^2 - xy + 4y = 3$
Points: $(0, 2), \left(-2, -\frac{1}{6}\right), (3, -6)$

63. For what values of k does the graph of $y = kx^3$ pass through the given point?
(a) $(1, 4)$ (b) $(-2, 1)$
(c) $(0, 0)$ (d) $(-1, -1)$

64. For what values of k does the graph of $y^2 = 4kx$ pass through the given point?
(a) $(1, 1)$ (b) $(2, 4)$
(c) $(0, 0)$ (d) $(3, 3)$

65. The Consumer Price Index (CPI) for selected years is given in the following table.

Year	1970	1975	1980	1985	1987
CPI	116.3	161.2	246.8	322.2	333.9

A mathematical model for the CPI during this time period is

$$y = 0.1t^2 + 11.9t + 111.4$$

where y represents the CPI and t represents the year, with $t = 0$ corresponding to 1970.
(a) Use a graph to compare the CPI with the model.
(b) Use the model to predict the CPI for 1995.

66. From the model in Exercise 65, we obtain the model

$$V = \frac{1000}{t^2 + 119t + 1114}$$

where V represents the purchasing power of the dollar (in terms of constant 1967 dollars) and t represents the year, with $t = 0$ corresponding to 1970. Use the model to complete the following table.

t	0	5	10	15	20	25
V						

67. The farm population in the United States as a percentage of the total population for selected years is given in the following table.

Year	1950	1960	1970	1980	1985
Percentage	15.3	8.7	4.8	2.7	2.2

A mathematical model for these data is given by

$$y = \frac{1000}{11t + 27}$$

where y represents the percentage and t represents the year, with $t = 0$ corresponding to 1950.

(a) Use a graph to compare the actual percentage with that given by the model.

(b) Use the model to predict the farm percentage of the population in 1995.

68. The average number of acres per farm in the United States for selected years is given in the following table.

Year	1950	1960	1970	1980	1985
Number of acres	213	297	374	427	446

A mathematical model for these data is given by

$$y = -0.08t^2 + 9.69t + 211.79$$

where y represents the average acreage and t represents the year, with $t = 0$ corresponding to 1950.

(a) Use a graph to compare the actual number of acres per farm with that given by the model.

(b) Use the model to predict the average number of acres per farm in the United States in 1995.

In Exercises 69 and 70, use a computer or graphics calculator to sketch the graph of the equation and find its intercepts.

69. $y = \frac{1}{16}(x^5 - 6x^4 + 9x^3 + 32)$

70. $y = \frac{5}{x^2 + 1} - 1$

71. Prove that if a graph is symmetric with respect to the x-axis and to the y-axis, then it is symmetric with respect to the origin. Give an example to show that the converse is not true.

72. Prove that if a graph is symmetric with respect to one axis and the origin, then it is symmetric with respect to the other axis also.

1.4 Lines in the Plane

The slope of a line ▪ Equations of lines ▪ Sketching the graph of a line ▪ Parallel lines ▪ Perpendicular lines

In Chapter 3, you will see that one of the primary problems in calculus is measuring (instantaneous) rates of change. In this section, we discuss the noncalculus version of this problem—measuring an *average* rate of change. We begin with an example.

Consider an automobile that is traveling at a *constant* rate on a straight highway. At 2:00 P.M. the car has traveled 20 miles from a particular city, and at 4:00 P.M. the car has traveled 132 miles, as shown in Figure 1.36. How fast is the car traveling?

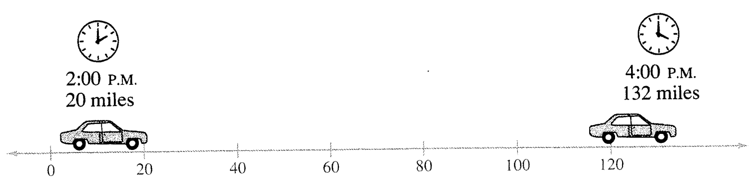


FIGURE 1.36

To measure the rate in *miles per hour*, we divide the distance traveled by the elapsed time.

$$\text{rate (mph)} = \frac{\text{distance (miles)}}{\text{time (hours)}} = \frac{132 - 20}{4 - 2} = \frac{112}{2} = 56 \text{ mph}$$

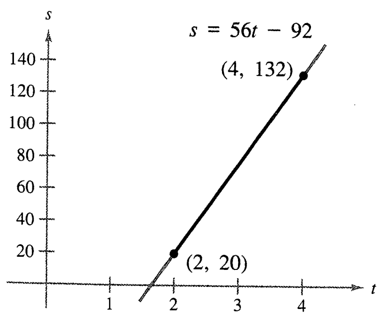


FIGURE 1.37

If we let s be the distance from the city in miles and t be the time in h
then the distance is related to the time by the **linear equation**

$$s = 56t - 92, \qquad 2 \le t \le 4.$$

The graph of the equation $s = 56t - 92$ is a line (in this text, we us
term *line* to mean *straight line*), as shown in Figure 1.37. For every uni
t increases, the distance s increases 56 units. Mathematically, we say
this line has a *slope* of 56.

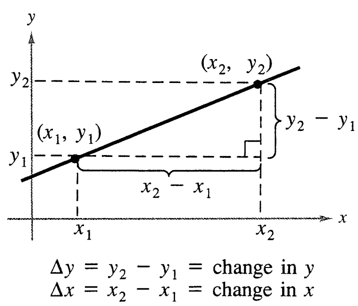


FIGURE 1.38

The slope of a line

By the **slope** of a (nonvertical) line, we mean the number of units a line
(or falls) vertically for each unit of horizontal change from left to right.
instance, consider the two points (x_1, y_1) and (x_2, y_2) on the line in Fi
1.38. As we move from left to right along this line, a vertical chang
$\Delta y = y_2 - y_1$ units corresponds to a horizontal change of $\Delta x = x_2$ -
units. (Δ is the Greek uppercase letter *delta*, and the symbols Δy and Δ
read "delta y" and "delta x.") We use the ratio of Δy to Δx to define the s
of a line as follows.

DEFINITION OF THE SLOPE OF A LINE

The **slope** m of a nonvertical line passing through the points (x_1, y_1) and $(x_2, y$
is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

REMARK Note that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

Hence, it does not matter in which order we subtract *as long as* we are consisten
both "subtracted coordinates" come from the same point.

EXAMPLE 1 The slope of a line passing through two points

(a) The slope of the line containing $(-2, 0)$ and $(3, 1)$ is

$$m = \frac{1 - 0}{3 - (-2)} = \frac{1}{3 + 2} = \frac{1}{5}.$$

(b) The slope of the line containing $(-1, 2)$ and $(2, 2)$ is

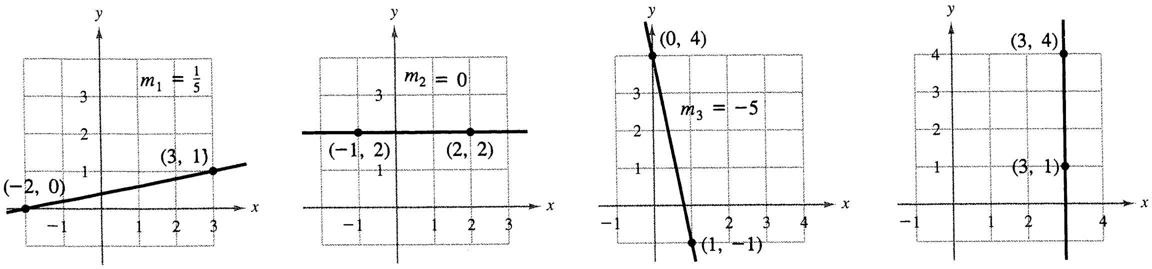
$$m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0.$$

(c) The slope of the line containing (0, 4) and (1, −1) is

$$m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5.$$

(d) We do not define the slope of the vertical line containing (3, 4) and (3, 1).

(See Figure 1.39.)



If m is positive, the line rises.

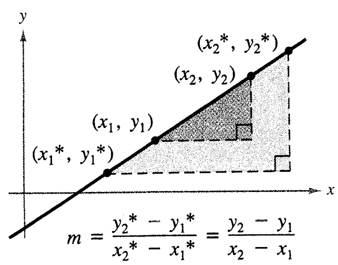
If m is zero, the line is horizontal.

If m is negative, the line falls.

If the line is vertical, the slope is undefined.

FIGURE 1.39

REMARK Note that we do not define the slope of a vertical line.



Any two points on a line can be used to determine its slope.

FIGURE 1.40

It is important to realize that *any* two points on a nonvertical line can be used to calculate its slope. This can be verified from the similar triangles shown in Figure 1.40. (See Exercise 77.) (Recall that the ratios of corresponding sides of similar triangles are equal.)

Equations of lines

If we know the slope of a line and one point on the line, how can we determine the equation of the line? Figure 1.40 leads us to the answer to this question. If (x_1, y_1) is a point lying on a line of slope m and (x, y) is any *other* point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation, involving the two variables x and y, can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

which is called the **point-slope equation of a line.**

THEOREM 1.7 POINT-SLOPE EQUATION OF A LINE

The equation of the line with slope m passing through the point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1).$$

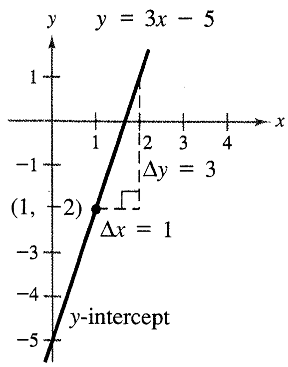


FIGURE 1.41

EXAMPLE 2 The point-slope equation of a line

Find an equation of the line that has a slope of 3 and passes through the p
(1, −2).

SOLUTION

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$
$$y - (-2) = 3(x - 1)$$
$$y + 2 = 3x - 3$$
$$y = 3x - 5$$

(See Figure 1.41.)

EXAMPLE 3 An application: total U.S. sales

The total U.S. sales (including inventories) during the first two quarte 1978 were 539.9 and 560.2 billion dollars, respectively. Assuming a *li growth pattern*, estimate the total sales during the fourth quarter of 197

SOLUTION

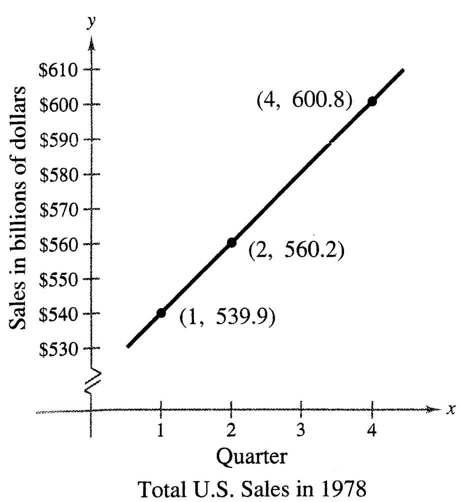


FIGURE 1.42

Referring to Figure 1.42, we let (1, 539.9) and (2, 560.2) be two point the line representing total U.S. sales. We let x represent the quarter a represent the sales in billions of dollars. The slope of the line passing thr these two points is

$$m = \frac{560.2 - 539.9}{2 - 1} = 20.3.$$

Thus, the equation of the line is

$$y - y_1 = m(x - x_1)$$
$$y - 539.9 = 20.3(x - 1)$$
$$y = 20.3(x - 1) + 539.9$$
$$y = 20.3x + 519.6.$$

Now, using this linear model, we estimate the fourth quarter sales ($x =$ be

$$y = (20.3)(4) + 519.6 = 600.8 \text{ billion dollars.}$$

(In this particular case, the estimate proves to be quite good. The actual fo quarter sales in 1978 were 600.5 billion dollars.)

REMARK The estimation method illustrated in Example 3 is called **linear ext olation.** Note that the estimated point does not lie between the two given pc When the estimated point lies between the two given points, we call the proce **linear interpolation.**

Sketching the graph of a line

In Section 1.2, we mentioned that many problems in analytic geometry can be classified in two basic categories: (1) Given a graph, what is its equation? and (2) Given an equation, what is its graph? The point-slope equation of a line fits in the first category. However, this form is *not* particularly useful for solving problems in the second category. The form that is best suited to sketching the graph of a line is called the **slope-intercept** form for the equation of a line.

THEOREM 1.8
THE SLOPE-INTERCEPT EQUATION OF A LINE

The graph of the equation

$$y = mx + b$$

is a line having a *slope* of m and a *y-intercept* at $(0, b)$.

EXAMPLE 4 Sketching lines in the plane

Sketch the graphs of the following linear equations.

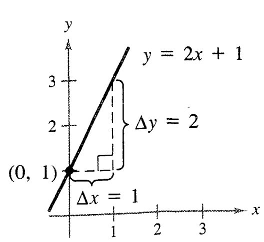
(a) $y = 2x + 1$ (b) $y = 2$ (c) $3y + x - 6 = 0$

SOLUTION

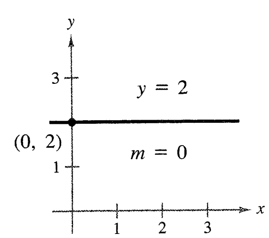
(a) Since $b = 1$, the y-intercept occurs at $(0, 1)$, and since the slope is $m = 2$, we know that this line rises 2 units for each unit it moves to the right. (See Figure 1.43(a).)
(b) Since $b = 2$, the y-intercept occurs at $(0, 2)$, and since the slope is $m = 0$, we know that the line is horizontal. That is, it doesn't rise or fall. (See Figure 1.43(b).)
(c) We begin by writing the equation in slope-intercept form.

$$3y + x - 6 = 0$$
$$3y = -x + 6$$
$$y = -\frac{1}{3}x + 2$$

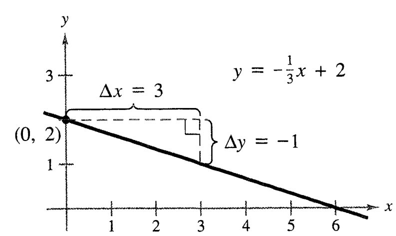
Thus, the y-intercept occurs at $(0, 2)$ and the slope is $m = -\frac{1}{3}$. This means that the line falls 1 unit for every 3 units it moves to the right. (See Figure 1.43(c).)



(a) $m = 2$, line rises.



(b) $m = 0$, line is horizontal.



(c) $m = -\frac{1}{3}$, line falls.

FIGURE 1.43

□

Since the slope of a vertical line is not defined, its equation cannot be written in the slope-intercept form. However, the equation of *any* line can be written in the **general form**

$$Ax + By + C = 0$$

where A and B are not *both* zero. For instance, the vertical line given by $x = a$ can be represented by the general form $x - a = 0$. We summarize the five most common forms of equations of lines in the following list.

SUMMARY OF EQUATIONS OF LINES

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Point-slope form: $y - y_1 = m(x - x_1)$
5. Slope-intercept form: $y = mx + b$

Parallel and perpendicular lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular. This is seen in the following two theorems.

THEOREM 1.9 PARALLEL LINES

Two distinct nonvertical lines are parallel if and only if their slopes are equal.

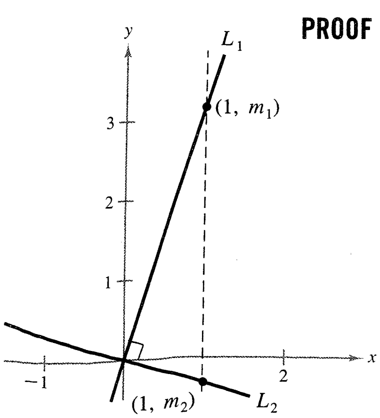
THEOREM 1.10 PERPENDICULAR LINES

Two nonvertical lines are perpendicular if and only if their slopes are related by the following equation.

$$m_1 = -\frac{1}{m_2}$$

PROOF We will prove only one direction of the theorem and leave the other direction as an exercise (see Exercise 78). Let us assume that we are given two nonvertical perpendicular lines L_1 and L_2 with slopes m_1 and m_2. For simplicity's sake let these two lines intersect at the origin, as shown in Figure 1.44. The vertical line $x = 1$ will intersect L_1 and L_2 at the respective points $(1, m_1)$ and $(1, m_2)$. Since the triangle formed by these two points and the origin is a right triangle, we can apply the Pythagorean Theorem and conclude that

$$\begin{pmatrix}\text{distance between}\\ (0, 0) \text{ and } (1, m_1)\end{pmatrix}^2 + \begin{pmatrix}\text{distance between}\\ (0, 0) \text{ and } (1, m_2)\end{pmatrix}^2 = \begin{pmatrix}\text{distance between}\\ (1, m_1) \text{ and } (1, m_2)\end{pmatrix}^2.$$



The slopes of perpendicular lines are negative reciprocals of each other.

FIGURE 1.44

Using the Distance Formula, we have

$$\begin{aligned}(\sqrt{1+m_1^2})^2 + (\sqrt{1+m_2^2})^2 &= (\sqrt{0^2 + (m_1 - m_2)^2})^2\\ 1 + m_1^2 + 1 + m_2^2 &= (m_1 - m_2)^2\\ 2 + m_1^2 + m_2^2 &= m_1^2 - 2m_1m_2 + m_2^2\\ 2 &= -2m_1m_2\\ -\frac{1}{m_2} &= m_1.\end{aligned}$$

EXAMPLE 5 Finding parallel and perpendicular lines

Find an equation for the line that passes through the point $(2, -1)$ and is
(a) parallel to the line $2x - 3y = 5$
(b) perpendicular to the line $2x - 3y = 5$.

SOLUTION

Writing the equation $2x - 3y = 5$ in slope-intercept form, we have

$$y = \frac{2}{3}x - \frac{5}{3}.$$

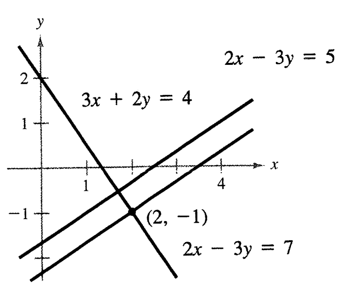


FIGURE 1.45

Therefore, the given line has a slope of $m = \frac{2}{3}$.

(a) The line through $(2, -1)$ that is parallel to the given line has an equation of the form

$$\begin{aligned}y - (-1) &= \frac{2}{3}(x - 2)\\ 3(y + 1) &= 2(x - 2)\\ 2x - 3y &= 7.\end{aligned}$$

(See Figure 1.45.) (Note the similarity to the original equation.)

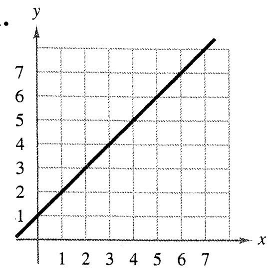
(b) Using the negative reciprocal of the slope of the given line, we find the slope of a line perpendicular to the given line to be $-\frac{3}{2}$. Therefore, the line through the point $(2, -1)$ that is perpendicular to the given line has the equation

$$\begin{aligned}y - (-1) &= -\frac{3}{2}(x - 2)\\ 2(y + 1) &= -3(x - 2)\\ 3x + 2y &= 4.\end{aligned}$$

(See Figure 1.45.)

EXERCISES for Section 1.4

In Exercises 1–6, estimate the slope of the given line from its graph.

1.


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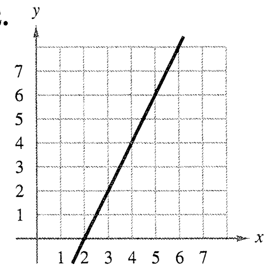

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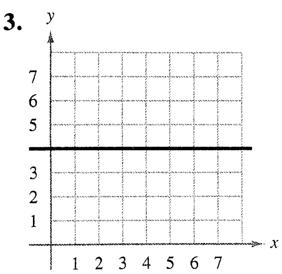

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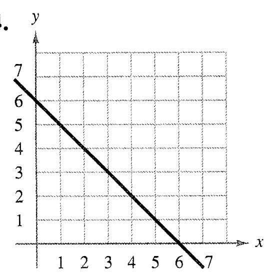
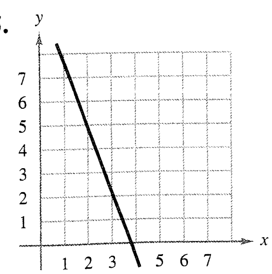


5.


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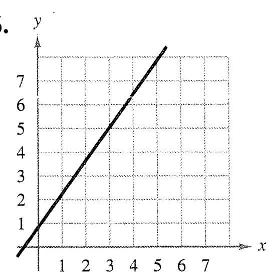


In Exercises 7–12, plot the given pair of points and find the slope of the line passing through them.

7. $(3, -4), (5, 2)$
8. $(-2, 1), (4, -3)$
9. $\left(\frac{1}{2}, 2\right), (6, 2)$
10. $(2, 1), (2, 5)$
11. $(1, 2), (-2, 4)$
12. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

In Exercises 13–16, use the given point on the line and the slope of the line to find three additional points that the line passes through. (The solution is not unique.)

	Point	Slope		Point	Slope
13.	$(2, 1)$	$m = 0$	**14.**	$(-3, 4)$	m undefined
15.	$(1, 7)$	$m = -3$	**16.**	$(-2, -2)$	$m = 2$

In Exercises 17–20, find the slope and y-intercept (if possible) of the line specified by the given equation.

17. $x + 5y = 20$
18. $6x - 5y = 15$
19. $x = 4$
20. $y = -1$

In Exercises 21–26, find an equation for the line that passes through the given points, and sketch the graph of the line.

21. $(2, 1), (0, -3)$
22. $(-3, -4), (1, 4)$
23. $(0, 0), (-1, 3)$
24. $(-3, 6), (1, 2)$
25. $(1, -2), (3, -2)$
26. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

In Exercises 27–32, find an equation of the line that passes through the given point and has the indicated slope. Sketch the line.

	Point	Slope		Point	Slope
27.	$(0, 3)$	$m = \frac{3}{4}$	**28.**	$(-1, 2)$	m undefined
29.	$(0, 0)$	$m = \frac{2}{3}$	**30.**	$(-2, 4)$	$m = -\frac{3}{5}$
31.	$(0, 2)$	$m = 4$	**32.**	$(0, 4)$	$m = 0$

33. Find an equation of the vertical line with x-intercept at 3.

34. Show that the line with intercepts $(a, 0)$ and $(0, b)$ has the following equation.

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, b \neq 0$$

In Exercises 35–40, use the result of Exercise 34 to write an equation of the indicated line.

35. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

36. x-intercept: $(-3, 0)$
y-intercept: $(0, 4)$

37. x-intercept: $\left(-\frac{1}{6}, 0\right)$
y-intercept: $\left(0, -\frac{2}{3}\right)$

38. x-intercept: $\left(-\frac{2}{3}, 0\right)$
y-intercept: $(0, -2)$

39. Point on line: $(1, 2)$
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

40. Point on line: $(-3, 4)$
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

In Exercises 41–46, write an equation of the line through the given point (a) parallel to the given line and (b) perpendicular to the given line.

	Point	*Line*
41.	(2, 1)	$4x - 2y = 3$
42.	(−3, 2)	$x + y = 7$
43.	$\left(\frac{7}{8}, \frac{3}{4}\right)$	$5x + 3y = 0$
44.	(−6, 4)	$3x + 4y = 7$
45.	(2, 5)	$x = 4$
46.	(−1, 0)	$y = -3$

In Exercises 47–52, sketch the graph of the equation.

47. $y = -3$

48. $x = 4$

49. $2x - y - 3 = 0$

50. $x + 2y + 6 = 0$

51. $y = -2x + 1$

52. $y - 1 = 3(x + 4)$

In Exercises 53 and 54, find an equation of the line determined by the points of intersection of the graphs of the parabolas.

53. $y = x^2$
$y = 4x - x^2$

54. $y = x^2 - 4x + 3$
$y = -x^2 + 2x + 3$

In Exercises 55 and 56, determine whether the three given points are collinear (lie on the same straight line).

55. (−2, 1), (−1, 0), (2, −2)

56. (0, 4), (7, −6), (−5, 11)

In Exercises 57–60, refer to the triangle in the accompanying figure.

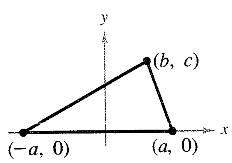


FIGURE FOR 57–60

57. Find the coordinates of the point of intersection of the perpendicular bisectors of the sides.

58. Find the coordinates of the point of intersection of the medians.

59. Find the coordinates of the point of intersection of the altitudes.

60. Show that the points of intersection of Exercises 57, 58, and 59 are collinear.

61. Find an equation of the line giving the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F. Use the fact that water freezes at 0° Celsius (32° Fahrenheit) and boils at 100° Celsius (212° Fahrenheit).

62. Use the result of Exercise 61 to complete the following table.

C		−10°	10°			177°
F	0°			68°	90°	

63. A company reimburses its sales representatives $95 per day for lodging and meals plus 25¢ per mile driven. Write a linear equation giving the daily cost C to the company in terms of x, the number of miles driven.

64. A manufacturing company pays its assembly line workers $9.50 per hour *plus* an additional piecework rate of $0.75 per unit produced. Find a linear equation for the hourly wages W in terms of x, the number of units produced per hour.

65. A small business purchases a piece of equipment for $875. After 5 years the equipment will be obsolete and have no value. Write a linear equation giving the value y of the equipment during the 5 years it will be used. (Let t represent the time in years.)

66. A company constructs a warehouse for $825,000. It has an estimated useful life of 25 years, after which its value is expected to be $75,000. Use straight-line depreciation to write a linear equation giving the value y of the warehouse during its 25 years of useful life. (Let t represent the time in years.)

67. A real estate office handles an apartment complex with 50 units. When the rent is $380 per month, all 50 units are occupied. However, when the rent is $425, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear. (Note: Here we use the term *demand* to refer to the number of occupied units.)
(a) Write a linear equation giving the quantity demanded x in terms of the rent p.
(b) (Linear extrapolation) Use this equation to predict the number of units occupied if the rent is raised to $455.
(c) (Linear interpolation) Predict the number of units occupied if the rent is lowered to $395.

68. The number of subscribers to cable TV for the years 1980 and 1986 were 16 million and 37.5 million, respectively. Assume that the relationship between the year t and the number of subscribers y is linear.
(a) Write the equation giving the number of subscribers y in terms of t. (Let $t = 0$ represent 1980.)
(b) [Linear extrapolation] Use this equation to estimate the number of subscribers in 1990.
(c) [Linear interpolation] Estimate the number of subscribers in 1985.
(d) What information is given by the slope of the line in part (a)?

In Exercises 69–74, find the distance between the given point and line (or two lines) using the following formula for the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$.

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

69. Point: (0, 0); line: $4x + 3y = 10$
70. Point: (2, 3); line: $4x + 3y = 10$
71. Point: $(-2, 1)$; line: $x - y - 2 = 0$
72. Point: (6, 2); line: $x = -1$
73. Lines: $x + y = 1$, $x + y = 5$
74. Lines: $3x - 4y = 1$, $3x - 4y = 10$

75. Prove that the diagonals of a rhombus intersect at right angles.

76. Prove that the figure formed by connecting consecutive midpoints of the sides of any quadrilateral is a parallelogram.

77. Prove that if the points (x_1, y_1) and (x_2, y_2) lie on the same line as (x_1^*, y_1^*) and (x_2^*, y_2^*), then

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Assume $x_1 \neq x_2$ and $x_2^* \neq x_1^*$.

78. Complete the proof of Theorem 1.10. That is, prove that if the slopes of two nonvertical lines are negative reciprocals of each other, then the lines are perpendicular.

1.5 Functions

Definition of function ▪ Function notation ▪ The graph of a function ▪ Transformations of graphs ▪ Classifications of functions ▪ Combinations of functions

Many common relationships involve two variables in such a way that the value of one of the variables depends on the value of the other. For example, the sales tax on an item depends on its selling price. The distance an object moves in a given time depends on its speed.

Consider the relationship between the area of a circle and its radius. This relationship can be expressed by the equation $A = \pi r^2$, where the value of A depends on the choice of r. We refer to A as the **dependent variable** and to r as the **independent variable.**

Of particular interest are relationships such that to every value of the independent variable there corresponds *one and only one* value of the dependent variable. We call this type of correspondence a **function.**

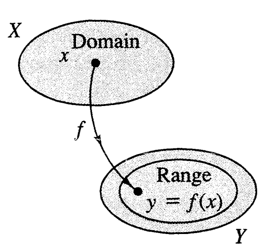


FIGURE 1.46

DEFINITION OF A FUNCTION

A **function** f from a set X into a set Y is a correspondence that assigns to each element x in X exactly one element y in Y. We call y the **image** of x under f and denote it by $f(x)$. The **domain** of f is the set X, and the **range** consists of all images of elements in X. (See Figure 1.46.)

If to each value in its range there corresponds exactly one value in its domain, the function is said to be **one-to-one.** Moreover, if the range of f consists of all of Y, then the function is said to be **onto.**

In the first twelve chapters of this text, we work with functions whose domains and ranges are sets of real numbers. We call such functions **real-valued functions of a real variable.** Other types of functions will be introduced in Chapters 13–16.

Functions can be specified in a variety of ways. We will, however, concentrate primarily on functions that are given by equations involving the dependent and independent variables. To evaluate a function described by an equation, we generally isolate the dependent variable on the left side of the equation. For instance, the equation $x + 2y = 1$, written as

$$y = \frac{1 - x}{2}$$

describes y as a function of x, and we can denote this function as

$$f(x) = \frac{1 - x}{2}.$$

This function notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time telling us that x is the independent variable and that the function itself will be called "f." The symbol $f(x)$ is read "f of x." The $f(x)$ notation also allows us to be less wordy. Instead of asking, "What is the value of y that corresponds to $x = 3$?" we can ask, "What is $f(3)$?" In general, to denote the value of the dependent variable when $x = a$, we use the symbol $f(a)$. For example, the value of f when $x = 3$ is

$$f(3) = \frac{1 - (3)}{2} = \frac{-2}{2} = -1.$$

In an equation that defines a function, the role of the variable x is simply that of a placeholder. For instance, the function given by

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\ ) = 2(\ )^2 - 4(\ ) + 1$$

where parentheses are used instead of x. Therefore, to evaluate $f(-2)$, we simply place -2 in each set of parentheses.

$$f(-2) = 2(-2)^2 - 4(-2) + 1 = 2(4) + 8 + 1 = 17$$

REMARK Although we generally use f as a convenient function name and x as the independent variable, we also can use other symbols. For instance, the following equations all define the same function.

$$f(x) = x^2 - 4x + 7 \qquad f(t) = t^2 - 4t + 7 \qquad g(s) = s^2 - 4s + 7$$

EXAMPLE 1 Evaluating a function

For the function f defined by $f(x) = x^2 + 7$, evaluate the following.

(a) $f(3a)$ (b) $f(b - 1)$ (c) $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$, $\Delta x \neq 0$

SOLUTION

(a) $f(3a) = (3a)^2 + 7$ Replace x with $3a$

$= 9a^2 + 7$

(b) $f(b - 1) = (b - 1)^2 + 7$ Replace x with $b - 1$

$= b^2 - 2b + 1 + 7$

$= b^2 - 2b + 8$

(c)
$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{[(x + \Delta x)^2 + 7] - [x^2 + 7]}{\Delta x}$$
$$= \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 7 - x^2 - 7}{\Delta x}$$
$$= \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$
$$= \frac{\Delta x(2x + \Delta x)}{\Delta x}$$
$$= 2x + \Delta x$$

REMARK The ratio in Example 1(c) is called a difference quotient and has a special significance in calculus. We will say more about this in Chapter 3.

The domain of a function may be described explicitly, or it may be described *implicitly* by an equation used to define the function. (The implied domain is the set of all real numbers for which the equation is defined.) For example, the function given by

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \le x \le 5$$

has an explicitly defined domain given by $\{x: 4 \le x \le 5\}$. On the other hand, the function given by

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain which is the set $\{x: x \neq \pm 2\}$. Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function given by

$$f(x) = \sqrt{x + 2}$$

has the implied domain $\{x: x \ge -2\}$.

EXAMPLE 2 Finding the domain and range of a function

Determine the domain and range for the function of x defined by

$$f(x) = \sqrt{x - 1}.$$

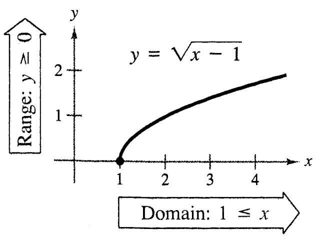


FIGURE 1.47

SOLUTION

Since $\sqrt{x-1}$ is not defined for $x - 1 < 0$ (that is, for $x < 1$), we must have $x \geq 1$. Therefore, the domain is the interval $[1, \infty)$.

To find the range, we observe that $f(x) = \sqrt{x-1}$ is never negative. Moreover, as x takes on the various values in the domain, $f(x)$ takes on all nonnegative values and we find the range to be the interval $[0, \infty)$.

The graph of the function is shown in Figure 1.47. ▭

EXAMPLE 3 A function defined by more than one equation

Determine the domain and range for the function of x given by

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1 \\ \sqrt{x-1}, & \text{if } x \geq 1. \end{cases}$$

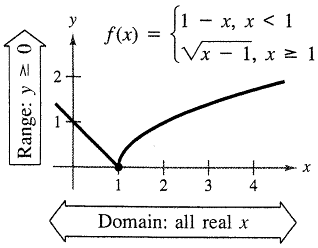


FIGURE 1.48

SOLUTION

Since f is defined for $x < 1$ and $x \geq 1$, the domain of the function is the entire set of real numbers.

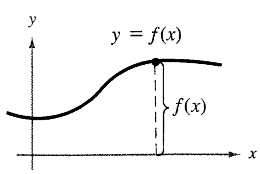
On the portion of the domain for which $x \geq 1$, the function behaves as in Example 2. For $x < 1$, the value of $1 - x$ is positive, and therefore the range of the function is the interval $[0, \infty)$. (See Figure 1.48.) ▭

REMARK Note that the function given in Example 2 is one-to-one whereas the function given in Example 3 is not one-to-one.

The graph of a function

As you study this section, remember that the graph of the function $y = f(x)$ consists of all points $(x, f(x))$ as shown in Figure 1.49, where

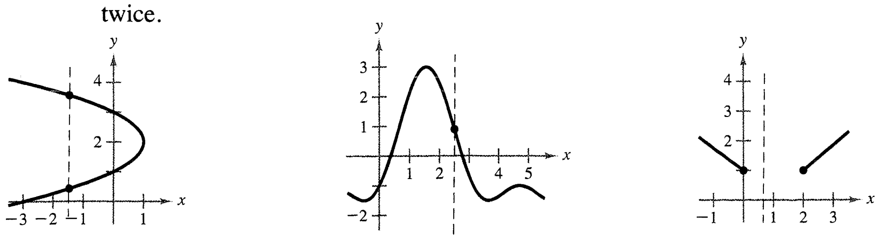
$$x = \text{the directed distance from the } y\text{-axis}$$
$$f(x) = \text{the directed distance from the } x\text{-axis}.$$



The Graph of a Function

FIGURE 1.49

Since, by the definition of a function, there is exactly one y-value for each x-value, it follows that a vertical line can intersect the graph of a function of x at most once. This observation provides us with a convenient visual test for functions. For example, in part (a) of Figure 1.50, we see that the graph does not define y as a function of x since a vertical line intersects the graph twice.



(a) Not a function of x (b) A function of x (c) A function of x

FIGURE 1.50 Vertical Line Test for Functions

Figure 1.51 shows the graphs of six basic functions. You need to know these graphs well.

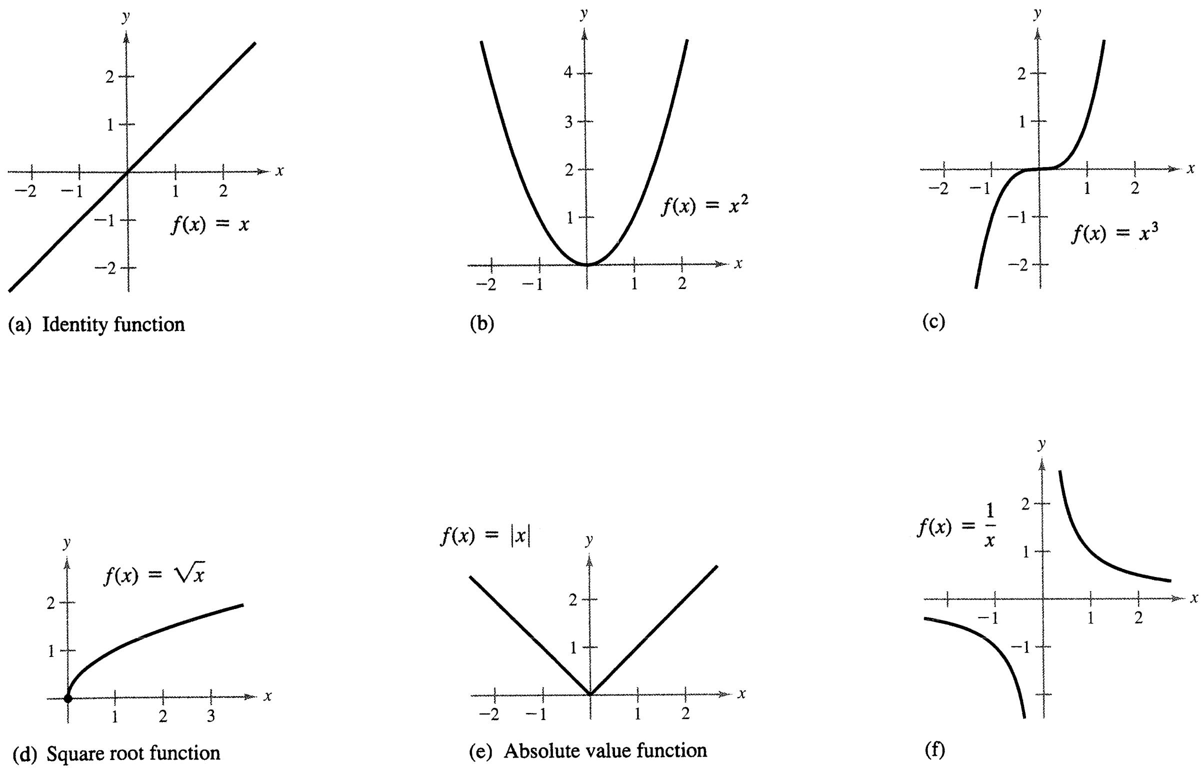
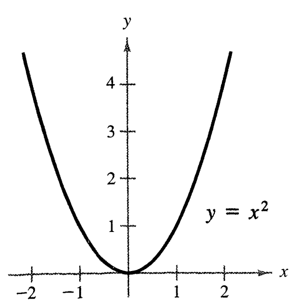


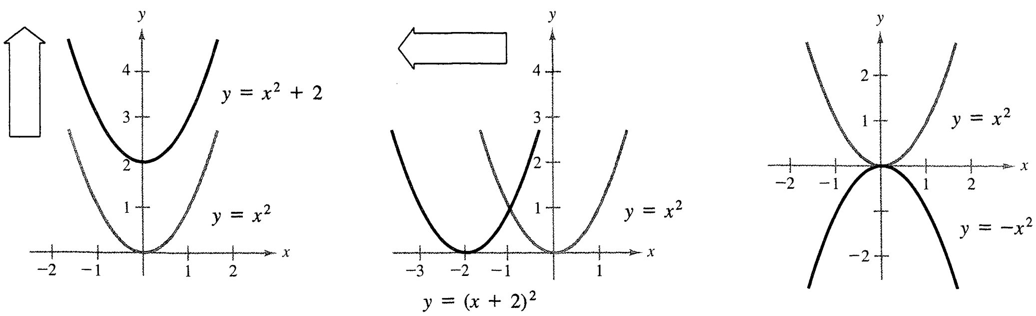
FIGURE 1.51

Function notation lends itself well to describing transformations of graphs in the plane. Some families of graphs all have the same basic shape. For example, consider the graph of $y = x^2$, as shown in Figure 1.52. Now compare this graph to those shown in Figure 1.53.



Original Graph

FIGURE 1.52



(a) Vertical shift upward (b) Horizontal shift to the left (c) Reflection

FIGURE 1.53

Each of the graphs in Figure 1.53 is a **transformation** of the graph of $y = x^2$. The three basic types of transformations illustrated by these graphs are (1) vertical shifts, (2) horizontal shifts, and (3) reflections.

BASIC TYPES OF TRANSFORMATIONS ($c > 0$)

Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$

Classifications and combinations of functions

The modern notion of a function is derived from the efforts of many seventeenth- and eighteenth-century mathematicians. Of particular note was Leonhard Euler (1707–1783), to whom we are indebted for the function notation $y = f(x)$. By the end of the eighteenth century, mathematicians and scientists had concluded that most real-world phenomena can be represented by mathematical models taken from a basic collection of functions called **elementary functions.** Elementary functions are divided into three categories: (1) algebraic, (2) trigonometric, and (3) logarithmic and exponential. We will review the trigonometric functions in Section 1.6 and introduce the remaining elementary functions in Chapter 6.

Leonhard Euler

The most common type of algebraic function is a **polynomial function**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \quad a_n \neq 0$$

where the positive integer n is the **degree** of the polynomial function. The numbers a_i are called **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function. It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree we often use the following simpler forms.

Zeroth degree:	$f(x) = a$	Constant function
First degree:	$f(x) = ax + b$	Linear function
Second degree:	$f(x) = ax^2 + bx + c$	Quadratic function
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	Cubic function

Although the graph of a polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the right or left. Whether the graph of

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

eventually rises or falls can be determined by the function's degree (odd or even) and by the leading coefficient a_n, as indicated in Figure 1.54. Note that the dashed portions of the graphs indicate that the **leading coefficient test** determines *only* the right and left behavior of the graph.

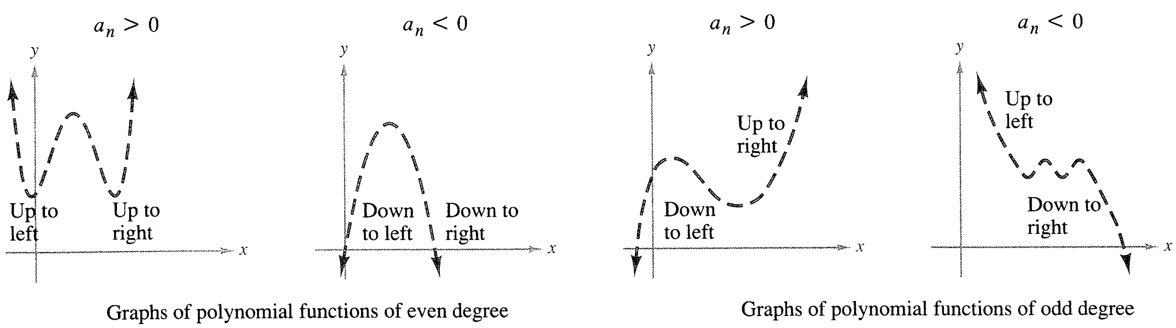


FIGURE 1.54 Leading Coefficient Test for Polynomial Functions

Just as a rational number can be written as the quotient of two integers, a **rational function** can be written as the quotient of two polynomials. Specifically, a function f is **rational** if it has the form

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

where $p(x)$ and $q(x)$ are polynomials.

Polynomial functions and rational functions are two examples of a larger class of functions called **algebraic functions.** An algebraic function is one that can be expressed as a finite number of sums, differences, multiples, quotients, and radicals involving x^n. For example, the following functions are algebraic.

$$f(x) = \sqrt{x + 1} \qquad \text{and} \qquad g(x) = x + \frac{1}{\sqrt[3]{x + 1}}$$

Functions that are not algebraic are called **transcendental.** For instance, the trigonometric functions discussed in Section 1.6 are transcendental.

Two functions can be combined in various ways to create new functions. For example, if

$$f(x) = 2x - 3 \qquad \text{and} \qquad g(x) = x^2 + 1$$

we can form the following functions.

$$f(x) + g(x) = (2x - 3) + (x^2 + 1) = x^2 + 2x - 2 \qquad \text{Sum}$$
$$f(x) - g(x) = (2x - 3) - (x^2 + 1) = -x^2 + 2x - 4 \qquad \text{Difference}$$
$$f(x)g(x) = (2x - 3)(x^2 + 1) = 2x^3 - 3x^2 + 2x - 3 \qquad \text{Product}$$
$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 + 1} \qquad \text{Quotient}$$

We can combine two functions in yet another way to form what is called a **composite function.**

DEFINITION OF COMPOSITE FUNCTION

Let f and g be functions. The function given by $(f \circ g)(x) = f(g(x))$ is called the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

It is important to realize that the composite of f with g may not be equal to the composite of g with f. This is illustrated in the following example.

EXAMPLE 4 Composition of functions

Given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, find $f \circ g$ and $g \circ f$.

SOLUTION

Since $f(x) = 2x - 3$, we have

$$(f \circ g)(x) = f(g(x)) = 2(g(x)) - 3 = 2(x^2 + 1) - 3 = 2x^2 - 1$$

and since $g(x) = x^2 + 1$, we have

$$(g \circ f)(x) = g(f(x)) = (f(x))^2 + 1 = (2x - 3)^2 + 1 = 4x^2 - 12x + 10.$$

Note that $(f \circ g)(x) \neq (g \circ f)(x)$. ▭

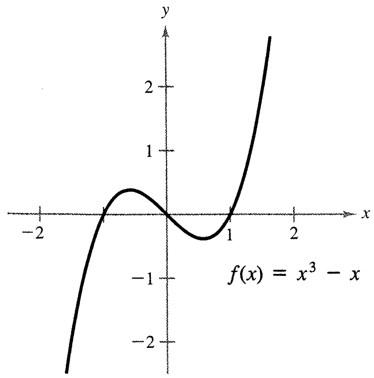
In Section 1.3, we defined an x-intercept of a graph to be a point $(a, 0)$ at which the graph crosses the x-axis. If the graph represents a function f, then the number a is called a **zero** of f. In other words, the zeros of a function f are the solutions of the equation $f(x) = 0$. For example, the function $f(x) = x - 4$ has a zero at $x = 4$ because $f(4) = 0$.

In Section 1.3 we also discussed different types of symmetry. In the terminology of functions, we say that a function is **even** if its graph is symmetric with respect to the y-axis, and a function is **odd** if its graph is symmetric with respect to the origin. Thus, the symmetry tests in Section 1.3 yield the following test for even and odd functions.

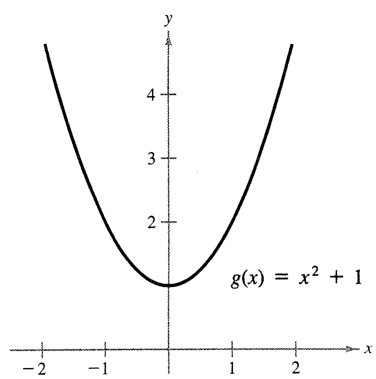
THEOREM 1.11 TEST FOR EVEN AND ODD FUNCTIONS

The function $y = f(x)$ is **even** if $f(-x) = f(x)$.
The function $y = f(x)$ is **odd** if $f(-x) = -f(x)$.

REMARK Except for such trivial cases as the constant function $f(x) = 0$, the graph of a function cannot have symmetry with respect to the x-axis because it then would fail the vertical line test for the graph of a function.



(a) Odd function



(b) Even function

FIGURE 1.55

EXAMPLE 5 Even and odd functions

Determine whether the following functions are even, odd, or neither. In each case find the zeros of the function.

(a) $f(x) = x^3 - x$ (b) $g(x) = x^2 + 1$

SOLUTION

(a) This function is odd since

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

The zeros of f are found as follows.

$$x^3 - x = 0 \qquad \text{Let } f(x) = 0$$
$$x(x^2 - 1) = x(x - 1)(x + 1) = 0 \qquad \text{Factor}$$
$$x = 0,\ 1,\ -1 \qquad \text{Zeros of } f$$

(b) This function is even since

$$g(-x) = (-x)^2 + 1 = x^2 + 1 = g(x).$$

It has no zeros since $x^2 + 1$ is positive for all x. (See Figure 1.55.)

▭

REMARK Each of the functions in Example 5 is either even or odd. However, some functions such as

$$f(x) = x^2 + x + 1$$

are neither even nor odd.

EXERCISES for Section 1.5

1. Given $f(x) = 2x - 3$, find the following.
   (a) $f(0)$ (b) $f(-3)$
   (c) $f(b)$ (d) $f(x - 1)$
2. Given $f(x) = x^2 - 2x + 2$, find the following.
   (a) $f\left(\frac{1}{2}\right)$ (b) $f(-1)$
   (c) $f(c)$ (d) $f(x + \Delta x)$
3. Given $f(x) = \sqrt{x + 3}$, find the following.
   (a) $f(-2)$ (b) $f(6)$
   (c) $f(c)$ (d) $f(x + \Delta x)$
4. Given $f(x) = 1/\sqrt{x}$, find the following.
   (a) $f(2)$ (b) $f\left(\frac{1}{4}\right)$
   (c) $f(x + \Delta x)$ (d) $f(x + \Delta x) - f(x)$
5. Given $f(x) = |x|/x$, find the following.
   (a) $f(2)$ (b) $f(-2)$
   (c) $f(x^2)$ (d) $f(x - 1)$
6. Given $f(x) = |x| + 4$, find the following.
   (a) $f(2)$ (b) $f(-2)$
   (c) $f(x^2)$ (d) $f(x + \Delta x) - f(x)$

7. Given $f(x) = x^2 - x + 1$, find

$$\frac{f(2 + \Delta x) - f(2)}{\Delta x}.$$

8. Given $f(x) = 1/x$, find

$$\frac{f(1 + \Delta x) - f(1)}{\Delta x}.$$

9. Given $f(x) = x^3$, find

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

10. Given $f(x) = 3x - 1$, find

$$\frac{f(x) - f(1)}{x - 1}.$$

11. Given $f(x) = 1/\sqrt{x - 1}$, find

$$\frac{f(x) - f(2)}{x - 2}.$$

12. Given $f(x) = x^3 - x$, find

$$\frac{f(x) - f(1)}{x - 1}.$$

In Exercises 13–22, find the domain and range of the given function, and sketch its graph.

13. $f(x) = 4 - x$ **14.** $f(x) = \frac{1}{3}x$

15. $f(x) = 4 - x^2$ **16.** $g(x) = \frac{4}{x}$

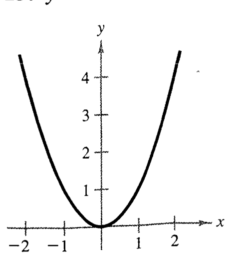
17. $h(x) = \sqrt{x - 1}$ **18.** $f(x) = \frac{1}{2}x^3 + 2$

19. $f(x) = \sqrt{9 - x^2}$ **20.** $h(x) = \sqrt{25 - x^2}$

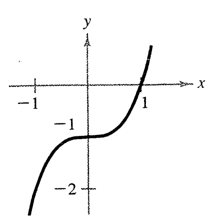
21. $f(x) = |x - 2|$ **22.** $f(x) = \frac{|x|}{x}$

In Exercises 23–28, use the vertical line test to determine whether y is a function of x.

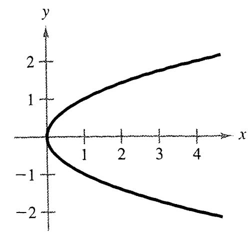
23. $y = x^2$



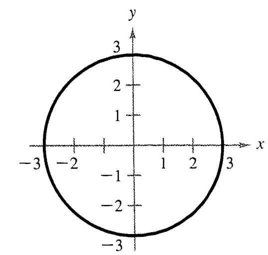
24. $y = x^3 - 1$



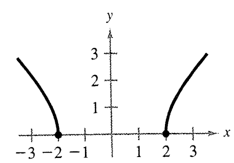
25. $x - y^2 = 0$



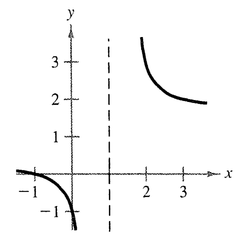
26. $x^2 + y^2 = 9$



27. $\sqrt{x^2 - 4} - y = 0$



28. $x - xy + y + 1 = 0$



In Exercises 29–36, determine whether y is a function of x.

29. $x^2 + y^2 = 4$ **30.** $x = y^2$

31. $x^2 + y = 4$ **32.** $x + y^2 = 4$

33. $2x + 3y = 4$ **34.** $x^2 + y^2 - 4y = 0$

35. $y^2 = x^2 - 1$ **36.** $x^2y - x^2 + 4y = 0$

37. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of each of the following.

(a) $y = \sqrt{x} + 2$ (b) $y = -\sqrt{x}$

(c) $y = \sqrt{x - 2}$ (d) $y = \sqrt{x + 3}$

(e) $y = \sqrt{x} - 4$ (f) $y = 2\sqrt{x}$

38. Use the graph of $f(x) = 1/x$ to sketch the graph of each of the following.

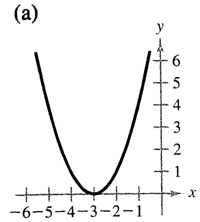
(a) $y = \frac{1}{x} - 1$ (b) $y = \frac{1}{x + 1}$

(c) $y = \frac{1}{x - 1}$ (d) $y = -\frac{1}{x}$

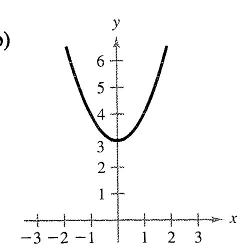
(e) $y = \frac{4}{x}$ (f) $y = -\frac{1}{x} + 2$

39. Use the graph of $f(x) = x^2$ to determine a formula for the indicated function.

(a)

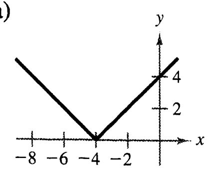


(b)

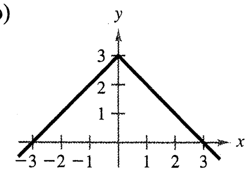


40. Use the graph of $f(x) = |x|$ to determine a formula for the indicated function.

(a)



(b)



41. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, find the following.

(a) $f(g(1))$ (b) $g(f(1))$
(c) $g(f(0))$ (d) $f(g(-4))$
(e) $f(g(x))$ (f) $g(f(x))$

42. Given $f(x) = 1/x$ and $g(x) = x^2 - 1$, find the following.

(a) $f(g(2))$ (b) $g(f(2))$
(c) $f\left(g\left(\frac{1}{\sqrt{2}}\right)\right)$ (d) $g\left(f\left(\frac{1}{\sqrt{2}}\right)\right)$
(e) $g(f(x))$ (f) $f(g(x))$

In Exercises 43–46, find the composite functions $(f \circ g)$ and $(g \circ f)$. What is the domain of each function? Are the two composite functions equal?

43. $f(x) = x^2,\quad g(x) = \sqrt{x}$

44. $f(x) = x^3,\quad g(x) = \sqrt[3]{x}$

45. $f(x) = x + 1,\quad g(x) = \dfrac{1}{x}$

46. $f(x) = x^2 - 1,\quad g(x) = x$

In Exercises 47–50, find the (real) zeros of the given function.

47. $f(x) = x^2 - 9$

48. $f(x) = x^3 - x$

49. $f(x) = \dfrac{3}{x-1} + \dfrac{4}{x-2}$

50. $f(x) = a + \dfrac{b}{x}$

In Exercises 51–54, determine whether the function is even, odd, or neither.

51. $f(x) = 4 - x^2$

52. $f(x) = \sqrt[3]{x}$

53. $f(x) = x(4 - x^2)$

54. $f(x) = 4x - x^2$

55. Show that the following function is odd.

$$f(x) = a_{2n+1}x^{2n+1} + \cdots + a_3x^3 + a_1x$$

56. Show that the following function is even.

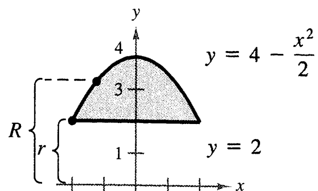
$$f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0$$

57. Show that the product of two even (or two odd) functions is even.

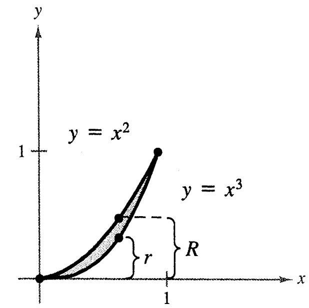
58. Show that the product of an odd function and an even function is odd.

In Exercises 59–62, express the indicated values as functions of x.

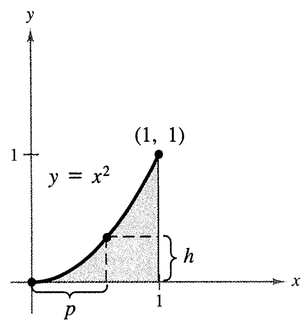
59. R and r



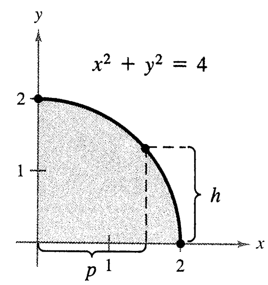
60. R and r



61. h and p



62. h and p



63. A rectangle has a perimeter of 100 feet (see figure). Express the area A of the rectangle as a function of x.

64. A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals (see figure). Express the area A of the enclosures as a function of x.

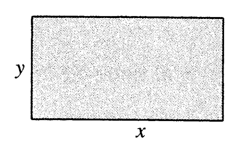


FIGURE FOR 63

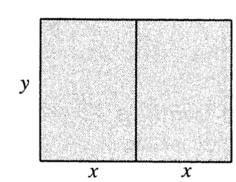


FIGURE FOR 64

65. An open box is to be made from a square piece of material 12 inches on a side, by cutting equal squares from each corner and turning up the sides (see figure). Express the volume V as a function of x.

66. A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). Write the area A of the rectangle as a function of x.

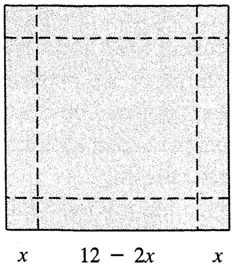


FIGURE FOR 65

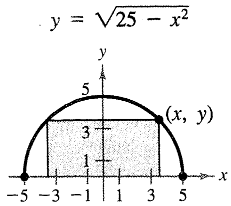


FIGURE FOR 66

67. A rectangular package with square cross sections has a combined length and girth (perimeter of a cross section) of 108 inches. Express the volume V as a function of x (see figure).

68. A closed box with a square base of side x has a surface area of 100 square feet (see figure). Express the volume V of the box as a function of x.

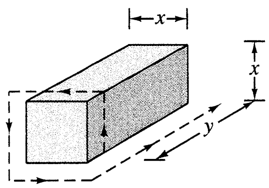


FIGURE FOR 67

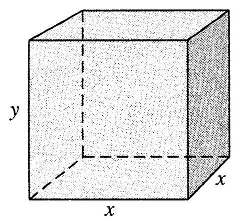


FIGURE FOR 68

69. A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, 3 miles down the coast and 1 mile inland (see figure). He can row at 2 mph and walk at 4 mph. Express the total time T of the trip as a function of x.

70. The portion of the vertical line through the point $(x, 0)$ that lies between the x-axis and the graph of $y = \sqrt{x}$ is revolved about the x-axis. Express the area A of the resulting disk as a function of x (see figure).

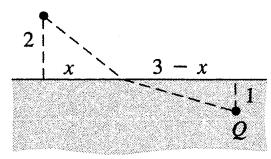


FIGURE FOR 69

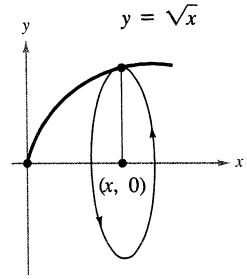


FIGURE FOR 70

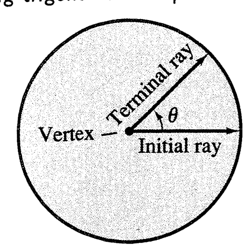
In Exercises 71 and 72, use a computer or graphics calculator to (a) sketch the graph of f, (b) find the zeros of f, and (c) determine the domain of f.

71. $f(x) = x\sqrt{9 - x^2}$

72. $f(x) = 2\left(\pi x^2 + \dfrac{6}{x}\right)$

1.6 Review of Trigonometric Functions

Angles and degree measure ▪ Radian measure ▪ The trigonometric functions ▪ Evaluation of trigonometric functions ▪ Solving trigonometric equations ▪ Graphs of trigonometric functions



Standard Position of an Angle

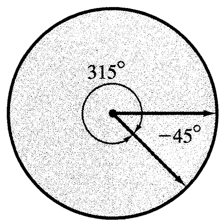
FIGURE 1.56

The concept of an angle is central to the study of trigonometry. As shown in Figure 1.56, an **angle** has three parts: an **initial ray**, a **terminal ray**, and a **vertex** (the point of intersection of the two rays). We say that an angle is in **standard position** if its initial ray coincides with the positive x-axis and its vertex is at the origin. We assume that you are familiar with the degree measure of an angle.* It is common practice to use θ (the Greek lowercase letter *theta*) to represent both an angle and its measure. We classify angles between 0° and 90° as **acute** and angles between 90° and 180° as **obtuse.** Positive angles are measured *counterclockwise*, beginning with the initial ray.

*For a more complete review of trigonometry, see *Algebra and Trigonometry*, 2nd edition, by Larson and Hostetler (Lexington, Mass., D. C. Heath and Company, 1989).

Negative angles are measured *clockwise*. For instance, Figure 1.57 shows an angle whose measure is $-45°$. We cannot assign a measure to an angle merely by knowing where its initial and terminal rays are located. To measure an angle, we must also know how the terminal ray was revolved. For example, Figure 1.57 shows that the angle measuring $-45°$ has the same terminal ray as the angle measuring $315°$. We call such angles **coterminal.**

An angle that is larger than $360°$ is one whose terminal ray has revolved more than one full revolution counterclockwise. Figure 1.58 shows an angle measuring more than $360°$. Similarly, we can generate angles whose measure is less than $-360°$ by revolving a terminal ray more than one full revolution clockwise.



Coterminal Angles

FIGURE 1.57

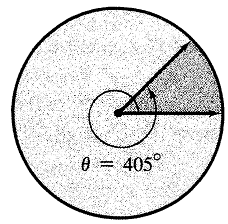


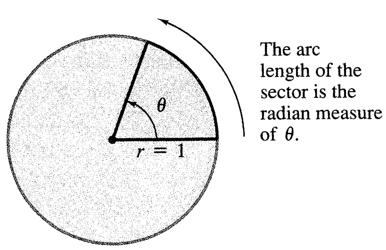
FIGURE 1.58

Radian measure

A second way to measure angles is by radian measure. To assign a radian measure to an angle θ, we consider θ to be the central angle of a circular sector of radius 1, as shown in Figure 1.59. The **radian measure** of θ is then defined to be the length of the arc of this sector. Recall that the total circumference of a circle is $2\pi r$. Thus, the circumference of a **unit circle** (that is, of radius 1) is simply 2π, and we may conclude that the radian measure of an angle measuring $360°$ is 2π. In other words, $360° = 2\pi$ radians.

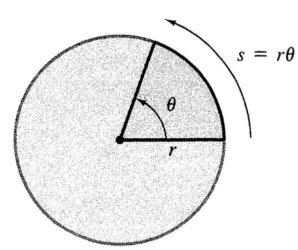
Using radian measure, we have a simple formula for the length s of a circular arc of radius r, as shown in Figure 1.60.

$$\text{arclength} = s = r\theta \qquad \theta \text{ measured in radians}$$



Unit Circle

FIGURE 1.59



Circle of Radius r

FIGURE 1.60

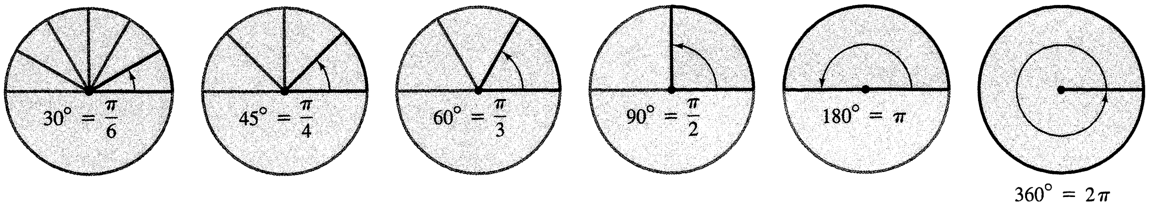


FIGURE 1.61 Radian and Degree Measure for Several Common Angles

It is helpful to memorize the conversions of the common angles pictured in Figure 1.61. For other angles, you can use one of the following conversion rules.

CONVERSION RULES

$180° = \pi$ radians

Degrees ⇨ Radians	**Radians ⇨ Degrees**
$1° = \frac{\pi}{180}$ radians	$1 \text{ radian} = \frac{180°}{\pi}$

EXAMPLE 1 Conversions between degrees and radians

(a) $40° = (40 \not{\text{deg}})\left(\frac{\pi \text{ rad}}{180 \not{\text{deg}}}\right) = \frac{2\pi}{9}$ radians

(b) $-270° = (-270 \not{\text{deg}})\left(\frac{\pi \text{ rad}}{180 \not{\text{deg}}}\right) = -\frac{3\pi}{2}$ radians

(c) $-\frac{\pi}{2}$ radians $= \left(-\frac{\pi}{2} \not{\text{rad}}\right)\left(\frac{180 \text{ deg}}{\pi \not{\text{rad}}}\right) = -90°$

(d) $\frac{9\pi}{2}$ radians $= \left(\frac{9\pi}{2} \not{\text{rad}}\right)\left(\frac{180 \text{ deg}}{\pi \not{\text{rad}}}\right) = 810°$ ▭

The trigonometric functions

There are two common approaches to the study of trigonometry. In one, the trigonometric functions are defined as ratios of two sides of a right triangle. In the other, these functions are defined in terms of a point on the terminal side of an angle in standard position. The first approach is generally used in surveying, navigation, and astronomy, where a typical problem involves a fixed triangle having three of its six parts (sides and angles) known and three to be determined. The second approach is normally used in physics, electronics, and biology, where the periodic nature of the trigonometric functions is emphasized. We define the six trigonometric functions, **sine, cosine, tangent, cotangent, secant,** and **cosecant** (abbreviated as sin, cos, etc.), from both viewpoints, as follows.

DEFINITION OF THE SIX TRIGONOMETRIC FUNCTIONS

Right triangle definitions, where $0 < \theta < \pi/2$. (Refer to Figure 1.62.)

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} \qquad \csc\theta = \frac{\text{hyp.}}{\text{opp.}}$$

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} \qquad \sec\theta = \frac{\text{hyp.}}{\text{adj.}}$$

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}} \qquad \cot\theta = \frac{\text{adj.}}{\text{opp.}}$$

Circular function definitions, where θ is any angle. (Refer to Figure 1.63.)

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

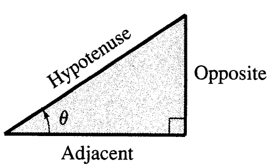


FIGURE 1.62

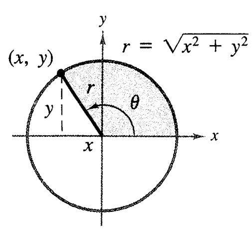


FIGURE 1.63

The following formulas are direct consequences of the definitions.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Furthermore, since

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

we can readily obtain the Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1.$$

Note that we use $\sin^2\theta$ to mean $(\sin\theta)^2$. Additional trigonometric identities are listed next. (ϕ is the Greek letter *phi*.)

TRIGONOMETRIC IDENTITIES

Pythagorean identities:

$\sin^2\theta + \cos^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$\cot^2\theta + 1 = \csc^2\theta$

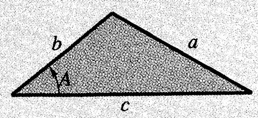
Sum or difference of two angles:

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$
$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Reduction formulas:

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$
$\sin\theta = -\sin(\theta - \pi)$
$\cos\theta = -\cos(\theta - \pi)$
$\tan\theta = \tan(\theta - \pi)$



Law of Cosines

Double angle formulas:

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
$= \cos^2\theta - \sin^2\theta$

Law of Cosines:

$a^2 = b^2 + c^2 - 2bc\cos A$

Half angle formulas:

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

REMARK All angles in the remainder of this text are measured in radians unless stated otherwise. For example, when we write sin 3, we mean the sine of three radians, and when we write sin 3°, we mean the sine of three degrees.

Evaluation of trigonometric functions

There are two common methods of evaluating trigonometric functions: (1) decimal approximations with a calculator (or a table of trigonometric values) and (2) exact evaluations using trigonometric identities and formulas from geometry. We demonstrate the second method first.

EXAMPLE 2 Evaluating trigonometric functions

Evaluate the sine, cosine, and tangent of $\pi/3$.

SOLUTION

We begin by drawing the angle $\theta = \pi/3$ in the standard position, as shown in Figure 1.64. Then, since $60° = \pi/3$ radians, we obtain an equilateral triangle with sides of length 1 and θ as one of its angles. Since the altitude of this triangle bisects its base, we know that $x = \frac{1}{2}$. Now, using the Pythagorean Theorem, we have

$$y = \sqrt{r^2 - x^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

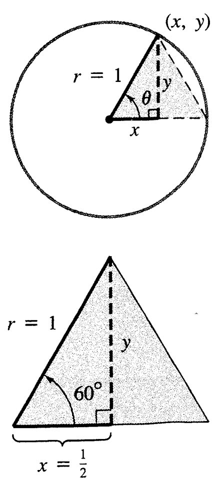


FIGURE 1.64

Thus,

$$\sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{3} = \frac{x}{r} = \frac{1/2}{1} = \frac{1}{2}$$

$$\tan\frac{\pi}{3} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

▭

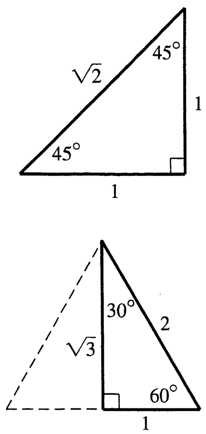


FIGURE 1.65

The degree and radian measures of several common angles are given in Table 1.6, along with the corresponding values of the sine, cosine, and tangent. (See Figure 1.65.)

TABLE 1.6 Common First Quadrant Angles

Degrees	0	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

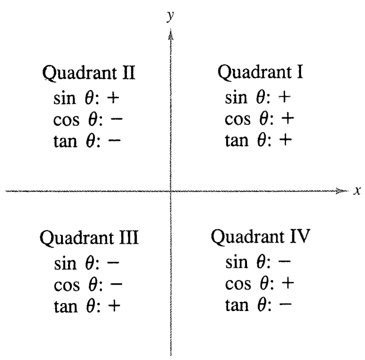


FIGURE 1.66

The quadrant signs of the various trigonometric functions are shown in Figure 1.66. To extend the use of Table 1.6 to angles in quadrants other than the first quadrant, we can use the concept of a **reference angle** (see Figure 1.67), with the appropriate quadrant sign.

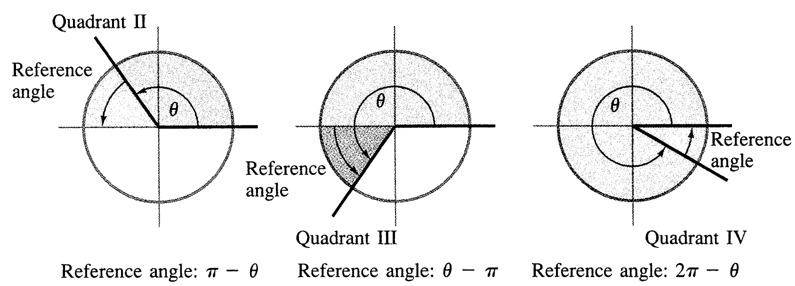


FIGURE 1.67

For instance, the reference angle for $3\pi/4$ is $\pi/4$, and since the sine is positive in the second quadrant, we can write

$$\sin\frac{3\pi}{4} = +\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Similarly, since the reference angle for 330° is 30°, and the tangent is negative in the fourth quadrant, we can write

$$\tan 330° = -\tan 30° = -\frac{\sqrt{3}}{3}.$$

EXAMPLE 3 Trigonometric identities and calculators

(a) Using the reduction formula $\sin(-\theta) = -\sin\theta$, we have

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

(b) Using the reciprocal formula $\sec\theta = 1/\cos\theta$, we have

$$\sec 60° = \frac{1}{\cos 60°} = \frac{1}{1/2} = 2.$$

(c) Using a calculator, we have

$$\cos(1.2) \approx 0.3624.$$

Remember that 1.2 is given in *radian* measure; consequently, your calculator must be set in radian mode. ▭

Solving trigonometric equations

In Examples 2 and 3, we looked at techniques for evaluating trigonometric functions for given values of θ. In the next two examples, we look at the reverse problem. That is, if we are given the value of a trigonometric function, how can we solve for θ? For example, consider the equation

$$\sin\theta = 0.$$

We know $\theta = 0$ is one solution. But this is not the only solution. Any one of the following values of θ are also solutions.

$$\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$$

We can write this infinite solution set as $\{n\pi: n \text{ is an integer}\}$.

EXAMPLE 4 Solving a trigonometric equation

Solve for θ in the following equation.

$$\sin\theta = -\frac{\sqrt{3}}{2}$$

SOLUTION

To solve the equation, we make two observations: the sine is negative in Quadrants III and IV *and* $\sin(\pi/3) = \sqrt{3}/2$. By combining these two observations, we conclude that we are seeking values of θ in the third and fourth quadrants that have a reference angle of $\pi/3$. In the interval $[0, 2\pi]$, the two angles fitting these criteria are

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \quad \text{and} \quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

Finally, we can add $2n\pi$ to either of these angles to obtain the solution set

$$\theta = \frac{4\pi}{3} + 2n\pi, \quad n \text{ is an integer}$$

or

$$\theta = \frac{5\pi}{3} + 2n\pi, \quad n \text{ is an integer.}$$ ▭

EXAMPLE 5 Solving a trigonometric equation

Solve the following equation for θ.

$$\cos 2\theta = 2 - 3 \sin \theta, \quad 0 \leq \theta \leq 2\pi$$

SOLUTION

Using the double angle identity $\cos 2\theta = 1 - 2 \sin^2 \theta$, we obtain the following polynomial (in $\sin \theta$).

$$1 - 2 \sin^2 \theta = 2 - 3 \sin \theta$$
$$0 = 2 \sin^2 \theta - 3 \sin \theta + 1$$
$$0 = (2 \sin \theta - 1)(\sin \theta - 1)$$

If $2 \sin \theta - 1 = 0$, we have $\sin \theta = 1/2$ and $\theta = \pi/6$ or $\theta = 5\pi/6$. If $\sin \theta - 1 = 0$, we have $\sin \theta = 1$ and $\theta = \pi/2$. Thus, for $0 \leq \theta \leq 2\pi$, there are three solutions to the given equation.

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \text{ or } \frac{\pi}{2}$$

Graphs of trigonometric functions

To sketch the graph of a trigonometric function in the *xy*-coordinate system, we usually use the variable x in place of θ. Moreover, when we write $y = \sin x$ or $y = \cos x$, we understand that x can have *any* real value and we evaluate the functions as if x were representing the radian measure of an angle.

REMARK This is not the same use of x as that given in the definition of the six trigonometric functions. Generally, the context of a problem will distinguish clearly between these two uses of x.

One of the first things we notice about the graphs of all six trigonometric functions is that they are periodic. We call a function f **periodic** if there exists a nonzero number p such that

$$f(x + p) = f(x)$$

for all x in the domain of f. The smallest such positive value of p is called the **period** of f. Both the sine and cosine functions have a period of 2π, and by plotting several values in the interval $0 \leq x \leq 2\pi$, we obtain the graphs shown in Figure 1.68.

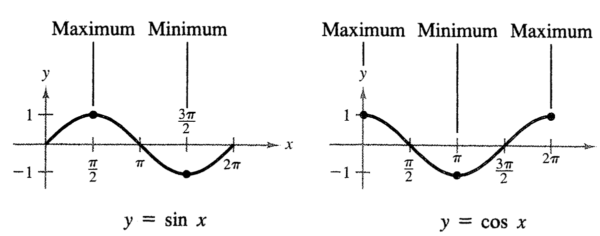


FIGURE 1.68

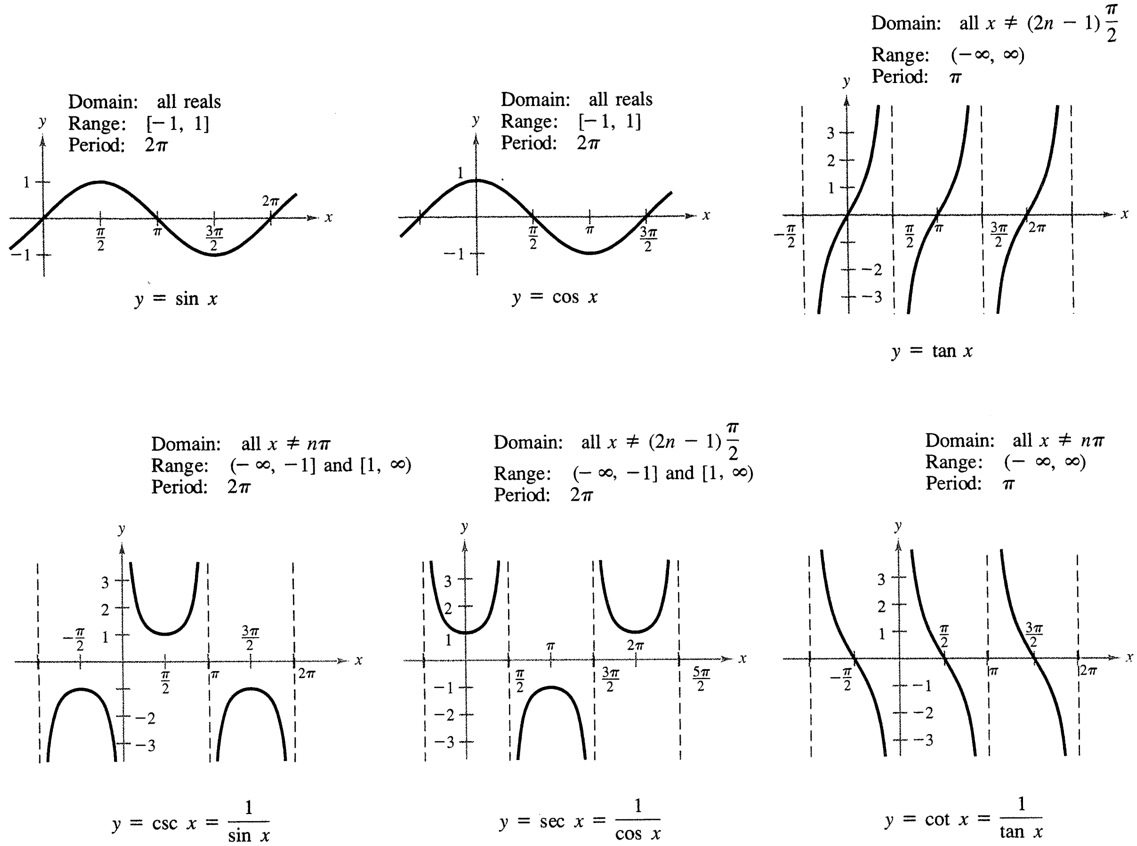


FIGURE 1.69 Graphs of the Six Trigonometric Functions

Note in Figure 1.68 that the maximum value of $\sin x$ is 1 and the minimum value is -1. Figure 1.69 shows the graphs of all six trigonometric functions. Familiarity with these six basic graphs will serve as a valuable aid in sketching the graphs of more complicated trigonometric functions.

The graph of the function $y = a \sin bx$ oscillates between $-a$ and a and hence has an **amplitude** of $|a|$. Furthermore, since $bx = 0$ when $x = 0$ and $bx = 2\pi$ when $x = 2\pi/b$, we may conclude that the function $y = a \sin bx$ has a period of $2\pi/|b|$. Table 1.7 summarizes the amplitudes and periods for some general types of trigonometric functions.

TABLE 1.7

Function	*Period*	*Amplitude*
$y = a \sin bx$ or $y = a \cos bx$	$\frac{2\pi}{\lvert b\rvert}$	$\lvert a\rvert$
$y = a \tan bx$ or $y = a \cot bx$	$\frac{\pi}{\lvert b\rvert}$	not applicable
$y = a \sec bx$ or $y = a \csc bx$	$\frac{2\pi}{\lvert b\rvert}$	not applicable

EXAMPLE 6 Sketching the graph of a trigonometric function

Sketch the graph of $f(x) = 3 \cos 2x$.

SOLUTION

The graph of $f(x) = 3 \cos 2x$ has the following characteristics.

amplitude: 3 period: $\frac{2\pi}{2} = \pi$

Using the basic shape of the graph of the cosine function, we sketch one period of the function on the interval $[0, \pi]$, using the following pattern.

maximum: $(0, 3)$ minimum: $\left(\frac{\pi}{2}, -3\right)$ maximum: $(\pi, 3)$

Then, by continuing this pattern, we sketch several cycles of the graph as shown in Figure 1.70.

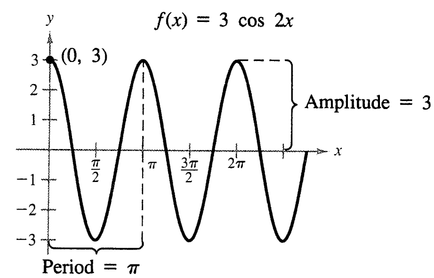


FIGURE 1.70

The discussion of horizontal shifts, vertical shifts, and reflections given in the previous section can be applied to the graphs of trigonometric functions. For instance, Figure 1.71 shows three different shifted graphs of sine functions.

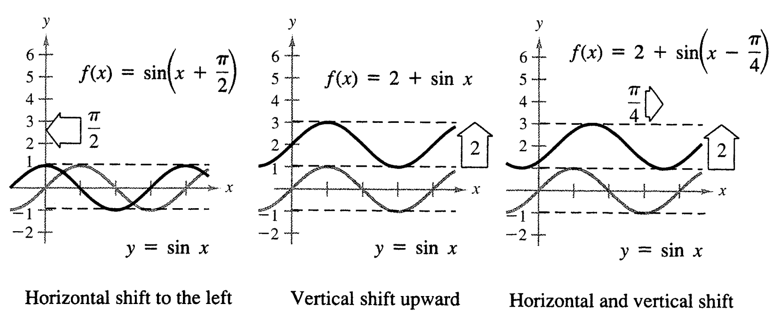
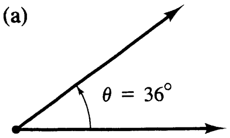


FIGURE 1.71

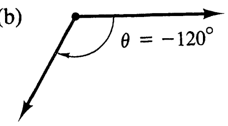
EXERCISES for Section 1.6

In Exercises 1 and 2, determine two coterminal angles (one positive and one negative) for the given angle. Give your answers in degrees.

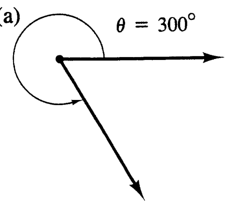
1. (a)



(b)



2. (a)



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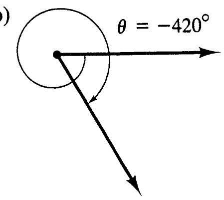
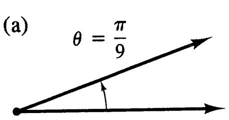
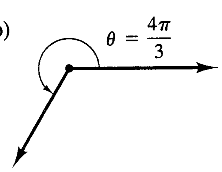


In Exercises 3 and 4, determine two coterminal angles (one positive and one negative) for the given angle. Give your answers in radians.

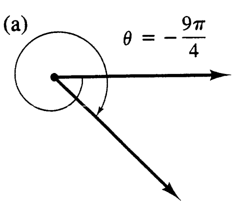
3. (a)



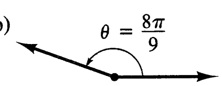
(b)



4. (a)



(b)



In Exercises 5 and 6, express the given angle in radian measure as a multiple of π.

5. (a) 30° (b) 150°
(c) 315° (d) 120°

6. (a) −20° (b) −240°
(c) −270° (d) 144°

In Exercises 7 and 8, express the given angle in degree measure.

7. (a) $\frac{3\pi}{2}$ (b) $\frac{7\pi}{6}$
(c) $-\frac{7\pi}{12}$ (d) $\frac{\pi}{9}$

8. (a) $\frac{7\pi}{3}$ (b) $-\frac{11\pi}{30}$
(c) $\frac{11\pi}{6}$ (d) $\frac{34\pi}{15}$

9. Let r represent the radius of a circle, θ the central angle (measured in radians), and s the length of the arc subtended by the angle. Use the relationship $\theta = s/r$ to complete the following table.

r	8 ft	15 in	85 cm		
s	12 ft			96 in	8642 mi
θ		1.6	$\frac{3\pi}{4}$	4	$\frac{2\pi}{3}$

10. The minute hand on a clock is $3\frac{1}{2}$ inches long (see figure). Through what distance does the tip of the minute hand move in 25 minutes?

11. A man bends his elbow through 75°. The distance from his elbow to the tip of his index finger is $18\frac{3}{4}$ inches (see figure).
(a) Find the radian measure of this angle.
(b) Find the distance the tip of the index finger moves.

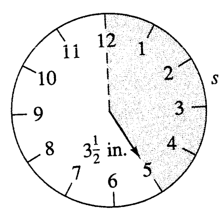


FIGURE FOR 10

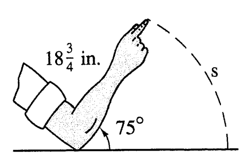
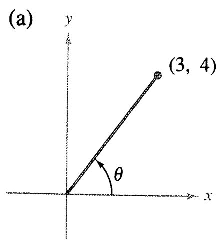


FIGURE FOR 11

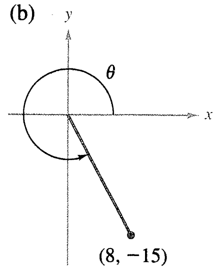
12. A tractor tire 5 feet in diameter is partially filled with a liquid ballast for additional traction. To check the air pressure, the tractor operator rotates the tire until the valve stem is at the top so that the liquid will not enter the gauge. On a given occasion, the operator notes that the tire must be rotated 80° to have the stem in the proper position.
(a) Find the radian measure of this rotation.
(b) How far must the tractor be moved to get the valve stem in the proper position?

In Exercises 13 and 14, determine all six trigonometric functions for the given angle θ.

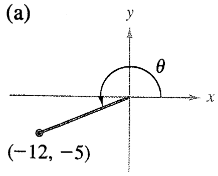
13. (a)



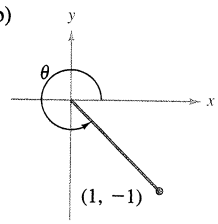
(b)



14. (a)



(b)

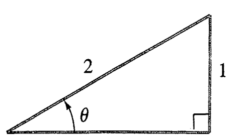


In Exercises 15 and 16, determine the quadrant in which θ lies.

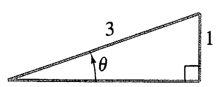
15. $\sin\theta < 0$ and $\cos\theta < 0$
16. $\sin\theta > 0$ and $\cos\theta < 0$

In Exercises 17–22, find the indicated trigonometric function from the given one. (Assume $0 < \theta < \pi/2$.)

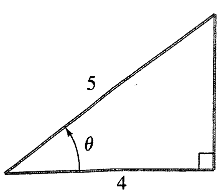
17. Given $\sin\theta = \frac{1}{2}$, find $\csc\theta$.



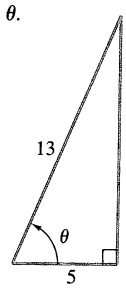
18. Given $\sin\theta = \frac{1}{3}$, find $\tan\theta$.



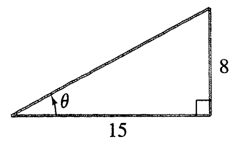
19. Given $\cos\theta = \frac{4}{5}$, find $\cot\theta$.



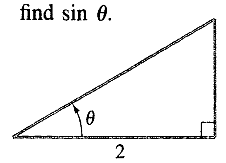
20. Given $\sec\theta = \frac{13}{5}$, find $\cot\theta$.



21. Given $\cot\theta = \frac{15}{8}$, find $\sec\theta$.



22. Given $\tan\theta = \frac{1}{2}$, find $\sin\theta$.



In Exercises 23–26, evaluate the sine, cosine, and tangent of the given angle *without* using a calculator.

23. (a) $60°$ (b) $\frac{2\pi}{3}$
(c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{4}$
24. (a) $-\frac{\pi}{6}$ (b) $150°$
(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$
25. (a) $225°$ (b) $-225°$
(c) $300°$ (d) $330°$
26. (a) $750°$ (b) $510°$
(c) $\frac{10\pi}{3}$ (d) $\frac{17\pi}{3}$

In Exercises 27–30, use a calculator to evaluate the given trigonometric function to four significant digits.

27. (a) $\sin 10°$ (b) $\csc 10°$
28. (a) $\sec 225°$ (b) $\sec 135°$
29. (a) $\tan\frac{\pi}{9}$ (b) $\tan\frac{10\pi}{9}$
30. (a) $\cot(1.35)$ (b) $\tan(1.35)$

In Exercises 31–34, find two values of θ corresponding to the given function. List the measure of θ in radians ($0 \le \theta < 2\pi$). Do not use a calculator.

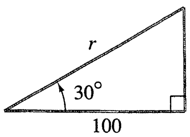
31. (a) $\cos\theta = \frac{\sqrt{2}}{2}$ (b) $\cos\theta = -\frac{\sqrt{2}}{2}$
32. (a) $\sec\theta = 2$ (b) $\sec\theta = -2$
33. (a) $\tan\theta = 1$ (b) $\cot\theta = -\sqrt{3}$
34. (a) $\sin\theta = \frac{\sqrt{3}}{2}$ (b) $\sin\theta = -\frac{\sqrt{3}}{2}$

In Exercises 35–42, solve the given equation for θ ($0 \le \theta < 2\pi$). For some of the equations, you should use the trigonometric identities listed in this section.

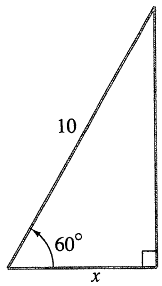
35. $2\sin^2\theta = 1$
36. $\tan^2\theta = 3$
37. $\tan^2\theta - \tan\theta = 0$
38. $2\cos^2\theta - \cos\theta = 1$
39. $\sec\theta\csc\theta = 2\csc\theta$
40. $\sin\theta = \cos\theta$
41. $\cos^2\theta + \sin\theta = 1$
42. $\cos(\theta/2) - \cos\theta = 1$

In Exercises 43–46, solve for x, y, or r as indicated.

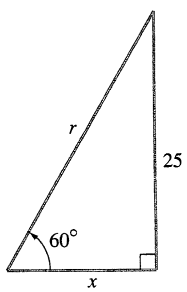
43. Solve for y.



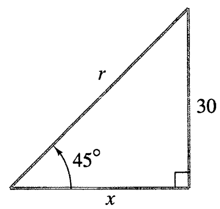
44. Solve for x.



45. Solve for x.



46. Solve for r.



47. A 20-foot ladder leaning against the side of a house makes a 75° angle with the ground (see figure). How far up the side of the house does the ladder reach?

48. A biologist wants to know the width w of a river in order to set instruments properly to study the pollutants in the water. From point A, the biologist walks downstream 100 feet and sights point C to determine that $\theta = 50°$ (see figure). How wide is the river?

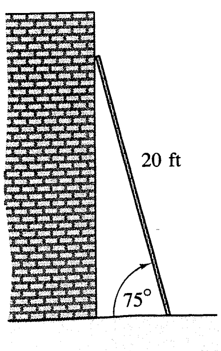


FIGURE FOR 47

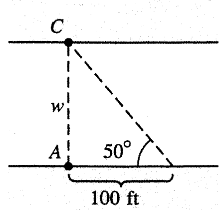


FIGURE FOR 48

49. From a 150-foot observation tower on the coast, a Coast Guard officer sights a boat in difficulty. The angle of depression of the boat is 4° (see figure). How far is the boat from the shoreline?

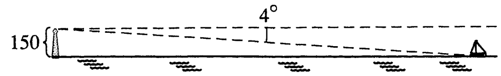


FIGURE FOR 49

50. A ramp $17\frac{1}{2}$ feet long rises to a loading platform that is $3\frac{1}{2}$ feet off the ground (see figure). Find the angle that the ramp makes with the ground.

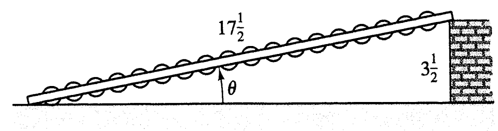
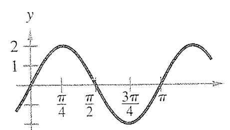
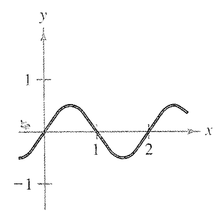


FIGURE FOR 50

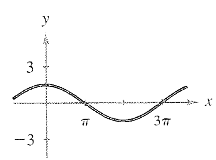
In Exercises 51–56, determine the period and amplitude of the given function.

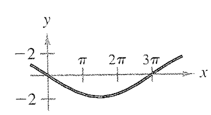
51. (a) $y = 2 \sin 2x$ (b) $y = \frac{1}{2} \sin \pi x$





52. (a) $y = \frac{3}{2} \cos \frac{x}{2}$ (b) $y = -2 \sin \frac{x}{3}$





53. $y = -2 \sin 10x$

54. $y = \frac{1}{2} \cos \frac{2x}{3}$

55. $y = 3 \sin 4\pi x$

56. $y = \frac{2}{3} \cos \frac{\pi x}{10}$

In Exercises 57–60, find the period of the given function.

57. $y = 5 \tan 2x$

58. $y = 7 \tan 2\pi x$

59. $y = \sec 5x$

60. $y = \csc 4x$

In Exercises 61–74, sketch the graph of the given function.

61. $y = \sin \frac{x}{2}$

62. $y = 2 \cos 2x$

63. $y = -2 \sin 6x$

64. $y = \cos 2\pi x$

65. $y = -\sin \frac{2\pi x}{3}$

66. $y = 2 \tan x$

67. $y = \csc \frac{x}{2}$

68. $y = \tan 2x$

69. $y = 2 \sec 2x$

70. $y = \csc 2\pi x$

71. $y = \sin (x + \pi)$

72. $y = \cos \left(x - \frac{\pi}{3}\right)$

73. $y = 1 + \cos \left(x - \frac{\pi}{2}\right)$

74. $y = 1 + \sin \left(x + \frac{\pi}{2}\right)$

75. For a person at rest, the rate of air intake v (in liters per second) during a respiratory cycle is

$$v = 0.85 \sin \frac{\pi t}{3}$$

where t is the time in seconds. Inhalation occurs when $v > 0$, and exhalation occurs when $v < 0$.
(a) Find the time for one full respiratory cycle.
(b) Find the number of cycles per minute.
(c) Sketch the graph of v as a function of t.

76. In the application at the beginning of this chapter, we developed the model

$$h = 50 + 50 \sin 8\pi t$$

for the height (in feet) of a Ferris wheel car, where t is measured in minutes. (The Ferris wheel has a radius of 50 feet.) This model yields a height of 50 feet when $t = 0$. Alter the model so that the height of the car is 0 feet when $t = 0$.

77. When tuning a piano, a technician strikes a tuning fork for the A above middle C, which creates a sound (a type of wave motion) that can be approximated by

$$y = 0.001 \sin 880\pi t$$

where y is measured in inches and t is the time in seconds.
(a) What is the period p of this function?
(b) What is the frequency f of this note ($f = 1/p$)?
(c) Sketch the graph of this function.

78. The function

$$P = 100 - 20 \cos \frac{5\pi t}{3}$$

approximates the blood pressure P in millimeters of mercury at time t in seconds for a person at rest.
(a) Find the period of the function.
(b) Find the number of heartbeats per minute.
(c) Sketch the graph of the pressure function.

In Exercises 79 and 80, use a computer or graphics calculator to sketch the graph of the given functions on the same coordinate axes where x is in the interval $[0, 2]$.

79. (a) $y = \frac{4}{\pi} \sin \pi x$

(b) $y = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3} \sin 3\pi x\right)$

80. (a) $y = \frac{1}{2} - \frac{4}{\pi^2} \cos \pi x$

(b) $y = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9} \cos 3\pi x\right)$

81. Use a computer or graphics calculator to sketch the functions $f(x) = x \sin x$ and $g(x) = x \cos x + \sin x$ on the same coordinate axes where x is in the interval $[0, \pi]$. The zero of $g(x)$ corresponds to what point on the graph of $f(x)$?

82. Sales S, in thousands of units, of a seasonal product is given by

$$S = 58.3 + 32.5 \cos \frac{\pi t}{6}$$

where t is the time in months (with $t = 1$ corresponding to January and $t = 12$ corresponding to December). Use a computer or graphics calculator to sketch the graph of S and determine the months when sales exceed 75,000 units.

REVIEW EXERCISES for Chapter 1

In Exercises 1–4, sketch the interval(s) defined by the given inequality.

1. $|x - 2| \le 3$
2. $|3x - 2| \le 0$
3. $4 < (x + 3)^2$
4. $\dfrac{1}{|x|} < 1$

In Exercises 5 and 6, find the midpoint of the given interval.

5. $\left[\dfrac{7}{8}, \dfrac{10}{4}\right]$
6. $\left[-1, \dfrac{3}{2}\right]$

7. Find the midpoints of the sides of the triangle whose vertices are (1, 4) (−3, 2), and (5, 0).
8. Find the vertices of the triangle whose sides have midpoints (0, 2), (1, −1), and (2, 1).

In Exercises 9–12, determine the radius and center of the given circle and sketch its graph.

9. $x^2 + y^2 + 6x - 2y + 1 = 0$
10. $4x^2 + 4y^2 - 4x + 8y = 11$
11. $x^2 + y^2 + 6x - 2y + 10 = 0$
12. $x^2 - 6x + y^2 + 8y = 0$

13. Determine the value of c so that the given circle has a radius of 2.

$$x^2 - 6x + y^2 + 8y = c$$

14. Find an equation in x and y such that the distance between (x, y) and $(-2, 0)$ is twice the distance between (x, y) and $(3, 1)$.
15. Find an equation for the circle whose center is (1, 2) and whose radius is 3. Then determine whether the following points are inside, outside, or on the circle.
    (a) (1, 5) (b) (0, 0)
    (c) (−2, 1) (d) (0, 4)
16. Find an equation for the circle whose center is (2, 1) and whose radius is 2. Then determine whether the following points are inside, outside, or on the circle.
    (a) (1, 1) (b) (4, 2)
    (c) (0, 1) (d) (3, 1)

In Exercises 17–20, sketch the graph of the given equation.

17. $y = \dfrac{-x + 3}{2}$
18. $y = 1 + \dfrac{1}{x}$
19. $y = 7 - 6x - x^2$
20. $y = 6x - x^2$

In Exercises 21 and 22, determine whether the given points lie on the same straight line.

21. (−1, 3), (2, 9), (3, 1)
22. (2, 5), (4, 10), (6, 20)

In Exercises 23–26, use the slope and y-intercept to sketch the graph of the given line.

23. $4x - 2y = 6$
24. $0.02x + 0.15y = 0.25$
25. $-\dfrac{1}{3}x + \dfrac{5}{6}y = 1$
26. $51x + 17y = 102$

27. Find equations of the lines passing through (−2, 4) and having the following characteristics.
    (a) Slope of $\frac{7}{16}$
    (b) Parallel to the line $5x - 3y = 3$
    (c) Passing through the origin
    (d) Parallel to the y-axis
28. Find equations of the lines passing through (1, 3) and having the following characteristics.
    (a) Slope of $-\frac{2}{3}$
    (b) Perpendicular to the line $x + y = 0$
    (c) Passing through the point (2, 4)
    (d) Parallel to the x-axis
29. The midpoint of a line segment is (−1, 4). If one end of the line segment is (2, 3), find the other end.
30. Find the point that is equidistant from (0, 0), (2, 3), and (3, −2).

In Exercises 31 and 32, find the point(s) of intersection of the graphs of the given equations.

31. $3x - 4y = 8,\ x + y = 5$
32. $x - y + 1 = 0,\ y - x^2 = 7$

In Exercises 33–38, find a formula for the given function and find the domain.

33. The value v of a farm at \$850 per acre, with buildings, livestock, and equipment worth \$300,000, is a function of the number of acres a.
34. The value v of wheat at \$3.25 per bushel is a function of the number of bushels b.
35. The surface area s of a cube is a function of the length of an edge x.
36. The surface area s of a sphere is a function of the radius r.
37. The distance d traveled by a car at a speed of 45 miles per hour is a function of the time traveled t.

38. The area a of an equilateral triangle is a function of the length of one of its sides x.

39. The sum of two positive numbers is 500. Let one of the numbers be x, and express the product P of the two numbers as a function of x.

40. The product of two positive numbers is 120. Let one of the numbers be x, and express the sum of the two numbers as a function of x.

In Exercises 41–46, sketch the graph of the given equation and use the vertical line test to determine whether the equation expresses y as a function of x.

41. $x^2 - y = 0$	**42.** $x^2 + 4y^2 = 16$
43. $x - y^2 = 0$	**44.** $x^3 - y^2 + 1 = 0$
45. $y = x^2 - 2x$	**46.** $y = 36 - x^2$

47. Given $f(x) = 1 - x^2$ and $g(x) = 2x + 1$, find the following.

(a) $f(x) + g(x)$ (b) $f(x) - g(x)$

(c) $f(x)g(x)$ (d) $\dfrac{f(x)}{g(x)}$

(e) $f(g(x))$ (f) $g(f(x))$

48. Given $f(x) = 2x - 3$ and $g(x) = \sqrt{x + 1}$, find the following.

(a) $f(x) + g(x)$ (b) $f(x) - g(x)$

(c) $f(x)g(x)$ (d) $\dfrac{f(x)}{g(x)}$

(e) $f(g(x))$ (f) $g(f(x))$

49. Consider a plane flying at a constant rate on a direct route between two cities. The distance s (in miles) it has traveled in t hours is given by $s = 560t$.
(a) Sketch the graph of this equation for $t \geq 0$.
(b) What information is given by the slope of the line?

50. Find an equation of the line that bisects the acute angle formed by the lines $y = \sqrt{3}x$ and $y = 2$.

51. Sales representatives for a certain company are required to use their own cars for transportation. The cost to the company is \$150 per day for lodging and meals, plus \$0.30 per mile driven. Write a linear equation expressing the daily cost C to the company in terms of x, the number of miles driven.

52. A contractor purchases a piece of equipment for \$36,500 that uses an average of \$9.25 per hour for fuel and maintenance. The equipment operator is paid \$13.50 per hour, and customers are charged \$30 per hour.
(a) Write an equation for the cost C of operating this equipment t hours.
(b) Write an equation for the revenue R derived from t hours of use.
(c) Find the break-even point for this equipment by finding the time at which $R = C$.

In Exercises 53 and 54, use a calculator to evaluate the given trigonometric function to four significant digits.

53. (a) $\tan 240°$ (b) $\cot 210°$
(c) $\sin(-0.65)$ (d) $\sec 3.1$

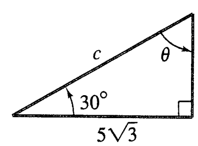
54. (a) $\sin 5.63$ (b) $\csc 2.62$
(c) $\csc 150°$ (d) $\cos(-110°)$

In Exercises 55–58, find two values of θ corresponding to the given function. List θ in degrees $(0 \leq \theta < 360°)$ and radians $(0 \leq \theta < 2\pi)$.

55. $\sin\theta = -\dfrac{1}{2}$	**56.** $\csc\theta = \sqrt{2}$
57. $\cos\theta = -\dfrac{\sqrt{3}}{2}$	**58.** $\tan\theta = -\dfrac{1}{\sqrt{3}}$

In Exercises 59–64, solve the given triangle for the indicated side and/or angle.

59.



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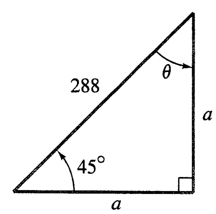

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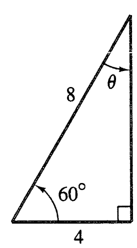

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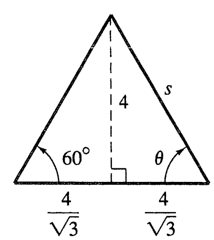

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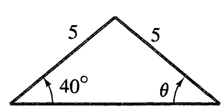

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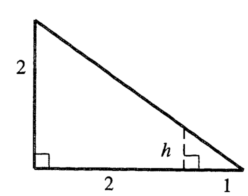


65. A 6-foot person standing 12 feet from a streetlight casts an 8-foot shadow (see figure). What is the height of the streetlight?

66. A guy wire is stretched from a broadcasting tower at a point 200 feet above the ground to an anchor 125 feet from the base of the tower (see figure). How long is the wire?

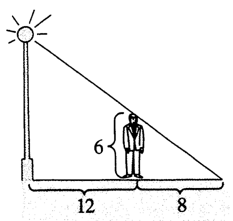


FIGURE FOR 65

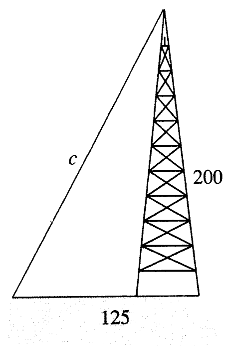


FIGURE FOR 66

In Exercises 67–74, sketch a graph showing two periods for the given function.

67. $f(x) = 2\sin\dfrac{2x}{3}$

68. $f(x) = \dfrac{1}{2}\cos\dfrac{x}{3}$

69. $f(x) = \cos\left(2x - \dfrac{\pi}{3}\right)$

70. $f(x) = -\sin\left(2x + \dfrac{\pi}{2}\right)$

71. $f(x) = \tan\dfrac{x}{2}$

72. $f(x) = \csc 2x$

73. $f(x) = \sec\left(x - \dfrac{\pi}{4}\right)$

74. $f(x) = \cot 3x$

75. The monthly sales S in thousands of units of a seasonal product are approximated by

$$S = 74.50 + 43.75\sin\frac{\pi t}{6}$$

where t is the time in months with $t = 1$ corresponding to January. Sketch the graph of this sales function over one year.

76. A company that produces a seasonal product forecasts monthly sales over the next two years to be

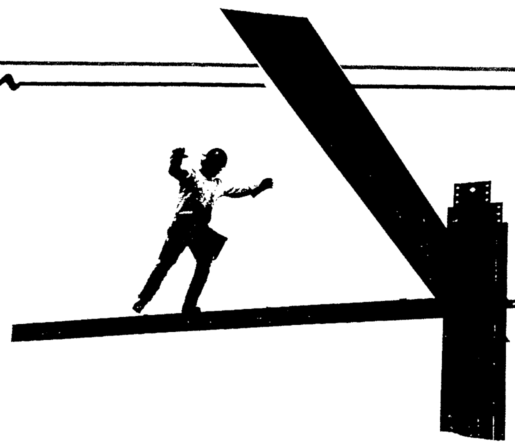
$$S = 23.1 + 0.442t + 4.3\sin\frac{\pi t}{6}$$

where S is measured in thousands of units and t is the time in months with $t = 1$ representing January 1989. Predict the sales for the following months.

(a) February 1989 (b) February 1990
(c) September 1989 (d) September 1990

Chapter 2 Application

***On construction sites,* otherwise harmless objects such as metal bolts can be life-threatening hazards. If a metal bolt is dropped from the hundredth floor of a skyscraper (approximately 1000 feet above ground), its velocity at ground level will be about 250 feet/second.**



Velocity of a Free-Falling Object

A classic problem in calculus concerns the velocity of a free-falling object. For instance, consider an object that is dropped from a height of 25 feet above the earth's surface. We let $s(t)$ represent the height (in feet) of the object at time t (measured in seconds). Assuming that the only force acting on the object is that due to gravity, and neglecting air resistance, the height is given by the position function

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$

where $g = -32$ ft/sec^2 is the acceleration due to gravity, $v_0 = 0$ is the initial velocity, and $s_0 = 25$ is the initial height. Thus, for this object, the position function is

$$s(t) = -16t^2 + 25.$$

Suppose we want to determine the velocity of the object at time $t = 1$ second. When $t = 1$, the height is $s(1) = 9$ feet and its *average* velocity during the time interval $[0, 1]$ is $(9 - 25)/(1 - 0) = -16$ ft/sec. Similarly, the average velocity during the time interval $[0.5, 1]$ is

$$\frac{s(1) - s(0.5)}{1 - 0.5} = \frac{9 - 21}{0.5} = -24 \text{ ft/sec.}$$

In general, the average velocity during the time interval $[t, 1]$ is $[s(1) - s(t)]/(1 - t)$. To find the *instantaneous* velocity when $t = 1$, we let t approach 1 and evaluate the resulting limit.

$$\begin{aligned} v = \lim_{t\to 1}\frac{s(1) - s(t)}{1 - t} &= \lim_{t\to 1}\frac{9 - (-16t^2 + 25)}{1 - t} \\ &= \lim_{t\to 1}\frac{16(t^2 - 1)}{1 - t} \\ &= \lim_{t\to 1}\frac{16(t + 1)(t - 1)}{-(t - 1)} \\ &= \lim_{t\to 1} -16(t + 1) \\ &= -32 \text{ ft/sec} \end{aligned}$$

See Exercises 45 and 46, Section 2.3.

Chapter Overview

The concept of the **limit** of a function is the primary idea that distinguishes calculus from algebra and analytic geometry. Section 2.1 begins with a brief discussion of the way a limit will be used later (in Chapter 3) to solve the *tangent line problem*. The section then gives an informal description of the idea of the limit

$$\lim_{x\to a} f(x) = L.$$

This is followed by a theoretical definition—the so-called "ε-δ definition."

Section 2.2 discusses properties of limits. In this section it is important that you become familiar with several types of functions whose limits are easily found. For instance, the limit of $f(x) = x^2$ as x approaches 2 is simply $f(2) = 4$. Then, in Section 2.3, we use the properties discussed in Section 2.2 to find limits that are not so straightforward.

Section 2.4 introduces the notion of continuity. Informally, when we say that a function f is *continuous* on an interval (a, b), we mean that the graph of f has no holes, gaps, or jumps on the interval.

The last section in the chapter discusses infinite limits and vertical asymptotes. (Limits at infinity and horizontal asymptotes are discussed later in the text, in Section 4.5.)

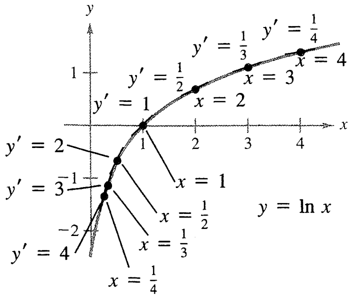


FIGURE 6.3

PROOF The domain of $f(x) = \ln x$ is $(0, \infty)$ by definition. Moreover, the function is continuous since it is differentiable. It is increasing since its derivative $f'(x) = 1/x$ is positive for $x > 0$, as shown in Figure 6.3. It is concave downward since $f''(x) = -1/x^2$ is negative. We leave the proof that f is one-to-one as an exercise (see Exercise 80). Since f is continuous, we can see by verifying the following limits that its range is the entire real line.

$$\lim_{x\to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty$$

Verification of these two limits is given in Appendix A.

By using the definition of the natural logarithmic function, we are able to prove some important properties involving operations with natural logarithms. If you are already familiar with logarithms, you will recognize that these properties are characteristic of all logarithms.

THEOREM 6.2
LOGARITHMIC PROPERTIES

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

PROOF We have already discussed the first property. The proof of the second property follows from the fact that two antiderivatives of the same function differ at most by a constant. From the Second Fundamental Theorem of Calculus and the definition of the natural logarithmic function, we know that

$$\frac{d}{dx}[\ln x] = \frac{1}{x}.$$

Thus, we consider the two derivatives

$$\frac{d}{dx}[\ln(ax)] = \frac{a}{ax} = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}[\ln a + \ln x] = 0 + \frac{1}{x} = \frac{1}{x}.$$

Since $\ln(ax)$ and $(\ln a + \ln x)$ are both antiderivatives of $1/x$, they must differ at most by a constant

$$\ln(ax) = \ln a + \ln x + C.$$

By letting $x = 1$, we see that $C = 0$. The third property can be proved similarly by comparing the derivatives of $\ln(x^n)$ and $n \ln x$. Finally, using the second and third properties, we prove the fourth property:

$$\ln\left(\frac{a}{b}\right) = \ln[a(b^{-1})] = \ln a + \ln(b^{-1}) = \ln a - \ln b.$$

EXAMPLE 1 Expanding logarithmic expressions

(a) $\ln \frac{10}{9} = \ln 10 - \ln 9$ — Property 4

(b) $\ln \sqrt{3x + 2} = \ln (3x + 2)^{1/2} = \frac{1}{2} \ln (3x + 2)$ — Property 3

(c) $\ln \frac{6x}{5} = \ln (6x) - \ln 5 = \ln 6 + \ln x - \ln 5$ — Properties 2 and 4

(d)
$$\begin{aligned}\ln \frac{(x^2 + 3)^2}{x\sqrt[3]{x^2 + 1}} &= \ln (x^2 + 3)^2 - \ln (x\sqrt[3]{x^2 + 1}) \\ &= 2 \ln (x^2 + 3) - [\ln x + \ln (x^2 + 1)^{1/3}] \\ &= 2 \ln (x^2 + 3) - \ln x - \ln (x^2 + 1)^{1/3} \\ &= 2 \ln (x^2 + 3) - \ln x - \frac{1}{3} \ln (x^2 + 1)\end{aligned}$$

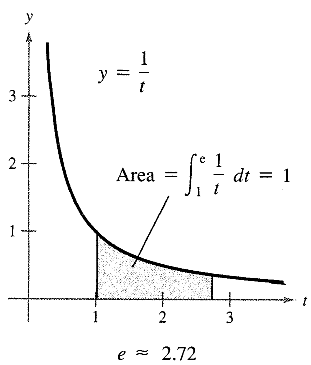


FIGURE 6.4

The number e

It is likely that you have previously studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a **base** number. For example, common logarithms have a base of 10 because $\log_{10} 10 = 1$. (We will say more about this in Section 6.5.)

To define the **base for the natural logarithm,** we use the fact that the natural logarithmic function is continuous, one-to-one, and has a range of $(-\infty, \infty)$. Hence, there must be a unique real number x such that $\ln x = 1$, as shown in Figure 6.4. We denote this number by the letter e. It can be shown that e is irrational and has the following decimal approximation.

$$e \approx 2.71828182846$$

DEFINITION OF e

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\, dt = 1.$$

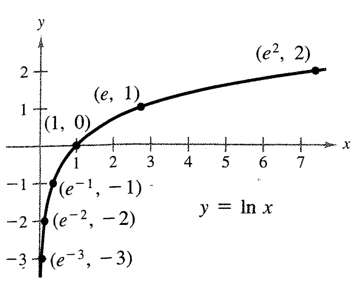


FIGURE 6.5

Once we know that $\ln e = 1$, we can use logarithmic properties to evaluate the natural logarithm of several other numbers. For example, by using the property

$$\ln (e^n) = n \ln e = n(1) = n$$

we can evaluate $\ln (e^n)$ for various powers of n, as shown in Table 6.1 and Figure 6.5.

EXAMPLE 5 Logarithmic differentiation

Find the derivative of

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}.$$

SOLUTION

We begin by taking the natural logarithm of both sides of the equation. Then we apply logarithmic properties and differentiate implicitly. Finally, we solve for y'.

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}} \qquad \text{Take ln of both sides}$$

$$\ln y = 2 \ln (x-2) - \frac{1}{2} \ln (x^2+1) \qquad \text{Logarithmic properties}$$

$$\frac{y'}{y} = 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right) \qquad \text{Differentiate}$$

$$= \frac{2}{x-2} - \frac{x}{x^2+1}$$

$$y' = y\left(\frac{2}{x-2} - \frac{x}{x^2+1}\right) \qquad \text{Solve for } y'$$

$$= \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Substitute for } y$$

$$= \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}} \qquad \text{Simplify}$$

Since the natural logarithm is undefined for negative numbers, we often encounter expressions of the form $\ln |u|$. The following theorem tells us that we can differentiate functions of the form $y = \ln |u|$ as if the absolute value sign were not present.

THEOREM 6.4 DERIVATIVE INVOLVING ABSOLUTE VALUE

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln |u|] = \frac{u'}{u}.$$

PROOF If $u > 0$, then $|u| = u$, and the result follows from Theorem 6.3. If $u < 0$, then $|u| = -u$, and we have

$$\frac{d}{dx}[\ln |u|] = \frac{d}{dx}[\ln (-u)] = \frac{-u'}{-u} = \frac{u'}{u}.$$

EXAMPLE 6 Derivative involving absolute value

Find the derivative of $f(x) = \ln|\cos x|$.

SOLUTION

Using Theorem 6.4, we let $u = \cos x$ and write

$$\frac{d}{dx}[\ln|\cos x|] = \frac{u'}{u} = \frac{-\sin x}{\cos x} = -\tan x.$$

EXAMPLE 7 Finding relative extrema

Locate the relative extrema of $y = \ln(x^2 + 2x + 3)$.

SOLUTION

Differentiating y, we obtain

$$\frac{dy}{dx} = \frac{2x+2}{x^2+2x+3}.$$

Now, since $dy/dx = 0$ when $x = -1$, we apply the First Derivative Test and conclude that the point $(-1, \ln 2)$ is a relative minimum. Since there are no other critical points, we conclude that this is the only relative extremum. (See Figure 6.6.)

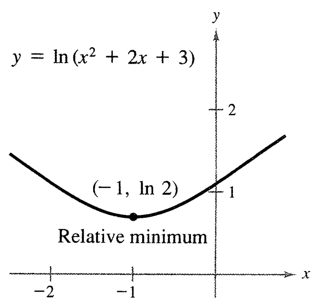


FIGURE 6.6

EXERCISES for Section 6.1

In Exercises 1–6, sketch the graph of the function.

1. $f(x) = 3 \ln x$
2. $f(x) = -2 \ln x$
3. $f(x) = \ln 2x$
4. $f(x) = \ln|x|$
5. $f(x) = \ln(x - 1)$
6. $g(x) = 2 + \ln x$

In Exercises 7 and 8, use the properties of logarithms to approximate the indicated logarithm given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.

7. (a) $\ln 6$ (b) $\ln \frac{2}{3}$ (c) $\ln 81$ (d) $\ln \sqrt{3}$
8. (a) $\ln 0.25$ (b) $\ln 24$ (c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

In Exercises 9–18, use the properties of logarithms to write each expression as a sum, difference, or multiple of logarithms.

9. $\ln \frac{2}{3}$
10. $\ln(xyz)$
11. $\ln \frac{xy}{z}$
12. $\ln \sqrt{a-1}$
13. $\ln \sqrt{2^3}$
14. $\ln \frac{1}{5}$
15. $\ln \left(\frac{x^2-1}{x^3}\right)^3$
16. $\ln 3e^2$
17. $\ln z(z-1)^2$
18. $\ln \frac{1}{e}$

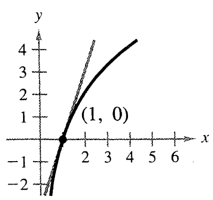
In Exercises 19–24, write each expression as a logarithm of a single quantity.

19. $\ln(x-2) - \ln(x+2)$
20. $3 \ln x + 2 \ln y - 4 \ln z$
21. $\frac{1}{3}[2 \ln(x+3) + \ln x - \ln(x^2-1)]$
22. $2[\ln x - \ln(x+1) - \ln(x-1)]$
23. $2 \ln 3 - \frac{1}{2} \ln(x^2+1)$

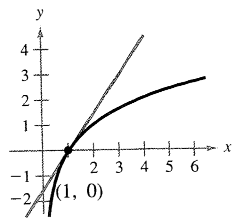
24. $\frac{3}{2}[\ln(x^2+1) - \ln(x+1) - \ln(x-1)]$

In Exercises 25–28, find the slope of the tangent line to the given logarithmic function at the point (1, 0).

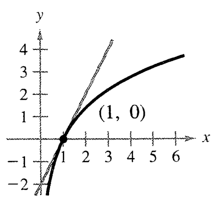
25. $y = \ln x^3$



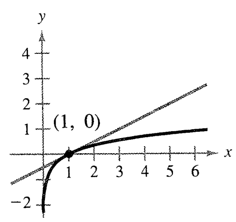
26. $y = \ln x^{3/2}$



27. $y = \ln x^2$



28. $y = \ln x^{1/2}$



In Exercises 29–54, find dy/dx.

29. $y = \ln x^2$

30. $y = \ln(x^2+3)$

31. $y = \ln\sqrt{x^4-4x}$

32. $y = \ln(1-x)^{3/2}$

33. $y = (\ln x)^4$

34. $y = x\ln x$

35. $y = \ln(x\sqrt{x^2-1})$

36. $y = \ln\left(\frac{x}{x+1}\right)$

37. $y = \ln\left(\frac{x}{x^2+1}\right)$

38. $y = \frac{\ln x}{x}$

39. $y = \frac{\ln x}{x^2}$

40. $y = \ln(\ln x)$

41. $y = \ln(\ln x^2)$

42. $y = \ln\sqrt{\frac{x-1}{x+1}}$

43. $y = \ln\sqrt{\frac{x+1}{x-1}}$

44. $y = \ln\sqrt{x^2-4}$

45. $y = \ln\left(\frac{\sqrt{4+x^2}}{x}\right)$

46. $y = \ln(x+\sqrt{4+x^2})$

47. $y = \frac{-\sqrt{x^2+1}}{x} + \ln(x+\sqrt{x^2+1})$

48. $y = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4}\ln\left(\frac{2+\sqrt{x^2+4}}{x}\right)$

49. $y = \ln|\sin x|$

50. $y = \ln|\sec x|$

51. $y = \ln\left|\frac{\cos x}{\cos x - 1}\right|$

52. $y = \ln|\sec x + \tan x|$

53. $y = \ln\left|\frac{-1+\sin x}{2+\sin x}\right|$

54. $y = \ln\sqrt{1+\sin^2 x}$

In Exercises 55–60, find dy/dx using logarithmic differentiation.

55. $y = x\sqrt{x^2-1}$

56. $y = \sqrt{(x-1)(x-2)(x-3)}$

57. $y = \frac{x^2\sqrt{3x-2}}{(x-1)^2}$

58. $y = \sqrt[3]{\frac{x^2+1}{x^2-1}}$

59. $y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$

60. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$

In Exercises 61 and 62, show that the given function is a solution to the differential equation.

Function	*Differential equation*
61. $y = 2\ln x + 3$	$xy'' + y' = 0$
62. $y = x\ln x - 4x$	$x + y - xy' = 0$

In Exercises 63 and 64, find dy/dx by using implicit differentiation.

63. $x^2 - 3\ln y + y^2 = 10$

64. $\ln xy + 5x = 30$

In Exercises 65 and 66, find an equation of the tangent line to the graph of the equation at the given point.

Equation	*Point*
65. $y = 3x^2 - \ln x$	$(1, 3)$
66. $x^2 + \ln(x+1) + y^2 = 4$	$(0, 2)$

In Exercises 67–72, find any relative extrema and inflection points, and sketch the graph of the function.

67. $y = \frac{x^2}{2} - \ln x$

68. $y = x - \ln x$

69. $y = x\ln x$

70. $y = \frac{\ln x}{x}$

71. $y = \frac{x}{\ln x}$

72. $y = x^2\ln x$

In Exercises 73 and 74, use Newton's Method to approximate, to three decimal places, the x-coordinate of the point of intersection of the graphs of the two equations.

73. $y = \ln x, \quad y = -x$

74. $y = \ln x, \quad y = 3 - x$

75. Apply the Mean Value Theorem to the function $f(x) = \ln x$ on the closed interval $[1, e]$. Find the value of c in the open interval $(1, e)$ such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1}.$$

76. Use Simpson's Rule with $n = 4$ to show that

$$\int_1^{2.7} \frac{1}{t}\, dt < 1 < \int_1^{2.8} \frac{1}{t}\, dt.$$

Then use this given inequality to show that

$$\ln 2.7 < \ln e < \ln 2.8$$

and therefore $2.7 < e < 2.8$.

77. Show that

$$f(x) = \frac{\ln x^n}{x}$$

is a decreasing function for $x > e$ and $n > 0$.

78. A person walking along a dock drags a boat by a 10-foot rope. The boat travels along a path known as a *tractrix* (see figure). The equation of this path is

$$y = 10 \ln \left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2}.$$

What is the slope of this path at the following x-values?

(a) $x = 10$ (b) $x = 5$

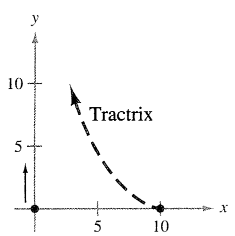


FIGURE FOR 78

79. There are 25 prime numbers less than 100. The **Prime Number Theorem** states that if $p(x)$ is the number of primes less than x, then the ratio of $p(x)$ to $x/(\ln x)$ approaches 1 as x approaches infinity. Compute $x/(\ln x)$ for $x = 1000$, $x = 1{,}000{,}000$, and $x = 1{,}000{,}000{,}000$. Then compute the ratio of $p(x)$ to $x/(\ln x)$ given that $p(1000) = 168$, $p(10^6) = 78{,}498$, and $p(10^9) = 50{,}847{,}478$.

80. Prove that the natural logarithmic function is one-to-one.

In Exercises 81 and 82, show that $f = g$ by using a computer to sketch the graph of f and g on the same coordinate axes. (Assume $x > 0$.)

81. $f(x) = \ln \dfrac{x^2}{4}$

$g(x) = 2 \ln x - \ln 4$

82. $f(x) = \ln \sqrt{x(x^2 + 1)}$

$g(x) = \dfrac{1}{2}[\ln x + \ln (x^2 + 1)]$

In Exercises 83 and 84, use a computer or calculator and Simpson's Rule with $n = 10$ to approximate the integral

$$\int_1^x \frac{1}{t}\, dt$$

and compare the result with $\ln x$ for the specified value of x.

83. $x = 3$ **84.** $x = 8.7$

6.2 The Natural Logarithmic Function and Integration

Log Rule for integration ▪ Integrals of trigonometric functions ▪ The six basic trigonometric integrals

In the previous section we defined the natural logarithmic function as an antiderivative of the function $y = 1/x$. With this definition, we are able to integrate several important functions that were not covered by previous integration rules. Specifically, the differentiation rules

$$\frac{d}{dx}[\ln |x|] = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}[\ln |u|] = \frac{u'}{u}$$

produce the following integration formulas.

THEOREM 6.5
LOG RULE FOR INTEGRATION

Let u be a differentiable function of x.

1. $\displaystyle\int \frac{1}{x}\, dx = \ln |x| + C$ 2. $\displaystyle\int \frac{1}{u}\, du = \ln |u| + C$

11. $\int_1^e \frac{(1 + \ln x)^2}{x}\,dx$
12. $\int_0^1 \frac{x - 1}{x + 1}\,dx$
13. $\int_0^2 \frac{x^2 - 2}{x + 1}\,dx$
14. $\int \frac{1}{(x + 1)^2}\,dx$
15. $\int \frac{1}{\sqrt{x + 1}}\,dx$
16. $\int \frac{x + 3}{x^2 + 6x + 7}\,dx$
17. $\int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x}\,dx$
18. $\int \frac{(\ln x)^2}{x}\,dx$
19. $\int \frac{1}{x^{2/3}(1 + x^{1/3})}\,dx$
20. $\int \frac{1}{x \ln (x^2)}\,dx$
21. $\int \frac{1}{1 + \sqrt{x}}\,dx$
22. $\int \frac{1 - \sqrt{x}}{1 + \sqrt{x}}\,dx$
23. $\int \frac{\sqrt{x}}{\sqrt{x} - 3}\,dx$
24. $\int_0^2 \frac{1}{1 + \sqrt{2x}}\,dx$
25. $\int \frac{\sqrt{x}}{1 - x\sqrt{x}}\,dx$
26. $\int \frac{2x}{(x - 1)^2}\,dx$
27. $\int \frac{x(x - 2)}{(x - 1)^3}\,dx$
28. $\int \tan 5x\,dx$
29. $\int \csc 2x\,dx$
30. $\int \sec \frac{x}{2}\,dx$
31. $\int \cos (1 - x)\,dx$
32. $\int \frac{\tan^2 2x}{\sec 2x}\,dx$
33. $\int \frac{\sec x \tan x}{\sec x - 1}\,dx$
34. $\int \frac{\sin x}{1 + \cos x}\,dx$
35. $\int \frac{\cos t}{1 + \sin t}\,dt$
36. $\int (\sec t + \tan t)\,dt$
37. $\int (\csc x - \sin x)\,dx$
38. $\int \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx$
39. $\int \frac{1 - \cos \theta}{\theta - \sin \theta}\,d\theta$
40. $\int (\csc 2\theta - \cot 2\theta)^2\,d\theta$

In Exercises 41–44, show the equivalence of the given pair of formulas.

41. $\int \tan x\,dx = -\ln |\cos x| + C$

$\int \tan x\,dx = \ln |\sec x| + C$

42. $\int \cot x\,dx = \ln |\sin x| + C$

$\int \cot x\,dx = -\ln |\csc x| + C$

43. $\int \sec x\,dx = \ln |\sec x + \tan x| + C$

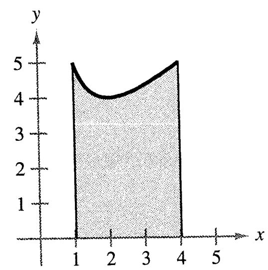
$\int \sec x\,dx = -\ln |\sec x - \tan x| + C$

44. $\int \csc x\,dx = -\ln |\csc x + \cot x| + C$

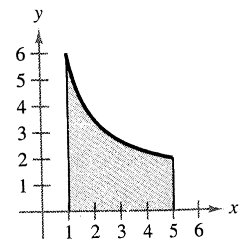
$\int \csc x\,dx = \ln |\csc x - \cot x| + C$

In Exercises 45 and 46, find the area of the indicated region.

45. $y = \frac{x^2 + 4}{x}$



46. $y = \frac{x + 5}{x}$



47. A population of bacteria is changing at the rate of

$$\frac{dP}{dt} = \frac{3000}{1 + 0.25t}$$

where t is the time in days. Assuming that the initial population (when $t = 0$) is 1000, write an equation that gives the population at any time t, and then find the population when $t = 3$ days.

48. Find the time required for an object to cool from 300° to 250° if that time is given by

$$t = \frac{10}{\ln 2} \int_{250}^{300} \frac{1}{T - 100}\,dT.$$

49. Find the average value of the function

$$f(x) = \frac{1}{x}$$

on the interval $1 \leq x \leq 5$.

50. The demand equation for a product is given by

$$p = \frac{90{,}000}{400 + 3x}.$$

Find the *average* price p on the interval $40 \leq x \leq 50$.

6.3 Inverse Functions

Inverse functions ▪ Existence of an inverse function ▪ Derivative of an inverse function

When we introduced the notion of a composite function in Section 1.5, we noted that composition is not commutative. That is, it is not necessarily true that $f(g(x))$ and $g(f(x))$ are equal. We now look at a special case for which composition is commutative—when f and g are inverses of each other.

DEFINITION OF INVERSE FUNCTION

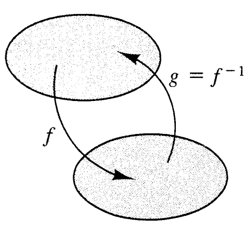
A function g is the **inverse** of the function f if

$$f(g(x)) = x \quad \text{for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for each } x \text{ in the domain of } f.$$

We denote g by f^{-1} (read "f inverse").



Domain of f = Range of g
Domain of g = Range of f

FIGURE 6.8

REMARK Although the notation used to denote an inverse function resembles *exponential notation,* it is a different use of -1 as a superscript. That is, in general, $f^{-1}(x) \neq 1/f(x)$.

Here are some important observations about this definition.

1. If g is the inverse of f, then f is also the inverse of g.
2. The domain of f^{-1} is equal to the range of f (and vice versa), as indicated in Figure 6.8.
3. A function need not possess an inverse, but if it does, the inverse is unique. (See Exercise 57.)

To understand the concept of an inverse function, it is helpful to think of f^{-1} as undoing what has been done by f. For example, subtraction can be used to undo addition, and division can be used to undo multiplication. Use the definition of an inverse function to check the following inverses.

1. $f(x) = x + c$ and $f^{-1}(x) = x - c$
2. $f(x) = cx$ and $f^{-1}(x) = \frac{x}{c}, \quad c \neq 0$

EXAMPLE 1 Verifying inverse functions

Show that the following functions are inverses of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

SOLUTION

First, note that both composite functions exist, since the domain and range of both f and g consist of the set of all real numbers. The composite of f with g is given by

$$f(g(x)) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 = 2\left(\frac{x+1}{2}\right) - 1 = x + 1 - 1 = x.$$

The composite of g with f is given by

$$g(f(x)) = \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x.$$

Since $f(g(x)) = g(f(x)) = x$, we conclude that f and g are inverses of each other. (See Figure 6.9). ▭

In Figure 6.9 the graphs of f and f^{-1} appear to be mirror images of each other with respect to the line $y = x$. We say that the graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$. This idea is generalized in the following theorem.

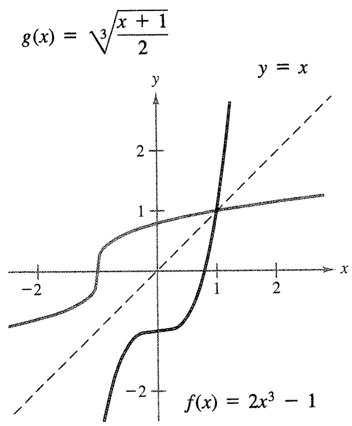


FIGURE 6.9

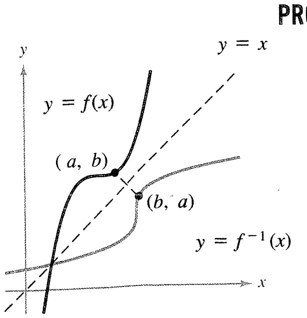
THEOREM 6.7
REFLECTIVE PROPERTY OF INVERSE FUNCTIONS

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

PROOF If (a, b) is on the graph of f, then $f(a) = b$ and we have

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

Thus, (b, a) is on the graph of f^{-1}, as shown in Figure 6.10. A similar argument will prove the theorem in the other direction.



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

FIGURE 6.10

Not every function has an inverse, and Theorem 6.7 suggests a graphical test for those that do. It is called the **horizontal line test** for an inverse function, and it follows directly from the vertical line test for functions together with the reflective property of the graphs of f and f^{-1}. The test states that a function f has an inverse if and only if every horizontal line intersects the graph of f at most once. The following theorem formally states why the horizontal line test is valid. (Recall from Section 4.3 that a function is *strictly monotonic* if it is either increasing on its entire domain or decreasing on its entire domain.)

THEOREM 6.8
THE EXISTENCE OF AN INVERSE FUNCTION

1. A function possesses an inverse if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and, hence, possesses an inverse.

PROOF We leave the proof of the first part as an exercise (see Exercise 59). To prove the second part, recall from Section 1.5 that f is one-to-one if for x_1, x_2 in its domain

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

The *contrapositive* of this implication is logically equivalent and it states that

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

Now, choose x_1 and x_2 in the given interval. If $x_1 \neq x_2$, and since f is strictly monotonic, it follows that either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$. In either case, $f(x_1) \neq f(x_2)$. Thus, f is one-to-one on the interval.

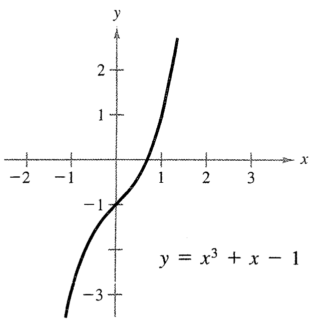


FIGURE 6.11

EXAMPLE 2 The existence of an inverse

Determine which of the following functions has an inverse.

(a) $f(x) = x^3 + x - 1$ (b) $f(x) = x^3 - x + 1$

SOLUTION

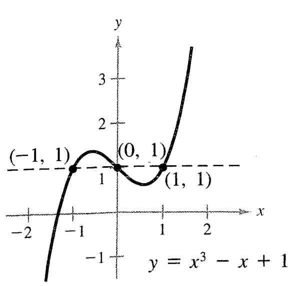


FIGURE 6.12

(a) From the graph of f given in Figure 6.11 it appears that f is increasing over its entire domain. To verify this, we note that the derivative, $f'(x) = 3x^2 + 1$, is positive for all real values of x. Therefore, f is strictly monotonic and it must have an inverse.
(b) From the graph given in Figure 6.12 we can see that the function does not pass the horizontal line test. In other words, it is not one-to-one. For instance, f has the same value when $x = -1$, 0, and 1.

$$f(-1) = f(1) = f(0) = 1 \quad \text{Not one-to-one}$$

Therefore, by Theorem 6.8, f does not have an inverse. ▭

REMARK Often it is easier to prove that a function has an inverse than to find the inverse. For instance, it would be difficult algebraically to determine the inverse of the function in Example 2(a).

GUIDELINES FOR FINDING THE INVERSE OF A FUNCTION

1. Use Theorem 6.8 to determine whether the function given by $y = f(x)$ has an inverse.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Define the domain of f^{-1} to be the range of f.
4. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

To avoid the confusion that could arise from using y as the independent variable for f^{-1}, it is customary to write f^{-1} as a *function of x* simply by interchanging the variables x and y after solving for x. This is illustrated in the next example.

EXAMPLE 3 Finding the inverse of a function

Find the inverse of the function given by $f(x) = \sqrt{2x - 3}$.

SOLUTION

By Theorem 6.8, this function has an inverse because it is increasing on its entire domain, as shown in Figure 6.13. To find an equation for this inverse, we let $y = f(x)$ and solve for x in terms of y.

$$\sqrt{2x - 3} = y \qquad \text{Let } y = f(x)$$

$$2x - 3 = y^2$$

$$x = \frac{y^2 + 3}{2} \qquad \text{Solve for } x$$

$$f^{-1}(y) = \frac{y^2 + 3}{2}$$

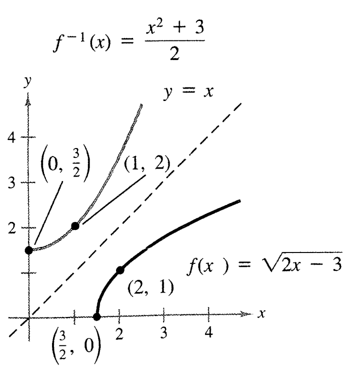


FIGURE 6.13

Since the range of f is $[0, \infty)$, we define this interval to be the domain of f^{-1}. Finally, using x as the independent variable, we have

$$f^{-1}(x) = \frac{x^2 + 3}{2}, \quad 0 \le x. \qquad \text{Determine domain}$$

REMARK Remember that any letter can be used to represent the independent variable. Thus,

$$f^{-1}(y) = \frac{y^2 + 3}{2}, \quad f^{-1}(x) = \frac{x^2 + 3}{2}, \quad \text{and} \quad f^{-1}(s) = \frac{s^2 + 3}{2}$$

all represent the same function.

Theorem 6.8 is useful in the following type of problem. Suppose you are given a function that is *not* one-to-one on its domain. By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function *is* one-to-one on the restricted domain. The next example illustrates this procedure.

EXAMPLE 4 Finding an interval on which a function is one-to-one

Show that the sine function, $f(x) = \sin x$, is not one-to-one on the entire real line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, for which f is strictly monotonic.

SOLUTION

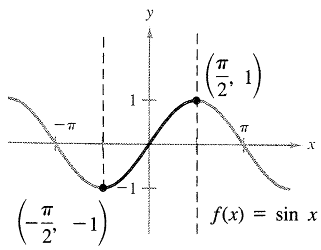


FIGURE 6.14

It is clear that f is not one-to-one, since many different x-values yield the same y-value. For instance, $\sin(0) = 0 = \sin(\pi)$. Moreover, f is increasing on the open interval $(-\pi/2, \pi/2)$, since its derivative

$$f'(x) = \cos x$$

is positive there. Finally, since the left and right endpoints correspond to relative extrema of the sine, we can conclude that f is increasing on the closed interval $[-\pi/2, \pi/2]$ *and* that in any larger interval the function would not be strictly monotonic. (See Figure 6.14.) ▭

The next two theorems discuss the derivative of an inverse function. The reasonableness of Theorem 6.9 follows from the reflective property of inverse functions as shown in Figure 6.10, and a formal proof of the theorem is given in Appendix A.

THEOREM 6.9 CONTINUITY AND DIFFERENTIABILITY OF INVERSE FUNCTIONS

Let f be a function that possesses an inverse.

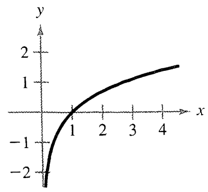
1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable at c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

THEOREM 6.10 THE DERIVATIVE OF AN INVERSE FUNCTION

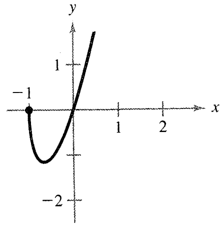
If f is differentiable on its domain and possesses an inverse function g, then the derivative of g is given by

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

35. $f(x) = \ln x$



36. $f(x) = 3x\sqrt{x+1}$



In Exercises 37–42, use the derivative to determine whether the given function is strictly monotonic on its entire domain and therefore has an inverse.

37. $f(x) = (x + a)^3 + b$

38. $f(x) = \cos \dfrac{3x}{2}$

39. $f(x) = \dfrac{x^4}{4} - 2x^2$

40. $f(x) = x^3 - 6x^2 + 12x$

41. $f(x) = 2 - x - x^3$

42. $f(x) = \ln (x - 3)$

In Exercises 43–48, show that f is strictly monotonic on the given interval and therefore has an inverse on that interval.

Function	*Interval*
43. $f(x) = (x - 4)^2$	$[4, \infty)$
44. $f(x) = \lvert x + 2 \rvert$	$[-2, \infty)$
45. $f(x) = \dfrac{4}{x^2}$	$(0, \infty)$
46. $f(x) = \tan x$	$\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
47. $f(x) = \cos x$	$[0, \pi]$
48. $f(x) = \sec x$	$\left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]$

In Exercises 49–52, show that the slopes of the graphs of f and f^{-1} are reciprocals at the given points.

Functions	*Point*
49. $f(x) = x^3$	$\left(\dfrac{1}{2}, \dfrac{1}{8}\right)$
$f^{-1}(x) = \sqrt[3]{x}$	$\left(\dfrac{1}{8}, \dfrac{1}{2}\right)$
50. $f(x) = 3 - 4x$	$(1, -1)$
$f^{-1}(x) = \dfrac{3 - x}{4}$	$(-1, 1)$
51. $f(x) = \sqrt{x - 4}$	$(5, 1)$
$f^{-1}(x) = x^2 + 4$	$(1, 5)$
52. $f(x) = \dfrac{1}{1 + x^2}$	$\left(1, \dfrac{1}{2}\right)$
$f^{-1}(x) = \sqrt{\dfrac{1 - x}{x}}$	$\left(\dfrac{1}{2}, 1\right)$

In Exercises 53 and 54, the derivative of the function has the same sign for all x in its domain, but the function is not one-to-one. Explain why.

53. $f(x) = \tan x$

54. $f(x) = \dfrac{x}{x^2 - 4}$

In Exercises 55 and 56, find the inverse function of f over the specified interval. Use a computer or graphics calculator to sketch the graph of f and f^{-1} on the same coordinate axes and observe that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

Function	*Interval*
55. $f(x) = \dfrac{x}{x^2 - 4}$	$(-2, 2)$
56. $f(x) = 2 - \dfrac{3}{x^2}$	$(0, 10)$

57. Prove that if a function has an inverse, then the inverse is unique.

58. Prove that if f has an inverse, then $(f^{-1})^{-1} = f$.

59. Prove that a function has an inverse if and only if it is one-to-one.

60. Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

6.4 Exponential Functions: Differentiation and Integration

The natural exponential function ▪ Differentiation ▪ Integration

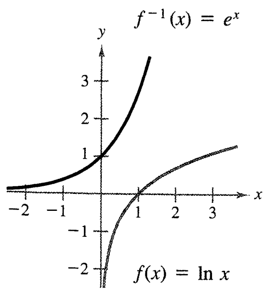


FIGURE 6.16

In Section 6.1 we saw that the function $f(x) = \ln x$ is increasing on its entire domain, and hence it has an inverse f^{-1}. Moreover, from the domain and range of f, we can conclude that the domain of f^{-1} is the set of all reals, and the range is the set of positive reals, as shown in Figure 6.16. Thus, for any real number x,

$$f(f^{-1}(x)) = \ln(f^{-1}(x)) = x. \qquad x \text{ is any real number}$$

We also know that if x happens to be rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number}$$

Since the natural logarithmic function is one-to-one, we conclude that $f^{-1}(x)$ and e^x agree for *rational* values of x. Because of this agreement, we extend the definition of e^x to cover *all* real numbers.

DEFINITION OF THE NATURAL EXPONENTIAL FUNCTION

The inverse of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$

We summarize the inverse relationship between the natural logarithmic function and the natural exponential function as follows.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x \qquad \text{Inverse relationships}$$

These two properties are useful in solving equations involving exponential and logarithmic functions, as demonstrated in the following example.

EXAMPLE 1 Solving exponential and logarithmic equations

Solve for x in the following equations.

(a) $7 = e^{x+1}$ (b) $\ln(2x - 3) = 5$

SOLUTION

(a) We can convert from exponential form to logarithmic form by *taking the natural log of both sides* of the exponential equation.

$$7 = e^{x+1} \Rightarrow \ln 7 = \ln(e^{x+1}) \Rightarrow \ln 7 = x + 1$$

Thus, $x = -1 + \ln 7 \approx 0.946$.

(b) To convert from logarithmic form to exponential form, we *apply the exponential function to both sides* of the logarithmic equation.

$$\ln(2x - 3) = 5 \Rightarrow e^{\ln(2x-3)} = e^5 \Rightarrow 2x - 3 = e^5$$

Thus, $x = \frac{1}{2}(e^5 + 3) \approx 75.707$.

The familiar rules for operating with rational exponents can be extended to the natural exponential function, as indicated in the following theorem.

THEOREM 6.11
OPERATIONS WITH EXPONENTIAL FUNCTIONS

Let a and b be any real numbers. Then the following properties are true.

1. $e^a e^b = e^{a+b}$ 2. $\dfrac{e^a}{e^b} = e^{a-b}$ 3. $(e^a)^b = e^{ab}$

PROOF We prove Property 1 and leave the proofs of the other two properties as exercises (see Exercises 87 and 88).

$$\ln(e^a e^b) = \ln(e^a) + \ln(e^b) = a + b = \ln(e^{a+b})$$

Thus, since the natural log function is one-to-one, we conclude that

$$e^a e^b = e^{a+b}.$$

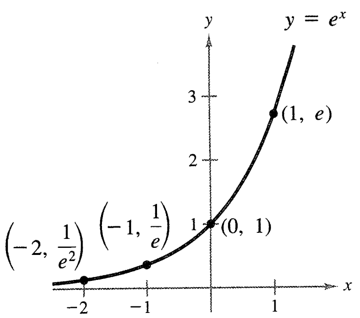


FIGURE 6.17

In Section 6.3, we saw that an inverse function f^{-1} shares many properties with f. Thus, the natural exponential function inherits the following properties from the natural logarithmic function. (See Figure 6.17.)

PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$

Derivative of the natural exponential function

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*. In other words, it is a solution to the equation $y' = y$. This result is stated in the next theorem.

THEOREM 6.12
THE DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[e^x] = e^x$ 2. $\dfrac{d}{dx}[e^u] = e^u \dfrac{du}{dx}$

PROOF Let $f(x) = \ln x$ and $g(x) = e^x$. Then $f'(x) = 1/x$, and by Theorem 6.10 we have

$$\frac{d}{dx}[e^x] = \frac{d}{dx}[g(x)] = \frac{1}{f'(g(x))} = \frac{1}{f'(e^x)} = \frac{1}{(1/e^x)} = e^x.$$

The derivative of e^u follows from the Chain Rule.

REMARK We can interpret this theorem geometrically by saying that the slope of the graph of $f(x) = e^x$ at any point (x, e^x) is equal to the y-coordinate of the point.

EXAMPLE 2 Differentiating exponential functions

(a) $\frac{d}{dx}[e^{2x-1}] = \frac{du}{dx}e^u = 2e^{2x-1}$ $\quad u = 2x - 1$

(b) $\frac{d}{dx}[e^{-3/x}] = \frac{du}{dx}e^u = \left(\frac{3}{x^2}\right)e^{-3/x} = \frac{3e^{-3/x}}{x^2}$ $\quad u = -\frac{3}{x}$

EXAMPLE 3 Locating relative extrema

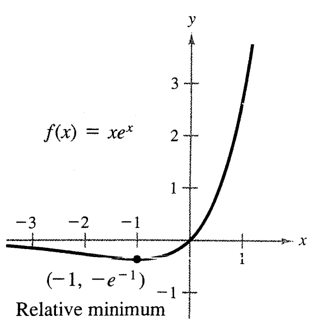


FIGURE 6.18

Find the relative extrema of $f(x) = xe^x$.

SOLUTION

The derivative of f is given by

$$f'(x) = x(e^x) + e^x(1) \qquad \text{Product Rule}$$
$$= e^x(x + 1).$$

Now, since e^x is never zero, the derivative is zero only when $x = -1$. Moreover, by the First Derivative Test, we can determine that this corresponds to a relative minimum, as shown in Figure 6.18. Since the derivative $f'(x) = e^x(x + 1)$ is defined for all x, there are no other critical points.

EXAMPLE 4 The normal probability density function

Show that the *normal probability density function*

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

has points of inflection when $x = \pm 1$.

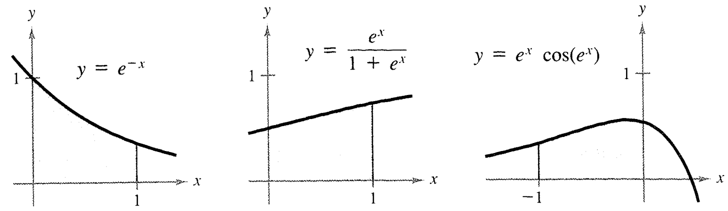


FIGURE 6.20 (a) (b) (c)

EXERCISES for Section 6.4

In Exercises 1–4, write the logarithmic equation as an exponential equation and vice versa.

1. (a) $e^0 = 1$ (b) $e^2 = 7.389\ldots$
2. (a) $e^{-2} = 0.1353\ldots$ (b) $e^{-1} = 0.3679\ldots$
3. (a) $\ln 2 = 0.6931\ldots$ (b) $\ln 8.4 = 2.128\ldots$
4. (a) $\ln 0.5 = -0.6931\ldots$
(b) $\ln 1 = 0$

In Exercises 5–8, solve for x.

5. (a) $e^{\ln x} = 4$ (b) $\ln e^{2x} = 3$
6. (a) $e^{\ln 2x} = 12$ (b) $\ln e^{-x} = 0$
7. (a) $\ln x = 2$ (b) $e^x = 4$
8. (a) $\ln x^2 = 10$ (b) $e^{-4x} = 5$

In Exercises 9–12, sketch the graph of the given function.

9. $y = e^{-x}$ **10.** $y = \frac{1}{2}e^x$
11. $y = e^{-x^2}$ **12.** $y = e^{-x/2}$

In Exercises 13–16, show that the given functions are inverses of each other by sketching their graphs on the same coordinate system.

13. $f(x) = e^{2x}$, $g(x) = \ln \sqrt{x}$
14. $f(x) = e^{x/3}$, $g(x) = \ln x^3$
15. $f(x) = e^x - 1$, $g(x) = \ln (x + 1)$
16. $f(x) = e^{x-1}$, $g(x) = 1 + \ln x$

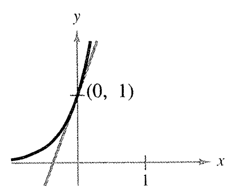
In Exercises 17 and 18, compare the given number to the natural number e.

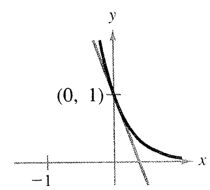
17. (a) $\frac{271{,}801}{99{,}990}$ (b) $\frac{299}{110}$

18. (a) $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$
(b) $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

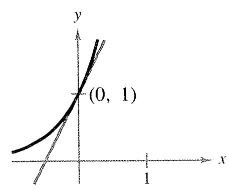
In Exercises 19 and 20, find the slope of the tangent line to the given exponential function at the point (0, 1).

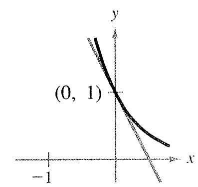
19. (a) $y = e^{3x}$ (b) $y = e^{-3x}$





20. (a) $y = e^{2x}$ (b) $y = e^{-2x}$





In Exercises 21–42, find dy/dx.

21. $y = e^{2x}$ **22.** $y = e^{1-x}$
23. $y = e^{-2x+x^2}$ **24.** $y = e^{-x^2}$
25. $y = e^{\sqrt{x}}$ **26.** $y = x^2e^{-x}$
27. $y = (e^{-x} + e^x)^3$ **28.** $y = e^{-1/x^2}$
29. $y = \ln (e^{x^2})$ **30.** $y = \ln \left(\frac{1 + e^x}{1 - e^x}\right)$
31. $y = \ln (1 + e^{2x})$ **32.** $y = \frac{2}{e^x + e^{-x}}$
33. $y = \ln \frac{e^x + e^{-x}}{2}$ **34.** $y = xe^x - e^x$
35. $y = x^2e^x - 2xe^x + 2e^x$
36. $y = \frac{e^x - e^{-x}}{2}$
37. $y = e^{-x} \ln x$ **38.** $y = e^3 \ln x$
39. $y = e^x(\sin x + \cos x)$ **40.** $y = e^{\tan x}$
41. $y = \tan^2 (e^x)$ **42.** $y = \ln e^x$

In Exercises 43 and 44, use implicit differentiation to find dy/dx.

43. $xe^y - 10x + 3y = 0$ **44.** $e^{xy} + x^2 - y^2 = 10$

In Exercises 45 and 46, show that the function $y = f(x)$ is a solution of the given differential equation.

45. $y = e^x(\cos \sqrt{2}x + \sin \sqrt{2}x)$
$y'' - 2y' + 3y = 0$

46. $y = e^x(3 \cos 2x - 4 \sin 2x)$
$y'' - 2y' + 5y = 0$

In Exercises 47–52, find the extrema and the points of inflection (if any exist), and sketch the graph of the function.

47. $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x^2/2)}$ **48.** $f(x) = \dfrac{e^x - e^{-x}}{2}$

49. $f(x) = \dfrac{e^x + e^{-x}}{2}$ **50.** $f(x) = xe^{-x}$

51. $f(x) = x^2e^{-x}$

52. $f(x) = -2 + e^{3x}(4 - 2x)$

53. Find an equation of the line normal to the graph of $y = e^{-x}$ at $(0, 1)$.

54. Find the point on the graph of $y = e^{-x}$ where the normal line to the curve will pass through the origin.

55. Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.

56. Find, to three decimal places, the value of x such that $e^{-x} = x$. [Use Newton's Method.]

In Exercises 57–76, evaluate the integral.

57. $\displaystyle\int_0^1 e^{-2x}\, dx$ **58.** $\displaystyle\int_1^2 e^{1-x}\, dx$

59. $\displaystyle\int_0^2 (x^2 - 1)e^{x^3-3x+1}\, dx$ **60.** $\displaystyle\int x^2e^{x^3}\, dx$

61. $\displaystyle\int \frac{e^{-x}}{1 + e^{-x}}\, dx$ **62.** $\displaystyle\int \frac{e^{2x}}{1 + e^{2x}}\, dx$

63. $\displaystyle\int xe^{ax^2}\, dx$ **64.** $\displaystyle\int_0^{\sqrt{2}} xe^{-(x^2/2)}\, dx$

65. $\displaystyle\int_1^3 \frac{e^{3/x}}{x^2}\, dx$ **66.** $\displaystyle\int (e^x - e^{-x})^2\, dx$

67. $\displaystyle\int e^x\sqrt{1 - e^x}\, dx$ **68.** $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

69. $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$ **70.** $\displaystyle\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\, dx$

71. $\displaystyle\int \frac{5 - e^x}{e^{2x}}\, dx$ **72.** $\displaystyle\int \frac{e^{2x} + 2e^x + 1}{e^x}\, dx$

73. $\displaystyle\int e^{\sin \pi x} \cos \pi x\, dx$ **74.** $\displaystyle\int e^{\tan 2x} \sec^2 2x\, dx$

75. $\displaystyle\int e^{-x} \tan (e^{-x})\, dx$ **76.** $\displaystyle\int \ln (e^{2x-1})\, dx$

In Exercises 77–80, find the area of the region bounded by the graphs of the equations.

77. $y = e^x, y = 0, x = 0, x = 5$

78. $y = e^{-x}, y = 0, x = a, x = b$

79. $y = xe^{-(x^2/2)}, y = 0, x = 0, x = \sqrt{2}$

80. $y = e^{-2x} + 2, y = 0, x = 0, x = 2$

81. Given $e^x \geq 1$ for $x \geq 0$, it follows that

$$\int_0^x e^t\, dt \geq \int_0^x 1\, dt.$$

Perform this integration to derive the inequality $e^x \geq 1 + x$ for $x \geq 0$.

82. Integrate each term of the following inequalities in a manner similar to that of Exercise 81 to obtain each succeeding inequality for $x \geq 0$. Then evaluate both sides of each inequality when $x = 1$.

(a) $e^x \geq 1 + x$

(b) $e^x \geq 1 + x + \dfrac{x^2}{2}$

(c) $e^x \geq 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$

(d) $e^x \geq 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$

In Exercises 83 and 84, find a function f that satisfies the given conditions.

83. $f''(x) = \dfrac{1}{2}(e^x + e^{-x}),$
$f(0) = 1, f'(0) = 0$

84. $f''(x) = \sin x + e^{2x},$
$f(0) = \dfrac{1}{4}, f'(0) = \dfrac{1}{2}$

85. Given the function

$$f(x) = \frac{2}{1 + e^{1/x}}$$

use a computer or graphics calculator to (a) sketch the graph of f, (b) find any horizontal asymptotes, and (c) find $\lim_{x\to 0} f(x)$ (if it exists).

86. The displacement from equilibrium of a mass oscillating on the end of a spring suspended from the ceiling is given by $y = 1.56e^{-0.22t} \cos 4.9t$ where y is the displacement in feet and t is the time in seconds. Use a computer or graphics calculator to sketch the graph of the displacement function on the interval $[0, 10]$ and find the time past which the displacement does not exceed 3 inches from equilibrium.

87. Prove that $\dfrac{e^a}{e^b} = e^{a-b}$.

88. Prove that $(e^a)^b = e^{ab}$.

6.5 Bases Other Than *e* and Applications

Bases other than *e* ▪ Differentiation and integration ▪ Power Rule for real exponents ▪ Applications

The **base** of the natural exponential function is e. Using the third property in Theorem 6.11, we now assign a meaning to the exponential function $y = a^x$.

DEFINITION OF EXPONENTIAL FUNCTION TO BASE *a*

If a is a positive real number ($a \neq 1$) and x is any real number, then $f(x) = a^x$ is given by

$$f(x) = a^x = (e^{\ln a})^x = e^{(\ln a)x}.$$

We call f the **exponential function to the base *a*.**

REMARK These functions obey the usual laws of exponents. For example, $a^x a^y = a^{x+y}$.

We can give a similar definition for logarithmic functions to other bases.

DEFINITION OF LOGARITHMIC FUNCTION TO BASE *a*

If a is a positive real number ($a \neq 1$) and x is any positive real number, then $f(x) = \log_a x$ is given by

$$f(x) = \log_a x = \frac{1}{\ln a} \ln x.$$

We call f the **logarithmic function to the base *a*.**

Logarithmic functions to the base a have properties similar to those of the natural logarithmic function given in Theorem 6.2. For instance, the following properties are valid.

$$\log_a 1 = 0$$

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

From the definition of the exponential and logarithmic functions to a base a, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other. Thus,

$$a^{\log_a x} = x \qquad \text{and} \qquad \log_a a^x = x.$$

Moreover,

$$y = a^x \qquad \text{if and only if} \qquad x = \log_a y.$$

The logarithmic function to the base 10 is called the **common logarithmic function.** Thus, for common logarithms, we have $y = 10^x$ if and only if $x = \log_{10} y$.

EXAMPLE 1 Bases other than e

Solve for x in the following equations.

(a) $3^x = \frac{1}{27}$ (b) $\log_2 x = -4$

SOLUTION

(a) Applying the logarithmic function to the base 3 to both sides produces

$$\log_3 3^x = \log_3 \frac{1}{27} = \log_3 3^{-3}$$
$$x = -3.$$

(b) We solve for x as follows.

$$\log_2 x = -4$$
$$2^{\log_2 x} = 2^{-4}$$
$$x = \frac{1}{2^4} = \frac{1}{16}$$

Differentiation and integration

To differentiate exponential and logarithmic functions to other bases, you have three options: (1) use the definition of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions, (2) use logarithmic differentiation, or (3) use the following differentiation rules for bases other than e.

THEOREM 6.14 DERIVATIVES FOR BASES OTHER THAN e

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x.

1. $\frac{d}{dx}[a^x] = (\ln a)a^x$
2. $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

PROOF By definition, $a^x = e^{(\ln a)x}$. Therefore, we can prove the first property by letting $u = (\ln a)x$ and differentiating with base e to obtain

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{(\ln a)x}] = e^u \frac{du}{dx} = e^{(\ln a)x} (\ln a) = (\ln a)a^x.$$

(c) When $t = 10$,

$$y = \frac{1.25}{1 + 0.25e^{-4}} \approx 1.244 \text{ g}.$$

(d) Finally, taking the limit as t approaches infinity, we have

$$\lim_{t \to \infty} \frac{1.25}{1 + 0.25e^{-0.4t}} = \frac{1.25}{1 + 0} = 1.25 \text{ g}.$$

(See Figure 6.21.)

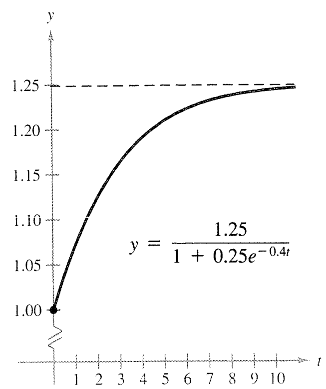


FIGURE 6.21

EXERCISES for Section 6.5

In Exercises 1–4, write the logarithmic equation as an exponential equation and vice versa.

1. (a) $2^3 = 8$ (b) $3^{-1} = \frac{1}{3}$

2. (a) $27^{2/3} = 9$ (b) $16^{3/4} = 8$

3. (a) $\log_{10} 0.01 = -2$ (b) $\log_{0.5} 8 = -3$

4. (a) $\log_3 \frac{1}{9} = -2$ (b) $49^{1/2} = 7$

In Exercises 5–10, solve for x (or b).

5. (a) $\log_{10} 1000 = x$ (b) $\log_{10} 0.1 = x$

6. (a) $\log_4 \frac{1}{64} = x$ (b) $\log_5 25 = x$

7. (a) $\log_3 x = -1$ (b) $\log_2 x = -4$

8. (a) $\log_b 27 = 3$ (b) $\log_b 125 = 3$

9. (a) $x^2 - x = \log_5 25$ (b) $3x + 5 = \log_2 64$

10. (a) $\log_3 x + \log_3 (x - 2) = 1$
(b) $\log_{10} (x + 3) - \log_{10} x = 1$

In Exercises 11–14, sketch the graph of the given function.

11. $y = 3^x$

12. $y = 3^{x-1}$

13. $y = \left(\frac{1}{3}\right)^x$

14. $y = 2^{x^2}$

In Exercises 15 and 16, show that the given functions are inverses of each other by sketching their graphs on the same coordinate system.

15. $f(x) = 4^x$, $g(x) = \log_4 x$

16. $f(x) = 3^x$, $g(x) = \log_3 x$

In Exercises 17–28, find dy/dx.

17. $y = 4^x$

18. $y = 2^{-x}$

19. $y = 5^{x-2}$

20. $y = x(7^{-3x})$

21. $y = x^2 2^x$

22. $y = 2^{x^2} 3^{-x}$

23. $y = \log_3 x$

24. $y = \log_{10} 2x$

25. $y = \log_2 \frac{x^2}{x - 1}$

26. $y = \log_3 \frac{x\sqrt{x - 1}}{2}$

27. $y = \log_5 \sqrt{x^2 - 1}$

28. $y = \log_{10} \frac{x^2 - 1}{x}$

In Exercises 29–32, use logarithmic differentiation to find dy/dx.

29. $y = x^{2/x}$ **30.** $y = x^{x-1}$

31. $y = (x - 2)^{x+1}$ **32.** $y = (1 + x)^{1/x}$

In Exercises 33–38, evaluate the given integral.

33. $\int 3^x\, dx$ **34.** $\int 4^{-x}\, dx$

35. $\int_{-1}^{2} 2^x\, dx$ **36.** $\int_{-2}^{0} (3^3 - 5^2)\, dx$

37. $\int x5^{-x^2}\, dx$ **38.** $\int (3 - x)7^{(3-x)^2}\, dx$

39. Find the area of the region bounded by the graphs of $y = 3^x$, $y = 0$, $x = 0$, and $x = 3$.

40. Show that

$$\log_{10} 2 = \frac{\ln 2}{\ln 10}.$$

41. Find the amount of time necessary for P dollars to double if it is compounded continuously at $7\frac{1}{2}$ percent interest. Find the time necessary for it to triple.

42. Find the amount of time necessary for P dollars to double if it is compounded continuously at 9 percent interest. Find the time necessary for it to triple.

43. Complete the accompanying table to demonstrate that e can also be defined as

$$\lim_{x \to 0^+} (1 + x)^{1/x}.$$

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$					

44. Complete the accompanying table to find the time t necessary for P dollars to double if interest is compounded continuously at the rate r.

r	2%	4%	6%	8%	10%	12%
t						

45. If \$1000 is invested at $7\frac{1}{2}$ percent interest, find the amount after 10 years if interest is compounded
(a) annually (b) semiannually
(c) quarterly (d) monthly
(e) daily (f) continuously

46. If \$2500 is invested at 12 percent interest, find the amount after 20 years if interest is compounded
(a) annually (b) semiannually
(c) quarterly (d) monthly
(e) daily (f) continuously

47. The yield V (in millions of cubic feet per acre) for a stand of timber at age t is given by

$$V = 6.7e^{(-48.1)/t}$$

where t is measured in years.
(a) Find the limiting volume of wood per acre as t approaches infinity.
(b) Find the rate at which the yield is changing when $t = 20$ and $t = 60$ years.

48. The average typing speed (in the number of words per minute) after t weeks of lessons is given by

$$N = \frac{157}{1 + 5.4e^{-0.12t}}.$$

(a) Find the limiting number of words per minute as t approaches infinity.
(b) Find the rate at which typing speed is changing when $t = 5$ and $t = 25$ weeks.

49. In a group project in learning theory, a mathematical model for the proportion P of correct responses after n trials was found to be

$$P = \frac{0.83}{1 + e^{-0.2n}}.$$

(a) Find the limiting proportion of correct responses as n approaches infinity.
(b) Find the rate at which P is changing after $n = 3$ and $n = 10$ trials.

50. A lake is stocked with 500 fish, and their population increases according to the logistics curve

$$p(t) = \frac{10{,}000}{1 + 19e^{-t/5}}$$

where t is measured in months. At what rate is the fish population changing at the end of 1 month and at the end of 10 months? After how many months is the population increasing most rapidly? (See figure.)

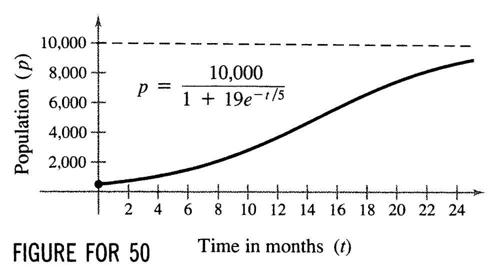


FIGURE FOR 50

51. Consider a deposit of \$100 placed in an account for 20 years at $r\%$ compounded continuously. Use a computer or graphics calculator to sketch the graphs on the same coordinate axes of the amount in the account if (a) $r = 6\%$, (b) $r = 9\%$, and (c) $r = 12\%$.

52. Repeat Exercise 51 for an account that earns interest for 50 years.

6.7 Inverse Trigonometric Functions and Differentiation

Inverse trigonometric functions ▪ Derivatives of inverse trigonometric functions ▪ Review of basic differentiation formulas

$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$

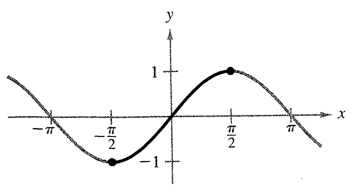


FIGURE 6.26

This section begins with a rather startling statement: *None of the six basic trigonometric functions has an inverse*. This statement is true because all six trigonometric functions are periodic, and hence not one-to-one. In this section we will examine these six functions to see whether we can redefine their domains in such a way that they will have inverses on the *restricted domains*.

For example, in Section 6.3, we demonstrated that the sine function is increasing (and therefore is one-to-one) on the interval $[-\pi/2, \pi/2]$, as shown in Figure 6.26. On this interval we define the inverse of the *restricted* sine function to be

$$y = \arcsin x \quad \text{if and only if} \quad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le \arcsin x \le \pi/2$.

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so possesses an inverse, as indicated in the following definition. (The term iff is used to represent the phrase "if and only if.")

DEFINITION OF INVERSE TRIGONOMETRIC FUNCTIONS

Function	*Domain*	*Range*
$y = \mathbf{arcsin}\ x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \mathbf{arccos}\ x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \mathbf{arctan}\ x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \mathbf{arccot}\ x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \mathbf{arcsec}\ x$ iff $\sec y = x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
$y = \mathbf{arccsc}\ x$ iff $\csc y = x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

REMARK The term arcsin x is read as the "inverse sine of x" or sometimes the "angle whose sine is x." An alternate notation for the inverse sine function is $\sin^{-1} x$.

The graphs of these six inverse functions are shown in Figure 6.27 on page 372. (Compare these to the graphs of the six trigonometric functions given in Section 1.6.)

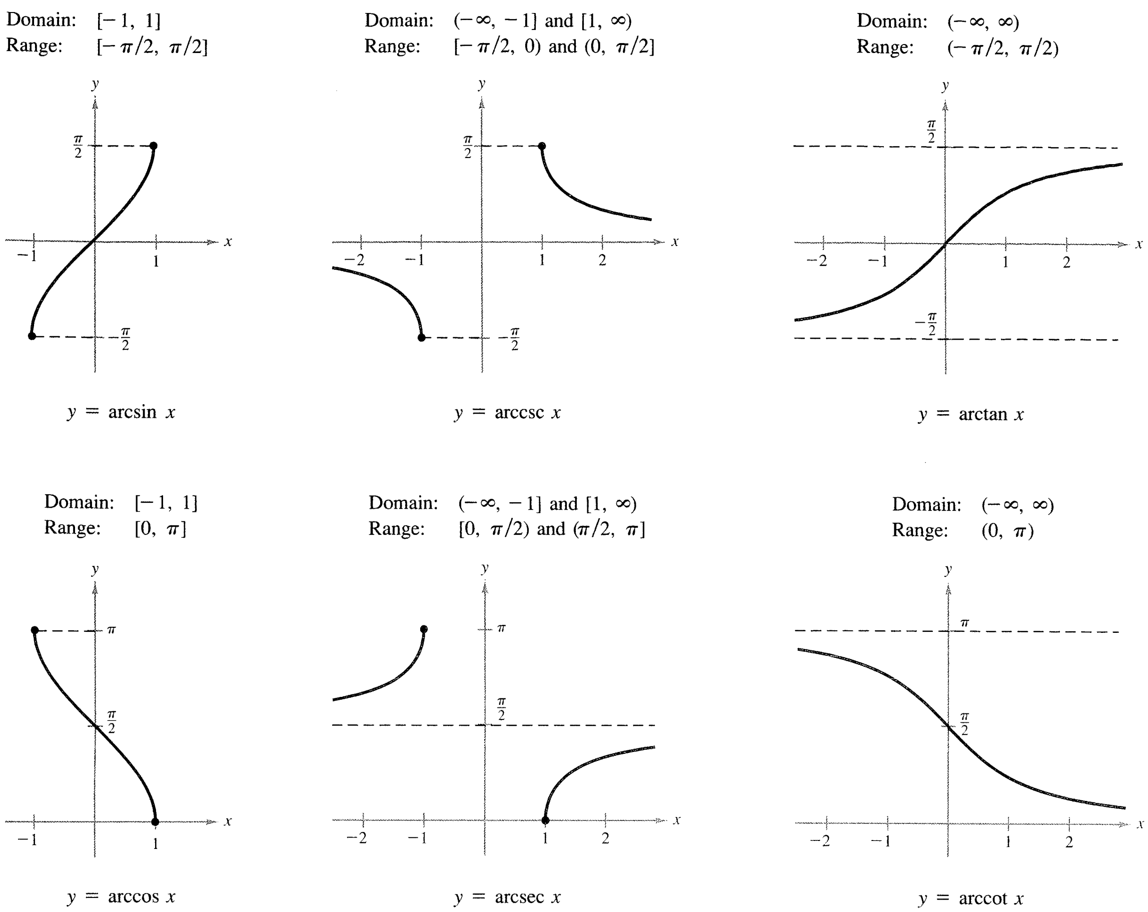


FIGURE 6.27

When evaluating inverse trigonometric functions, remember that they denote *angles in radian measure*.

EXAMPLE 1 Evaluating inverse trigonometric functions

Evaluate the following.

(a) $\arcsin\left(-\frac{1}{2}\right)$ (b) $\arccos 0$

(c) $\arctan \sqrt{3}$ (d) $\arcsin (0.3)$

SOLUTION

(a) By definition, $y = \arcsin\left(-\frac{1}{2}\right)$ implies that $\sin y = -\frac{1}{2}$. In the interval $[-\pi/2, \pi/2]$, we choose $y = -\pi/6$. Therefore,

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

(b) By definition, $y = \arccos 0$ implies that $\cos y = 0$. In the interval $[0, \pi]$, we choose $y = \pi/2$. Therefore,

$$\arccos 0 = \frac{\pi}{2}.$$

(c) By definition, $y = \arctan \sqrt{3}$ implies that $\tan y = \sqrt{3}$. In the interval $(-\pi/2, \pi/2)$, we choose $y = \pi/3$. Therefore,

$$\arctan \sqrt{3} = \frac{\pi}{3}.$$

(d) By using a calculator set in *radian mode*, we obtain

$$\arcsin(0.3) \approx 0.3047.$$

Inverse functions possess the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

When applying these properties to inverse trigonometric functions, remember that the trigonometric functions possess inverses only in restricted domains. For x-values outside these domains, these two properties do not hold. For example,

$$\arcsin(\sin \pi) = \arcsin 0 = 0 \neq \pi.$$

INVERSE PROPERTIES

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\text{arcsec } x) = x \quad \text{and} \quad \text{arcsec}(\sec y) = y.$$

REMARK Similar properties hold for the other three inverse trigonometric functions.

Notice how we use one of these inverse properties to solve the equation in the next example.

EXAMPLE 2 Solving an equation

Solve for x in the equation $\arctan(2x - 3) = \pi/4$.

SOLUTION

$$\arctan(2x - 3) = \frac{\pi}{4}$$

$$\tan[\arctan(2x - 3)] = \tan\frac{\pi}{4}$$

$$2x - 3 = 1$$

$$x = 2$$

There are some important types of problems in calculus in which we evaluate expressions like sec (arctan x). To solve this type of problem, it helps to use right triangles, as demonstrated in the next example.

EXAMPLE 3 Using right triangles

(a) Given $y = \arcsin x$, where $0 < y < \pi/2$, find $\cos y$.
(b) Given $y = \operatorname{arcsec}(\sqrt{5}/2)$, find $\tan y$.

SOLUTION

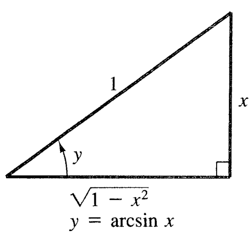


FIGURE 6.28

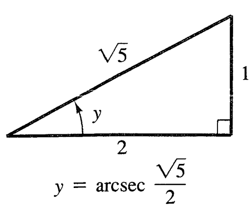


FIGURE 6.29

(a) Since y is the angle whose sine is x, we form a right triangle having an acute angle $y = \arcsin x$, as shown in Figure 6.28. Therefore,

$$\cos y = \cos(\arcsin x) = \sqrt{1 - x^2}. \qquad \cos y = \frac{\text{adj.}}{\text{hyp.}}$$

It can also be shown that for $-\pi/2 < y \leq 0$, $y = \arcsin x$ implies $\cos y = \sqrt{1 - x^2}$.

(b) Since y is the angle whose secant is $\sqrt{5}/2$, we can sketch this angle as part of a triangle, as shown in Figure 6.29. Therefore,

$$\tan y = \tan\left[\operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)\right] = \frac{1}{2}. \qquad \tan y = \frac{\text{opp.}}{\text{adj.}}$$

Derivatives of inverse trigonometric functions

In Section 6.1 we saw that the derivative of the *transcendental* function $f(x) = \ln x$ is the *algebraic* function $f'(x) = 1/x$. We will now see that the derivatives of the inverse trigonometric functions also are algebraic, even though the inverse trigonometric functions are themselves transcendental.

EXERCISES for Section 6.7

In Exercises 1–10, evaluate the given expression.

1. $\arcsin \frac{1}{2}$ **2.** $\arcsin 0$

3. $\arccos \frac{1}{2}$ **4.** $\arccos 0$

5. $\arctan \frac{\sqrt{3}}{3}$ **6.** $\text{arccot}(-1)$

7. $\text{arccsc} \sqrt{2}$ **8.** $\arcsin(-0.39)$

9. $\text{arcsec}\, 1.269$ **10.** $\arctan(-3)$

In Exercises 11–16, evaluate the given expression without a calculator. [Hint: Make a sketch of a right triangle, as illustrated in Example 3.]

11. (a) $\sin\left(\arcsin \frac{1}{2}\right)$ (b) $\cos\left(2 \arcsin \frac{1}{2}\right)$

12. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right)$ (b) $\cos\left(\arcsin \frac{5}{13}\right)$

13. (a) $\sin\left(\arctan \frac{3}{4}\right)$ (b) $\sec\left(\arcsin \frac{4}{5}\right)$

14. (a) $\tan(\text{arccot}\, 2)$ (b) $\cos(\text{arcsec} \sqrt{5})$

15. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right]$ (b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right]$

16. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$ (b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right]$

In Exercises 17–26, write the given expression in algebraic form.

17. $\tan(\arctan x)$ **18.** $\sin(\arccos x)$

19. $\cos(\arcsin 2x)$ **20.** $\sec(\arctan 3x)$

21. $\sin(\text{arcsec}\, x)$ **22.** $\cos(\text{arccot}\, x)$

23. $\tan\left(\text{arcsec} \frac{x}{3}\right)$ **24.** $\sec[\arcsin(x - 1)]$

25. $\csc\left(\arctan \frac{x}{\sqrt{2}}\right)$ **26.** $\cos\left(\arcsin \frac{x - h}{r}\right)$

In Exercises 27 and 28, fill in the blank.

27. $\arctan \frac{9}{x} = \arcsin(\quad)$

28. $\arcsin \frac{\sqrt{36 - x^2}}{6} = \arccos(\quad)$

In Exercises 29 and 30, verify each identity.

29. (a) $\text{arccsc}\, x = \arcsin \frac{1}{x}, |x| \geq 1$

(b) $\text{arccot}\, x = \arctan \frac{1}{x}, x > 0$

30. (a) $\arcsin(-x) = -\arcsin x, |x| \leq 1$

(b) $\arccos(-x) = \pi - \arccos x, |x| \leq 1$

In Exercises 31–34, sketch the graph of the function.

31. $f(x) = \arcsin(x - 1)$ **32.** $f(x) = \arctan x + \frac{\pi}{2}$

33. $f(x) = \text{arcsec}\, 2x$ **34.** $f(x) = \arccos \frac{x}{4}$

In Exercises 35–38, solve the given equation for x.

35. $\arcsin(3x - \pi) = \frac{1}{2}$ **36.** $\arctan 2x = -1$

37. $\arcsin \sqrt{2x} = \arccos \sqrt{x}$

38. $\arccos x = \text{arcsec}\, x$

In Exercises 39–58, find the derivative of the given function.

39. $f(x) = \arcsin 2x$ **40.** $f(x) = \arcsin x^2$

41. $f(x) = 2 \arcsin(x - 1)$ **42.** $f(x) = \arccos \sqrt{x}$

43. $f(x) = 3 \arccos \frac{x}{2}$ **44.** $f(x) = \arctan \sqrt{x}$

45. $f(x) = \arctan 5x$ **46.** $f(x) = x \arctan x$

47. $f(x) = \arccos \frac{1}{x}$ **48.** $f(x) = \text{arcsec}\, 2x$

49. $f(x) = \arcsin x + \arccos x$

50. $f(x) = \text{arcsec}\, x + \text{arccsc}\, x$

51. $h(t) = \sin(\arccos t)$ **52.** $g(t) = \tan(\arcsin t)$

53. $f(t) = \frac{1}{\sqrt{6}} \arctan \frac{\sqrt{6}t}{2}$

54. $f(x) = \frac{1}{2}\left(\frac{1}{2} \ln \frac{x + 1}{x - 1} - \arctan x\right)$

55. $f(x) = \frac{1}{2}\left(\frac{1}{2} \ln \frac{x + 1}{x - 1} + \arctan x\right)$

56. $f(x) = \frac{1}{2}(x\sqrt{1 - x^2} + \arcsin x)$

57. $f(x) = x \arcsin x + \sqrt{1 - x^2}$

58. $f(x) = x \arctan 2x - \frac{1}{4} \ln(1 + 4x^2)$

In Exercises 59 and 60, find the point of inflection of the graph of the given function.

59. $f(x) = \arcsin x$ **60.** $f(x) = \text{arccot}\, 2x$

In Exercises 61 and 62, find any relative extrema of the given function.

61. $f(x) = \text{arcsec}\, x - x$ **62.** $f(x) = \arcsin x - 2x$

In Exercises 63 and 64, find the point of intersection of the graphs of the given functions.

63. $y = \arccos x$, $y = \arctan x$

64. $y = \arcsin x$, $y = \arccos x$

65. A small boat is being pulled toward a dock that is 10 feet above the water. The rope is being pulled in at a rate of 1.5 feet per second. Find the rate at which the angle the rope makes with the horizontal is changing when 20 feet of rope is out.

66. An observer is standing 300 feet from the point at which a balloon is released. The balloon rises at a rate of 5 feet per second. How fast is the angle of elevation of the observer's line of sight increasing when the balloon is 100 feet high?

67. Verify the following differentiation formulas.

(a) $\frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2}$

(b) $\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$

(c) $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}$

(d) $\frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1 + u^2}$

(e) $\frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$

68. Show that the function

$$f(x) = \arcsin\left(\frac{x - 2}{2}\right) - 2 \arcsin \frac{\sqrt{x}}{2}$$

is constant for $0 \le x \le 4$.

6.8 Inverse Trigonometric Functions: Integration and Completing the Square

Integrals involving inverse trigonometric functions ▪ Completing the square ▪ Review of basic integration formulas

The derivatives of the six inverse trigonometric functions occur in three pairs. In each pair the derivative of one function is the negative of the other. For example,

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1 - x^2}} \quad \text{and} \quad \frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1 - x^2}}.$$

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, we need use only one member from each pair. For example, we choose to use $\arcsin x$ as the antiderivative of $1/\sqrt{1 - x^2}$, rather than $-\arccos x$. The next theorem gives one antiderivative formula for each of the three pairs.

THEOREM 6.19
INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a > 0$.

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \text{arcsec} \frac{|u|}{a} + C$$

PROOF We prove the first formula and leave the remaining proofs as an exercise (see Exercise 52). Let $y = \arcsin(u/a)$. Then

$$y' = \frac{1}{\sqrt{1 - (u/a)^2}}\left(\frac{u'}{a}\right) = \frac{u'}{a\sqrt{(a^2 - u^2)/a^2}} = \frac{u'}{\sqrt{a^2 - u^2}}.$$

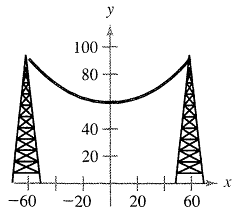


FIGURE FOR 70

71. Repeat Exercise 70, if the equation for the catenary is given by

$$y = a \cosh \frac{x}{a}, \quad -b \le x \le b$$

where a is measured in feet and the distance between the towers is $2b$ feet.

72. A barn is 100 feet long and 40 feet wide (see figure). A cross section of the roof is the inverted catenary

$$y = 31 - 20 \cosh \frac{x}{20}.$$

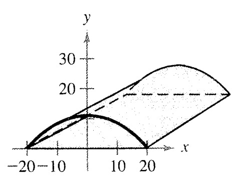


FIGURE FOR 72

Find the number of cubic feet of storage space in the barn.

73. In the Chapter 6 Application, we introduced the following equation for the catenary for the Gateway Arch,

$$y = 757.71 - 127.71 \cosh \frac{x}{127.71}.$$

Show that the height of the Gateway Arch is the same as the distance between its two legs.

74. Use the formula given in Exercise 73 to find the arc length of the Gateway Arch.

In Exercises 75–79, verify the given derivative formula.

75. $\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x$

76. $\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$

77. $\frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2 - 1}}$

78. $\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{x^2 + 1}}$

79. $\frac{d}{dx}[\operatorname{sech}^{-1} x] = \frac{-1}{x\sqrt{1 - x^2}}$

REVIEW EXERCISES for Chapter 6

In Exercises 1–6, (a) find the inverse of the given function, (b) sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes, and (c) verify that $f^{-1}[f(x)] = f[f^{-1}(x)] = x$.

1. $f(x) = \frac{1}{2}x - 3$ **2.** $f(x) = 5x - 7$

3. $f(x) = \sqrt{x + 1}$ **4.** $f(x) = x^3 + 2$

5. $f(x) = x^2 - 5, x \ge 0$ **6.** $f(x) = \sqrt[3]{x + 1}$

In Exercises 7 and 8, the function does not have an inverse. Give a restriction on the domain so that the restricted function has an inverse, and then find the inverse.

7. $f(x) = 2(x - 4)^2$ **8.** $f(x) = |x - 2|$

In Exercises 9–12, solve the given equation for x.

9. $e^{\ln x} = 3$ **10.** $\ln x + \ln (x - 3) = 0$

11. $\log_3 x + \log_3 (x - 1) - \log_3 (x - 2) = 2$

12. $\log_x 125 = 3$

In Exercises 13–40, find dy/dx.

13. $y = \ln \sqrt{x}$ **14.** $y = \ln \frac{x(x - 1)}{x - 2}$

15. $y = x\sqrt{\ln x}$ **16.** $y = \ln [x(x^2 - 2)^{2/3}]$

17. $y \ln x + y^2 = 0$ **18.** $\ln (x + y) = x$

19. $y = \frac{1}{b^2}\left[\ln (a + bx) + \frac{a}{a + bx}\right]$

20. $y = \frac{1}{b^2}[a + bx - a \ln (a + bx)]$

21. $y = -\frac{1}{a} \ln \frac{a + bx}{x}$

22. $y = -\frac{1}{ax} + \frac{b}{a^2} \ln \frac{a + bx}{x}$

23. $y = \ln (e^{-x^2})$ **24.** $y = \ln \frac{e^x}{1 + e^x}$

25. $y = x^2 e^x$ **26.** $y = e^{-x^2/2}$

27. $y = \sqrt{e^{2x} + e^{-2x}}$ **28.** $y = x^{2x+1}$

29. $y = 3^{x-1}$ **30.** $y = (4e)^x$

31. $ye^x + xe^y = xy$ **32.** $y = \frac{x^2}{e^x}$

33. $\cos x^2 = xe^y$ **34.** $y = \frac{1}{2}e^{\sin 2x}$

35. $y = \tan(\arcsin x)$ **36.** $y = \arctan(x^2 - 1)$

37. $y = x \operatorname{arcsec} x$ **38.** $y = \frac{1}{2}\arctan e^{2x}$

39. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1 - x^2}\arcsin x$

40. $y = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \frac{x}{2}, \ 0 < x < \frac{\pi}{4}$

41. Find the derivative of each of the following, given that a is constant.
(a) $y = x^a$ (b) $y = a^x$
(c) $y = x^x$ (d) $y = a^a$

42. Show that

$$y = e^x(a \cos 3x + b \sin 3x)$$

satisfies the differential equation $y'' - 2y' + 10y = 0$.

In Exercises 43–70, evaluate the integral.

43. $\int \frac{1}{7x - 2}\,dx$ **44.** $\int \frac{x}{x^2 - 1}\,dx$

45. $\int \frac{\sin x}{1 + \cos x}\,dx$ **46.** $\int \frac{\ln \sqrt{x}}{x}\,dx$

47. $\int \frac{x^2 + 3}{x}\,dx$ **48.** $\int \frac{x + 2}{2x + 3}\,dx$

49. $\int_1^4 \frac{x + 1}{x}\,dx$ **50.** $\int_1^e \frac{\ln x}{x}\,dx$

51. $\int_0^{\pi/3} \sec \theta\,d\theta$ **52.** $\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right) dx$

53. $\int xe^{-3x^2}\,dx$ **54.** $\int \frac{e^{1/x}}{x^2}\,dx$

55. $\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx$ **56.** $\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

57. $\int \frac{e^x}{e^x - 1}\,dx$ **58.** $\int x^2 e^{x^3 + 1}\,dx$

59. $\int xe^{-x^2/2}\,dx$ **60.** $\int \frac{x - 1}{3x^2 - 6x - 1}\,dx$

61. $\int \frac{e^{-2x}}{1 + e^{-2x}}\,dx$ **62.** $\int \frac{\tan(1/x)}{x^2}\,dx$

63. $\int \frac{1}{e^{2x} + e^{-2x}}\,dx$ **64.** $\int \frac{1}{3 + 25x^2}\,dx$

65. $\int \frac{x}{\sqrt{1 - x^4}}\,dx$ **66.** $\int \frac{1}{16 + x^2}\,dx$

67. $\int \frac{x}{16 + x^2}\,dx$ **68.** $\int \frac{\arcsin x}{\sqrt{1 - x^2}}\,dx$

69. $\int \frac{\arctan(x/2)}{4 + x^2}\,dx$ **70.** $\int \frac{4 - x}{\sqrt{4 - x^2}}\,dx$

In Exercises 71 and 72, sketch the graph of the region whose area is given by the integral and find the area.

71. $\int_0^{\pi/3} \tan x\,dx$ **72.** $\int_0^1 \frac{1}{x + 1}\,dx$

In Exercises 73 and 74, find the area of the region bounded by the graphs of the equations.

73. $y = xe^{-x^2}, \ y = 0, \ x = 0, \ x = 4$
74. $y = 3e^{-x/2}, \ y = 0, \ x = 0, \ x = 4$

In Exercises 75 and 76, use Simpson's Rule to approximate the definite integral.

75. $\int_0^1 e^{x^3}\,dx, \ n = 4$

76. $\int_0^1 e^{-x^2}\,dx, \ n = 4$

77. A deposit of \$500 earns interest at the rate of 5 percent compounded continuously. Find its value after each of the following time periods.
(a) 1 year
(b) 10 years
(c) 100 years

78. A deposit earns interest at the rate of r percent compounded continuously and doubles in value in 10 years. Find r.

79. How large a deposit, at 7 percent interest compounded continuously, must be made to obtain a balance of \$10,000 in 15 years?

80. A deposit of \$2500 is made in a savings account at an annual interest rate of 12 percent compounded continuously. Find the average balance in this account during the first five years.

81. A population is growing continuously at the rate of $2\frac{1}{2}$ percent per year. Find the time necessary for the population to (a) double and (b) triple in size.

82. Under ideal conditions, air pressure decreases continuously with height above sea level at a rate proportional to the pressure at that height. If the barometer reads 30 inches at sea level and 15 inches at 18,000 feet, find the barometric pressure at 35,000 feet.

In Exercises 83 and 84, use the following model for human memory.

$$p(t) = 80e^{-0.5t} + 20$$

where $p(t)$ is the percentage retained after t weeks (see figure).

83. At what rate is information being retained after (a) 1 week and (b) 2 weeks?

84. Find the average percentage retained during (a) the first 2 weeks and (b) the second 2 weeks.