IB Mathematics Analysis and Approaches HL Class of 2022
Summer Assignment to Prepare for Year Two

1. Take some time off from school completely

2. Solve some problems
   - Mathematics is a language, a skill set, and a way of thinking. It's good to take a break from it, but it's also essential to stay in practice.
   - There are 17 practice problems in this assignment, space them out as you wish.
   - Hopefully you'll find them to be interesting, a little challenging, but accessible and not too many!
   - They are attached below. Solve them to exercise your mathematical knowledge and prepare for year two.
   - Bring your solutions to school on the first day of classes. (I hope we'll be back!!)

3. Lay the groundwork for your mathematical exploration (IA)

The IB Math Internal Assessment is a mathematical exploration. It's not about finding especially advanced mathematics. Rather it's about exploring something that is personally interesting to you (either pure mathematics or an application), then communicating your work in a report (12-20 pages, including diagrams and graphs). Your mathematical explanation should be at a level your peers would find accessible, interesting, and informative. Full details and criteria are in the course guide starting on p.78 (SL and HL have the same criteria except Criterion E). But the best advice comes from the seniors who wrote IAs this year!

Some additional information from the IB about how to write a good IA:
- Students should be advised to try to present a personal example and try to obtain a solution rather than reproduce general information found in sources.
- The mathematics used need only be what is required to support the development of the exploration. This could be a few small topics or even a single topic from the syllabus. It will be better to do a few things well, rather than a lot of things not so well.
- The student should always explain why every step is done and how results are obtained.
- If the mathematics used is relevant to the topic being explored, commensurate with the course, and understood by the student, then it can achieve a high level.
- Mathematical understanding must be *demonstrated*. Obtaining a correct answer is not sufficient to demonstrate understanding.
- The exploration must meet the stated aim, and be concise and complete.

Think about what you might be interested in exploring and writing on for your Math IA in the fall.
- If you google IA topics, one of first hits is this mind map of ideas for explorations. But this is generic. What would a mind map look like based on **your own interests**?
- You don't need to turn anything in this summer. But it will be key for you to write things down for yourself.
- The IA expectations have changed very little from the previous IB math courses. Here are examples of real IAs with Criterion Marking and Comments from previous IB Math SL and HL students.
- Feel free to email me with ideas over the summer. I would love to hear from you and give you feedback.

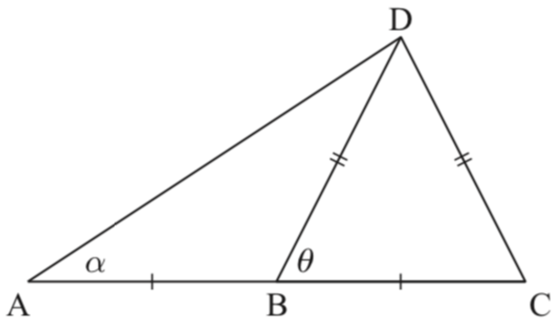
I wish you an excellent summer, and I look forward to a successful Year Two of Math Analysis HL!

Ashton Betancourt (betancourta@asf.edu.mx)

Summer Practice Problems for Math Analysis HL:

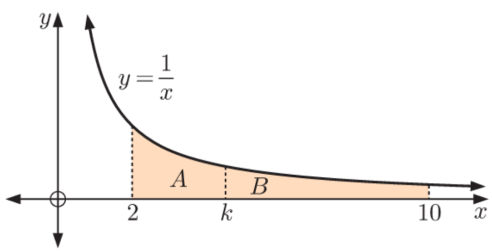
No calculator needed for any of these problems.

1. AB = BC. Show that $\tan(\theta) = 3\tan(\alpha)$



2.

Find k such that area A and area B are equal.



3. For this question, you may assume that the domain of f is restricted to $x > 0$.

The function $f(x)$ has the following properties: $f'(x) > 0$ and $f''(x) < 0$ for all x, $f(2) = 1$, and $f'(2) = 2$.

a Find the equation of the tangent to $f(x)$ at $x = 2$, and sketch it on a graph.

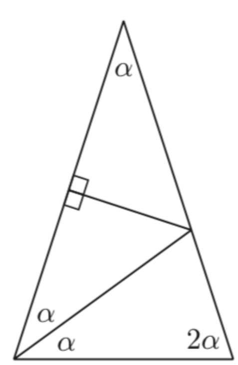
b Sketch a possible graph of $y = f(x)$ on the same set of axes.

c Explain why $f(x)$ has exactly one zero.

d Estimate an interval in which the zero of $f(x)$ lies.

4.

Use the figure alongside to show that $\cos 36° = \frac{1+\sqrt{5}}{4}$.



5.

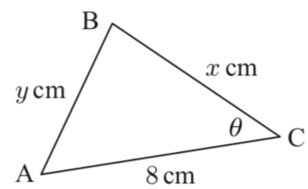
If $x^2 + y^2 = 52xy$, and $0 < y < x$, show that $\log\left(\frac{x-y}{5}\right) = \frac{1}{2}(\log x + \log 2y)$.

6.

$x^n + ax^2 - 6$ leaves a remainder of -3 when divided by $(x-1)$, and a remainder of -15 when divided by $(x+3)$. Find the values of a and n.

7.

Triangle ABC has perimeter 20 cm.



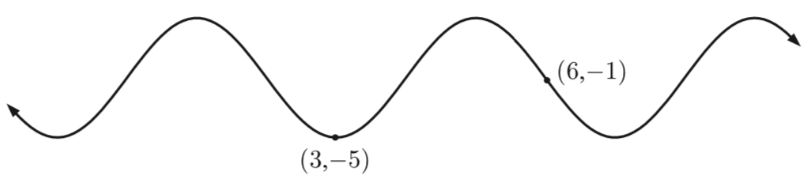
a Find y^2 in terms of x and θ and hence write $\cos\theta$ in terms of x only.

b If the triangle has area A, show that $A^2 = -20(x^2 - 12x + 20)$.

c Find the maximum area of the triangle. Comment on the triangle's shape when its area is a maximum.

8.

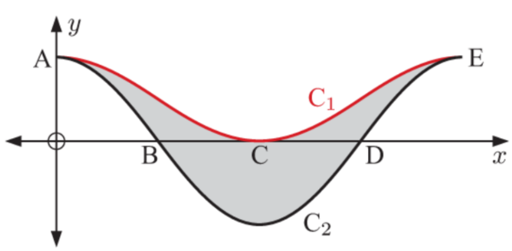
Find a trigonometric equation of the form $y = a\sin(b(x+c)) + d$ that represents the following graph with the information given below.
You may assume that $(3, -5)$ is a minimum point and $(6, -1)$ lies on the principal axis.



9.

If $f(2x+3) = 5x - 7$, find $f^{-1}(x)$.

10.



The illustrated curves are $y = \cos 2x$ and $y = \cos^2 x$.

a Identify each curve.

b Determine the coordinates of A, B, C, D, and E.

c Show that the shaded area is $\frac{\pi}{2}$ units^2.

11.

Find exact solutions for the following:

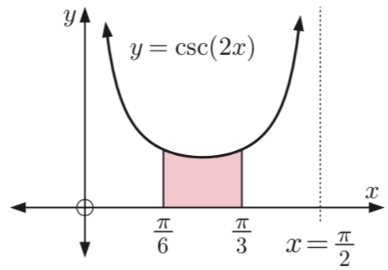
a $|1 - 4x| > \frac{1}{3}|2x - 1|$

b $\frac{x-2}{6 - 5x - x^2} \leqslant 0$

12. When you take the derivative of $y = \ln(\tan x)$, you should get $\sin x \cos x$ in the denominator. Part (a) of this problem asks you to use a trigonometric identity to change the form of your answer.

a Given that $y = \ln(\tan x)$, $x \in \left]0, \frac{\pi}{2}\right[$, show that $\frac{dy}{dx} = k\csc(2x)$ for some constant k.

b The graph of $y = \csc(2x)$ is illustrated on the interval $\left]0, \frac{\pi}{2}\right[$.
Find the area of the shaded region. Give your answer in the form $a \ln b$ where $a \in \mathbb{Q}$ and $b \in \mathbb{Z}^+$.



13.

a Show that $\sqrt{14 - 4\sqrt{6}}$ cannot be written in the form $a + b\sqrt{6}$ where $a, b \in \mathbb{Z}$.

b Write $\sqrt{14 - 4\sqrt{6}}$ in the form $a\sqrt{m} + b\sqrt{n}$ where $a, b, m, n \in \mathbb{Z}$.

14. Feel free to leave your answer in whatever form you like:

The real polynomial $P(z)$ of degree 4 has one complex zero of the form $1 - 2i$, and another of the form ai, where $a \neq 0$, $a \in \mathbb{R}$.
Find $P(z)$ if $P(0) = 10$ and the coefficient of z^4 is 1. Leave your answer in factorised form.

15. In triangle ABC,

$$3\sin B + 4\cos C = 6 \text{ and}$$
$$4\sin C + 3\cos B = 1.$$

(a) Show that $\sin(B + C) = \frac{1}{2}$. *[6]*

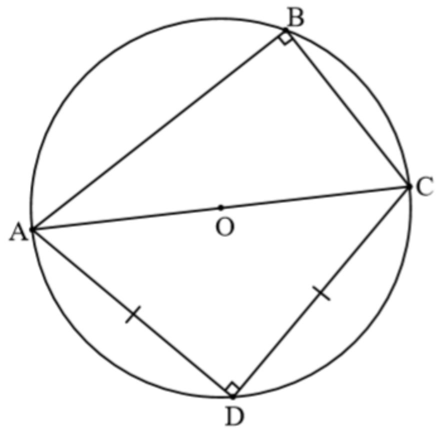
Robert conjectures that $C\hat{A}B$ can have two possible values.

(b) Show that Robert's conjecture is incorrect by proving that $C\hat{A}B$ has only one possible value. *[5]*

16.

(a) Given that $\cos 75° = q$, show that $\cos 105° = -q$. [1]

In the following diagram, the points A, B, C and D are on the circumference of a circle with centre O and radius r. [AC] is a diameter of the circle. $BC = r$, $AD = CD$ and $A\hat{B}C = A\hat{D}C = 90°$.



(b) Show that $B\hat{A}D = 75°$. [3]

(c) (i) By considering triangle ABD, show that $BD^2 = 5r^2 - 2r^2 q\sqrt{6}$.

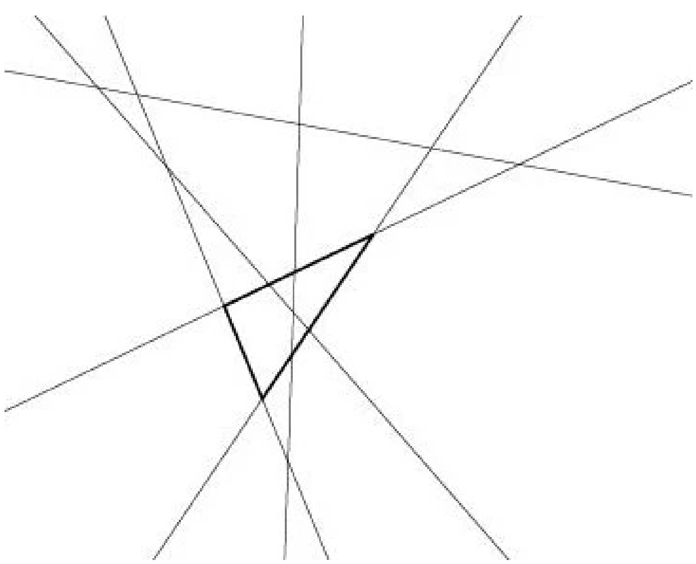
(ii) By considering triangle CBD, find another expression for BD^2 in terms of r and q. [7]

(d) Use your answers to part (c) to show that $\cos 75° = \dfrac{1}{\sqrt{6}+\sqrt{2}}$. [3]

17. Try out this problem to start thinking about probability, the big new topic we'll see next fall:
Two sticks have length a and b with a > b. You break the longer stick at a random point. What is the probability that the resulting three sticks form a triangle? (Hint: Try some examples, perhaps start with a=20cm and b=10cm.)

<u>An optional question looking towards year two - Counting</u>

How many triangles are there in the picture below? (One triangle is highlighted as an example.)



(From Po-Shen Loh)

Another optional question looking towards year two - An Introduction to Probability:

Perhaps you are familiar with a neurodegenerative disorder called Huntington's Disease.

There is a very accurate test that can diagnose whether you will develop Huntington's Disease. According to this study, the test is approximately 99.5% sensitive and 99.2% specific.[1] That means that 99.5% of people who will develop the disease test positive, and 99.2% of those who will not develop the disease test negative (0.8% false positive rate).

Huntington's disease is inherited in an autosomal dominant pattern.[2] "In this type of inheritance pattern, each offspring of an affected individual has a 50% risk of inheriting the mutant allele and therefore being affected with the disorder."[3] If you have a family history of the disease, you may well wish to take this test. However, "up to 10% of cases are due to a new mutation."[4]

What if you wish to take the test even though you do not have a family history, perhaps to rule out the possibility that you will develop the disease?

To get an idea of how common the disease is, "About 30,000 people in the United States have Huntington's disease and another 200,000 are at risk of developing the condition."[5] Out of a population of 330 million[6], this means that the prevalence of this disease is approximately 9 cases per 100,000 people.

Question: Suppose that you do not have a family history of Huntington's disease, but you are curious. You take the test. It comes back positive. **What is the probability that you will develop the disease?**

- Think about how you would diagram and calculate this probability.
- Feel free to use a calculator for the operations.

[1] https://pubmed.ncbi.nlm.nih.gov/22237433/
[2] https://ghr.nlm.nih.gov/condition/huntington-disease#inheritance
[3] https://en.wikipedia.org/wiki/Huntington%27s_disease#:~:text=It%20is%20rare%20for%20Huntington's,all%20children%20will%20be%20affected).
[4] https://en.wikipedia.org/wiki/Huntington%27s_disease#Inheritance
[5] https://rarediseases.org/rare-diseases/huntingtons-disease/
[6] https://www.census.gov/popclock/