IB MATHEMATICS: ANALYSIS AND APPROACHES HL 2021-2023

Ashton Betancourt (betancourta@asf.edu.mx)

Welcome to IB Mathematics Analysis and Approaches HL! This course was created for students who love mathematical problem-solving and proof. You will have the opportunity to solve challenging problems, share your reasoning, and demonstrate why your solutions are correct.

A desire to learn, however, is not enough by itself. To be successful, it is required that you start the course with a substantial amount of background knowledge. This is material that you have studied in previous courses. Now it is your responsibility to own this material - to know it and know how to use it.

Summer task #1:

- Do you like solving problems? Do you like challenges?
- Starting on page two of this document you will find 25 math competition problems, all of them solvable with math that you have learned in your previous courses.
- Get a notebook in which you will be proud of presenting your work. Write your name on it.
- Out of the 25 problems, choose 10 that interest you. Solve them as best you can, and write your solutions in your notebook.
- Bring your notebook to the first day of classes back in school! Be ready to discuss your ideas.

Summer task #2:

- Review the Haese textbook sections in this background knowledge document, to see the knowledge and skills expected to start the IB Math HL course. You will find topics that you know well, and perhaps others that you may have to study or review in detail - either because you never learned the material well, or perhaps you learned it a long time ago.

- Solve the following problems, and write your solutions in your notebook:
    - Background Knowledge: Exercise H (p.14) #5e, Ex. I (p.17) #5h, 9, Ex. M #23, 24, Ex. N.1 #2b, 4, 6, Ex. N.3 #5, and Ex. N.4 #3, 5
    - Statistics: Ex. C.1 (p.14) #3 (check statistical vocabulary and concepts)
    - Chapter 1 Functions: Review set 1B (p.34) #1, 3, 4, 6, 7, 9.
    - Chapter 7 Quadratics: Ex. 7D.1 #3d, Ex. 7G #9, 10, 11, Ex. 7I.2 #4, Ex. 7K #3d

It is desirable that your solutions to these problems will be correct, yet I am not interested in seeing your answers per se. (In fact the answers to all of the problems are given at the end of the chapters in this document.) The purpose of solving these problems is rather for you to practice and demonstrate good mathematical reasoning, communication, and mathematical writing. I already know how to solve these problems, what I care about is how **you** solve them!

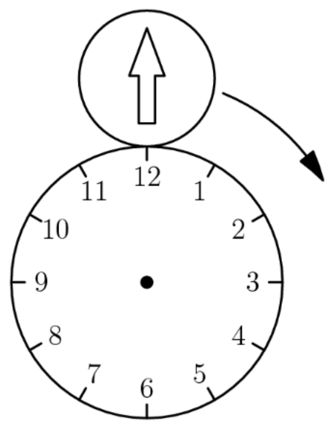
- You will be evaluated on your knowledge and understanding of this background knowledge material on the second day of classes.
- Read this sage advice from the graduating class of 2021 on how to succeed in Math HL.

I wish you an excellent summer, and I look forward to having you in my class for a successful two year course in higher level mathematics!

25 Multiple Choice Math Competition Problems

1.

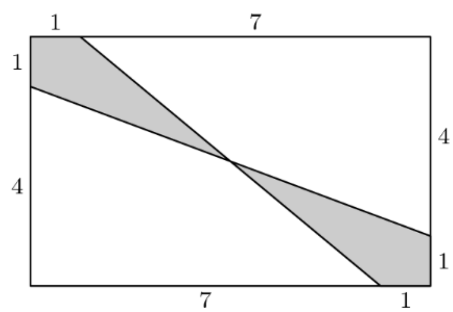
The diagram below shows the circular face of a clock with radius 20 cm and a circular disk with radius 10 cm externally tangent to the clock face at 12 o'clock. The disk has an arrow painted on it, initially pointing in the upward vertical direction. Let the disk roll clockwise around the clock face. At what point on the clock face will the disk be tangent when the arrow is next pointing in the upward vertical direction?



(A) $2o'$clock **(B)** $3o'$clock **(C)** $4o'$clock **(D)** $6o'$clock **(E)** $8o'$clock

2.

Find the area of the shaded region.



(A) $4\frac{3}{5}$ **(B)** 5 **(C)** $5\frac{1}{4}$ **(D)** $6\frac{1}{2}$ **(E)** 8

3.

Claudia has 12 coins, each of which is a 5-cent coin or a 10-cent coin. There are exactly 17 different values that can be obtained as combinations of one or more of her coins. How many 10-cent coins does Claudia have?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

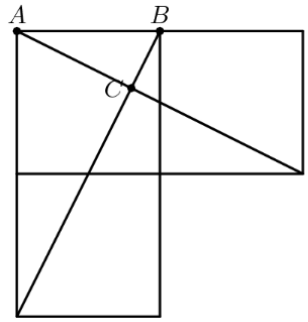
4.

Two different positive numbers a and b each differ from their reciprocals by 1. What is $a+b$?

(A) 1 (B) 2 (C) $\sqrt{5}$ (D) $\sqrt{6}$ (E) 3

5.

Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of $\triangle ABC$?



(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{\sqrt{2}}{4}$

6.

Older television screens have an aspect ratio of $4:3$. That is, the ratio of the width to the height is $4:3$. The aspect ratio of many movies is not $4:3$, so they are sometimes shown on a television screen by "letterboxing" - darkening strips of equal height at the top and bottom of the screen, as shown. Suppose a movie has an aspect ratio of $2:1$ and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip?

(A) 2 (B) 2.25 (C) 2.5 (D) 2.7 (E) 3

7.

Three distinct integers are selected at random between 1 and 2016, inclusive. Which of the following is a correct statement about the probability p that the product of the three integers is odd?

(A) $p < \frac{1}{8}$ **(B)** $p = \frac{1}{8}$ **(C)** $\frac{1}{8} < p < \frac{1}{3}$ **(D)** $p = \frac{1}{3}$ **(E)** $p > \frac{1}{3}$

8.

Three fair six-sided dice are rolled. What is the probability that the values shown on two of the dice sum to the value shown on the remaining die?

(A) $\frac{1}{6}$ **(B)** $\frac{13}{72}$ **(C)** $\frac{7}{36}$ **(D)** $\frac{5}{24}$ **(E)** $\frac{2}{9}$

9.

Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 P.M. How long, in minutes, was each day's lunch break?

(A) 30 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 60

10.

Two points on the circumference of a circle of radius r are selected independently and at random. From each point a chord of length r is drawn in a clockwise direction. What is the probability that the two chords intersect?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

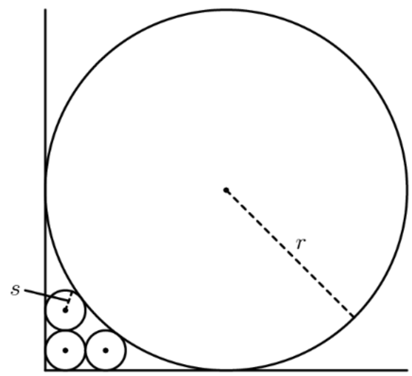
11.

Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20 (B) 40 (C) 60 (D) 160 (E) 320

12.

Three circles of radius s are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?



(A) 5 (B) 6 (C) 8 (D) 9 (E) 10

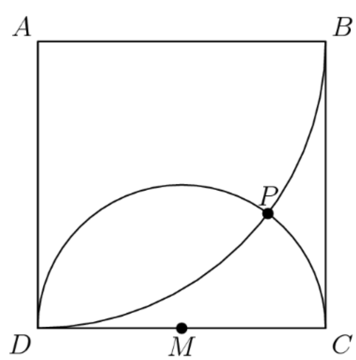
13.

A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. If the odometer now reads 002005, how many miles has the car actually traveled?

(A) 1404 (B) 1462 (C) 1604 (D) 1605 (E) 1804

14.

Square $ABCD$ has sides of length 4, and M is the midpoint of $\overline{CD}$. A circle with radius 2 and center M intersects a circle with radius 4 and center A at points P and D. What is the distance from P to $\overline{AD}$?



(A) 3 **(B)** $\frac{16}{5}$ **(C)** $\frac{13}{4}$ **(D)** $2\sqrt{3}$ **(E)** $\frac{7}{2}$

15.

The numbers from 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum?

(A) 14 (B) 16 (C) 18 (D) 20 (E) 24

16.

A piece of cheese is located at $(12, 10)$ in a coordinate plane. A mouse is at $(4, -2)$ and is running up the line $y = -5x + 18$. At the point (a, b) the mouse starts getting farther from the cheese rather than closer to it. What is $a + b$?

(A) 6 (B) 10 (C) 14 (D) 18 (E) 22

17.

Two lines with slopes $\frac{1}{2}$ and 2 intersect at $(2, 2)$. What is the area of the triangle enclosed by these two lines and the line $x + y = 10$?

(A) 4 **(B)** $4\sqrt{2}$ **(C)** 6 **(D)** 8 **(E)** $6\sqrt{2}$

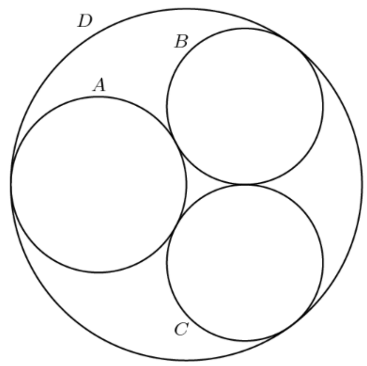
18.

Given that x and y are distinct nonzero real numbers such that $x + \frac{2}{x} = y + \frac{2}{y}$, what is xy?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** 4

19.

Circles A, B, and C are externally tangent to each other and internally tangent to circle D. Circles B and C are congruent. Circle A has radius 1 and passes through the center of D. What is the radius of circle B?



(A) $\frac{2}{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{7}{8}$ (D) $\frac{8}{9}$ (E) $\frac{1+\sqrt{3}}{3}$

20.

Consider the statement, "If n is not prime, then $n - 2$ is prime." Which of the following values of n is a counterexample to this statement?

(A) 11 **(B)** 15 **(C)** 19 **(D)** 21 **(E)** 27

21.

For how many integral values of x can a triangle of positive area be formed having side lengths $\log_2 x, \log_4 x, 3$?

(A) 57 **(B)** 59 **(C)** 61 **(D)** 62 **(E)** 63

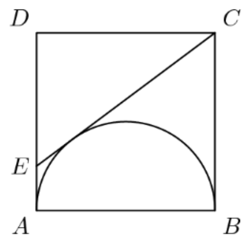
22.

A traffic light runs repeatedly through the following cycle: green for 30 seconds, then yellow for 3 seconds, and then red for 30 seconds. Leah picks a random three-second time interval to watch the light. What is the probability that the color changes while she is watching?

(A) $\frac{1}{63}$ (B) $\frac{1}{21}$ (C) $\frac{1}{10}$ (D) $\frac{1}{7}$ (E) $\frac{1}{3}$

23.

Square $ABCD$ has side length 2. A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from C intersects side $\overline{AD}$ at E. What is the length of $\overline{CE}$?



(A) $\frac{2+\sqrt{5}}{2}$ (B) $\sqrt{5}$ (C) $\sqrt{6}$ (D) $\frac{5}{2}$ (E) $5-\sqrt{5}$

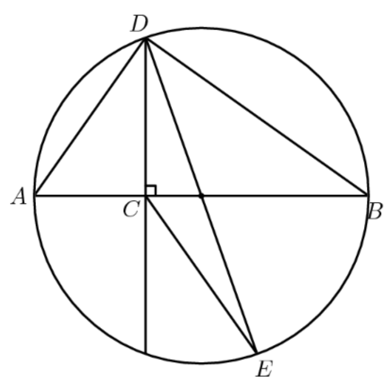
24.

On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of dots?

(A) $\frac{5}{11}$ (B) $\frac{10}{21}$ (C) $\frac{1}{2}$ (D) $\frac{11}{21}$ (E) $\frac{6}{11}$

25.

Let $\overline{AB}$ be a diameter of a circle and C be a point on $\overline{AB}$ with $2 \cdot AC = BC$. Let D and E be points on the circle such that $\overline{DC} \perp \overline{AB}$ and $\overline{DE}$ is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?



(A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

26. (An extra challenge, if you're looking for it.)

A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

(A) $\frac{11}{20}$ **(B)** $\frac{4}{7}$ **(C)** $\frac{81}{140}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{17}{28}$

(Source: AMC problems from http://artofproblemsolving.com/)

Background knowledge

A SETS

> A **set** is a collection of distinct numbers or objects.
> Each object is called an **element** or **member** of the set.

When we record a set, we write its members within curly brackets, with commas between them.

We often use a capital letter to represent a set so that we can refer to it easily.

SET NOTATION

$\in$	means	"is an element of" or "is in"
$\notin$	means	"is not an element of" or "is not in"
$n(A)$	means	"the number of elements in set A".

For example, if $P = \{\text{prime numbers less than } 20\} = \{2, 3, 5, 7, 11, 13, 17, 19\}$ then $11 \in P$, $15 \notin P$, and $n(P) = 8$.

Following is a list of some special number sets you should be familiar with. They are all endless, so they are infinite sets.

- $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, ...\}$ is the set of all **natural** or **counting numbers**.

- $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \pm4, ...\}$ is the set of all **integers**.

- $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, 6, 7, ...\}$ is the set of all **positive integers**.

- $\mathbb{Q}$ is the set of all **rational numbers**, or numbers which can be written in the form $\frac{p}{q}$ where p and q are integers, $q \neq 0$.

  For example:
  - $\frac{3}{8}$ and $\frac{-7}{5}$ are rational
  - $-3\frac{2}{3}$ is rational as $-3\frac{2}{3} = \frac{-11}{3}$
  - $0.\overline{3}$ is rational as $0.\overline{3} = \frac{3}{9} = \frac{1}{3}$
  - $\sqrt{16}$ is rational as $\sqrt{16} = \frac{4}{1}$
  - All decimal numbers that terminate or recur are rational numbers.

- $\mathbb{Q}'$ is the set of all **irrational numbers**, or numbers which cannot be written in rational form.

  For example: $\sqrt{3}$ and π are irrational.

- $\mathbb{R}$ is the set of all **real numbers**, which are all numbers which can be placed on the number line.

Notice that $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.

$\frac{1}{0}$ and $\sqrt{-2}$ cannot be placed on a number line, and so are not real.

1 Copy and complete:

Number	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
6	✓	✓	✓	✓
$-\frac{3}{8}$				
1.8				
$1.\overline{8}$				
-17				
$\sqrt{64}$				
$\frac{\pi}{2}$				
$\sqrt{-3}$				
$-\sqrt{3}$				

2 Determine whether each statement is true or false:

a $-7 \in \mathbb{Z}^+$ **b** $\frac{2}{3} \notin \mathbb{Z}$ **c** $\sqrt{3} \in \mathbb{Q}$ **d** $\frac{7}{9} \in \mathbb{Q}$

e $0.201 \in \mathbb{Z}$ **f** $\dfrac{7}{0.31} \in \mathbb{Q}$ **g** $\sqrt{|-1|} \in \mathbb{R}$ **h** $\sqrt{-9} \in \mathbb{R}$

3 Determine whether each statement is true or false:

a $\mathbb{Z}^+ \subseteq \mathbb{N}$ **b** $\mathbb{N} \subset \mathbb{Z}$ **c** $\mathbb{N} = \mathbb{Z}^+$ **d** $\mathbb{Z}^- \subseteq \mathbb{Z}$

e $\mathbb{Q} \subset \mathbb{Z}$ **f** $\{0\} \subseteq \mathbb{Z}$ **g** $\mathbb{Z} \subseteq \mathbb{Q}$ **h** $\mathbb{Z}^+ \cup \mathbb{Z}^- = \mathbb{Z}$

THEORY OF KNOWLEDGE

A number is *rational* if and only if its decimal expansion eventually terminates or recurs.

Equivalently, a number is *irrational* if and only if its decimal expansion never terminates nor recurs.

If we begin to write the decimal expansion of $\sqrt{2}$, there is no indication that it will terminate or recur, and we might therefore suspect that $\sqrt{2}$ is irrational.

$$1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\,671\,875\,376\,948\,073\ldots.$$

However, we cannot *prove* that $\sqrt{2}$ is irrational by writing out its decimal expansion, as we would have to write an infinite number of decimal places. We might therefore *believe* that $\sqrt{2}$ is irrational, but it may also seem impossible to *prove* it.

1 If something has not yet been proven, does that make it untrue?

2 Is the state of an idea being true or false dependent on our ability to prove it?

SYMBOLS AND NOTATION USED IN THIS BOOK

$\mathbb{N}$	the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm1, \pm2, \pm3, ...\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, ...\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers, $\{x \mid x > 0,\ x \in \mathbb{Q}\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, $\{x \mid x > 0,\ x \in \mathbb{R}\}$
$\mathbb{C}$	the set of complex numbers, $\{a + bi \mid a, b \in \mathbb{R}\}$
i	$\sqrt{-1}$
z	a complex number
z^*	the complex conjugate of z
$\|z\|$	the modulus of z
$\arg z$	the argument of z
$\mathcal{Re}\, z$	the real part of z
$\mathcal{Im}\, z$	the imaginary part of z
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in the finite set A
$\{x \mid ....$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$	the empty (null) set
U	the universal set
$\cup$	union
$\cap$	intersection
$\subset$	is a proper subset of
$\subseteq$	is a subset of
A'	the complement of the set A
$a^{\frac{1}{n}}$, $\sqrt[n]{a}$	a to the power of $\frac{1}{n}$, nth root of a (if $a \geqslant 0$ then $\sqrt[n]{a} \geqslant 0$)
$a^{\frac{1}{2}}$, $\sqrt{a}$	a to the power $\frac{1}{2}$, square root of a (if $a \geqslant 0$ then $\sqrt{a} \geqslant 0$)
$\|x\|$	the modulus or absolute value of x, that is $\begin{cases} x \text{ for } x \geqslant 0, & x \in \mathbb{R} \\ -x \text{ for } x < 0, & x \in \mathbb{R} \end{cases}$

$\equiv$	identity or is equivalent to
$\approx$	is approximately equal to
$>$	is greater than
$\geq$ or $\geqslant$	is greater than or equal to
$<$	is less than
$\leq$ or $\leqslant$	is less than or equal to
$\ngtr$	is not greater than
$\nless$	is not less than
$[a, b]$	the closed interval $a \leqslant x \leqslant b$
$]a, b[$	the open interval $a < x < b$
u_n	the nth term of a sequence or series
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence
S_n	the sum of the first n terms of a sequence, $u_1 + u_2 + ... + u_n$
S_∞ or S	the sum to infinity of a sequence, $u_1 + u_2 + ...$
$\sum_{i=1}^{n} u_i$	$u_1 + u_2 + ... + u_n$
$\binom{n}{r}$	$\frac{n!}{r!(n-r)!}$
$f : A \to B$	f is a function under which each element of set A has an image in set B
$f : x \mapsto y$	f is a function under which x is mapped to y
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f
$f \circ g$	the composite function of f and g
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
$\frac{dy}{dx}$	the derivative of y with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$\frac{d^2y}{dx^2}$	the second derivative of y with respect to x
$f''(x)$	the second derivative of $f(x)$ with respect to x
$\frac{d^ny}{dx^n}$	the nth derivative of y with respect to x
$f^{(n)}(x)$	the nth derivative of $f(x)$ with respect to x
$\int y\, dx$	the indefinite integral of y with respect to x

$\int_a^b y\,dx$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
e^x	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	the natural logarithm of x, $\log_e x$
sin, cos, tan	the circular functions
arcsin, arccos, arctan	the inverse circular functions
csc, sec, cot	the reciprocal circular functions
cis θ	$\cos\theta + i\sin\theta$
$\mathrm{A}(x, y)$	the point A in the plane with Cartesian coordinates x and y
[AB]	the line segment with end points A and B
AB	the length of [AB]
(AB)	the line containing points A and B
$\widehat{\mathrm{A}}$	the angle at A
$\widehat{\mathrm{CAB}}$ or $\mathrm{C\widehat{A}B}$	the angle between [CA] and [AB]
$\triangle$ABC	the triangle whose vertices are A, B, and C
$\mathbf{v}$	the vector $\mathbf{v}$
$\overrightarrow{\mathrm{AB}}$	the vector represented in magnitude and direction by the directed line segment from A to B
$\mathbf{a}$	the position vector $\overrightarrow{\mathrm{OA}}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the Cartesian coordinate axes
$\lvert \mathbf{a} \rvert$	the magnitude of vector $\mathbf{a}$
$\lvert \overrightarrow{\mathrm{AB}} \rvert$	the magnitude of $\overrightarrow{\mathrm{AB}}$
$\mathbf{v} \bullet \mathbf{w}$	the scalar product of $\mathbf{v}$ and $\mathbf{w}$
$\mathbf{v} \times \mathbf{w}$	the vector product of $\mathbf{v}$ and $\mathbf{w}$
$\mathbf{I}$	the identity matrix

$\mathrm{P}(A)$	probability of event A
$\mathrm{P}'(A)$	probability of the event "not A"
$\mathrm{P}(A \mid B)$	probability of the event A given B
$x_1, x_2, ...$	observations of a variable
$f_1, f_2, ...$	frequencies with which the observations $x_1, x_2, x_3, ...$ occur
p_x	probability distribution function $\mathrm{P}(X = x)$ of the discrete random variable X
$f(x)$	probability density function of the continuous random variable X
$\mathrm{E}(x)$	the expected value of the random variable X
$\mathrm{Var}(X)$	the variance of the random variable X
μ	population mean
σ	population standard deviation
σ^2	population variance
$\overline{x}$	sample mean
${s_n}^2$	sample variance
s_n	standard deviation of the sample
s_{n-1}^2	unbiased estimate of the population variance
$\mathrm{B}(n, p)$	binomial distribution with parameters n and p
$\mathrm{Po}(m)$	Poisson distribution with mean m
$\mathrm{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$X \sim \mathrm{B}(n, p)$	the random variable X has a binomial distribution with parameters n and p
$X \sim \mathrm{Po}(m)$	the random variable X has a Poisson distribution with mean m
$X \sim \mathrm{N}(\mu, \sigma^2)$	the random variable X has a normal distribution with mean μ and variance σ^2

Background knowledge

Contents:

This chapter contains material that is assumed knowledge for the course. It does not cover all assumed knowledge, as other necessary work is revised within the chapters.

A SURDS AND RADICALS

A **radical** is any number which is written with the **radical sign** $\sqrt{}$.

A **surd** is a real, irrational radical such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ or $\sqrt{6}$. Surds are present in solutions to some quadratic equations. $\sqrt{4}$ is a radical but is not a surd as it simplifies to 2.

$\sqrt{a}$ is the non-negative number such that $\sqrt{a} \times \sqrt{a} = a$.

Properties:
- $\sqrt{a}$ is never negative, so $\sqrt{a} \geqslant 0$.
- $\sqrt{a}$ is meaningful only for $a \geqslant 0$.
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ for $a \geqslant 0$ and $b \geqslant 0$.
- $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ for $a \geqslant 0$ and $b > 0$.

Example 1 — Self Tutor

Write as a single surd: **a** $\sqrt{2} \times \sqrt{3}$ **b** $\dfrac{\sqrt{18}}{\sqrt{6}}$

a
$$\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6}$$

b
$$\frac{\sqrt{18}}{\sqrt{6}} = \sqrt{\frac{18}{6}} = \sqrt{3}$$

or

$$\frac{\sqrt{18}}{\sqrt{6}} = \frac{\sqrt{6} \times \sqrt{3}}{\sqrt{6}} = \sqrt{3}$$

EXERCISE A

1 Write as a single surd or rational number:

a $\sqrt{3} \times \sqrt{5}$ **b** $(\sqrt{3})^2$ **c** $2\sqrt{2} \times \sqrt{2}$ **d** $3\sqrt{2} \times 2\sqrt{2}$

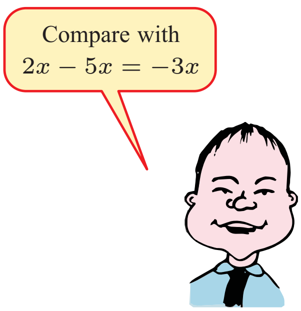
e $3\sqrt{7} \times 2\sqrt{7}$ **f** $\dfrac{\sqrt{12}}{\sqrt{2}}$ **g** $\dfrac{\sqrt{12}}{\sqrt{6}}$ **h** $\dfrac{\sqrt{18}}{\sqrt{3}}$

Example 2 — Self Tutor

Simplify: **a** $3\sqrt{3} + 5\sqrt{3}$ **b** $2\sqrt{2} - 5\sqrt{2}$

a
$$3\sqrt{3} + 5\sqrt{3} = (3+5)\sqrt{3} = 8\sqrt{3}$$

b
$$2\sqrt{2} - 5\sqrt{2} = (2-5)\sqrt{2} = -3\sqrt{2}$$



2 Simplify the following mentally:

a $2\sqrt{2}+3\sqrt{2}$ **b** $2\sqrt{2}-3\sqrt{2}$ **c** $5\sqrt{5}-3\sqrt{5}$ **d** $5\sqrt{5}+3\sqrt{5}$

e $3\sqrt{5}-5\sqrt{5}$ **f** $7\sqrt{3}+2\sqrt{3}$ **g** $9\sqrt{6}-12\sqrt{6}$ **h** $\sqrt{2}+\sqrt{2}+\sqrt{2}$

Example 3 Self Tutor

Write $\sqrt{18}$ in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible.

$\sqrt{18}$

$= \sqrt{9\times 2}$ {9 is the largest perfect square factor of 18}

$= \sqrt{9}\times\sqrt{2}$

$= 3\sqrt{2}$

3 Write the following in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible:

a $\sqrt{8}$ **b** $\sqrt{12}$ **c** $\sqrt{20}$ **d** $\sqrt{32}$

e $\sqrt{27}$ **f** $\sqrt{45}$ **g** $\sqrt{48}$ **h** $\sqrt{54}$

i $\sqrt{50}$ **j** $\sqrt{80}$ **k** $\sqrt{96}$ **l** $\sqrt{108}$

Example 4 Self Tutor

Simplify: $2\sqrt{75}-5\sqrt{27}$

$2\sqrt{75}-5\sqrt{27}$

$= 2\sqrt{25\times 3}-5\sqrt{9\times 3}$

$= 2\times 5\times\sqrt{3}-5\times 3\times\sqrt{3}$

$= 10\sqrt{3}-15\sqrt{3}$

$= -5\sqrt{3}$

4 Simplify:

a $4\sqrt{3}-\sqrt{12}$ **b** $3\sqrt{2}+\sqrt{50}$ **c** $3\sqrt{6}+\sqrt{24}$

d $2\sqrt{27}+2\sqrt{12}$ **e** $\sqrt{75}-\sqrt{12}$ **f** $\sqrt{2}+\sqrt{8}-\sqrt{32}$

Example 5 Self Tutor

Write $\frac{9}{\sqrt{3}}$ without a radical in the denominator.

$\frac{9}{\sqrt{3}}$

$= \frac{9}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{9\sqrt{3}}{3}$

$= 3\sqrt{3}$

5 Write without a radical in the denominator:

a $\frac{1}{\sqrt{2}}$ **b** $\frac{6}{\sqrt{3}}$ **c** $\frac{7}{\sqrt{2}}$ **d** $\frac{10}{\sqrt{5}}$

e $\frac{10}{\sqrt{2}}$ **f** $\frac{18}{\sqrt{6}}$ **g** $\frac{12}{\sqrt{3}}$ **h** $\frac{5}{\sqrt{7}}$

i $\frac{14}{\sqrt{7}}$ **j** $\frac{2\sqrt{3}}{\sqrt{2}}$

B SCIENTIFIC NOTATION (STANDARD FORM)

Scientific notation (or **standard form**) involves writing any given number as a number between 1 and 10, multiplied by a power of 10,

i.e., $a \times 10^k$ where $1 \leqslant a < 10$ and $k \in \mathbb{Z}$.

Example 6 Self Tutor

Write in standard form: **a** 37 600 **b** 0.000 86

a $37\,600 = 3.76 \times 10\,000$ {shift decimal point 4 places to the left and $\times$ 10 000}
$= 3.76 \times 10^4$

b $0.000\,86 = 8.6 \div 10^4$ {shift decimal point 4 places to the right and $\div$ 10 000}
$= 8.6 \times 10^{-4}$

EXERCISE B

1 Express the following in scientific notation:

a 259 **b** 259 000 **c** 2.59 **d** 0.259

e 0.000 259 **f** 40.7 **g** 4070 **h** 0.0407

i 407 000 **j** 407 000 000 **k** 0.000 040 7

2 Express the following in scientific notation:

a The distance from the Earth to the Sun is 149 500 000 000 m.

b Bacteria are single cell organisms, some of which have a diameter of 0.0003 mm.

c A speck of dust has width smaller than 0.001 mm.

d The core temperature of the Sun is 15 million degrees Celsius.

e A single red blood cell lives for about four months. During this time it will circulate around the body 300 000 times.

Example 7 **Self Tutor**

Write as an ordinary number:

a 3.2×10^2 **b** 5.76×10^{-5}

a 3.2×10^2
$= 3.20 \times 100$
$= 320$

b 5.76×10^{-5}
$= 000005.76 \div 10^5$
$= 0.000\,057\,6$

3 Write as an ordinary decimal number:

a 4×10^3 **b** 5×10^2 **c** 2.1×10^3 **d** 7.8×10^4
e 3.8×10^5 **f** 8.6×10^1 **g** 4.33×10^7 **h** 6×10^7

4 Write as an ordinary decimal number:

a 4×10^{-3} **b** 5×10^{-2} **c** 2.1×10^{-3} **d** 7.8×10^{-4}
e 3.8×10^{-5} **f** 8.6×10^{-1} **g** 4.33×10^{-7} **h** 6×10^{-7}

5 Write as an ordinary decimal number:

a The wavelength of light is 9×10^{-7} m.
b The estimated world population for the year 2000 was 6.130×10^9.
c The diameter of our galaxy, the Milky Way, is 1×10^5 light years.
d The smallest viruses are 1×10^{-5} mm in size.

6 Find, correct to 2 decimal places:

a $(3.42 \times 10^5) \times (4.8 \times 10^4)$ **b** $(6.42 \times 10^{-2})^2$ **c** $\dfrac{3.16 \times 10^{-10}}{6 \times 10^7}$

d $(9.8 \times 10^{-4}) \div (7.2 \times 10^{-6})$ **e** $\dfrac{1}{3.8 \times 10^5}$ **f** $(1.2 \times 10^3)^3$

7 If a missile travels at 5400 km h^{-1}, how far will it travel in:

a 1 day **b** 1 week **c** 2 years?

Give your answers in scientific notation correct to 2 decimal places, and assume that 1 year $\approx$ 365.25 days.


©iStockphoto.com

8 Light travels at a speed of 3×10^8 metres per second. How far will light travel in:

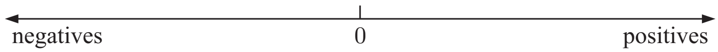
a 1 minute **b** 1 day **c** 1 year?

Give your answers with decimal part correct to 2 decimal places, and assume that 1 year $\approx$ 365.25 days.

C NUMBER SYSTEMS AND SET NOTATION

NUMBER SYSTEMS

We use:
- $\mathbb{R}$ to represent the set of all **real numbers**. These include all of the numbers on the number line.



- $\mathbb{N}$ to represent the set of all **natural numbers**. $\mathbb{N} = \{0, 1, 2, 3, 4, 5, ...\}$
- $\mathbb{Z}$ to represent the set of all **integers**. $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \pm4, ...\}$
- $\mathbb{Z}^+$ is the set of all positive integers. $\mathbb{Z}^+ = \{1, 2, 3, 4, ...\}$
- $\mathbb{Q}$ to represent the set of all **rational numbers** which are any numbers of the form $\frac{p}{q}$ where p and q are integers, $q \neq 0$.

SET NOTATION

$\{x \mid -3 < x < 2\}$ reads "the set of all values that x can be such that x lies between -3 and 2".

Unless stated otherwise, we assume that x is real.

EXERCISE C

1 Write verbal statements for the meaning of:

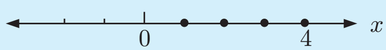
a $\{x \mid x > 5,\ x \in \mathbb{R}\}$ **b** $\{x \mid x \leqslant 3,\ x \in \mathbb{Z}\}$ **c** $\{y \mid 0 < y < 6\}$

d $\{x \mid 2 \leqslant x \leqslant 4,\ x \in \mathbb{Z}\}$ **e** $\{t \mid 1 < t < 5\}$ **f** $\{n \mid n < 2 \text{ or } n \geqslant 6\}$

Example 8 Self Tutor

Write in set notation:

a



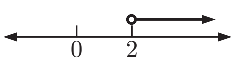
b

a $\{x \mid 1 \leqslant x \leqslant 4,\quad x \in \mathbb{N}\}$
or $\{x \mid 1 \leqslant x \leqslant 4,\quad x \in \mathbb{Z}\}$

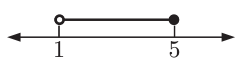
b $\{x \mid -3 \leqslant x < 4,\ x \in \mathbb{R}\}$

2 Write in set notation:

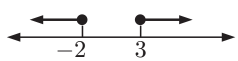
a



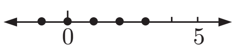
b



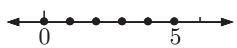
c



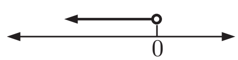
d



e



f



3 Sketch the following number sets:

a $\{x \mid 4 \leqslant x < 10,\ x \in \mathbb{N}\}$
b $\{x \mid -4 < x \leqslant 5,\ x \in \mathbb{Z}\}$
c $\{x \mid -5 < x \leqslant 4,\ x \in \mathbb{R}\}$
d $\{x \mid x > -4,\ x \in \mathbb{Z}\}$
e $\{x \mid x \leqslant 8,\ x \in \mathbb{R}\}$

D ALGEBRAIC SIMPLIFICATION

To answer the following questions, you will need to remember:

- the distributive law $\boldsymbol{a(b+c) = ab + ac}$
- power notation $a^2 = a \times a$, $a^3 = a \times a \times a$, $a^4 = a \times a \times a \times a$, and so on.

EXERCISE D

1 Simplify if possible:

a $3x + 7x - 10$
b $3x + 7x - x$
c $2x + 3x + 5y$
d $8 - 6x - 2x$
e $7ab + 5ba$
f $3x^2 + 7x^3$

2 Remove the brackets and then simplify:

a $3(2x + 5) + 4(5 + 4x)$
b $6 - 2(3x - 5)$
c $5(2a - 3b) - 6(a - 2b)$
d $3x(x^2 - 7x + 3) - (1 - 2x - 5x^2)$

3 Simplify:

a $2x(3x)^2$
b $\dfrac{3a^2b^3}{9ab^4}$
c $\sqrt{16x^4}$
d $(2a^2)^3 \times 3a^4$

E LINEAR EQUATIONS AND INEQUALITIES

When dealing with inequalities:

- multiplying or dividing both sides by a negative reverses the inequality sign.
- do not multiply or divide both sides by the unknown or a term involving the unknown.

EXERCISE E

1 Solve for x:

a $2x + 5 = 25$
b $3x - 7 > 11$
c $5x + 16 = 20$
d $\dfrac{x}{3} - 7 = 10$
e $6x + 11 < 4x - 9$
f $\dfrac{3x - 2}{5} = 8$
g $1 - 2x \geqslant 19$
h $\frac{1}{2}x + 1 = \frac{2}{3}x - 2$
i $\frac{2}{3} - \dfrac{3x}{4} = \frac{1}{2}(2x - 1)$

2 Solve simultaneously for x and y:

a $\begin{cases} x + 2y = 9 \\ x - y = 3 \end{cases}$

b $\begin{cases} 2x + 5y = 28 \\ x - 2y = 2 \end{cases}$

c $\begin{cases} 7x + 2y = -4 \\ 3x + 4y = 14 \end{cases}$

d $\begin{cases} 5x - 4y = 27 \\ 3x + 2y = 9 \end{cases}$

e $\begin{cases} x + 2y = 5 \\ 2x + 4y = 1 \end{cases}$

f $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 5 \\ \dfrac{x}{3} + \dfrac{y}{4} = 1 \end{cases}$

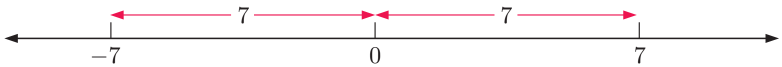
F MODULUS OR ABSOLUTE VALUE

The **modulus** or **absolute value** of a real number is its size, ignoring its sign.

For example: the modulus of 7 is 7, and
the modulus of -7 is also 7.

GEOMETRIC DEFINITION

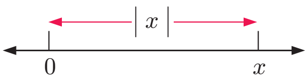
The modulus of a real number is its *distance* from zero on the number line. Because the modulus is a distance, it cannot be negative.



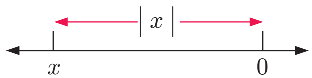
We denote the modulus of x as $|x|$.

$|x|$ is the distance of x from 0 on the real number line.

If $x > 0$



If $x < 0$



$|x - a|$ can be considered as 'the distance of x from a'.

ALGEBRAIC DEFINITION

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{or} \quad |x| = \sqrt{x^2}$$

MODULUS EQUATIONS

It is clear that $|x| = 2$ has two solutions, $x = 2$ and $x = -2$, since $|2| = 2$ and $|-2| = 2$.

If $|x| = a$ where $a > 0$, then $x = \pm a$.

EXERCISE F

1 Find the value of:

a $5-(-11)$ **b** $|5|-|-11|$ **c** $|5-(-11)|$

d $\left|(-2)^2+11(-2)\right|$ **e** $|-6|-|-8|$ **f** $|-6-(-8)|$

2 If $a=-2$, $b=3$, and $c=-4$ find the value of:

a $|a|$ **b** $|b|$ **c** $|a|\,|b|$ **d** $|ab|$

e $|a-b|$ **f** $|a|-|b|$ **g** $|a+b|$ **h** $|a|+|b|$

i $|a|^2$ **j** a^2 **k** $\left|\frac{c}{a}\right|$ **l** $\frac{|c|}{|a|}$

3 Solve for x:

a $|x|=3$ **b** $|x|=-5$ **c** $|x|=0$

d $|x-1|=3$ **e** $|3-x|=4$ **f** $|x+5|=-1$

g $|3x-2|=1$ **h** $|3-2x|=3$ **i** $|2-5x|=12$

G PRODUCT EXPANSION

$y=2(x-1)(x+3)$ can be expanded into the general form $y=ax^2+bx+c$.

Likewise, $y=2(x-3)^2+7$ can also be expanded into this form.

Following is a **list of expansion rules** you can use:

- $(a+b)(c+d)=ac+ad+bc+bd$
- $(a+b)(a-b)=a^2-b^2$ {**difference of two squares**}
- $(a+b)^2=a^2+2ab+b^2$ {**perfect squares**}

Example 9 Self Tutor

Expand and simplify:

a $(2x+1)(x+3)$ **b** $(3x-2)(x+3)$

a $(2x+1)(x+3)$
$=2x^2+6x+x+3$
$=2x^2+7x+3$

b $(3x-2)(x+3)$
$=3x^2+9x-2x-6$
$=3x^2+7x-6$

EXERCISE G

1 Expand and simplify using $(a+b)(c+d) = ac+ad+bc+bd$:

a $(2x+3)(x+1)$ **b** $(3x+4)(x+2)$ **c** $(5x-2)(2x+1)$
d $(x+2)(3x-5)$ **e** $(7-2x)(2+3x)$ **f** $(1-3x)(5+2x)$
g $(3x+4)(5x-3)$ **h** $(1-3x)(2-5x)$ **i** $(7-x)(3-2x)$
j $(5-2x)(3-2x)$ **k** $-(x+1)(x+2)$ **l** $-2(x-1)(2x+3)$

Example 10 **Self Tutor**

Expand using the rule $(a+b)(a-b) = a^2 - b^2$:

a $(5x-2)(5x+2)$ **b** $(7+2x)(7-2x)$

a $(5x-2)(5x+2)$
$= (5x)^2 - 2^2$
$= 25x^2 - 4$

b $(7+2x)(7-2x)$
$= 7^2 - (2x)^2$
$= 49 - 4x^2$

2 Expand using the rule $(a+b)(a-b) = a^2 - b^2$:

a $(x+6)(x-6)$ **b** $(x+8)(x-8)$ **c** $(2x-1)(2x+1)$
d $(3x-2)(3x+2)$ **e** $(4x+5)(4x-5)$ **f** $(5x-3)(5x+3)$
g $(3-x)(3+x)$ **h** $(7-x)(7+x)$ **i** $(7+2x)(7-2x)$
j $(x+\sqrt{2})(x-\sqrt{2})$ **k** $(x+\sqrt{5})(x-\sqrt{5})$ **l** $(2x-\sqrt{3})(2x+\sqrt{3})$

Example 11 **Self Tutor**

Expand using the perfect square expansion rule:

a $(x+2)^2$ **b** $(3x-1)^2$

a $(x+2)^2$
$= x^2 + 2(x)(2) + 2^2$
$= x^2 + 4x + 4$

b $(3x-1)^2$
$= (3x)^2 + 2(3x)(-1) + (-1)^2$
$= 9x^2 - 6x + 1$

3 Expand and simplify using the perfect square expansion rule:

a $(x+5)^2$ **b** $(x+7)^2$ **c** $(x-2)^2$ **d** $(x-6)^2$
e $(3+x)^2$ **f** $(5+x)^2$ **g** $(11-x)^2$ **h** $(10-x)^2$
i $(2x+7)^2$ **j** $(3x+2)^2$ **k** $(5-2x)^2$ **l** $(7-3x)^2$

4 Expand the following into the general form $y = ax^2+bx+c$:

a $y = 2(x+2)(x+3)$ **b** $y = 3(x-1)^2+4$ **c** $y = -(x+1)(x-7)$
d $y = -(x+2)^2 - 11$ **e** $y = 4(x-1)(x-5)$ **f** $y = -\frac{1}{2}(x+4)^2 - 6$
g $y = -5(x-1)(x-6)$ **h** $y = \frac{1}{2}(x+2)^2 - 6$ **i** $y = -\frac{5}{2}(x-4)^2$

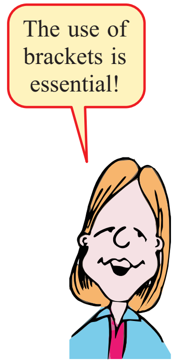
Example 12 Self Tutor

Expand and simplify:

a $1 - 2(x+3)^2$ **b** $2(3+x) - (2+x)(3-x)$

a
$$\begin{aligned} &1 - 2(x+3)^2 \\ &= 1 - 2[x^2 + 6x + 9] \\ &= 1 - 2x^2 - 12x - 18 \\ &= -2x^2 - 12x - 17 \end{aligned}$$

b
$$\begin{aligned} &2(3+x) - (2+x)(3-x) \\ &= 6 + 2x - [6 - 2x + 3x - x^2] \\ &= 6 + 2x - 6 + 2x - 3x + x^2 \\ &= x^2 + x \end{aligned}$$



5 Expand and simplify:

a $1 + 2(x+3)^2$ **b** $2 + 3(x-2)(x+3)$

c $3 - (3-x)^2$ **d** $5 - (x+5)(x-4)$

e $1 + 2(4-x)^2$ **f** $x^2 - 3x - (x+2)(x-2)$

g $(x+2)^2 - (x+1)(x-4)$ **h** $(2x+3)^2 + 3(x+1)^2$

i $x^2 + 3x - 2(x-4)^2$ **j** $(3x-2)^2 - 2(x+1)^2$

H FACTORISATION

Algebraic **factorisation** is the reverse process of expansion.

For example, $(2x+1)(x-3)$ is **expanded** to $2x^2 - 5x - 3$, whereas
$2x^2 - 5x - 3$ is **factorised** to $(2x+1)(x-3)$.

Notice that $2x^2 - 5x - 3 = (2x+1)(x-3)$ has been factorised into two **linear factors**.

Flow chart for factorising:

Expression → Take out any common factors → Recognise type

Difference of two squares
$a^2 - b^2 = (a+b)(a-b)$

Perfect square
$a^2 + 2ab + b^2 = (a+b)^2$

Sum and Product type
$x^2 + bx + c$

$x^2 + bx + c = (x+p)(x+q)$
where $p + q = b$ and $pq = c$

Sum and Product type
$ax^2 + bx + c, \quad a \neq 0$

- find ac
- find the factors of ac which add to b
- if these factors are p and q, replace bx by $px + qx$
- complete the factorisation

Example 13 Self Tutor

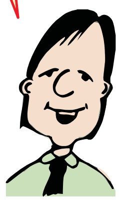
Fully factorise:

a $3x^2 - 12x$ **b** $4x^2 - 1$ **c** $x^2 - 12x + 36$

a $3x^2 - 12x$ {$3x$ is a common factor}
$= 3x(x - 4)$

b $4x^2 - 1$ {difference of two squares}
$= (2x)^2 - 1^2$
$= (2x + 1)(2x - 1)$

c $x^2 - 12x + 36$ {perfect square form}
$= x^2 + 2(x)(-6) + (-6)^2$
$= (x - 6)^2$



EXERCISE H

1 Fully factorise:

a $3x^2 + 9x$ **b** $2x^2 + 7x$ **c** $4x^2 - 10x$
d $6x^2 - 15x$ **e** $9x^2 - 25$ **f** $16x^2 - 1$
g $2x^2 - 8$ **h** $3x^2 - 9$ **i** $4x^2 - 20$
j $x^2 - 8x + 16$ **k** $x^2 - 10x + 25$ **l** $2x^2 - 8x + 8$
m $16x^2 + 40x + 25$ **n** $9x^2 + 12x + 4$ **o** $x^2 - 22x + 121$

Example 14 Self Tutor

Fully factorise:

a $3x^2 + 12x + 9$ **b** $-x^2 + 3x + 10$

a $3x^2 + 12x + 9$ {3 is a common factor}
$= 3(x^2 + 4x + 3)$ {sum $= 4$, product $= 3$}
$= 3(x + 1)(x + 3)$

b $-x^2 + 3x + 10$
$= -[x^2 - 3x - 10]$ {removing -1 as a common factor to make the coefficient of x^2 be 1}
$= -(x - 5)(x + 2)$ {sum $= -3$, product $= -10$}

2 Fully factorise:

a $x^2 + 9x + 8$ **b** $x^2 + 7x + 12$ **c** $x^2 - 7x - 18$
d $x^2 + 4x - 21$ **e** $x^2 - 9x + 18$ **f** $x^2 + x - 6$
g $-x^2 + x + 2$ **h** $3x^2 - 42x + 99$ **i** $-2x^2 - 4x - 2$

j $2x^2+6x-20$ **k** $2x^2-10x-48$ **l** $-2x^2+14x-12$

m $-3x^2+6x-3$ **n** $-x^2-2x-1$ **o** $-5x^2+10x+40$

FACTORISATION BY 'SPLITTING' THE x-TERM

Using the distributive law to expand we see that:
$$\begin{aligned}&(2x+3)(4x+5)\\&=8x^2+10x+12x+15\\&=8x^2+22x+15\end{aligned}$$

We will now **reverse** the process to **factorise** the quadratic expression $8x^2+22x+15$.

$8x^2+22x+15$

Step 1: 'Split' the middle term $=8x^2+10x+12x+15$

Step 2: Group in pairs $=(8x^2+10x)+(12x+15)$

Step 3: Factorise each pair separately $=2x(4x+5)+3(4x+5)$

Step 4: Factorise fully $=(4x+5)(2x+3)$

The 'trick' in factorising these types of quadratic expressions is in *Step 1*. The middle term is 'split' into two so the rest of the factorisation can proceed smoothly.

Rules for splitting the x-term:

The following procedure is recommended for factorising ax^2+bx+c:

- Find ac.
- Find the factors of ac which add to b.
- If these factors are p and q, replace bx by $px+qx$.
- Complete the factorisation.

Example 15 Self Tutor

Fully factorise:

a $2x^2-x-10$ **b** $6x^2-25x+14$

a $2x^2-x-10$ has $ac=2\times-10=-20$.
The factors of -20 which add to -1 are -5 and $+4$.
$$\begin{aligned}\therefore\quad 2x^2-x-10&=2x^2-5x+4x-10\\&=x(2x-5)+2(2x-5)\\&=(2x-5)(x+2)\end{aligned}$$

b $6x^2-25x+14$ has $ac=6\times14=84$.
The factors of 84 which add to -25 are -21 and -4.
$$\begin{aligned}\therefore\quad 6x^2-25x+14&=6x^2-21x-4x+14\\&=3x(2x-7)-2(2x-7)\\&=(2x-7)(3x-2)\end{aligned}$$

3 Fully factorise:

a $2x^2+5x-12$	**b** $3x^2-5x-2$	**c** $7x^2-9x+2$
d $6x^2-x-2$	**e** $4x^2-4x-3$	**f** $10x^2-x-3$
g $2x^2-11x-6$	**h** $3x^2-5x-28$	**i** $8x^2+2x-3$
j $10x^2-9x-9$	**k** $3x^2+23x-8$	**l** $6x^2+7x+2$
m $-4x^2-2x+6$	**n** $12x^2-16x-3$	**o** $-6x^2-9x+42$
p $21x-10-9x^2$	**q** $8x^2-6x-27$	**r** $12x^2+13x+3$
s $12x^2+20x+3$	**t** $15x^2-22x+8$	**u** $14x^2-11x-15$

Example 16 Self Tutor

Fully factorise: $3(x+2)+2(x-1)(x+2)-(x+2)^2$

$$\begin{aligned} & 3(x+2)+2(x-1)(x+2)-(x+2)^2 \\ &= (x+2)[3+2(x-1)-(x+2)] \quad \{\text{as } (x+2) \text{ is a common factor}\} \\ &= (x+2)[3+2x-2-x-2] \\ &= (x+2)(x-1) \end{aligned}$$

4 Fully factorise:

a $3(x+4)+2(x+4)(x-1)$	**b** $8(2-x)-3(x+1)(2-x)$
c $6(x+2)^2+9(x+2)$	**d** $4(x+5)+8(x+5)^2$
e $(x+2)(x+3)-(x+3)(2-x)$	**f** $(x+3)^2+2(x+3)-x(x+3)$
g $5(x-2)-3(2-x)(x+7)$	**h** $3(1-x)+2(x+1)(x-1)$

Example 17 Self Tutor

Fully factorise using the 'difference of two squares':

a $(x+2)^2-9$ **b** $(1-x)^2-(2x+1)^2$

a
$$\begin{aligned} & (x+2)^2-9 \\ &= (x+2)^2-3^2 \\ &= [(x+2)+3][(x+2)-3] \\ &= (x+5)(x-1) \end{aligned}$$

b
$$\begin{aligned} & (1-x)^2-(2x+1)^2 \\ &= [(1-x)+(2x+1)][(1-x)-(2x+1)] \\ &= [1-x+2x+1][1-x-2x-1] \\ &= -3x(x+2) \end{aligned}$$

5 Fully factorise:

a $(x+3)^2-16$	**b** $4-(1-x)^2$	**c** $(x+4)^2-(x-2)^2$
d $16-4(x+2)^2$	**e** $(2x+3)^2-(x-1)^2$	**f** $(x+h)^2-x^2$
g $3x^2-3(x+2)^2$	**h** $5x^2-20(2-x)^2$	**i** $12x^2-27(3+x)^2$

INVESTIGATION ANOTHER FACTORISATION TECHNIQUE

What to do:

1 By expanding, show that $\dfrac{(ax+p)(ax+q)}{a} = ax^2 + [p+q]x + \left[\dfrac{pq}{a}\right]$.

2 If $ax^2 + bx + c = \dfrac{(ax+p)(ax+q)}{a}$, show that $p+q=b$ and $pq=ac$.

3 Using **2** on $8x^2+22x+15$, we have

$$8x^2+22x+15 = \frac{(8x+p)(8x+q)}{8} \quad \text{where} \quad \begin{cases} p+q=22 \\ pq = 8\times 15 = 120 \end{cases}$$

$\therefore \quad p=12, \; q=10$, or vice versa

$$\begin{aligned} \therefore \quad 8x^2+22x+15 &= \frac{(8x+12)(8x+10)}{8} \\ &= \frac{\cancel{4}(2x+3)\cancel{2}(4x+5)}{\cancel{8}_1} \\ &= (2x+3)(4x+5) \end{aligned}$$

a Use the method shown to factorise:

i $3x^2+14x+8$ **ii** $12x^2+17x+6$ **iii** $15x^2+14x-8$

b Check your answers to **a** by expansion.

I FORMULA REARRANGEMENT

In the formula $D = xt + p$ we say that D is the **subject**. This is because D is expressed in terms of the other variables x, t and p.

We can rearrange the formula to make one of the other variables the subject. We do this using the usual rules for solving equations. Whatever we do to one side of the equation we must also do to the other side.

Example 18 **Self Tutor**

Make x the subject of $D = xt + p$.

$$\begin{aligned} \text{If} \quad D &= xt + p \\ \therefore \quad xt + p &= D \\ \therefore \quad xt + p - p &= D - p \quad \{\text{subtracting } p \text{ from both sides}\} \\ \therefore \quad xt &= D - p \\ \therefore \quad \frac{xt}{t} &= \frac{D-p}{t} \\ \therefore \quad x &= \frac{D-p}{t} \quad \{\text{dividing both sides by } t\} \end{aligned}$$

EXERCISE I

1 Make x the subject of:

a $a+x=b$ **b** $ax=b$ **c** $2x+a=d$

d $c+x=t$ **e** $5x+2y=20$ **f** $2x+3y=12$

g $7x+3y=d$ **h** $ax+by=c$ **i** $y=mx+c$

Example 19 Self Tutor

Make z the subject of $c=\dfrac{m}{z}$.

$$c=\frac{m}{z}$$

$$c\times z=\frac{m}{z}\times z \quad \{\text{multiplying both sides by } z\}$$

$$\therefore \quad cz=m$$

$$\therefore \quad \frac{cz}{c}=\frac{m}{c} \quad \{\text{dividing both sides by } c\}$$

$$\therefore \quad z=\frac{m}{c}$$

2 Make z the subject of:

a $az=\dfrac{b}{c}$ **b** $\dfrac{a}{z}=d$ **c** $\dfrac{3}{d}=\dfrac{2}{z}$ **d** $\dfrac{z}{2}=\dfrac{a}{z}$

3 Make:

a a the subject of $F=ma$ **b** r the subject of $C=2\pi r$

c d the subject of $V=ldh$ **d** K the subject of $A=\dfrac{b}{K}$.

Example 20 Self Tutor

Make t the subject of $s=\frac{1}{2}gt^2$ where $t>0$.

$$\tfrac{1}{2}gt^2=s \quad \{\text{writing with } t^2 \text{ on LHS}\}$$

$$\therefore \quad 2\times\tfrac{1}{2}gt^2=2\times s \quad \{\text{multiplying both sides by } 2\}$$

$$\therefore \quad gt^2=2s$$

$$\therefore \quad \frac{gt^2}{g}=\frac{2s}{g} \quad \{\text{dividing both sides by } g\}$$

$$\therefore \quad t^2=\frac{2s}{g}$$

$$\therefore \quad t=\sqrt{\frac{2s}{g}} \quad \{\text{as } t>0\}$$

4 Make:

a r the subject of $A = \pi r^2$ if $r > 0$

b x the subject of $N = \dfrac{x^5}{a}$

c r the subject of $V = \frac{4}{3}\pi r^3$

d x the subject of $D = \dfrac{n}{x^3}$.

5 Make:

a a the subject of $d = \dfrac{\sqrt{a}}{n}$

b l the subject of $T = \frac{1}{5}\sqrt{l}$

c a the subject of $c = \sqrt{a^2 - b^2}$

d l the subject of $T = 2\pi\sqrt{\dfrac{l}{g}}$

e a the subject of $P = 2(a + b)$

f h the subject of $A = \pi r^2 + 2\pi r h$

g r the subject of $I = \dfrac{E}{R + r}$

h q the subject of $A = \dfrac{B}{p - q}$.

6 **a** Make a the subject of the formula $k = \dfrac{d^2}{2ab}$.

b Find the value for a when $k = 112$, $d = 24$, $b = 2$.

7 The formula for determining the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

a Make r the subject of the formula.

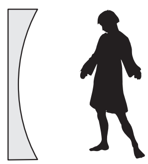
b Find the radius of a sphere which has a volume of 40 cm^3.

8 The distance travelled by an object accelerating from a stationary position is given by the formula $S = \frac{1}{2}at^2$ cm where a is the acceleration in cm s^{-2} and t is the time in seconds.

a Make t the subject of the formula. Consider $t > 0$ only.

b Find the time taken for an object accelerating at 8 cm s^{-2} to travel 10 m.

9 The relationship between the object and image distances (in cm) for a concave mirror can be written as $\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v}$ where f is the focal length, u is the object distance and v is the image distance.



a Make v the subject of the formula.

b Given a focal length of 8 cm, find the image distance for the following object distances: **i** 50 cm **ii** 30 cm.

10 According to Einstein's theory of relativity, the mass of a particle is given by the formula $m = \dfrac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$, where m_0 is the mass of the particle at rest, v is the speed of the particle, and c is the speed of light in a vacuum.

a Make v the subject of the formula given $v > 0$.

b Find the speed necessary to increase the mass of a particle to three times its rest mass, i.e., $m = 3m_0$. Give the value for v as a fraction of c.

c A cyclotron increased the mass of an electron to $30m_0$. With what velocity must the electron have been travelling, given $c = 3 \times 10^8\ \text{m s}^{-1}$?

J ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

To add or subtract algebraic fractions, we combine them into a single fraction with the **least common denominator** (LCD).

For example, $\dfrac{x-1}{3}-\dfrac{x+3}{2}$ has LCD of 6, so we write each fraction with denominator 6.

Example 21 Self Tutor

Write as a single fraction: **a** $2+\dfrac{3}{x}$ **b** $\dfrac{x-1}{3}-\dfrac{x+3}{2}$

a
$$2+\frac{3}{x}$$
$$=2\left(\frac{x}{x}\right)+\frac{3}{x}$$
$$=\frac{2x+3}{x}$$

b
$$\frac{x-1}{3}-\frac{x+3}{2}$$
$$=\frac{2}{2}\left(\frac{x-1}{3}\right)-\frac{3}{3}\left(\frac{x+3}{2}\right)$$
$$=\frac{2(x-1)-3(x+3)}{6}$$
$$=\frac{2x-2-3x-9}{6}$$
$$=\frac{-x-11}{6}$$

EXERCISE J

1 Write as a single fraction:

a $3+\dfrac{x}{5}$ **b** $1+\dfrac{3}{x}$ **c** $3+\dfrac{x-2}{2}$

d $3-\dfrac{x-2}{4}$ **e** $\dfrac{2+x}{3}+\dfrac{x-4}{5}$ **f** $\dfrac{2x+5}{4}-\dfrac{x-1}{6}$

Example 22 Self Tutor

Write $\dfrac{3x+1}{x-2}-2$ as a single fraction.

$$\frac{3x+1}{x-2}-2$$
$$=\left(\frac{3x+1}{x-2}\right)-2\left(\frac{x-2}{x-2}\right) \quad \{\text{LCD}=(x-2)\}$$
$$=\frac{(3x+1)-2(x-2)}{x-2}$$
$$=\frac{3x+1-2x+4}{x-2}$$
$$=\frac{x+5}{x-2}$$

2 Write as a single fraction:

a $1 + \frac{3}{x+2}$ **b** $-2 + \frac{3}{x-4}$ **c** $-3 - \frac{2}{x-1}$

d $\frac{2x-1}{x+1} + 3$ **e** $3 - \frac{x}{x+1}$ **f** $-1 + \frac{4}{1-x}$

3 Write as a single fraction:

a $\frac{3x}{2x-5} + \frac{2x+5}{x-2}$ **b** $\frac{1}{x-2} - \frac{1}{x-3}$

c $\frac{5x}{x-4} + \frac{3x-2}{x+4}$ **d** $\frac{2x+1}{x-3} - \frac{x+4}{2x+1}$

K CONGRUENCE AND SIMILARITY

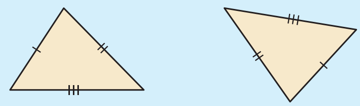
CONGRUENCE

Two triangles are **congruent** if they are identical in every respect apart from position. The triangles have the same shape and size.

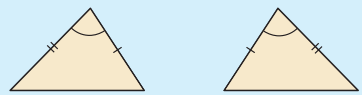
There are four acceptable tests for the **congruence of two triangles**.

Two triangles are congruent if one of the following is true:

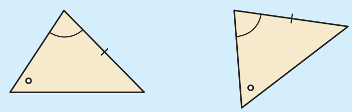
- corresponding sides are equal in length (**SSS**)



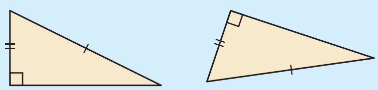
- two sides and the included angle are equal (**SAS**)



- two angles and a pair of corresponding sides are equal (**AAcorS**)



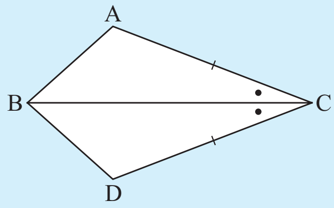
- for right angled triangles, the hypotenuse and one other pair of sides are equal (**RHS**).

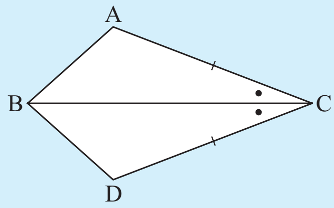


If congruence can be proven then all corresponding lengths, angles and areas must be equal.

Example 23 **Self Tutor**

Explain why ΔABC and ΔDBC are congruent:



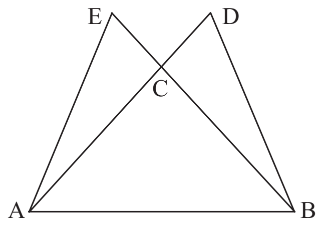


Δs ABC and DBC are congruent (SAS) as:

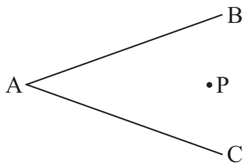
- AC = DC
- $A\widehat{C}B = D\widehat{C}B$, and
- [BC] is common to both.

EXERCISE K.1

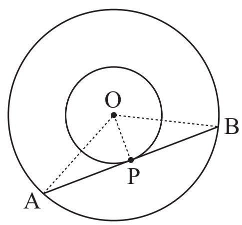
1 Triangle ABC is isosceles with AC = BC. [BC] and [AC] are produced to E and D respectively so that CE = CD. Prove that AE = BD.



2 Point P is equidistant from both [AB] and [AC]. Use congruence to show that P lies on the bisector of $B\widehat{A}C$.



3 Two concentric circles are drawn. At P on the inner circle, a tangent is drawn which meets the other circle at A and B. Use triangle congruence to prove that P is the midpoint of [AB].



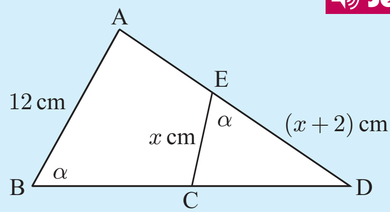
SIMILARITY

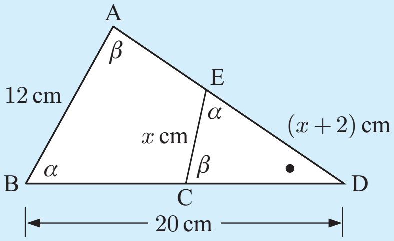
Two triangles are **similar** if one is an enlargement of the other.

Similar triangles are **equiangular**, and have corresponding sides in the same **ratio**.

Example 24 | **Self Tutor**

Establish that a pair of triangles is similar, then find x given BD = 20 cm.





α	β	$\bullet$	
-	$x+2$	x	small Δ
-	20	12	large Δ

The triangles are equiangular and hence similar.

$$\therefore \quad \frac{x+2}{20} = \frac{x}{12} \quad \{\text{sides in the same ratio}\}$$

$$\therefore \quad 12(x+2) = 20x$$

$$\therefore \quad 12x + 24 = 20x$$

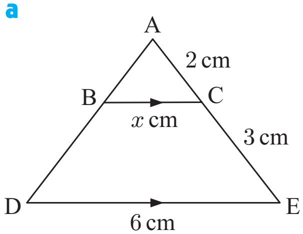
$$\therefore \quad 24 = 8x$$

$$\therefore \quad x = 3$$

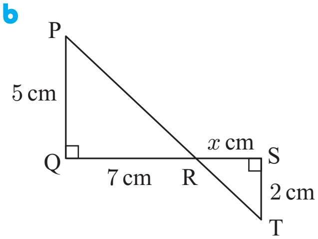
EXERCISE K.2

1 In each of the following, establish that a pair of triangles is similar, and hence find x:

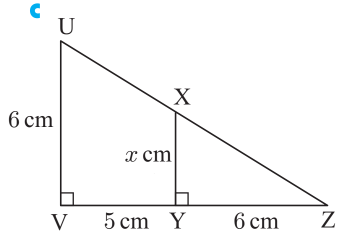
a



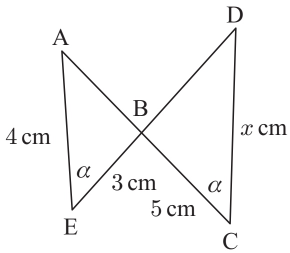
b



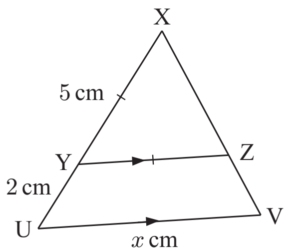
c



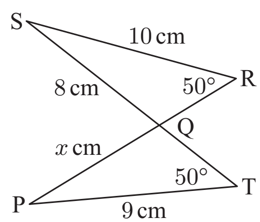
d



e



f



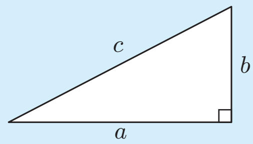
2 A father and son are standing side-by-side. The father is 1.8 m tall and casts a shadow 3.2 m long, while his son's shadow is 2.4 m long. How tall is the son?

L PYTHAGORAS' THEOREM

The **hypotenuse** is the longest side of a right angled triangle. It is opposite the right angle.

Pythagoras' Theorem is:

> In a right angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.
>
> $$c^2 = a^2 + b^2$$



This theorem, known to the ancient Greeks, is valuable because:

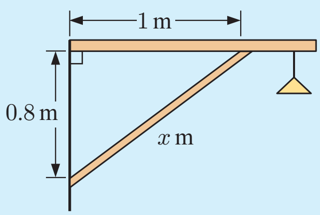
- if we know the lengths of any two sides of a right angled triangle then we can calculate the length of the third side
- if we know the lengths of the three sides then we can determine whether or not the triangle is right angled.

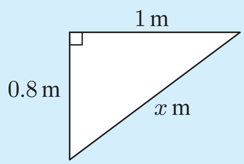
The second statement here relies on the **converse of Pythagoras' Theorem**, which is:

> If a triangle has sides of length a, b and c units and $a^2 + b^2 = c^2$ then the triangle is right angled and its hypotenuse is c units long.

Example 25 — Self Tutor

Find the unknown length in:





$x^2 = 0.8^2 + 1^2$

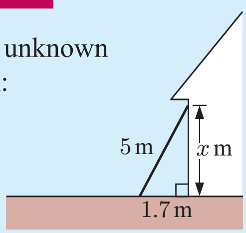
$\therefore \quad x = \sqrt{(0.8^2 + 1^2)}$

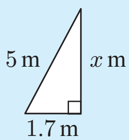
$\therefore \quad x \approx 1.2806$

So, the length is about 1.28 m.

Example 26 — Self Tutor

Find the unknown length in:





$x^2 + 1.7^2 = 5^2$

$\therefore \quad x^2 = 5^2 - 1.7^2$

$\therefore \quad x = \sqrt{(5^2 - 1.7^2)}$

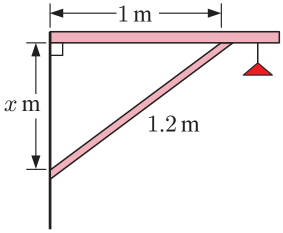
$\therefore \quad x \approx 4.7021$

So, the length is about 4.70 m.

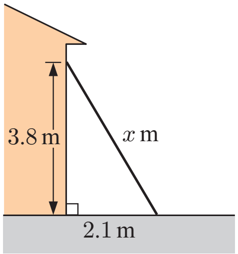
EXERCISE L.1

1 Find, correct to 3 significant figures, the value of x in:

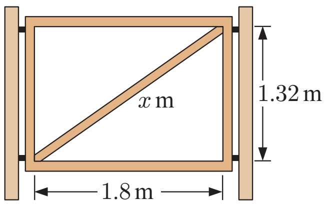
a



b

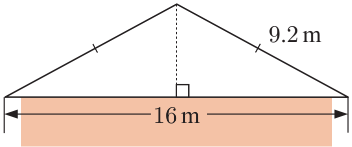


c

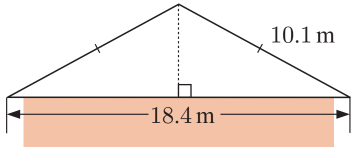


2 How high is the roof above the walls in the following roof structures?

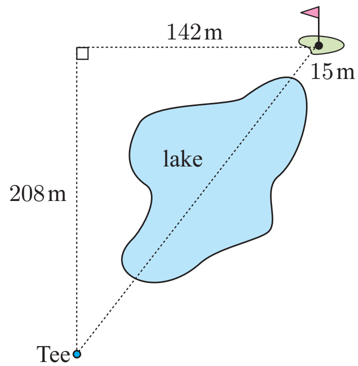
a



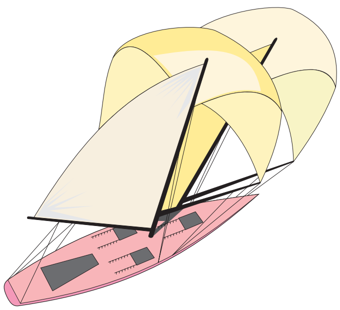
b



3 Bob is about to tee off on the sixth, a par 4 at the Royal Golf Club. He chooses to hit over the lake, directly at the flag. If the pin is 15 m from the water's edge, how far must he hit the ball to clear the lake?



4



A sailing ship sails 46 km north then 74 km east.

a Draw a fully labelled diagram of the ship's course.

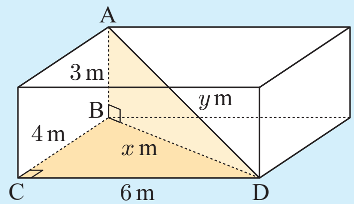
b How far is the ship from its starting point?

PYTHAGORAS' THEOREM IN 3-D PROBLEMS

The theorem of Pythagoras is often used twice in 3-D problem solving.

Example 27 Self Tutor

The floor of a room is 6 m by 4 m, and the floor to ceiling height is 3 m. Find the distance from a corner point on the floor to the opposite corner point on the ceiling.



The required distance is [AD]. We join [BD].

In ΔBCD, $x^2 = 4^2 + 6^2$ {Pythagoras}

$\therefore$ $x^2 = 16 + 36 = 52$

In ΔABD, $y^2 = x^2 + 3^2$

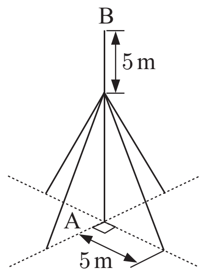
$\therefore$ $y^2 = 52 + 9 = 61$

$\therefore$ $y = \sqrt{61} \approx 7.81$

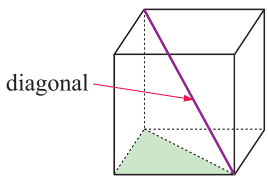
So, the distance is about 7.81 m.

EXERCISE L.2

1 A pole [AB] is 16 m tall. At a point 5 m below B, four wires are connected from the pole to the ground.
Each wire is pegged to the ground 5 m from the base of the pole.
What is the total length of wire needed if a total of 2 m extra is needed for tying?



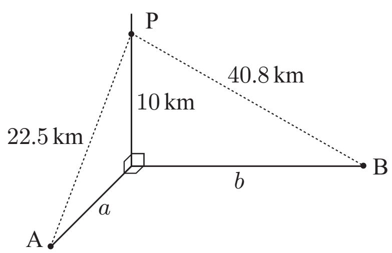
2 A cube has sides of length 10 cm.
Find the length of a diagonal of the cube.



3 A room is 7 m by 4 m and has a height of 3 m. Find the distance from a corner point on the floor to the opposite corner of the ceiling.

4 A pyramid of height 40 m has a square base with edges of length 50 m. Determine the length of the slant edges.

5



An aeroplane P is flying at an altitude of 10 000 m. The pilot spots two ships A and B. Ship A is due south of P and 22.5 km away in a direct line. Ship B is due east and 40.8 km from P in a direct line. Find the distance between the two ships.

M COORDINATE GEOMETRY

THE NUMBER PLANE

The position or location of any point in the **number plane** can be specified in terms of an **ordered pair** of numbers (x, y), where x is the **horizontal step** from a fixed point O, and y is the **vertical step** from O.

The point O is called the **origin**. Once O has been specified, we draw two perpendicular axes through it.

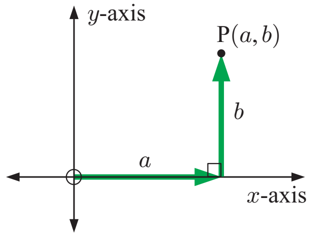
The ***x*-axis** is horizontal and the ***y*-axis** is vertical.

DEMO

The **number plane** is also known as either:

- the **2-dimensional plane**, or
- the **Cartesian plane**, named after **René Descartes**.

(a, b) is called an **ordered pair**, where a and b are the coordinates of the point.
a is called the ***x*-coordinate**.
b is called the ***y*-coordinate**.



THE DISTANCE FORMULA

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in a plane, then the distance between these points is given by $\mathbf{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 28 Self Tutor

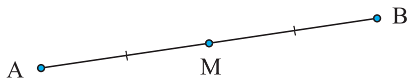
Find the distance between A(−2, 1) and B(3, 4).

A(−2, 1) B(3, 4)
x_1 y_1 x_2 y_2

$$\begin{aligned} AB &= \sqrt{(3 - -2)^2 + (4 - 1)^2} \\ &= \sqrt{5^2 + 3^2} \\ &= \sqrt{25 + 9} \\ &= \sqrt{34} \text{ units} \end{aligned}$$

THE MIDPOINT FORMULA

If M is halfway between points A and B then M is the **midpoint** of [AB].



If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points then the **midpoint** M of [AB] has coordinates $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

Example 29 Self Tutor

Find the coordinates of the midpoint of [AB] for A(−1, 3) and B(4, 7).

The x-coordinate of the midpoint $= \dfrac{-1 + 4}{2} = \frac{3}{2} = 1\frac{1}{2}$

The y-coordinate of the midpoint $= \dfrac{3 + 7}{2} = 5$

$\therefore$ the midpoint of [AB] is $(1\frac{1}{2}, 5)$.

THE GRADIENT OR SLOPE OF A LINE

When looking at line segments drawn on a set of axes, it is clear that different line segments are inclined to the horizontal at different angles. Some appear to be steeper than others.

The **gradient** or **slope** of a line is a measure of its steepness.

If A is (x_1, y_1) and B is (x_2, y_2) then the **gradient** of [AB] is $\boldsymbol{\dfrac{y_2 - y_1}{x_2 - x_1}}$.

Example 30 — Self Tutor

Find the gradient of the line through $(3, -2)$ and $(6, 4)$.

$(3, -2)$ $(6, 4)$ with $x_1 = 3$, $y_1 = -2$, $x_2 = 6$, $y_2 = 4$

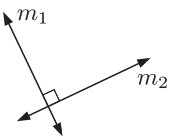
gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4 - -2}{6 - 3} = 2$

PROPERTIES OF GRADIENT

- horizontal lines have a gradient of **0 (zero)**
- vertical lines have an **undefined** gradient
-  forward sloping lines have **positive** gradients
-  backward sloping lines have **negative** gradients
- **parallel lines** have equal gradients
- the gradients of two **perpendicular lines** are *negative reciprocals* of each other.

  If the gradients are m_1 and m_2 then $\boldsymbol{m_2 = \dfrac{-1}{m_1}}$ or $\boldsymbol{m_1 m_2 = -1}$.

  This is true except when the lines are parallel to the axes.



EQUATIONS OF LINES

The **equation of a line** states the connection between the x and y values for every point on the line, and only for points on the line

Equations of lines have various forms:

- All **vertical lines** have equations of the form $x = a$ where a is a constant.
- All **horizontal lines** have equations of the form $y = c$ where c is a constant.
- If a straight line has gradient m and passes through (a, b)

  then it has equation $\boldsymbol{\dfrac{y - b}{x - a} = m}$ or $\boldsymbol{y - b = m(x - a)}$ {**point-gradient** form}

  which can be rearranged into $\boldsymbol{y = mx + c}$ {**gradient-intercept** form}

- If a straight line has gradient $\frac{A}{B}$ and passes through (x_1, x_2) then it has equation $\boldsymbol{Ax - By = Ax_1 - By_1}$ or $\boldsymbol{Ax - By = C}$ {**general** form}

Example 31 — Self Tutor

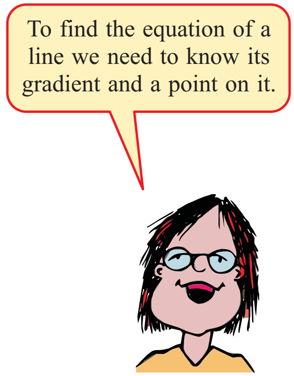
Find, in gradient-intercept form, the equation of the line through $(-1, 3)$ with a gradient of 5.

The equation of the line is $\frac{y-3}{x--1} = 5$

$\therefore \frac{y-3}{x+1} = 5$

$\therefore y - 3 = 5(x+1)$

$\therefore y = 5x + 8$



Example 32 — Self Tutor

Find, in general form, the equation of the line through $(1, -5)$ and $(5, -2)$.

The gradient $= \frac{-2 - -5}{5-1}$

$= \frac{3}{4}$

So, the equation is $\frac{y - -2}{x-5} = \frac{3}{4}$

$\therefore \frac{y+2}{x-5} = \frac{3}{4}$

$\therefore 4y + 8 = 3x - 15$

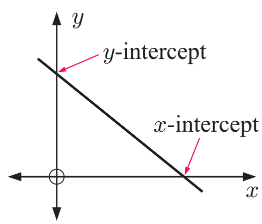
$\therefore 3x - 4y = 23$

AXES INTERCEPTS

Axes intercepts are the x- and y-values where a graph cuts the coordinate axes.

The x-intercept is found by letting $y = 0$.

The y-intercept is found by letting $x = 0$.



Example 33 — Self Tutor

For the line with equation $2x - 3y = 12$, find the axes intercepts.

When $x = 0$, $-3y = 12$
$\therefore y = -4$

When $y = 0$, $2x = 12$
$\therefore x = 6$

So, the y-intercept is -4 and the x-intercept is 6.

DOES A POINT LIE ON A LINE?

A point lies on a line if its coordinates satisfy the equation of the line.

Example 34 Self Tutor

Does $(3, -2)$ lie on the line with equation $5x - 2y = 20$?

Substituting $(3, -2)$ into $5x - 2y = 20$ gives

$$\text{LHS} = 5(3) - 2(-2) = 19$$

$$\therefore \text{LHS} \neq \text{RHS}$$

$\therefore$ $(3, -2)$ does not lie on the line.

FINDING WHERE GRAPHS MEET

Example 35 Self Tutor

Use graphical methods to find where the lines $x + y = 6$ and $2x - y = 6$ meet.

For $x + y = 6$:

when $x = 0$, $y = 6$
when $y = 0$, $x = 6$

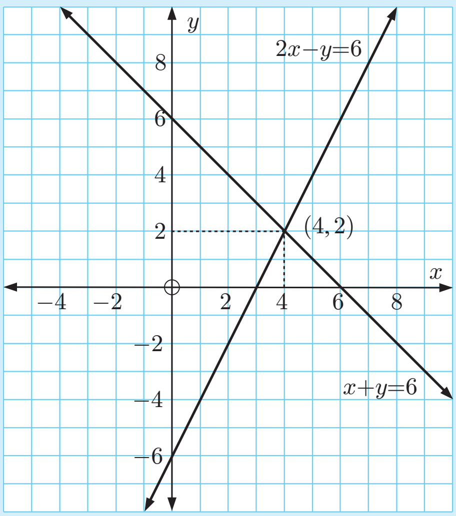
x	0	6
y	6	0

For $2x - y = 6$:

when $x = 0$, $-y = 6$
$\therefore$ $y = -6$

when $y = 0$, $2x = 6$
$\therefore$ $x = 3$

x	0	3
y	-6	0

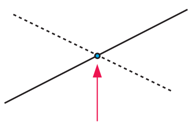


The graphs meet at $(4, 2)$.

Check: $4 + 2 = 6$ ✓
$2 \times 4 - 2 = 6$ ✓

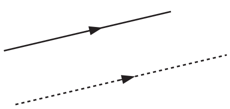
When determining whether two lines meet, there are three possible situations which may occur. These are:

Case 1:



The lines meet in a single **point of intersection**.

Case 2:



The lines are **parallel** and **never meet**. There is no point of intersection.

Case 3:

The lines are **coincident**. There are infinitely many points of intersection.

EXERCISE M

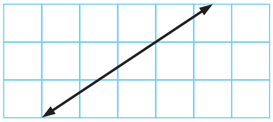
1 Use the distance formula to find the distance between the following pairs of points:

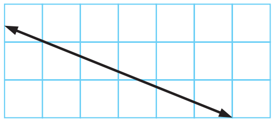
a A(1, 3) and B(4, 5)
b O(0, 0) and C(3, −5)
c P(5, 2) and Q(1, 4)
d S(0, −3) and T(−1, 0).

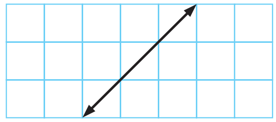
2 Find the midpoint of [AB] for:

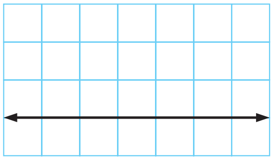
a A(3, 6) and B(1, 0)
b A(5, 2) and B(−1, −4)
c A(7, 0) and B(0, 3)
d A(5, −2) and B(−1, −3).

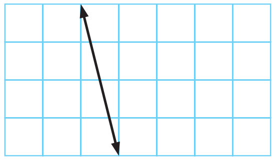
3 By finding a y-step and an x-step, determine the gradient of each of the following lines:

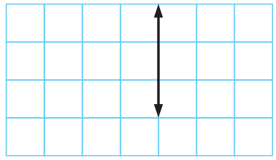
a


b


c


d


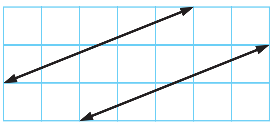
e


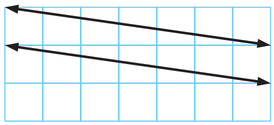
f


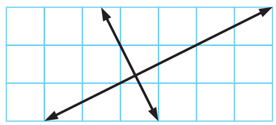
4 Find the gradient of the line passing through:

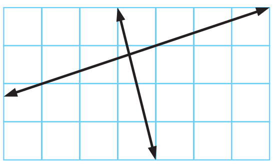
a (2, 3) and (4, 7)
b (3, 2) and (5, 8)
c (−1, 2) and (−1, 5)
d (4, −3) and (−1, −3)
e (0, 0) and (−1, 4)
f (3, −1) and (−1, −2).

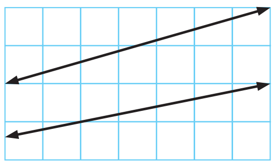
5 Classify the following pairs of lines as parallel, perpendicular, or neither. Give reasons for your answers.

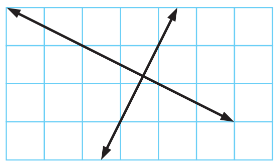
a


b


c


d


e


f


6 State the gradient of the line which is perpendicular to the line with gradient:

a $\frac{3}{4}$ **b** $\frac{11}{3}$ **c** 4 **d** $-\frac{1}{3}$ **e** −5 **f** 0

7 Find, in gradient-intercept form, the equation of the line through:

a (4, 1) with gradient 2
b (1, 2) with gradient −2
c (5, 0) with gradient 3
d (−1, 7) with gradient −3
e (1, 5) with gradient −4
f (2, 7) with gradient 1.

8 Find, in general form, the equation of the line through:

a $(2,\ 1)$ with gradient $\frac{3}{2}$
b $(1,\ 4)$ with gradient $-\frac{3}{2}$
c $(4,\ 0)$ with gradient $\frac{1}{3}$
d $(0,\ 6)$ with gradient -4
e $(-1,\ -3)$ with gradient 3
f $(4,\ -2)$ with gradient $-\frac{4}{9}$.

9 Find the equations of the lines through:

a $(0,\ 1)$ and $(3,\ 2)$
b $(1,\ 4)$ and $(0,\ -1)$
c $(2,\ -1)$ and $(-1,\ -4)$
d $(0,\ -2)$ and $(5,\ 2)$
e $(3,\ 2)$ and $(-1,\ 0)$
f $(-1,\ -1)$ and $(2,\ -3)$

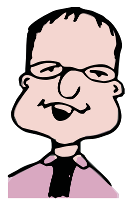
10 Find the equations of the lines through:

a $(3,\ -2)$ and $(5,\ -2)$
b $(6,\ 7)$ and $(6,\ -11)$
c $(-3,\ 1)$ and $(-3,\ -3)$

11 Copy and complete:

	Equation of line	*Gradient*	*x-intercept*	*y-intercept*
a	$2x - 3y = 6$			
b	$4x + 5y = 20$			
c	$y = -2x + 5$			
d	$x = 8$			
e	$y = 5$			
f	$x + y = 11$			
g	$4x + y = 8$			
h	$x - 3y = 12$			

If a line has equation $y = mx + c$ then the gradient of the line is m and the y-intercept is c.



12 **a** Does $(3,\ 4)$ lie on the line with equation $3x - 2y = 1$?
b Does $(-2,\ 5)$ lie on the line with equation $5x + 3y = -5$?
c Does $(6,\ -\frac{1}{2})$ lie on the line $3x - 8y = 22$?

13 Use graphical methods to find where the following lines meet:

a $x + 2y = 8$, $y = 2x - 6$
b $y = -3x - 3$, $3x - 2y = -12$
c $3x + y = -3$, $2x - 3y = -24$
d $2x - 3y = 8$, $3x + 2y = 12$
e $x + 3y = 10$, $2x + 6y = 11$
f $5x + 3y = 10$, $10x + 6y = 20$

GRAPHING PACKAGE

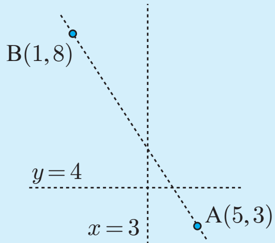
Example 36 **Self Tutor**

A straight road is to pass through the points $A(5,\ 3)$ and $B(1,\ 8)$.

a Find where this road meets the road given by:
 i $x = 3$ **ii** $y = 4$

b If we wish to refer to the points on the road (AB) *between* A and B, how can we indicate this?

c Does $C(23,\ -20)$ lie on the road (AB)?



The line representing the road has gradient $m = \frac{3-8}{5-1} = -\frac{5}{4}$.

So, its equation is $\frac{y-3}{x-5} = -\frac{5}{4}$

$\therefore \quad 4(y-3) = -5(x-5)$

$\therefore \quad 4y - 12 = -5x + 25$

$\therefore \quad 5x + 4y = 37$

a **i** When $x = 3$, $5(3) + 4y = 37$
$\therefore \quad 15 + 4y = 37$
$\therefore \quad 4y = 22$
$\therefore \quad y = 5\frac{1}{2}$
$\therefore$ they meet at $(3, 5\frac{1}{2})$.

ii When $y = 4$, $5x + 4(4) = 37$
$\therefore \quad 5x + 16 = 37$
$\therefore \quad 5x = 21$
$\therefore \quad x = \frac{21}{5}$
$\therefore$ they meet at $(\frac{21}{5}, 4)$.

b We restrict the possible x-values to $1 \leqslant x \leqslant 5$.

c If C(23, -20) lies on the line, its coordinates must satisfy the line's equation.

Now LHS $= 5(23) + 4(-20)$
$= 115 - 80$
$= 35 \neq 37$ $\quad \therefore$ C does not lie on the road.

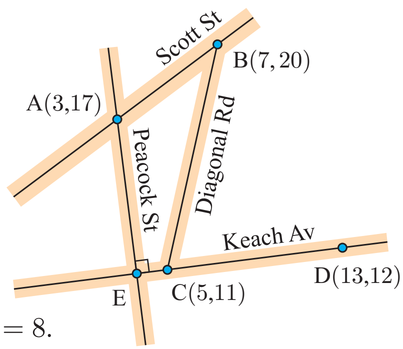
14 Find the equation of the:

a horizontal line through $(3, -4)$
b vertical line with x-intercept 5
c vertical line through $(-1, -3)$
d horizontal line with y-intercept 2
e x-axis
f y-axis.

15 Find the equation of the line:

a through A$(-1, 4)$ which has gradient $\frac{3}{4}$
b through P$(2, -5)$ and Q$(7, 0)$
c parallel to the line with equation $y = 3x - 2$ and passing through $(0, 0)$
d parallel to the line with equation $2x + 3y = 8$ and passing through $(-1, 7)$
e perpendicular to the line with equation $y = -2x + 5$ and passing through $(3, -1)$
f perpendicular to the line with equation $3x - y = 11$ and passing through $(-2, 5)$.

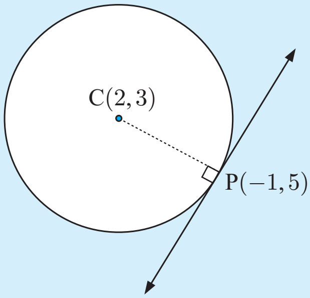
16 A is the town hall on Scott Street and D is a Post Office on Keach Avenue. Diagonal Road intersects Scott Street at B and Keach Avenue at C.



a Find the equation of Keach Avenue.
b Find the equation of Peacock Street.
c Find the equation of Diagonal Road. (Be careful!)
d Plunkit Street lies on the map reference line $x = 8$. Where does Plunkit Street intersect Keach Avenue?

Example 37 **Self Tutor**

Find the equation of the tangent to the circle with centre $(2,\ 3)$ at the point $(-1,\ 5)$.



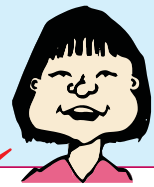
The gradient of [CP] is $\dfrac{3-5}{2-(-1)} = \dfrac{-2}{3} = -\frac{2}{3}$

$\therefore$ the gradient of the tangent at P is $\frac{3}{2}$.

Since the tangent passes through $(-1,\ 5)$,

its equation is $\dfrac{y-5}{x--1} = \dfrac{3}{2}$

$$\therefore \quad 2(y-5) = 3(x+1)$$
$$\therefore \quad 2y - 10 = 3x + 3$$
$$\therefore \quad 3x - 2y = -13$$



The tangent is perpendicular to the radius at the point of contact.

17 Find the equation of the tangent to the circle with centre:

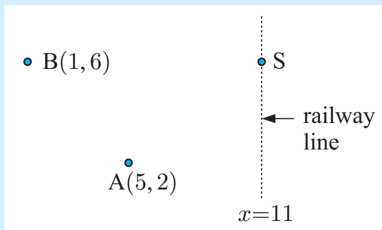
a $(0,\ 2)$ at the point $(-1,\ 5)$

b $(3,\ -1)$ at the point $(-1,\ 1)$

c $(2,\ -2)$ at the point $(5,\ -2)$.

Example 38 **Self Tutor**

Mining towns are situated at $B(1,\ 6)$ and $A(5,\ 2)$. Where should the railway siding S be located so that ore trucks from either A or B would travel equal distances to a railway line with equation $x = 11$?

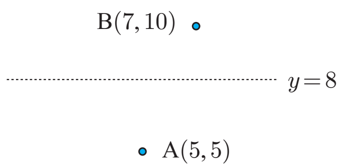


Suppose S has the coordinates $(11,\ a)$.

$$\text{Now} \quad BS = AS$$
$$\therefore \quad \sqrt{(11-1)^2 + (a-6)^2} = \sqrt{(11-5)^2 + (a-2)^2}$$
$$\therefore \quad 10^2 + (a-6)^2 = 6^2 + (a-2)^2 \qquad \{\text{squaring both sides}\}$$
$$\therefore \quad 100 + a^2 - 12a + 36 = 36 + a^2 - 4a + 4$$
$$\therefore \quad -12a + 4a = 4 - 100$$
$$\therefore \quad -8a = -96$$
$$\therefore \quad a = 12$$

So, the railway siding should be located at $(11,\ 12)$.

18 A(5, 5) and B(7, 10) are houses and $y = 8$ is a gas pipeline. Where should the one outlet from the pipeline be placed so that it is the same distance from both houses?

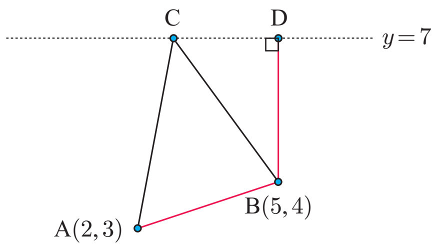


19 (CD) is a water pipeline. A and B are two towns. A pumping station is to be located on the pipeline to pump water to A and B. Each town is to pay for their own service pipes and they insist on equality of costs.

a Where should C be located to ensure equality of costs?

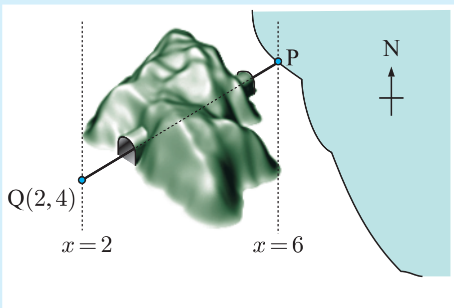
b What is the total length of service pipe required?

c If the towns agree to pay equal amounts, would it be cheaper to install the service pipeline from D to B to A?



Scale: 1 unit ≡ 1 km

Example 39 **Self Tutor**



Scale: each grid unit is 1 km.

A tunnel through the mountains connects town Q(2, 4) to the port at P.

P is on grid reference $x = 6$ and the distance between the town and the port is 5 km.

Assuming the diagram is reasonably accurate, find the horizontal grid reference of the port.

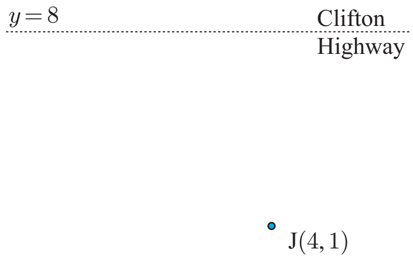
Suppose P is at $(6, a)$.

$$
\begin{aligned}
\text{Now} \quad PQ &= 5 \\
\therefore \quad \sqrt{(6-2)^2 + (a-4)^2} &= 5 \\
\therefore \quad \sqrt{16 + (a-4)^2} &= 5 \\
\therefore \quad 16 + (a-4)^2 &= 25 \\
\therefore \quad (a-4)^2 &= 9 \\
\therefore \quad a - 4 &= \pm 3 \\
\therefore \quad a &= 4 \pm 3 = 7 \text{ or } 1
\end{aligned}
$$

From the diagram, P is further north than Q, and so $a > 4$

So, P is at (6, 7) and the horizontal grid reference is $y = 7$.

20



Jason's girlfriend lives in a house on Clifton Highway which has equation $y = 8$. The distance 'as the crow flies' from Jason's house to his girlfriend's house is 11.73 km. If Jason lives at $(4, 1)$, what are the coordinates of his girlfriend's house?

Scale: 1 unit $\equiv$ 1 km.

21 **a** A circle has centre (a, b) and radius r units. P(x, y) moves on the circle. Show that $(x-a)^2+(y-b)^2=r^2$.

b Find the equation of the circle with:

i centre $(4, 3)$ and radius 5 units **ii** centre $(-1, 5)$ and radius 2 units

iii centre $(0, 0)$ and radius 10 units

iv ends of a diameter $(-1, 5)$ and $(3, 1)$.

22 Find the centre and radius of the circle:

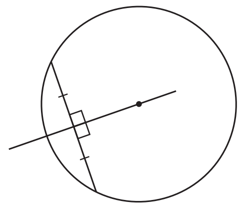
a $(x-1)^2+(y-3)^2=4$ **b** $x^2+(y+2)^2=16$ **c** $x^2+y^2=7$

23 Consider the circle with equation $(x-2)^2+(y+3)^2=20$.

a State the circle's centre and radius. **b** Show that $(4, 1)$ lies on the circle.

c Find the equation of the tangent to the circle at the point $(4, 1)$.

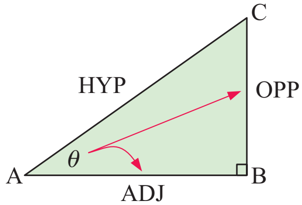
24 The perpendicular bisector of a chord of a circle passes through the centre of the circle.
Find the centre of a circle passing through points P$(5, 7)$, Q$(7, 1)$ and R$(-1, 5)$ by finding the perpendicular bisectors of [PQ] and [QR] and solving them simultaneously.



N RIGHT ANGLED TRIANGLE TRIGONOMETRY

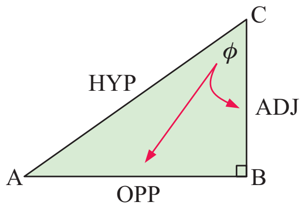
LABELLING RIGHT ANGLED TRIANGLES

The **hypotenuse (HYP)** is the side which is opposite the right angle. It is the longest side of the triangle.



For the angle marked θ:

- [BC] is the side **opposite (OPP)** angle θ
- [AB] is the side **adjacent (ADJ)** angle θ.

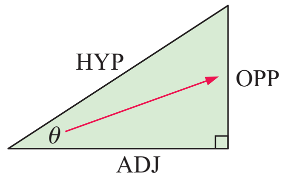


For the angle marked ϕ:

- [AB] is the side **opposite (OPP)** angle ϕ
- [BC] is the side **adjacent (ADJ)** angle ϕ.

THE THREE BASIC TRIGONOMETRIC RATIOS

$$\sin\theta = \frac{\text{OPP}}{\text{HYP}}, \qquad \cos\theta = \frac{\text{ADJ}}{\text{HYP}}, \qquad \tan\theta = \frac{\text{OPP}}{\text{ADJ}}$$

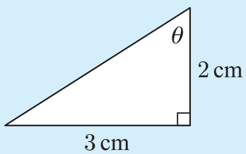


$\sin\theta$, $\cos\theta$ and $\tan\theta$ are abbreviations for sine θ, cosine θ and tangent θ.

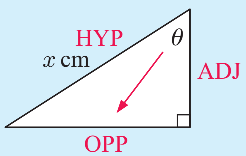
The three formulae above are called the **trigonometric ratios** and are the tools we use for finding side lengths and angles of right angled triangles.

However, before doing this we will calculate the trigonometric ratios in right angled triangles where we know two of the sides.

Example 40 Self Tutor



Find, without using a calculator, $\sin\theta$, $\cos\theta$ and $\tan\theta$.



If the hypotenuse is x cm long

$x^2 = 2^2 + 3^2$ {Pythagoras}

$\therefore\ x^2 = 13$

$\therefore\ x = \sqrt{13}$ {as $x > 0$}

So, $\sin\theta = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{3}{\sqrt{13}}$, $\cos\theta = \dfrac{\text{ADJ}}{\text{HYP}} = \dfrac{2}{\sqrt{13}}$, $\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{3}{2}$.

Example 41 Self Tutor

If θ is an acute angle and $\sin\theta = \frac{1}{3}$, find $\cos\theta$ and $\tan\theta$ without using a calculator.

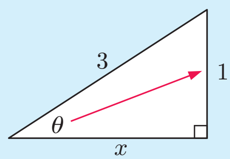
We draw a right angled triangle and mark on angle θ so that OPP $= 1$ unit and HYP $= 3$ units.

Now $x^2 + 1^2 = 3^2$ {Pythagoras}

$\therefore\ x^2 + 1 = 9$

$\therefore\ x^2 = 8$

$\therefore\ x = \sqrt{8}$ {as $x > 0$}

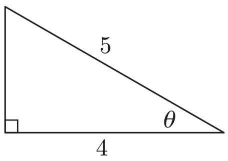


$\therefore\ \cos\theta = \dfrac{\text{ADJ}}{\text{HYP}} = \dfrac{\sqrt{8}}{3}$ and $\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{1}{\sqrt{8}}$.

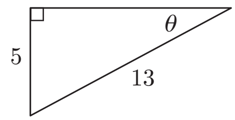
EXERCISE N.1

1 For the following triangles, find the length of the third side and hence find $\sin\theta$, $\cos\theta$ and $\tan\theta$:

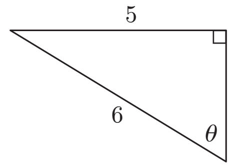
a



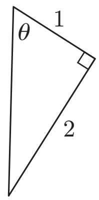
b



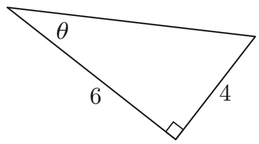
c



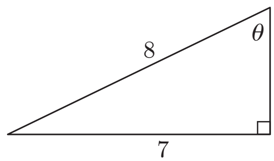
d



e



f

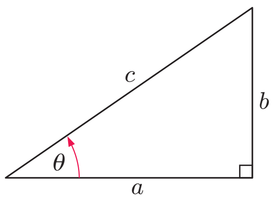


2 **a** If θ is an acute angle and $\cos\theta = \frac{1}{2}$, find $\sin\theta$ and $\tan\theta$.

b If α is an acute angle and $\sin\alpha = \frac{2}{3}$, find $\cos\alpha$ and $\tan\alpha$.

c If β is an acute angle and $\tan\beta = \frac{4}{3}$, find $\sin\beta$ and $\cos\beta$.

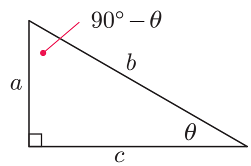
3



a For the triangle given, write down expressions for $\sin\theta$, $\cos\theta$ and $\tan\theta$.

b Write $\dfrac{\sin\theta}{\cos\theta}$ in terms of a, b and c and hence show that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$.

4 The angle marked $90^o - \theta$ is the **complement** of θ.



a Find:

i $\sin\theta$ **ii** $\cos\theta$

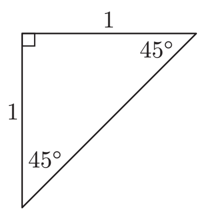
iii $\sin(90^o - \theta)$ **iv** $\cos(90^o - \theta)$

b Use your results of **a** to complete the following statements:

i *The sine of an angle is the cosine of its ......*

ii *The cosine of an angle is the sine of its ......*

5



a Find the length of the remaining side.

b Find $\sin 45^o$, $\cos 45^o$, $\tan 45^o$ using the figure.

c Use your calculator to check your answers.

6 Triangle ABC is equilateral. [AN] is the altitude corresponding to side [BC].

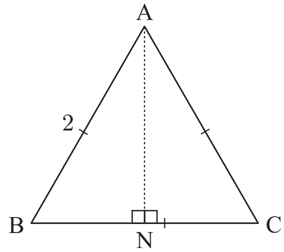
a State the measures of $A\widehat{B}N$ and $B\widehat{A}N$.

b Find the lengths of [BN] and [AN].

c Without using a calculator, find:

i $\sin 60^o$, $\cos 60^o$ and $\tan 60^o$

ii $\sin 30^o$, $\cos 30^o$ and $\tan 30^o$.



COMMON TRIGONOMETRIC RATIOS

We can summarise the ratios for special angles in table form. Try to learn them.

θ	0^o	30^o	45^o	60^o	90^o
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

FINDING SIDES AND ANGLES

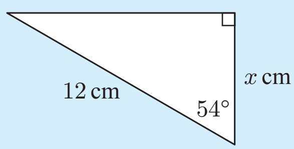
Before commencing calculations, check that the MODE on your calculator is set on **degrees**. In this chapter all angle measure is in degrees.

In a right angled triangle, if we wish to find the **length of a side**, we first need to know one angle and one other side.

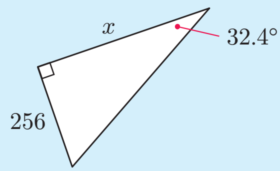
Example 42 Self Tutor

Find, correct to 3 significant figures, the value of x in:

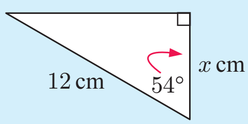
a



b



a



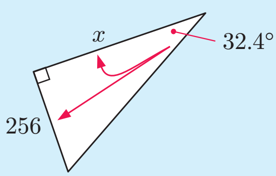
For the 54^o angle, HYP $= 12$, ADJ $= x$.

$\therefore \quad \cos 54^o = \dfrac{x}{12}$

$\therefore \quad 12\cos 54^o = x$

$\therefore \quad x \approx 7.05$

b



For the 32.4^o angle, OPP $= 12$, ADJ $= x$.

So, $\quad \tan 32.4^o = \dfrac{256}{x}$

$\therefore \quad x = \dfrac{256}{\tan 32.4^o} \approx 403.4$

$\therefore \quad x \approx 403$

In a right angled triangle, if we wish to find the **size of an acute angle** we need to know the lengths of two sides. We then need to find the appropriate **inverse** trigonometric ratio:

- If $\sin\theta = \frac{a}{b}$ then $\theta = \sin^{-1}\left(\frac{a}{b}\right)$ which reads 'the angle with a sine of $\frac{a}{b}$'.
- If $\cos\theta = \frac{a}{b}$ then $\theta = \cos^{-1}\left(\frac{a}{b}\right)$ which reads 'the angle with a cosine of $\frac{a}{b}$'.
- If $\tan\theta = \frac{a}{b}$ then $\theta = \tan^{-1}\left(\frac{a}{b}\right)$ which reads 'the angle with a tangent of $\frac{a}{b}$'.

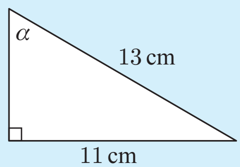
An alternative notation for the three inverse trigonometric functions is:

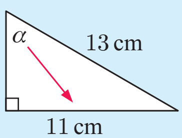
- $\arcsin\theta$ for $\sin^{-1}\theta$
- $\arccos\theta$ for $\cos^{-1}\theta$
- $\arctan\theta$ for $\tan^{-1}\theta$

Find help using your calculator to find inverse trigonometric ratios, consult the graphics calculator instructions chapter.

Example 43 **Self Tutor**

Find α in degrees, correct to 3 significant figures:





For angle α, OPP $= 11$, HYP $= 13$.

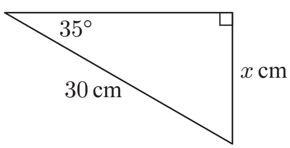
$\therefore \quad \sin\alpha = \frac{11}{13}$

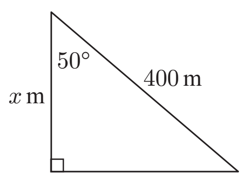
$\therefore \quad \alpha = \sin^{-1}\left(\frac{11}{13}\right)$

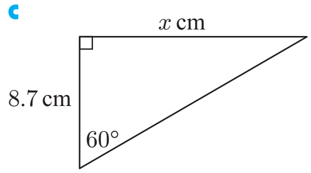
$\therefore \quad \alpha \approx 57.8^o$

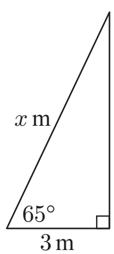
EXERCISE N.2

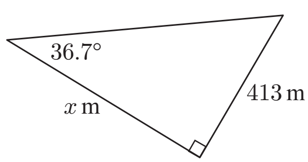
1 Find, correct to 3 significant figures, the value of the unknown in each of the following:

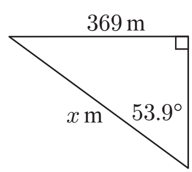
a


b


c


d


e


f


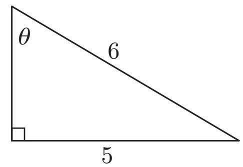
2 Use your calculator to find the acute angle θ, to 3 significant figures, if:

a $\sin\theta = 0.9364$ **b** $\cos\theta = 0.2381$ **c** $\tan\theta = 1.7321$ **d** $\cos\theta = \frac{2}{7}$

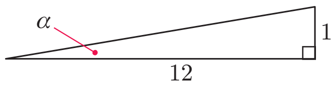
e $\sin\theta = \frac{1}{3}$ **f** $\tan\theta = \frac{14}{3}$ **g** $\sin\theta = \frac{\sqrt{3}}{11}$ **h** $\cos\theta = \frac{5}{\sqrt{37}}$

3 Find, correct to 3 significant figures, the measure of the unknown angle in each of the following:

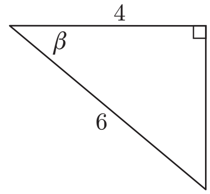
a



b

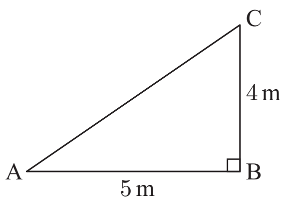


c

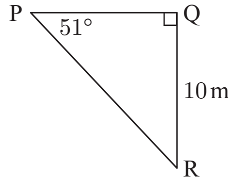


4 Find all unknown side lengths and angles of the following triangles:

a

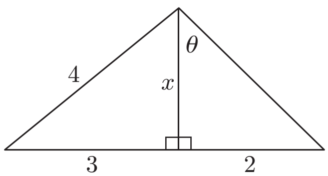


b

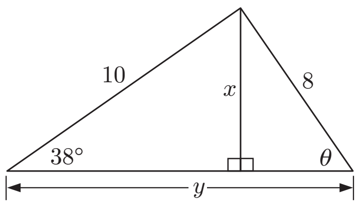


5 Find all unknown sides and angles in:

a



b



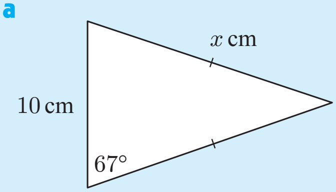
ISOSCELES TRIANGLES

To use trigonometry with isosceles triangles we invariably draw the **perpendicular** from the apex to the base. This altitude **bisects** the base.

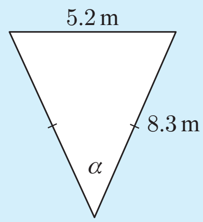
Example 44 **Self Tutor**

Find the unknowns in the following diagrams:

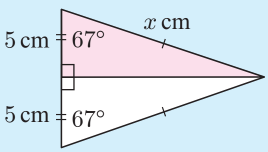
a



b



a

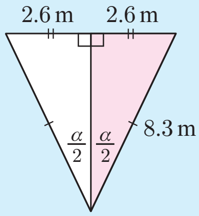


In the shaded right angled triangle,

$$\cos 67^o = \frac{5}{x}$$

$$\therefore \quad x = \frac{5}{\cos 67^o} \approx 12.8$$

b



In the shaded right angled triangle,

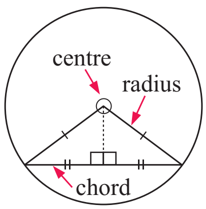
$$\sin\left(\frac{\alpha}{2}\right) = \frac{2.6}{8.3}$$

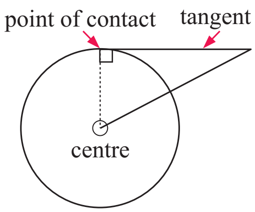
$$\therefore \quad \frac{\alpha}{2} = \sin^{-1}\left(\frac{2.6}{8.3}\right)$$

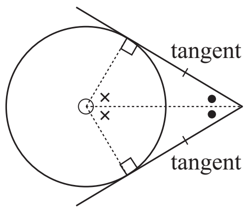
$$\therefore \quad \alpha = 2\sin^{-1}\left(\frac{2.6}{8.3}\right) \approx 36.5^o$$

CHORDS AND TANGENTS

Right angled triangles occur in chord and tangent problems.

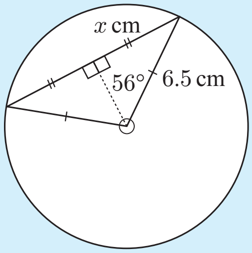






Example 45 **Self Tutor**

A chord of a circle subtends an angle of 112^o at its centre. Find the length of the chord if the radius of the circle is 6.5 cm.



We complete an isosceles triangle and draw the line from the apex to the base.

For the 56^o angle, HYP $= 6.5$, OPP $= x$

$$\therefore \quad \sin 56^o = \frac{x}{6.5}$$

$$\therefore \quad 6.5 \times \sin 56^o = x$$

$$\therefore \quad x \approx 5.389$$

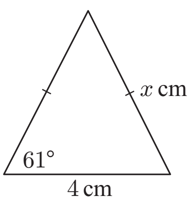
$$\therefore \quad 2x \approx 10.78$$

$\therefore$ the chord is about 10.8 cm long.

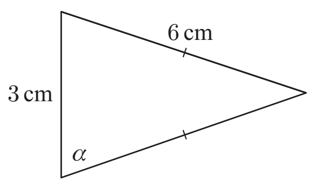
EXERCISE N.3

1 Find, correct to 4 significant figures, the unknowns in the following:

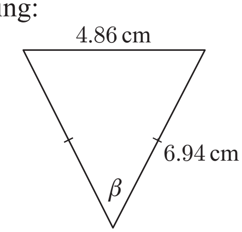
a



b

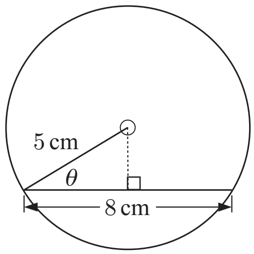


c

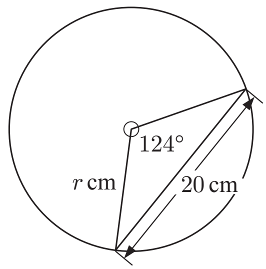


2 Find the value of the unknown in:

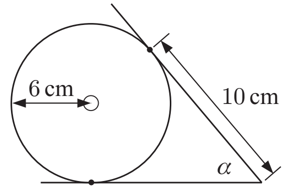
a



b



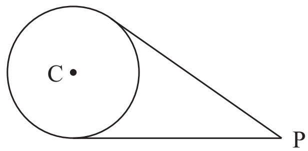
c



3 A chord of a circle subtends an angle of 89^o at its centre. Find the length of the chord given that the circle's diameter is 11.4 cm.

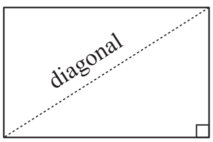
4 A chord of a circle is 13.2 cm long and the circle's radius is 9.4 cm. Find the angle subtended by the chord at the centre of the circle.

5 Point P is 10 cm from the centre of a circle of radius 4 cm. Tangents are drawn from P to the circle. Find the angle between the tangents.

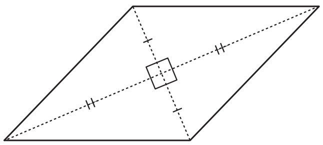


OTHER FIGURES

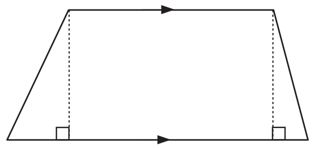
Right angled triangles can also be found in other geometric figures such as rectangles, rhombi, and trapezia.



rectangle



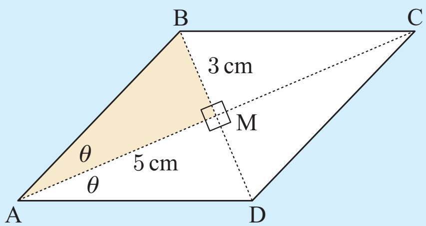
rhombus



trapezium or trapezoid

Example 46 **Self Tutor**

A rhombus has diagonals of length 10 cm and 6 cm respectively.
Find the smaller angle of the rhombus.



The diagonals bisect each other at right angles, so AM = 5 cm and BM = 3 cm.

In ΔABM, θ will be the smallest angle as it is opposite the shortest side.

$$\tan\theta = \tfrac{3}{5}$$

$$\therefore \quad \theta = \tan^{-1}(\tfrac{3}{5})$$

$$\therefore \quad \theta \approx 30.964^o$$

The required angle is 2θ as the diagonals bisect the angles at each vertex.
So, the angle is about 61.9^o.

EXERCISE N.4

1 A rectangle is 9.2 m by 3.8 m. What angle does its diagonal make with its longer side?

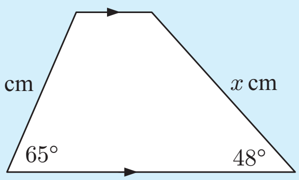
2 The diagonal and the longer side of a rectangle make an angle of 43.2^o. If the longer side is 12.6 cm, find the length of the shorter side.

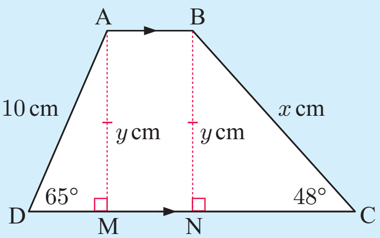
3 A rhombus has diagonals of length 12 cm and 7 cm respectively. Find the larger angle of the rhombus.

4 The smaller angle of a rhombus measures 21.8^o and the shorter diagonal has length 13.8 cm. Find the lengths of the sides of the rhombus.

Example 47 Self Tutor

Find x given:





We draw perpendiculars [AM] and [BN] to [DC], creating right angled triangles and the rectangle ABNM.

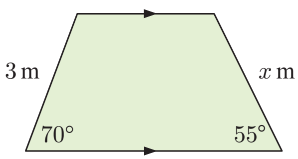
In ΔADM, $\sin 65^o = \dfrac{y}{10}$

$\therefore \quad y = 10 \sin 65^o$

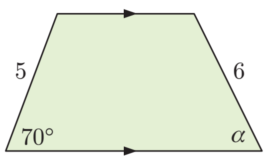
In ΔBCN, $\sin 48^o = \dfrac{y}{x} = \dfrac{10 \sin 65^o}{x}$

$\therefore \quad x = \dfrac{10 \sin 65^o}{\sin 48^o} \approx 12.2$

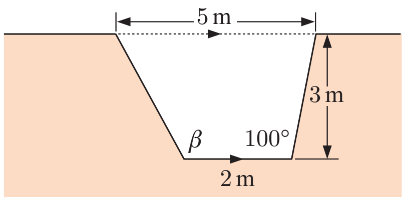
5 **a** Find the value of x in:



b Find the unknown angle in:

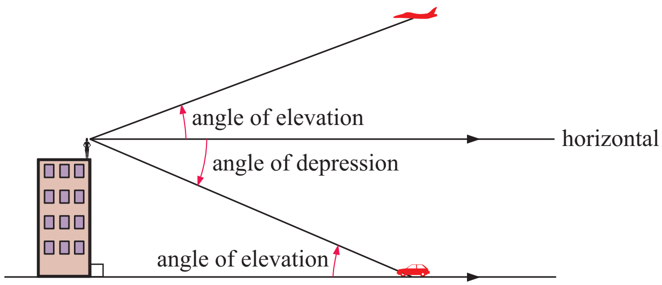


6 A stormwater drain is to have the shape illustrated. Determine the angle β the left hand side makes with the bottom of the drain.



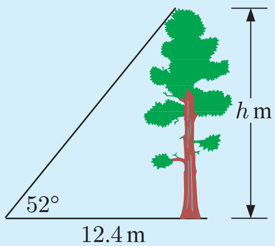
PROBLEM SOLVING USING TRIGONOMETRY

Trigonometry is a very useful branch of mathematics. **Heights** and **distances** which are very difficult or even impossible to measure can often be found using **trigonometry**.



Example 48 | Self Tutor

Find the height of a tree which casts a shadow of 12.4 m when the sun makes an angle of 52^o to the horizon.



Let h m be the tree's height.

For the 52^o angle, OPP $= h$, ADJ $= 12.4$

$$\therefore \quad \tan 52^o = \frac{h}{12.4}$$

$$\therefore \quad 12.4 \times \tan 52^o = h$$

$$\therefore \quad h \approx 15.9$$

$\therefore$ the tree is 15.9 m high.

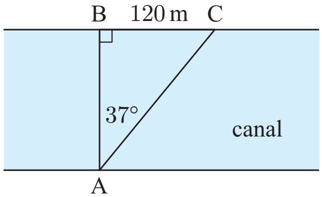
EXERCISE N.5

1 Find the height of a flagpole which casts a shadow of 9.32 m when the sun makes an angle of 63^o to the horizontal.

2 A hill is inclined at 18^o to the horizontal. It runs down to the beach so its base is at sea level.

- **a** If I walk 1.2 km up the hill, what is my height above sea level?
- **b** If I am 500 metres above sea level, how far have I walked up the hill?

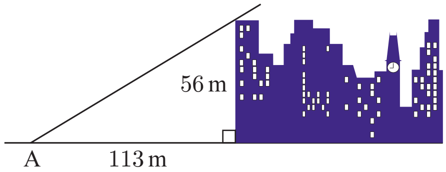
3



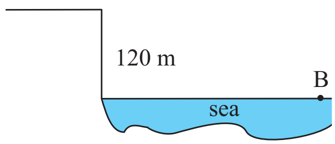
A surveyor standing at A notices two posts B and C on the opposite side of a canal. The posts are 120 m apart. If the angle of sight between the posts is 37^o, how wide is the canal?

4 A train must climb a constant gradient of 5.5 m for every 200 m of track. Find the angle of incline.

5 Find the angle of elevation to the top of a 56 m high building from point A which is 113 m from its base. What is the angle of depression from the top of the building to A?



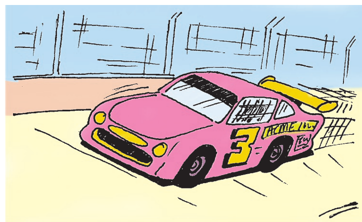
6 The angle of depression from the top of a 120 m high vertical cliff to a boat B is 16^o.
Find how far the boat is from the base of the cliff.



7 Sarah measures the angle of elevation to the top of a tree as 23.6^o from a point which is 250 m from its base. Her eye level, from which the angle measurement is taken, is 1.5 m above the ground. Assuming the ground is horizontal, find the height of the tree.

8 Kylie measures the angle of elevation from a point on level ground to the top of a building 120 metres high to be 32^o. She walks towards the building until the angle of elevation is 45^o. How far did she walk?

9 A circular track of radius r m is banked at an angle of θ to the horizontal. The ideal speed for the bend is given by the formula $s = \sqrt{gr\tan\theta}$ where $g = 9.8\ \text{m}\,\text{s}^{-2}$.

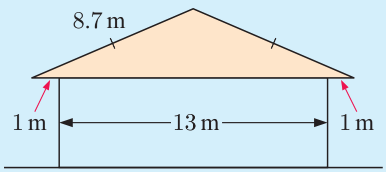


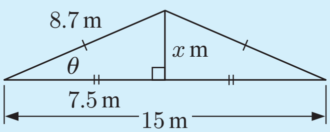
a What is the ideal speed for a vehicle travelling on a circular track of radius 100 m which is banked at an angle of 15^o?

b At what angle should a track of radius 200 m be banked if it is designed for a vehicle travelling at $20\ \text{m}\,\text{s}^{-1}$?

Example 49 **Self Tutor**

A builder has designed the roof structure illustrated. The pitch of the roof is the angle that the roof makes with the horizontal. Find the pitch of this roof.





By constructing an altitude of the isosceles triangle, we form two right angled triangles.
For angle θ:

ADJ = 7.5, HYP = 8.7

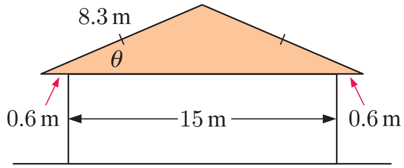
$$\therefore\quad \cos\theta = \frac{7.5}{8.7}$$

$$\therefore\quad \theta = \cos^{-1}\left(\frac{7.5}{8.7}\right)$$

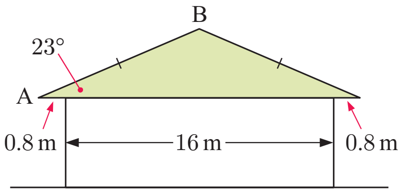
$$\therefore\quad \theta \approx 30.450^o$$

$\therefore$ the pitch is approximately $30\frac{1}{2}^o$.

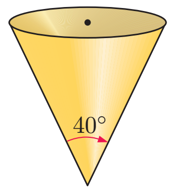
10 Find θ, the pitch of the roof.



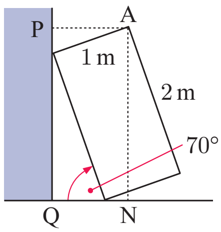
11 If the pitch of the given roof is 23^o, find the length of the timber beam [AB].



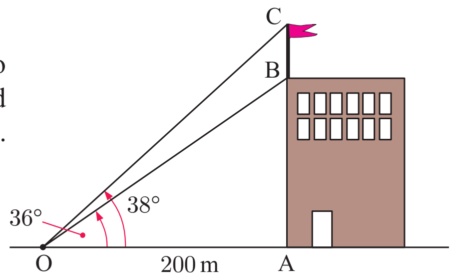
12 An open right-circular cone has a vertical angle measuring 40^o and a base radius of 30 cm.
Find the capacity of the cone in litres.



13 A refrigerator is tipped against a vertical wall so it can be serviced. It makes an angle of 70^o with the horizontal floor. How high is point A above the floor?



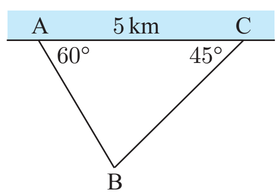
14 From an observer O, the angles of elevation to the bottom and the top of a flagpole are 36^o and 38^o respectively. Find the height of the flagpole.



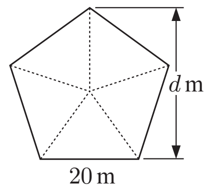
15 The angle of depression from the top of a 150 m high cliff to a boat at sea is 7^o. How much closer to the cliff must the boat move for the angle of depression to become 19^o?

16 A helicopter flies horizontally at 100 km h^{-1}. An observer notices that it takes 20 seconds for the helicopter to fly from directly overhead to being at an angle of elevation of 60^o. Find the height of the helicopter above the ground.

17 [AC] is a straight shore line and B is a boat out at sea. Find the shortest distance from the boat to the shore if A and C are 5 km apart.



18



A regular pentagonal garden plot is to be constructed with sides of length 20 m. Find the width of land d m required for the plot.

EXERCISE A

1 **a** $\sqrt{15}$ **b** 3 **c** 4 **d** 12 **e** 42
f $\sqrt{6}$ **g** $\sqrt{2}$ **h** $\sqrt{6}$

2 **a** $5\sqrt{2}$ **b** $-\sqrt{2}$ **c** $2\sqrt{5}$ **d** $8\sqrt{5}$ **e** $-2\sqrt{5}$
f $9\sqrt{3}$ **g** $-3\sqrt{6}$ **h** $3\sqrt{2}$

3 **a** $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $2\sqrt{5}$ **d** $4\sqrt{2}$ **e** $3\sqrt{3}$
f $3\sqrt{5}$ **g** $4\sqrt{3}$ **h** $3\sqrt{6}$ **i** $5\sqrt{2}$ **j** $4\sqrt{5}$
k $4\sqrt{6}$ **l** $6\sqrt{3}$

4 **a** $2\sqrt{3}$ **b** $8\sqrt{2}$ **c** $5\sqrt{6}$ **d** $10\sqrt{3}$ **e** $3\sqrt{3}$
f $-\sqrt{2}$

5 **a** $\frac{\sqrt{2}}{2}$ **b** $2\sqrt{3}$ **c** $\frac{7\sqrt{2}}{2}$ **d** $2\sqrt{5}$ **e** $5\sqrt{2}$
f $3\sqrt{6}$ **g** $4\sqrt{3}$ **h** $\frac{5\sqrt{7}}{7}$ **i** $2\sqrt{7}$ **j** $\sqrt{6}$

EXERCISE B

1 **a** 2.59×10^{2} **b** 2.59×10^{5} **c** 2.59×10^{0}
d 2.59×10^{-1} **e** 2.59×10^{-4} **f** 4.07×10^{1}
g 4.07×10^{3} **h** 4.07×10^{-2} **i** 4.07×10^{5}
j 4.07×10^{8} **k** 4.07×10^{-5}

2 **a** 1.495×10^{11} m **b** 3×10^{-4} mm **c** 1×10^{-3} mm
d 1.5×10^{7} °C **e** 3×10^{5}

3 **a** 4000 **b** 500 **c** 2100 **d** 78 000
e 380 000 **f** 86 **g** 43 300 000 **h** 60 000 000

4 **a** 0.004 **b** 0.05 **c** 0.0021 **d** 0.000 78
e 0.000 038 **f** 0.86 **g** 0.000 000 433
h 0.000 000 6

5 **a** 0.000 000 9 m **b** 6 130 000 000
c 100 000 light years **d** 0.000 01 mm

6 **a** 1.64×10^{10} **b** 4.12×10^{-3} **c** 5.27×10^{-18}
d 1.36×10^{2} **e** 2.63×10^{-6} **f** 1.73×10^{9}

7 **a** 1.30×10^{5} km **b** 9.07×10^{5} km **c** 9.47×10^{7} km

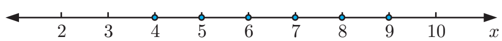
8 **a** 1.8×10^{10} m **b** 2.59×10^{13} m **c** 9.47×10^{15} m

EXERCISE C

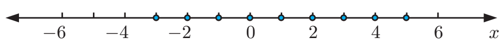
1 **a** The set of all real x such that x is greater than 5.
b The set of all integers x such that x is less than or equal to 3.
c The set of all y such that y lies between 0 and 6.
d The set of all integers x such that x is greater than or equal to 2, but less than or equal to 4. x is 2, 3 or 4.
e The set of all t such that t lies between 1 and 5.
f The set of all n such that n is less than 2 or greater than or equal to 6.

2 **a** $\{x \mid x > 2\}$ **b** $\{x \mid 1 < x \leqslant 5\}$
c $\{x \mid x \leqslant -2$ or $x \geqslant 3\}$ **d** $\{x \mid -1 \leqslant x \leqslant 3, x \in \mathbb{Z}\}$
e $\{x \mid 0 \leqslant x \leqslant 5, x \in \mathbb{Z}\}$ **f** $\{x \mid x < 0\}$

3 **a**

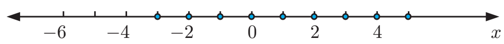


b

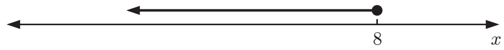


c

d



e



EXERCISE D

1 **a** $10x - 10$ **b** $9x$ **c** $5x + 5y$ **d** $8 - 8x$
e $12ab$ **f** cannot be simplified

2 **a** $22x + 35$ **b** $16 - 6x$ **c** $4a - 3b$
d $3x^3 - 16x^2 + 11x - 1$

3 **a** $18x^3$ **b** $\frac{a}{3b}$ **c** $4x^2$ **d** $24a^{10}$

EXERCISE E

1 **a** $x = 10$ **b** $x > 6$ **c** $x = \frac{4}{5}$ **d** $x = 51$
e $x < -10$ **f** $x = 14$ **g** $x \leqslant -9$ **h** $x = 18$
i $x = \frac{2}{3}$

2 **a** $x = 5,\ y = 2$ **b** $x = \frac{22}{3},\ y = \frac{8}{3}$ **c** $x = -2,\ y = 5$
d $x = \frac{45}{11},\ y = -\frac{18}{11}$ **e** no solution **f** $x = 66,\ y = -84$

EXERCISE F

1 **a** 16 **b** −6 **c** 16 **d** 18 **e** −2 **f** 2

2 **a** 2 **b** 3 **c** 6 **d** 6 **e** 5 **f** −1
g 1 **h** 5 **i** 4 **j** 4 **k** 2 **l** 2

3 **a** $x = \pm 3$ **b** no solution **c** $x = 0$
d $x = 4$ or -2 **e** $x = -1$ or 7 **f** no solution
g $x = 1$ or $\frac{1}{3}$ **h** $x = 0$ or 3 **i** $x = -2$ or $\frac{14}{5}$

EXERCISE G

1 **a** $2x^2 + 5x + 3$ **b** $3x^2 + 10x + 8$
c $10x^2 + x - 2$ **d** $3x^2 + x - 10$
e $-6x^2 + 17x + 14$ **f** $-6x^2 - 13x + 5$
g $15x^2 + 11x - 12$ **h** $15x^2 - 11x + 2$
i $2x^2 - 17x + 21$ **j** $4x^2 - 16x + 15$
k $-x^2 - 3x - 2$ **l** $-4x^2 - 2x + 6$

2 **a** $x^2 - 36$ **b** $x^2 - 64$ **c** $4x^2 - 1$ **d** $9x^2 - 4$
e $16x^2 - 25$ **f** $25x^2 - 9$ **g** $9 - x^2$ **h** $49 - x^2$
i $49 - 4x^2$ **j** $x^2 - 2$ **k** $x^2 - 5$ **l** $4x^2 - 3$

3 **a** $x^2 + 10x + 25$ **b** $x^2 + 14x + 49$ **c** $x^2 - 4x + 4$
d $x^2 - 12x + 36$ **e** $x^2 + 6x + 9$ **f** $x^2 + 10x + 25$
g $x^2 - 22x + 121$ **h** $x^2 - 20x + 100$ **i** $4x^2 + 28x + 49$
j $9x^2 + 12x + 4$ **k** $4x^2 - 20x + 25$ **l** $9x^2 - 42x + 49$

4 **a** $y = 2x^2 + 10x + 12$ **b** $y = 3x^2 - 6x + 7$
c $y = -x^2 + 6x + 7$ **d** $y = -x^2 - 4x - 15$
e $y = 4x^2 - 24x + 20$ **f** $y = -\frac{1}{2}x^2 - 4x - 14$
g $y = -5x^2 + 35x - 30$ **h** $y = \frac{1}{2}x^2 + 2x - 4$
i $y = -\frac{5}{2}x^2 + 20x - 40$

5 **a** $2x^2 + 12x + 19$ **b** $3x^2 + 3x - 16$ **c** $-x^2 + 6x - 6$
d $-x^2 - x + 25$ **e** $2x^2 - 16x + 33$ **f** $-3x + 4$ **g** $7x + 8$
h $7x^2 + 18x + 12$ **i** $-x^2 + 19x - 32$ **j** $7x^2 - 16x + 2$

EXERCISE H

1 **a** $3x(x+3)$ **b** $x(2x+7)$ **c** $2x(2x-5)$ **d** $3x(2x-5)$
e $(3x-5)(3x+5)$ **f** $(4x+1)(4x-1)$ **g** $2(x-2)(x+2)$
h $3(x+\sqrt{3})(x-\sqrt{3})$ **i** $4(x+\sqrt{5})(x-\sqrt{5})$ **j** $(x-4)^2$
k $(x-5)^2$ **l** $2(x-2)^2$ **m** $(4x+5)^2$ **n** $(3x+2)^2$
o $(x-11)^2$

2 **a** $(x+8)(x+1)$ **b** $(x+4)(x+3)$ **c** $(x-9)(x+2)$
d $(x+7)(x-3)$ **e** $(x-6)(x-3)$ **f** $(x+3)(x-2)$
g $-(x-2)(x+1)$ **h** $3(x-11)(x-3)$ **i** $-2(x+1)^2$
j $2(x+5)(x-2)$ **k** $2(x-8)(x+3)$ **l** $-2(x-6)(x-1)$
m $-3(x-1)^2$ **n** $-(x+1)^2$ **o** $-5(x-4)(x+2)$

3 **a** $(2x-3)(x+4)$ **b** $(3x+1)(x-2)$ **c** $(7x-2)(x-1)$
d $(3x-2)(2x+1)$ **e** $(2x-3)(2x+1)$ **f** $(5x-3)(2x+1)$
g $(2x+1)(x-6)$ **h** $(3x+7)(x-4)$ **i** $(4x+3)(2x-1)$
j $(5x+3)(2x-3)$ **k** $(3x-1)(x+8)$ **l** $(3x+2)(2x+1)$
m $-2(2x+3)(x-1)$ **n** $(6x+1)(2x-3)$
o $-3(2x+7)(x-2)$ **p** $-(3x-2)(3x-5)$
q $(4x-9)(2x+3)$ **r** $(4x+3)(3x+1)$ **s** $(6x+1)(2x+3)$
t $(5x-4)(3x-2)$ **u** $(7x+5)(2x-3)$

4 **a** $(x+4)(2x+1)$ **b** $(2-x)(5-3x)$
c $3(x+2)(2x+7)$ **d** $4(x+5)(2x+11)$
e $2x(x+3)$ **f** $5(x+3)$
g $(x-2)(3x+26)$ **h** $(x-1)(2x-1)$

5 **a** $(x+7)(x-1)$ **b** $(x+1)(3-x)$ **c** $12(x+1)$
d $-4x(x+4)$ **e** $(3x+2)(x+4)$ **f** $h(2x+h)$
g $-12(x+1)$ **h** $-5(3x-4)(x-4)$
i $-3(x+9)(5x+9)$

EXERCISE I

1 **a** $x=b-a$ **b** $x=\dfrac{b}{a}$ **c** $x=\dfrac{d-a}{2}$
d $x=t-c$ **e** $x=\dfrac{20-2y}{5}$ **f** $x=\dfrac{12-3y}{2}$
g $x=\dfrac{d-3y}{7}$ **h** $x=\dfrac{c-by}{a}$ **i** $x=\dfrac{y-c}{m}$

2 **a** $z=\dfrac{b}{ac}$ **b** $z=\dfrac{a}{d}$ **c** $z=\dfrac{2d}{3}$ **d** $z=\pm\sqrt{2a}$

3 **a** $a=\dfrac{F}{m}$ **b** $r=\dfrac{C}{2\pi}$ **c** $d=\dfrac{V}{lh}$ **d** $K=\dfrac{b}{A}$

4 **a** $r=\sqrt{\dfrac{A}{\pi}}$ **b** $x=\sqrt[5]{aN}$ **c** $r=\sqrt[3]{\dfrac{3V}{4\pi}}$
d $x=\sqrt[3]{\dfrac{n}{D}}$

5 **a** $a=d^2n^2$ **b** $l=25T^2$ **c** $a=\pm\sqrt{b^2+c^2}$
d $l=\dfrac{gT^2}{4\pi^2}$ **e** $a=\dfrac{P}{2}-b$ **f** $h=\dfrac{A-\pi r^2}{2\pi r}$
g $r=\dfrac{E}{I}-R$ **h** $q=p-\dfrac{B}{A}$

6 **a** $a=\dfrac{d^2}{2kb}$ **b** 1.29 **7** **a** $r=\sqrt[3]{\dfrac{3V}{4\pi}}$ **b** 2.122 cm

8 **a** $t=\sqrt{\dfrac{2S}{a}}$ **b** 15.81 s

9 **a** $v=\dfrac{uf}{u-f}$ **b** **i** 9.52 cm **ii** 10.9 cm

10 **a** $v=\sqrt{c^2\left(1-\dfrac{m_0^2}{m^2}\right)}=\dfrac{c}{m}\sqrt{m^2-m_0^2}$
b $v=\dfrac{\sqrt{8}}{3}c$ **c** $2.998\times10^8\ \text{m s}^{-1}$

EXERCISE J

1 **a** $\dfrac{15+x}{5}$ **b** $\dfrac{x+3}{x}$ **c** $\dfrac{x+4}{2}$
d $\dfrac{14-x}{4}$ **e** $\dfrac{8x-2}{15}$ **f** $\dfrac{4x+17}{12}$

2 **a** $\dfrac{x+5}{x+2}$ **b** $\dfrac{11-2x}{x-4}$ **c** $\dfrac{1-3x}{x-1}$
d $\dfrac{5x+2}{x+1}$ **e** $\dfrac{2x+3}{x+1}$ **f** $\dfrac{x+3}{1-x}$

3 **a** $\dfrac{7x^2-6x-25}{(2x-5)(x-2)}$ **b** $\dfrac{-1}{(x-2)(x-3)}$
c $\dfrac{8x^2+6x+8}{x^2-16}$ **d** $\dfrac{3x^2+3x+13}{(x-3)(2x+1)}$

EXERCISE K.1

1 **Hint:** Consider Δs AEC, BDC

2 **Hint:** Let M be on [AB] so that [PM] $\perp$ [AB].
Let N be on [AC] so that [PN] $\perp$ [AC].
Join [PM], [PN] and consider the two triangles formed.

EXERCISE K.2

1 **a** $x=2.4$ **b** $x=2.8$ **c** $x=3\frac{3}{11}$ **d** $x=6\frac{2}{3}$
e $x=7$ **f** $x=7.2$

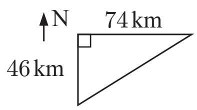
2 1.35 m tall

EXERCISE L.1

1 **a** $x=0.663$ **b** $x=4.34$ **c** $x=2.23$

2 **a** 4.54 m **b** 4.17 m **3** 237 m

4 **a**



b 87.1 km

EXERCISE L.2

1 50.3 m **2** 17.3 cm **3** 8.60 m **4** 53.4 m **5** 44.4 km

EXERCISE M

1 **a** $\sqrt{13}$ units **b** $\sqrt{34}$ units **c** $\sqrt{20}$ units **d** $\sqrt{10}$ units

2 **a** (2, 3) **b** (2, −1) **c** $(3\frac{1}{2}, 1\frac{1}{2})$ **d** $(2, -2\frac{1}{2})$

3 **a** $\frac{2}{3}$ **b** $-\frac{2}{5}$ **c** 1 **d** 0 **e** -4 **f** undefined

4 **a** 2 **b** 3 **c** undefined **d** 0 **e** -4 **f** $\frac{1}{4}$

5 **a** parallel, gradients $\frac{2}{5}$ **b** parallel, gradients $-\frac{1}{7}$
c perpendicular, gradients $\frac{1}{2}, -2$
d neither, gradients $-4, \frac{1}{3}$ **e** neither, gradients $\frac{2}{7}, \frac{1}{5}$
f perpendicular, gradients $2, -\frac{1}{2}$

6 **a** $-\frac{4}{3}$ **b** $-\frac{3}{11}$ **c** $-\frac{1}{4}$ **d** 3 **e** $\frac{1}{5}$ **f** undefined

7 **a** $y=2x-7$ **b** $y=-2x+4$ **c** $y=3x-15$
d $y=-3x+4$ **e** $y=-4x+9$ **f** $y=x+5$

8 **a** $3x-2y=4$ **b** $3x+2y=11$ **c** $x-3y=4$
d $4x+y=6$ **e** $3x-y=0$ **f** $4x+9y=-2$

9 **a** $x-3y=-3$ **b** $5x-y=1$ **c** $x-y=3$
d $4x-5y=10$ **e** $x-2y=-1$ **f** $2x+3y=-5$

10 **a** $y=-2$ **b** $x=6$ **c** $x=-3$

11

	Equation of line	Gradient	x-int.	y-int.
a	$2x-3y=6$	$\frac{2}{3}$	3	-2
b	$4x+5y=20$	$-\frac{4}{5}$	5	4
c	$y=-2x+5$	-2	$\frac{5}{2}$	5
d	$x=8$	undef.	8	no y-int.
e	$y=5$	0	no x-int.	5
f	$x+y=11$	-1	11	11
g	$4x+y=8$	-4	2	8
h	$x-3y=12$	$\frac{1}{3}$	12	-4

12 **a** yes **b** no **c** yes

13 **a** (4, 2) **b** (−2, 3) **c** (−3, 6) **d** (4, 0)
e parallel lines do not meet **f** coincident lines

14 **a** $y=-4$ **b** $x=5$ **c** $x=-1$ **d** $y=2$
e $y=0$ **f** $x=0$

15 **a** $3x-4y=-19$ **b** $x-y=7$ **c** $y=3x$
d $2x+3y=19$ **e** $x-2y=5$ **f** $x+3y=13$

16 **a** $x-8y=-83$ **b** $8x+y=41$
c $9x-2y=23$ for $5\leqslant x\leqslant 7$ **d** $(8, 11\frac{3}{8})$

17 **a** $x-3y=-16$ **b** $2x-y=-3$ **c** $x=5$

18 $(4\frac{3}{4}, 8)$ **19** **a** $(2\frac{1}{3}, 7)$ **b** 8.03 km **c** yes (6.16 km)

20 (13.41, 8) or (−5.41, 8)

21 **a** **Hint:** Use the distance formula to find the distance from the centre of the circle to point P.
b **i** $(x-4)^2+(y-3)^2=25$ **ii** $(x+1)^2+(y-5)^2=4$
iii $x^2+y^2=100$ **iv** $(x-1)^2+(y-3)^2=8$

22 **a** centre (1, 3), radius 2 units
b centre (0, −2), radius 4 units
c centre (0, 0), radius $\sqrt{7}$ units

23 **a** centre (2, −3), radius $\sqrt{20}$ units
b **Hint:** Substitute (4, 1) into equation of circle.
c $x+2y=6$

24 (3, 3)

EXERCISE N.1

1 **a** 3, $\sin\theta=\frac{3}{5}$, $\cos\theta=\frac{4}{5}$, $\tan\theta=\frac{3}{4}$
b 12, $\sin\theta=\frac{5}{13}$, $\cos\theta=\frac{12}{13}$, $\tan\theta=\frac{5}{12}$
c $\sqrt{11}$, $\sin\theta=\frac{5}{6}$, $\cos\theta=\frac{\sqrt{11}}{6}$, $\tan\theta=\frac{5}{\sqrt{11}}$
d $\sqrt{5}$, $\sin\theta=\frac{2}{\sqrt{5}}$, $\cos\theta=\frac{1}{\sqrt{5}}$, $\tan\theta=2$
e $\sqrt{52}$, $\sin\theta=\frac{4}{\sqrt{52}}$, $\cos\theta=\frac{6}{\sqrt{52}}$, $\tan\theta=\frac{2}{3}$
f $\sqrt{15}$, $\sin\theta=\frac{7}{8}$, $\cos\theta=\frac{\sqrt{15}}{8}$, $\tan\theta=\frac{7}{\sqrt{15}}$

2 **a** $\sin\theta=\frac{\sqrt{3}}{2}$, $\tan\theta=\sqrt{3}$ **b** $\cos\alpha=\frac{\sqrt{5}}{3}$, $\tan\alpha=\frac{2}{\sqrt{5}}$
c $\sin\beta=\frac{4}{5}$, $\cos\beta=\frac{3}{5}$

3 **a** $\sin\theta=\dfrac{b}{c}$, $\cos\theta=\dfrac{a}{c}$, $\tan\theta=\dfrac{b}{a}$

4 **a** **i** $\dfrac{a}{b}$ **ii** $\dfrac{c}{b}$ **iii** $\dfrac{c}{b}$ **iv** $\dfrac{a}{b}$
b **i** complement **ii** complement

5 **a** $\sqrt{2}$ **b** $\sin 45^o=\frac{1}{\sqrt{2}}$, $\cos 45^o=\frac{1}{\sqrt{2}}$, $\tan 45^o=1$

6 **a** $\widehat{ABN}=60^o$, $\widehat{BAN}=30^o$ **b** BN = 1, AN = $\sqrt{3}$
c **i** $\sin 60^o=\frac{\sqrt{3}}{2}$, $\cos 60^o=\frac{1}{2}$, $\tan 60^o=\sqrt{3}$
ii $\sin 30^o=\frac{1}{2}$, $\cos 30^o=\frac{\sqrt{3}}{2}$, $\tan 30^o=\frac{1}{\sqrt{3}}$

EXERCISE N.2

1 **a** $x=17.2$ **b** $x=257$ **c** $x=15.1$
d $x=7.10$ **e** $x=554$ **f** $x=457$

2 **a** $\theta=69.5^o$ **b** $\theta=76.2^o$ **c** $\theta=60.0^o$
d $\theta=73.4^o$ **e** $\theta=19.5^o$ **f** $\theta=77.9^o$
g $\theta=9.06^o$ **h** $\theta=34.7^o$

3 **a** $\theta=56.4$ **b** $\alpha=4.76$ **c** $\beta=48.2$

4 **a** AC = 6.40 m, $\widehat{A}=38.7^o$, $\widehat{C}=51.3^o$
b $\widehat{R}=39^o$, PQ = 8.10 m, PR = 12.9 m

5 **a** $x=2.65$, $\theta=37.1$
b $x=6.16$, $\theta=50.3$, $y=13.0$

EXERCISE N.3

1 **a** $x=4.13$ **b** $\alpha=75.5$ **c** $\beta=41.0$
2 **a** $\theta=36.9$ **b** $r=11.3$ **c** $\alpha=61.9$ **3** 7.99 cm
4 89.2^o **5** 47.2^o

EXERCISE N.4

1 22.4^o **2** 11.8 cm **3** 119^o **4** 36.5 cm
5 **a** $x=3.44$ **b** $\alpha=51.5$ **6** 129^o

EXERCISE N.5

1 18.3 m **2** **a** 371 m **b** 1.62 km **3** 159 m **4** 1.58^o
5 angle of elevn. = 26.4^o, angle of depn. = 26.4^o
6 418.5 m **7** 111 m **8** 72.0 m
9 **a** 16.2 m s^{-1} **b** 11.5^o **10** $\theta=12.6$ **11** 9.56 m
12 77.7 litres **13** 2.22 m **14** 10.95 m **15** 786 m **16** 962 m
17 3.17 km **18** 30.8 m

Background knowledge in statistics

Contents:

Facts and figures are part of the flow of information we are bombarded with each day.

For example,

- The median house price has increased by 50% in the last 10 years.
- If an election was held tomorrow the two major parties would get 89% of the vote.
- Yesterday was the coldest August day since August 4th 1897.

Figures like those given above tell us something about the topic being discussed.

Some statements, particularly those about weather, cannot be made without gathering a large amount of information over a long period of time. We also have to know if the statements are reliable.

The facts or pieces of information are called **data**.

Data may be collected by counting, measuring or asking questions.

[One piece of information is known as one piece of *datum* (singular), whereas lots of pieces of information are known as *data* (plural).]

If we collect information about the weights (to the nearest kg) of students in our school, we have a number list such as 53, 57, 69, 63, 48, 56, 56, 43, 57, 57, ...

This number list is called a **data set** and because it is not in organised form it is called **raw data**.

STATISTICS

> **Statistics** is the art of solving problems and answering questions by collecting and analysing data.

Statistics are used by governments, businesses and sports organisations so that they can make informed decisions when they are providing services such as in health, transport and commerce or developing new tactics. They are also interested in using statistics as a means of analysing the effects of certain changes that may have been made, or in predicting what may happen in the future.

Businesses use market research surveys to determine consumer preferences of goods and services.

Statistics are used by manufacturers for quality control and in medical research to test new drugs.

A medical researcher may believe that a newly discovered drug could prolong the life of heart attack patients. To prove this, the drug would have to be given to a group of heart attack sufferers. Statistics which compared their quality of life and life expectancy to another group where the drug was not given would be obtained. If the effect of the drug improved life expectancy and quality of life with no bad side effects it would almost certainly become a legal drug.

In statistical work we use tables, graphs and diagrams showing data collected from scientific experiments, and figures from economics, public opinion polls, census returns and many other situations.

HISTORICAL NOTE

- The **Babylonians** (before 3000 BC) recorded yields for their crops on small clay tablets.
- Pharoahs in ancient **Egypt** recorded their wealth on walls of stone.
- Censuses were conducted by the **Ancient Greeks** so that taxes could be collected.
- The book of Numbers in the Old Testament records the results of two censuses of the **Israelites** taken by **Moses**.
- At the time of the birth of Jesus, **Emperor Augustus** ordered a census to be taken throughout the **Roman Empire**. This is why Mary and Joseph travelled to Bethlehem.
- After **William the Conqueror** invaded and conquered England in 1066, his followers overtook estates previously occupied by Saxons. Confusion reigned over who owned what.

  In 1086, William ordered that a census be conducted to record population, wealth and land ownership. A person's wealth was recorded in terms of land, animals, farm implements and number of peasants on the estate. All this information was collated and has become known as the "Domesday Book". It is regarded as the greatest public record of Medieval Europe.

  The Domesday Book is displayed in the Public Record Office in London.

OPENING PROBLEM

A city school can be easily accessed by train and school bus as well as by walking and riding in a private car. The school is interested in finding out how students get to and go home from school as there are local traffic problems in the area during these times. For safety reasons parents and bus drivers are requesting that the school provides a bus/car pull-in bay.

An initial survey of 75 students was carried out and the results in coded form were:

T T B W W	T W C W C	C C B C B	W T W C C	W C C B B
B C B B W	W B T B B	W B C B C	C W B W T	T C B B T
B W T C B	B C C C C	W C W T B	T T T C C	C C C T W

(T = by train, B = by school bus, C = by car, W = by walking)

Questions to consider:

- There are 859 students attending the school. How were the students selected for the survey? Would the survey represent the whole school if the students were only selected from Year 12?
- Is the sample large enough to reflect the method of travel for all students in the school?
- How could we best organise this raw data?
- How could we display this information graphically?
- What calculations could we perform on the organised data to make it more meaningful?
- What likely conclusions could we make and report to the school?

A STATISTICAL ENQUIRIES

Let us consider the steps involved in a statistical investigation.

1 Examining a problem

There is no interest in gathering data just for the sake of it. Which of the following problems may possibly be solved by examining data?

- If I open a shoe store, how many of each size shoe should I keep in stock?
- Is it true that there is global warming and the Earth's temperature is increasing?
- If the maximum speed limit is reduced by 10 km/h will this reduce the number of fatal accidents?
- What is the likelihood that a particular netball team will win next Saturday?

Once we have recognised the problem and written down a question we may proceed.

2 Collection of data (information)

Data for a statistical investigation can be collected from records, from surveys (either face-to-face, telephone, or postal), by direct observation or by measuring or counting. Data can be collected for the whole **population**, which generally means all the people or things that the conclusions of a statistical investigation would apply to. This is called a **census**. Or data can be collected for a **sample** of the population.

Collection of data is the first, and most important task in an investigation, because unless the correct data is collected, valid conclusions cannot be made.

3 Organisation of data

This process involves organising data into tables.

4 Summarising and display of data

We summarise data by counting it in some way and then we display the data with a suitable graph so that some of the features of the data are clearly visible.

For the two types of data that we study in detail (categorical and numerical) some types of graphs are appropriate and some are not.

5 Analysing data and making a conclusion in the form of a conjecture

There are some calculated quantities that are universally used to *describe* a set of data.

Calculating quantities that indicate the centre of the data (mean, median and mode) and the spread of the data (range, interquartile range and standard deviation) gives us a picture of the sample or population under investigation. Using these quantities gives a more satisfactory way of comparing two or more data sets and making a conclusion.

TYPES OF DATA

Data are individual observations of a **variable**. A variable is a quantity that can have a value recorded for it or to which we can assign an attribute or quality.

There are two types of variable that we commonly deal with:

CATEGORICAL VARIABLES

A **categorical variable** is one which describes a particular quality or characteristic. It can be divided into **categories**. The information collected is called **categorical data**.

Examples of categorical variables are:

- *Getting to school:* the categories could be train, bus, car and walking.
- *Colour of eyes:* the categories could be blue, brown, hazel, green, grey.
- *Gender:* male and female.

QUANTITATIVE (NUMERICAL) VARIABLES

A **quantitative (numerical) variable** is one which has a numerical value and is often called a numerical variable. The information collected is called **numerical data**.

Quantitative variables can be either discrete or continuous.

A **quantitative discrete variable** takes exact number values and is often a result of **counting**.

Examples of discrete quantitative variables are:

- *The number of people in a household:* the variable could take the values 1, 2, 3, ...
- *The score out of* 30 *on a test:* the variable could take the values 0, 1, 2, 3, ... 30.

A **quantitative continuous variable** takes numerical values within a certain continuous range. It is usually a result of **measuring**.

Examples of quantitative continuous variables are:

- *The weight of newborn babies:* the variable could take any value on the number line but is likely to be in the range 0.5 kg to 8 kg.
- *The heights of Year 8 students:* the variable would be measured in centimetres. A student whose height is recorded as 145 cm could have exact height between 144.5 cm and 145.5 cm.

Example 1

Classify these variables as categorical, quantitative discrete or quantitative continuous:

a the number of heads when 3 coins are tossed
b the brand of toothpaste used by the students in a class
c the heights of a group of 15 year old children.

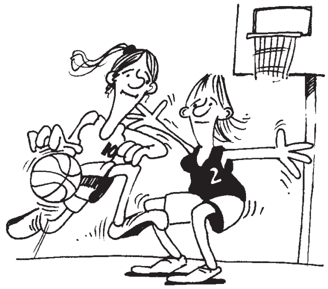
a The values of the variables are obtained by counting the number of heads. The result can only be one of the values 0, 1, 2 or 3. It is quantitative discrete data.

b The variable describes the brands of toothpaste. It is categorical data.

c This is numerical data obtained by measuring. The results can take any value between certain limits determined by the degree of accuracy of the measuring device. It is quantitative continuous data.

EXERCISE A

1 For each of the following possible investigations, classify the variable as categorical, quantitative discrete or quantitative continuous:



- **a** the number of goals scored each week by a netball team
- **b** the heights of the members of a football team
- **c** the most popular radio station
- **d** the number of children in a Japanese family
- **e** the number of loaves of bread bought each week by a family
- **f** the pets owned by students in a year 8 class
- **g** the number of leaves on the stems of plants
- **h** the amount of sunshine in a day
- **i** the number of people who die from cancer each year in the USA
- **j** the amount of rainfall in each month of the year
- **k** the countries of origin of immigrants
- **l** the most popular colours of cars
- **m** the gender of school principals
- **n** the time spent doing homework
- **o** the marks scored in a class test
- **p** the items sold at the school canteen
- **q** the number of matches in a box
- **r** the reasons people use taxis
- **s** the sports played by students in high schools
- **t** the stopping distances of cars doing 60 km/h
- **u** the pulse rates of a group of athletes at rest.

2 **a** For the categorical variables in question **1**, write down two or three possible categories. (In all cases but one, there will be more than three categories possible.) Discuss your answers.

b For each of the quantitative variables (discrete and continuous) identified in question **1**, discuss as a class the range of possible values you would expect.

INVESTIGATION — STATISTICS FROM THE INTERNET

In this investigation you will be exploring the web sites of a number of organisations to find out the topics and the types of data that they collect and analyse.

Note that the web addresses given here were operative at the time of writing but there is a chance that they will have changed in the meantime. If the address does not work, try using a search engine to find the site of the organisation.

What to do:

Visit the site of a world organisation such as the United Nations (www.un.org) or the World Health Organisation (www.who.int) and see the available types of data and statistics.

B POPULATIONS AND SAMPLES

When a statistical investigation is to be conducted, there is always a target *population* about which information is required.

The *population* might be the entire *population* of the country, the entire population of a school, an entire animal species, or the complete output of a machine making a particular item.

CENSUS OR SAMPLE

One of the first decisions to be made when collecting data is from whom, or what, the information is to be collected. There are two ways in which this can be done.
These are: a census or a sample.

A **census** is a method which involves collecting data about every individual in the *whole population.*

The individuals may be people or objects. A census is detailed and accurate but is expensive, time consuming and often impractical.

A **sample** is a method which involves collecting data about a *part of the population* only.

A sample is cheaper and quicker than a census but is not as detailed or as accurate. Conclusions drawn from samples always involve some error.

However, the aim is to make the sample an accurate representation of the population so that the features of the sample's distribution are close to the features of the population's distribution. Hence, if the method of sample selection was successful, statistics such as median and range should have similar values for both the sample and the population.

Hence two types of **statistics** exist:

- **sample statistics**
- **population statistics** (more commonly called **population parameters**).

Example 2

Would a census or sample be used to investigate:

a the length of time an electric light globe will last
b the causes of car accidents in a particular state
c the number of people who use White-brite toothpaste?

a *Sample.* It is impractical to test every light globe produced as there would be none left for sale!

b *Census.* An accurate analysis of all accidents would be required.

c *Sample.* It would be very time consuming to interview the whole population to find out who uses, or does not use, White-brite toothpaste.

EXERCISE B.1

1. State whether a census or a sample would be used for each of the possible investigations in question **1** on page **6**.

2. Give three examples of data which would be collected by using a:
   a census **b** sample

BIAS IN SAMPLING

The most common way of collecting information is by using a sample. The purpose of a sample is to provide an estimate of a particular characteristic of the whole population. Therefore the challenge in selecting a sample is to make it as free from prejudice as possible and large enough to be representative of the whole population.

A **biased sample** is one in which the data has been unfairly influenced by the collection process and is not truly representative of the whole population.

Example 3

Suggest the possible bias in each of the following samples:

a a phone survey during the day
b a survey of people on a train station
c a survey of a football crowd

a The sample would be biased towards people who are at home during the day, i.e., it does not include people who go to work.

b The sample would be biased towards people who catch the train, i.e., it does not include people who use other forms of transport or work at home.

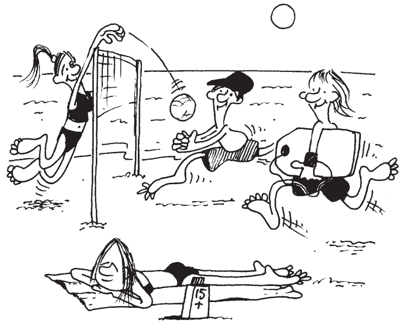
c The sample would be biased towards people who attend football matches. For example, there would probably be more males than females at football matches.

Sometimes people use **biased samples** to enhance their claims for their products or to support a particular point of view.

For example, a person wanted the local council to upgrade its swimming pool, and sampled the views of swimmers who used the pool.

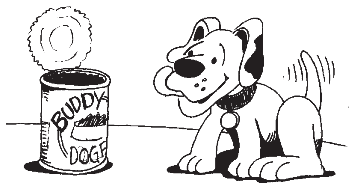
In this case you would expect the people who use the pool to be biased very favourably towards this proposal, so the person taking the sample could be accused of producing an unfair report.

EXERCISE B.2



1. Explain and discuss any possible bias in the following samples:
   a. a phone survey on a Saturday night
   b. a survey of the people at a bus stop
   c. a survey of the people in a supermarket carpark
   d. a survey of people at the beach
   e. a survey of people in your street
   f. people selected from the electoral roll

2. Comment on any possible bias in the following situations:
   a. Year 12 students are interviewed about changes to the school uniform.
   b. Motorists stopped in peak hour are interviewed about traffic problems.
   c. Real estate agents are interviewed about the prices of houses.
   d. Politicians are interviewed about the state of the country's economy.
   e. People are asked to phone in to register their vote on an issue.
   f. An opinion poll is conducted by posting a questionnaire to people.
   g. An advertisement claims that "Dog breeders recommend Buddy dog food."



A NOTE ON VARIABLES

Two variables under consideration are usually linked by the fact that one of them is *dependent* on the other.

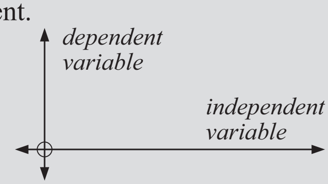
For example, the *total cost of a dinner* depends on *the number of guests present*.

We say that *the total cost of a dinner* is the **dependent variable**, and *the number of guests present* is the **independent variable**.

DISCUSSION

- Discuss the following sentences and find the dependent and independent variables:
  'The number of hours worked by a plumber affects the total charge.'
  'The amount received by each person in a Lottery syndicate is linked to the number of people in the syndicate.'
  'The diameter of a circular table top determines its area.'
- Discuss and write down *two* sentences which contain variables, and identify which of the variables is the dependent and which is independent.

Generally, when drawing **graphs** involving two variables, the *independent variable* is on the **horizontal axis** and the *dependent variable* is on the **vertical axis**.
An exception to this is when we draw a horizontal bar chart.



C PRESENTING AND INTERPRETING DATA

ORGANISING CATEGORICAL DATA

A **tally and frequency table** can be used to organise categorical data.

For example, a survey was conducted on the type of fuel used by 50 randomly selected vehicles.

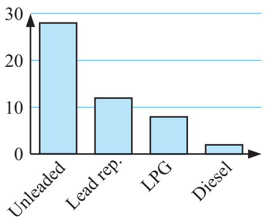
The variable 'type of fuel' is a categorical variable because the information collected for each vehicle can only be one of the four categories: Unleaded, Lead Replacement, LPG or Diesel. The data has been tallied and organised in the given frequency table:

Fuel type	*Tally*	*Freq.*
Unleaded	卌 卌 卌 卌 卌 |||	28
Lead Rep	卌 卌 ||	12
LPG	卌 |||	8
Diesel	||	2
	Total	50

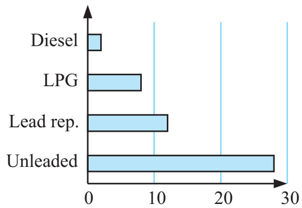
DISPLAYING CATEGORICAL DATA

Acceptable graphs to display the 'type of fuel' categorical data are:

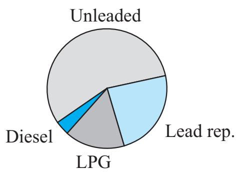
Vertical column graph



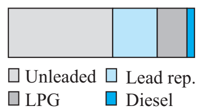
Horizontal bar chart



Pie chart



Segment bar chart



For categorical data, the **mode** is the category which occurs most frequently.

ORGANISING DISCRETE NUMERICAL DATA

Discrete numerical data can be organised:

- in a **tally and frequency table**
- using a **dot plot**
- using a **stem-and-leaf plot** (also called a **stemplot**).

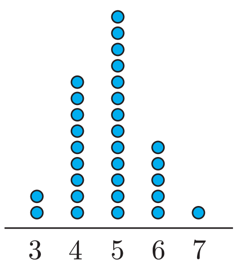
Stemplots are used when there are many possible data values. The stemplot is a form of grouping of the data which displays frequencies but retains the actual data values.

Examples:

- **frequency table**

Number	*Tally*	*Freq.*
3	||	2
4	卌 ||||	9
5	卌 卌 |||	13
6	卌	5
7	|	1

- **dot plot**



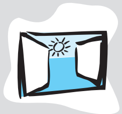
- **stemplot**

Example:

Stem	*Leaf*
0	9
1	7 1
2	8 3 6 7 6 4
3	9 3 5 5 6 8 2 1
4	7 9 3 4 2
5	1

As data is collected it can be entered directly into a carefully set up tally table, dot plot or stemplot blank sheet.

THE PEA PROBLEM



A farmer wishes to investigate the effect of a new organic fertiliser on his crops of peas. He is hoping to improve the crop yield by using the fertiliser. He set up a small garden which was subdivided into two equal plots and planted many peas. Both plots were treated the same except for the use of the fertiliser on one, but not the other. All other factors such as watering were as normal.

A random sample of 150 pods was harvested from each plot at the same time and the number of peas in each pod counted. The results were:

Without fertiliser

4 6 5 6 5 6 4 6 4 9 5 3 6 8 5 4 6 8 6 5 6 7 4 6 5 2 8 6 5 6 5 5 5 4 4 4 6 7 5 6
7 5 5 6 4 8 5 3 7 5 3 6 4 7 5 6 5 7 5 7 6 7 5 4 7 5 5 5 6 6 5 6 7 5 8 6 8 6 7 6
6 3 7 6 8 3 3 4 4 7 6 5 6 4 5 7 3 7 7 6 7 7 4 6 6 5 6 7 6 3 4 6 6 3 7 6 7 6 8 6
6 6 6 4 7 6 6 5 3 8 6 7 6 8 6 7 6 6 6 8 4 4 8 6 6 2 6 5 7 3

With fertiliser

6 7 7 4 9 5 5 5 8 9 8 9 7 7 5 8 7 6 6 7 9 7 7 7 8 9 3 7 4 8 5 10 8 6 7 6 7 5 6 8
7 9 4 4 9 6 8 5 8 7 7 4 7 8 10 6 10 7 7 7 9 7 7 8 6 8 6 8 7 4 8 6 8 7 3 8 7 6 9 7
6 9 7 6 8 3 9 5 7 6 8 7 9 7 8 4 8 7 7 7 6 6 8 6 3 8 5 8 7 6 7 4 9 6 6 6 8 4 7 8
9 7 7 4 7 5 7 4 7 6 4 6 7 7 6 7 8 7 6 6 7 8 6 7 10 5 13 4 7 7

For you to consider:

- Can you state clearly the problem that the farmer wants to solve?
- How has the farmer tried to make a fair comparison?
- How could the farmer make sure that his selection is at random?
- What is the best way of organising this data?
- What are suitable methods of display?
- Are there any abnormally high or low results and how should they be treated?
- How can we best indicate the most typical pod size?
- How can we best indicate the spread of possible pod sizes?
- What is the best way to show 'typical pod size' and the spread?
- Can a satisfactory conclusion be made?

ORGANISATION AND DISPLAY OF DISCRETE DATA

In **The Pea Problem**, the **discrete quantitative variable is**: *The number of peas in a pod.*

To organise the data a tally/frequency table could be used.

We count the data systematically and use a '|' to indicate each data value.

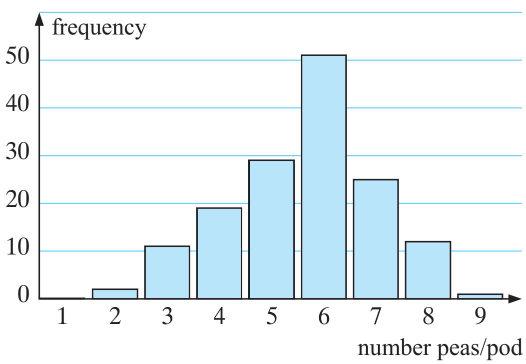
Remember that 𝍸 represents 5.

Below is the table for *Without fertiliser*:

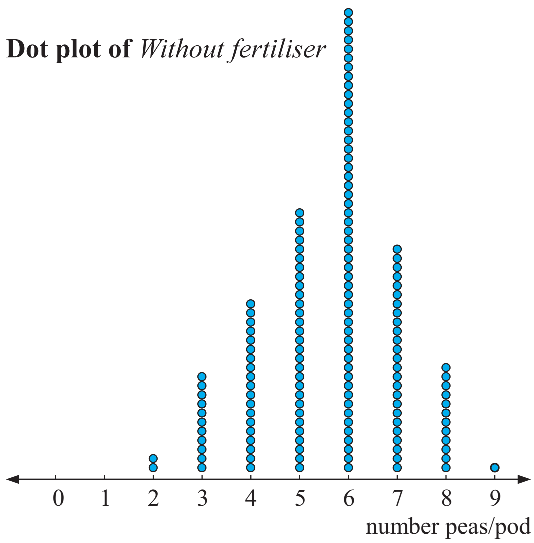
Number of peas/pod	*Tally*	*Frequency*
1		0
2	||	2
3	𝍸 𝍸 |	11
4	𝍸 𝍸 𝍸 ||||	19
5	𝍸 𝍸 𝍸 𝍸 𝍸 ||||	29
6	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 |	51
7	𝍸 𝍸 𝍸 𝍸 𝍸	25
8	𝍸 𝍸 ||	12
9	|	1

A **dot plot** could be used to organise and display the results, or a **column graph** could be used to display the results.

Column graph of ***Without fertiliser***



Dot plot of ***Without fertiliser***



DISCUSSION

Are there any advantages/disadvantages in using a dot plot rather than a column graph?

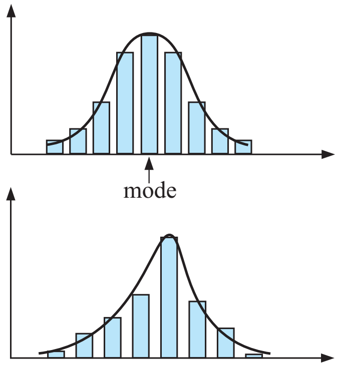
From both graphs we can make observations and calculations such as:

- 6 peas per pod is the mode of the *Without fertiliser* data.
- 8.7% of the pods had fewer than 4 peas in them, etc.

DESCRIBING THE DISTRIBUTION OF THE DATA SET

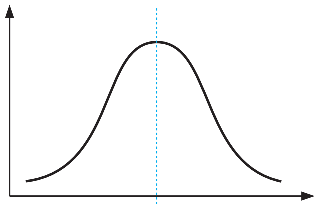
Many data sets show **symmetry** or **partial symmetry** about the mode.

If we place a curve over the column graph (or dot plot) we see that this curve shows symmetry and we say that we have a **symmetrical distribution** of the data.

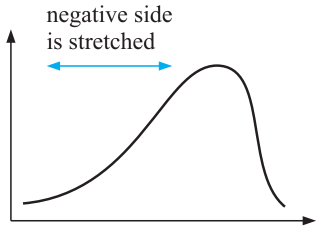


For the *Without fertiliser* data we have: This distribution is said to be **negatively skewed** as if we compare it with the symmetrical distribution it has been 'stretched' on the left (or negative) side of the mode.

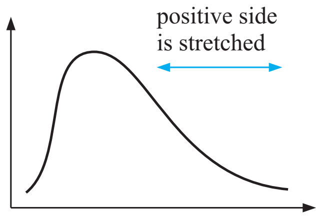
So we have:



symmetrical distribution



negatively skewed distribution

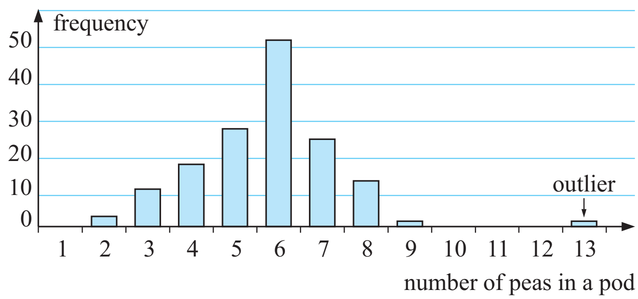


positively skewed distribution

OUTLIERS

Outliers are data values that are either much larger or much smaller than the general body of data. Outliers appear separated from the body of data on a frequency graph.

For example, if the farmer in **The Pea Problem** (page **11**) found one pod in the *Without fertiliser* sample contained 13 peas, then the data value 13 would be considered an outlier. It is much larger than the other data in the sample. On the column graph it would appear separated.



EXERCISE C.1

1 State whether the following quantitative (or numerical) variables are discrete or continuous:

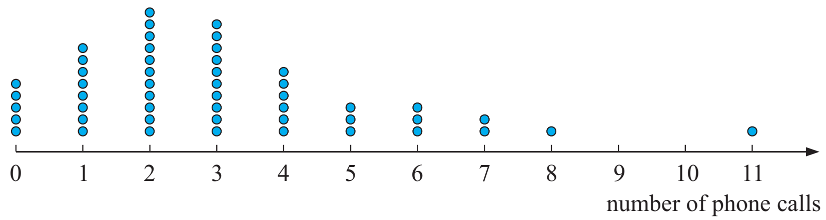
- **a** the time taken to run 100 metres
- **b** the maximum temperature reached on a January day
- **c** the number of matches in a box
- **d** the weight of luggage taken on an aircraft
- **e** the time taken for a battery to run down
- **f** the number of bricks needed to build a house
- **g** the number of passengers on a bus
- **h** the number of minutes spent on the internet per day

2 A class of 20 students was asked "How many pets do you have in your household?" and the following data was collected:

0 1 2 2 1 3 4 3 1 2 0 0 1 0 2 1 0 1 0 1

- **a** What is the variable in this investigation?
- **b** Is the data discrete or continuous? Why?
- **c** Construct a dotplot to display the data. Use a heading for the graph, and scale and label the axes.
- **d** How would you describe the distribution of the data? (Is it symmetrical, positively skewed or negatively skewed? Are there any outliers?)
- **e** What percentage of the households had no pets?
- **f** What percentage of the households had three or more pets?

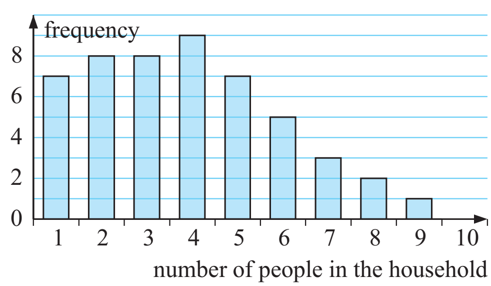
3 For an investigation into the number of phone calls made by teenagers, a sample of 50 fifteen-year-olds were asked the question "How many phone calls did you make yesterday?" The following dotplot was constructed from the data:



- **a** What is the variable in this investigation?
- **b** Explain why the data is discrete numerical data.
- **c** What percentage of the fifteen-year-olds did not make any phone calls?
- **d** What percentage of the fifteen-year-olds made 5 or more phone calls?
- **e** Copy and complete:
  "The most frequent number of phone calls made was ...."
- **f** Describe the distribution of the data.
- **g** How would you describe the data value '11'?

4 A randomly selected sample of households has been asked, 'How many people live in your household?' A column graph has been constructed for the results.

- **a** How many households gave data in the survey?
- **b** How many of the households had only one or two occupants?
- **c** What percentage of the households had five or more occupants?
- **d** Describe the distribution of the data.



5 The number of matches in a box is stated as 50 but the actual number of matches has been found to vary. To investigate this, the number of matches in a box has been counted for a sample of 60 boxes:

51 50 50 51 52 49 50 48 51 50 47 50 52 48 50 49 51 50 50 52
52 51 50 50 52 50 53 48 50 51 50 50 49 48 51 49 52 50 49 50
50 52 50 51 49 52 52 50 49 50 49 51 50 50 51 50 53 48 49 49

- **a** What is the variable in this investigation?
- **b** Is the data continuous or discrete numerical data?
- **c** Construct a frequency table for this data.
- **d** Display the data using a bar chart.
- **e** Describe the distribution of the data.
- **f** What percentage of the boxes contained exactly 50 matches?

6 **Revisiting The Pea Problem**

For the *With fertiliser* data on page **11**:

- **a** Organise the data in a tally-frequency table.
- **b** Draw a column graph of the data.
- **c** Are there any outliers?
- **d** Is the data skewed?
- **e** What evidence is there that the fertiliser 'increases the number of peas in a pod'?
- **f** Can it be said that the fertiliser will increase the farmer's pea crop and therefore his profits?

GROUPED DISCRETE DATA

It is not sensible to organise some discrete data by using a frequency table. Also graphing by dot plot or column graph is not appropriate.

For example, a local kindergarten is concerned about the number of vehicles passing by between 8.45 am and 9.00 am.

Over 30 consecutive week days they recorded data.

The results were: 27, 30, 17, 13, 46, 23, 40, 28, 38, 24, 23, 22, 18, 29, 16,
35, 24, 18, 24, 44, 32, 52, 31, 39, 32, 9, 41, 38, 24, 32

In situations like this grouping the data into **class intervals** is appropriate.

It seems sensible to use class intervals of length 10 in this case.

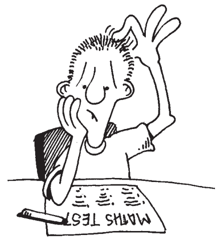
The tally/frequency table is:

Number of cars	*Tally*	*Frequency*
0 to 9	|	1
10 to 19	~~||||~~	5
20 to 29	~~||||~~ ~~||||~~	10
30 to 39	~~||||~~ ||||	9
40 to 49	||||	4
50 to 59	|	1
	Total	30

EXERCISE C.2

1 The data set below is the test scores (out of 100) for a Maths test for 50 students.

56	29	78	67	68	69	80	89	92	71
58	66	56	88	81	70	73	63	74	38
67	64	62	55	56	75	90	92	47	44
59	64	89	62	51	87	89	76	59	88
72	80	95	68	80	64	53	43	61	39



a Construct a tally and frequency table for this data using class intervals 0 - 9, 10 - 19, 20 - 29, ..., 90 - 100.

b What percentage of the students scored 80 or more for the test?

c What percentage of students scored less than 50 for the test?

d Copy and complete the following:
More students had a test score in the interval .... than in any other interval.

2 The number of points scored by the eight winning teams in the first five rounds of the Year 2001 AFL season are to be investigated. The variable 'the number of points scored by a winning team' is a discrete numerical variable that can theoretically take the values 1, 2, 3, ..., 101, 102, 103, ..., 201, ... etc.

The winning scores are:

94	156	154	131	129	134	152	140	124	162
103	139	82	170	110	111	116	160	104	110
98	106	187	149	165	88	118	123	137	128
113	130	145	139	125	154	126	141	122	106

a Construct a frequency table for this data using class intervals 80-89, 90-99, 100-109, ..., 180-189.

b Which class interval has the most number of winning scores?

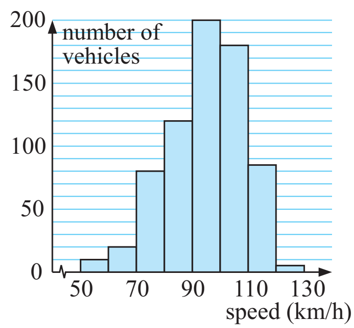
c How many matches were won with a winning score of 99 points or less?

d What percentage of the matches had winning scores of 150 points or more?

e Describe the distribution of the data.

f Are there any outliers in this data set?

3 The speed of vehicles travelling along a section of highway has been recorded and displayed using the histogram shown:



a How many vehicles were included in this survey?

b What percentage of the vehicles were travelling at a speed from 100 up to 110 km/h?

c What percentage of the vehicles were travelling at speeds equal to or greater than 100 km/h?

d What percentage of the vehicles were travelling at a speed less than 80 km/h?

e If the owners of the vehicles travelling at 110 km/h or more were fined $165 each, what amount would be collected in fines?

EXERCISE A

1 **a** quantitative discrete **b** quantitative continuous
c categorical **d** quantitative discrete
e quantitative discrete **f** categorical
g quantitative discrete **h** quantitative continuous
i quantitative discrete **j** quantitative continuous
k categorical **l** categorical **m** categorical
n quantitative continuous **o** quantitative discrete
p categorical **q** quantitative discrete **r** categorical
s categorical **t** quantitative continuous
u quantitative discrete

2 Answers will vary

EXERCISE B.1

1 **a** census **b** census **c** sample **d** sample or census
e sample **f** census **g** sample **h** census **i** census
j census **k** census **l** sample **m** sample **n** sample
o census **p** census **q** sample **r** sample **s** sample
t sample **u** census

2 **a** age structure of Australia's population,
marks in a class test,
the uniform preferences of students at a high school
b favourite television program,
favourite football team,
favourite model of motor car

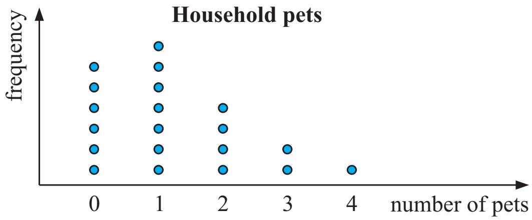
EXERCISE B.2

This exercise is for discussion. Answers will vary. None are given.

EXERCISE C.1

1 **a** continuous **b** continuous **c** discrete **d** continuous
e continuous **f** discrete **g** discrete **h** continuous

2 **a** number of pets
b Discrete since you can't have part of a pet.
c



d positively skewed, no outliers **e** 30% **f** 15%

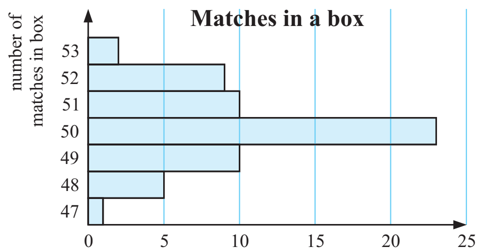
3 **a** the number of phone calls made in a day
b You can only make whole phone calls. **c** 10%
d 20% **e** ...... two calls per day
f positively skewed with an outlier
g Data value 11 is an outlier.

4 **a** 50 households **b** 15 households **c** 36%
d positively skewed, no outliers

5 **a** number of matches in a box **b** discrete
c

No. matches	*Tally*	*Freq.*
47	|	1
48	卌	5
49	卌 卌	10
50	卌 卌 卌 卌 |||	23
51	卌 卌	10
52	卌 ||||	9
53	||	2

d

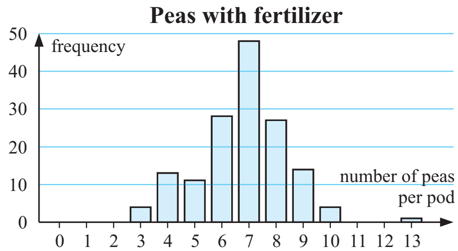


e approximately symmetrical **f** 38.3%

6 **a**

No. peas	*Tally*	*Freq.*
3	||||	4
4	卌 卌 |||	13
5	卌 卌 |	11
6	卌 卌 卌 卌 卌 |||	28
7	卌 卌 卌 卌 卌 卌 卌 卌 卌 |||	48
8	卌 卌 卌 卌 卌 ||	27
9	卌 卌 ||||	14
10	||||	4
11		0
12		0
13	|	1
	Total	150

b



c Yes, data value 13 is an outlier. **d** negatively skewed
e On average the number of peas is higher in the "with fertiliser" group. The mode has increased from 6 to 7.
f Yes, assuming the fertiliser is not too expensive and the peas are as big as they were previously.

EXERCISE C.2

1 **a**

Test Score	*Tally*	*Freq.*
0 - 9		0
10 - 19		0
20 - 29	|	1
30 - 39	||	2
40 - 49	|||	3
50 - 59	卌 ||||	9
60 - 69	卌 卌 |||	13
70 - 79	卌 |||	8
80 - 89	卌 卌	10
90 - 100	||||	4
	Total	50

b 28% **c** 12%
d More students had a test score in the interval 60 - 69 than in any other interval.

2 **a**

Winning Score	*Frequency*
80 - 89	2
90 - 99	2
100 - 109	4
110 - 119	6
120 - 129	7
130 - 139	6
140 - 149	4
150 - 159	4
160 - 169	3
170 - 179	1
180 - 189	1

b 120 - 129
c 4
d 22.5%

e approximately symmetric distribution
f yes, data value 187 is an outlier

3 **a** 700 **b** $\doteq$ 25.7% **c** $\doteq$ 38.6% **d** $\doteq$ 15.7%
e $14 850

Chapter 1

Functions

Contents:

A RELATIONS AND FUNCTIONS

The charges for parking a car in a short-term car park at an Airport are given in the table shown alongside.

Car park charges	
Period (h)	*Charge*
0 - 1 hours	\$5.00
1 - 2 hours	\$9.00
2 - 3 hours	\$11.00
3 - 6 hours	\$13.00
6 - 9 hours	\$18.00
9 - 12 hours	\$22.00
12 - 24 hours	\$28.00

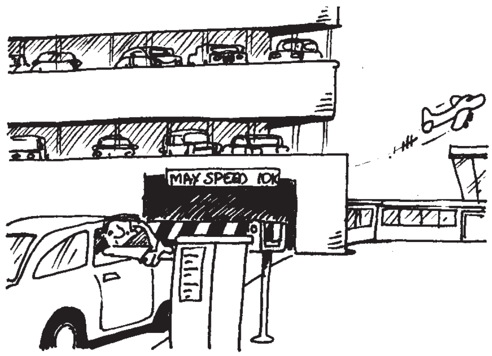
There is an obvious relationship between time spent and the cost. The cost is dependent on the length of time the car is parked.

Looking at this table we might ask: How much would be charged for exactly one hour? Would it be \$5 or \$9?

To make the situation clear, and to avoid confusion, we could adjust the table and draw a graph. We need to indicate that 2-3 hours really means for time over 2 hours up to and including 3 hours i.e., $2 < t \leqslant 3$.

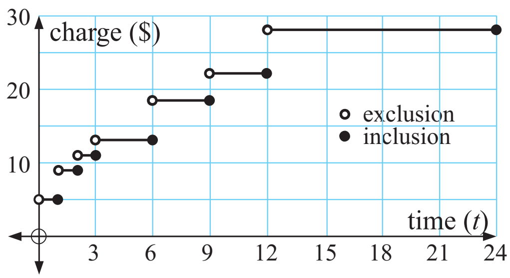
So, we now have

Car park charges	
Period	*Charge*
$0 < t \leqslant 1$ hours	\$5.00
$1 < t \leqslant 2$ hours	\$9.00
$2 < t \leqslant 3$ hours	\$11.00
$3 < t \leqslant 6$ hours	\$13.00
$6 < t \leqslant 9$ hours	\$18.00
$9 < t \leqslant 12$ hours	\$22.00
$12 < t \leqslant 24$ hours	\$28.00



In mathematical terms, because we have a relationship between two variables, time and cost, the schedule of charges is an example of a **relation**.

A relation may consist of a finite number of ordered pairs, such as $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ or an infinite number of ordered pairs.



The parking charges example is clearly the latter as any real value of time (t hours) in the interval $0 < t \leqslant 24$ is represented.

The set of possible values of the variable on the horizontal axis is called the **domain** of the relation.

For example:
- $\{t: \ 0 < t \leqslant 24\}$ is the domain for the car park relation
- $\{-2, 1, 4\}$ is the domain of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$.

The set which describes the possible y-values is called the **range** of the relation.

For example:
- the range of the car park relation is $\{5, 9, 11, 13, 18, 22, 28\}$
- the range of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{3, 5, 6\}$.

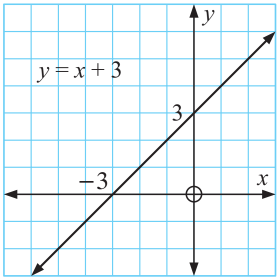
We will now look at relations and functions more formally.

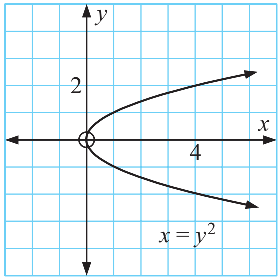
RELATIONS

A **relation** is any set of points on the Cartesian plane.

A relation is often expressed in the form of an **equation** connecting the **variables** x and y.
For example $y = x + 3$ and $x = y^2$ are the equations of two relations.
These equations generate sets of ordered pairs.

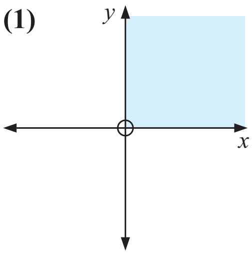
Their graphs are:





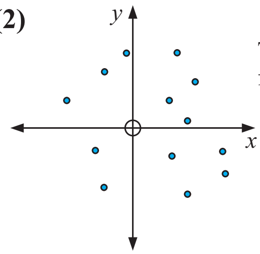
However, a relation may not be able to be defined by an equation. Below are two examples which show this:

(1)



All points in the first quadrant are a relation. $x > 0$, $y > 0$

(2)



These 13 points form a relation.

FUNCTIONS

A **function**, sometimes called a **mapping**, is a relation in which no two different ordered pairs have the same x-coordinate (first member).

We can see from the above definition that a function is a special type of relation.

TESTING FOR FUNCTIONS

Algebraic Test:

If a relation is given as an equation, and the substitution of any value for x results in one and only one value of y, we have a function.

For example:
- $y = 3x - 1$ is a function, as for any value of x there is only one value of y
- $x = y^2$ is not a function since if $x = 4$, say, then $y = \pm 2$.

Geometric Test ("Vertical Line Test"):

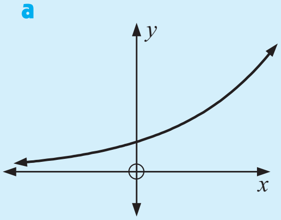
If we draw all possible vertical lines on the graph of a relation, the relation:
- is a function if each line cuts the graph no more than once
- is not a function if one line cuts the graph more than once.

DEMO

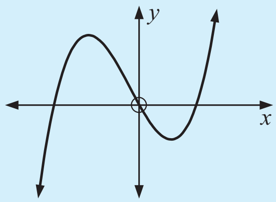
Example 1

Which of the following relations are functions?

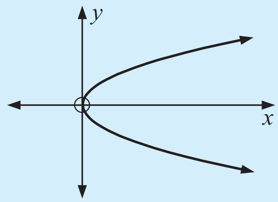
a



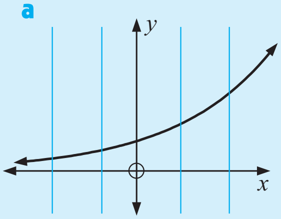
b



c

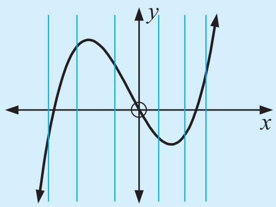


a



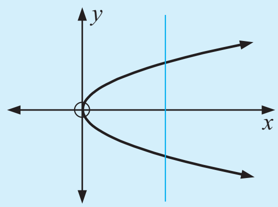
a function

b



a function

c



not a function

GRAPHICAL NOTE

- If a graph contains a small **open circle** end point such as ——o , the end point is **not included**.
- If a graph contains a small **filled-in circle** end point such as ——● , the end point **is included**.
- If a graph contains an **arrow head** at an end such as ——► then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

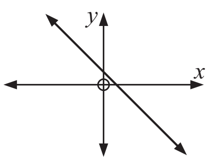
EXERCISE 1A

1 Which of the following sets of ordered pairs are functions? Give reasons.

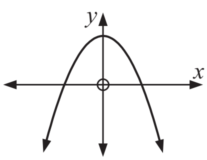
a (1, 3), (2, 4), (3, 5), (4, 6)
b (1, 3), (3, 2), (1, 7), (−1, 4)
c (2, −1), (2, 0), (2, 3), (2, 11)
d (7, 6), (5, 6), (3, 6), (−4, 6)
e (0, 0), (1, 0), (3, 0), (5, 0)
f (0, 0), (0, −2), (0, 2), (0, 4)

2 Use the vertical line test to determine which of the following relations are functions:

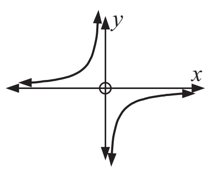
a



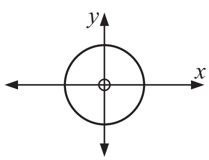
b



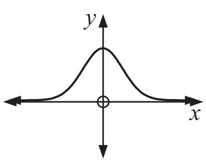
c



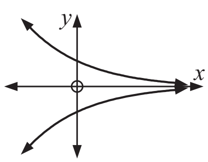
d



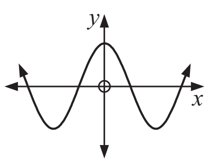
e



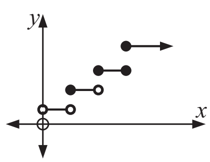
f



g



h



3 Will the graph of a straight line always be a function? Give evidence.

4 Give algebraic evidence to show that the relation $x^2 + y^2 = 9$ is not a function.

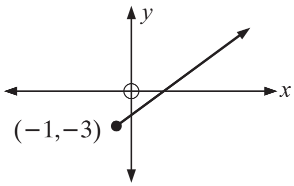
B INTERVAL NOTATION, DOMAIN AND RANGE

DOMAIN AND RANGE

The **domain** of a relation is the set of permissible values that x may have.
The **range** of a relation is the set of permissible values that y may have.

For example:

(1)



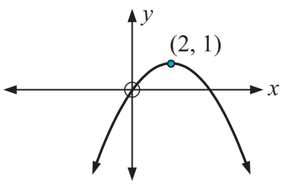
All values of $x \geqslant -1$ are permissible.

So, the domain is $\{x: x \geqslant -1\}$ or $x \in [-1, \infty[$.

All values of $y \geqslant -3$ are permissible.

So, the range is $\{y: y \geqslant -3\}$ or $y \in [-3, \infty[$.

(2)



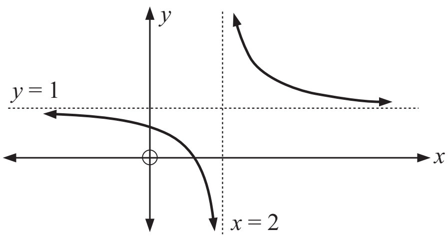
x can take any value.

So, the domain is $\{x: x \text{ is in } \mathcal{R}\}$ or $x \in \mathcal{R}$.

y cannot be > 1

$\therefore$ range is $\{y: y \leqslant 1\}$ or $y \in]-\infty, 1]$.

(3)



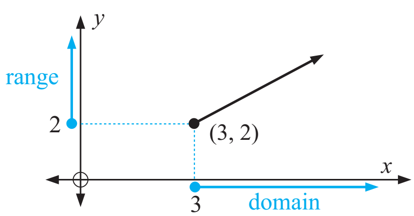
x can take all values except $x = 2$.

So, the domain is $\{x: x \neq 2\}$.

Likewise, the range is $\{y: y \neq 1\}$.

The domain and range of a relation are best described where appropriate using **interval notation**.

For example:



The domain consists of all real x such that $x \geqslant 3$ and we write this as

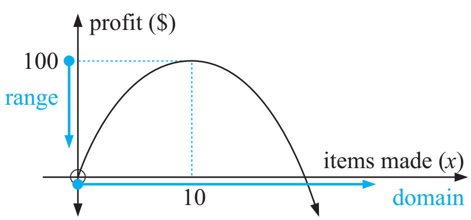
$\{x: x \geqslant 3\}$ or $x \in [3, \infty[$.

the set of all x such that

Likewise the range would be $\{y: y \geqslant 2\}$ or $y \in [2, \infty[$.

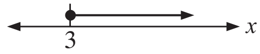
For this profit function:

- the domain is $\{x: x \geqslant 0\}$ or $x \in [0, \infty[$
- the range is $\{y: y \leqslant 100\}$ or $y \in]-\infty, 100]$.

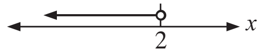


Intervals have corresponding graphs. For example:

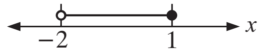
$\{x: x \geqslant 3\}$ or $x \in [3, \infty[$ is read "the set of all x such that x is greater than or equal to 3" and has number line graph



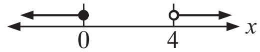
$\{x: x < 2\}$ or $x \in]-\infty, 2[$ has number line graph

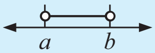


$\{x: -2 < x \leqslant 1\}$ or $x \in]-2, 1]$ has number line graph

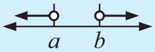


$\{x: x \leqslant 0$ or $x > 4\}$
i.e., $x \in]-\infty, 0]$ or $]4, \infty[$ has number line graph



Note: 

for numbers *between* a and b we write $a < x < b$ or $x \in]a, b[$.

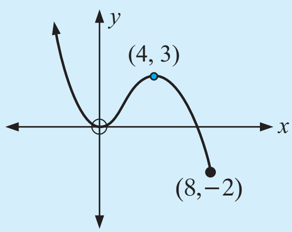


for numbers '*outside*' a and b we write $x < a$ or $x > b$
i.e., $x \in]-\infty, a[$ or $]b, \infty[$.

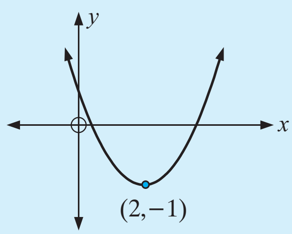
Example 2

For each of the following graphs state the domain and range:

a



b



a Domain is $\{x: x \leqslant 8\}$
or $x \in]-\infty, 8]$

Range is $\{y: y \geqslant -2\}$
or $y \in [-2, \infty[$

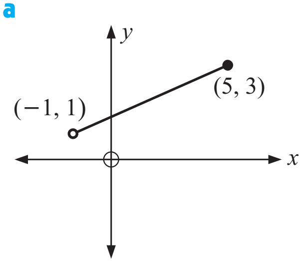
b Domain is $\{x: x \text{ is in } \mathcal{R}\}$
or $x \in \mathcal{R}$

Range is $\{y: y \geqslant -1\}$
or $y \in [-1, \infty[$

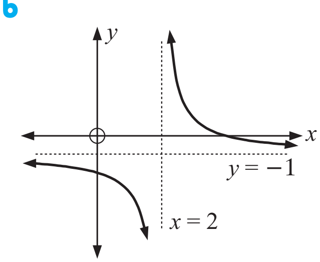
EXERCISE 1B

1 For each of the following graphs find the domain and range:

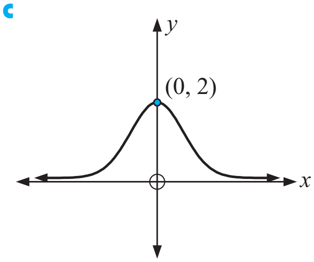
a



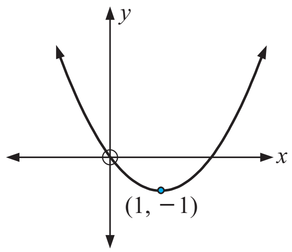
b



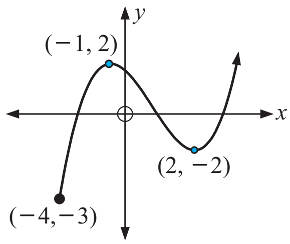
c



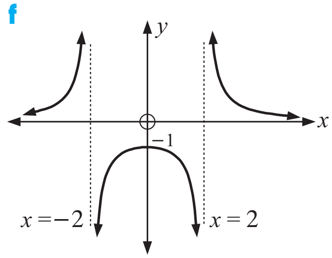
d



e



f



2 Use a graphics calculator to help sketch carefully the graphs of the following functions and find the domain and range of each:

a $f(x) = \sqrt{x}$ **b** $f(x) = \frac{1}{x^2}$ **c** $f(x) = \sqrt{4 - x}$

d $y = x^2 - 7x + 10$ **e** $y = 5x - 3x^2$ **f** $y = x + \frac{1}{x}$

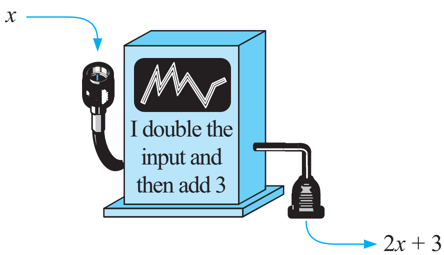
g $y = \frac{x+4}{x-2}$ **h** $y = x^3 - 3x^2 - 9x + 10$ **i** $y = \frac{3x - 9}{x^2 - x - 2}$

j $y = x^2 + x^{-2}$ **k** $y = x^3 + \frac{1}{x^3}$ **l** $y = x^4 + 4x^3 - 16x + 3$

C FUNCTION NOTATION

Function machines are sometimes used to illustrate how functions behave.

For example:



So, if 4 is fed into the machine, $2(4) + 3 = 11$ comes out.

The above 'machine' has been programmed to perform a particular function.
If f is used to represent that particular function we can write:

f is the function that will convert x into $2x + 3$.

So, f would convert 2 into $2(2) + 3 = 7$ and
−4 into $2(-4) + 3 = -5$.

This function can be written as:

$$\boldsymbol{f : x \longmapsto 2x + 3}$$

function f — such that — x is converted into $2x + 3$

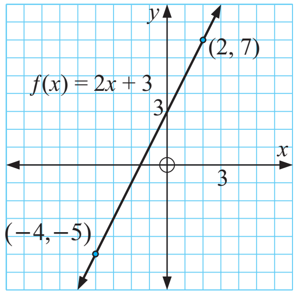
Two other equivalent forms we use are: $f(x) = 2x + 3$ or $y = 2x + 3$

So, $f(x)$ is the value of y for a given value of x, i.e., $y = f(x)$.

Notice that for $f(x) = 2x + 3$, $f(2) = 2(2) + 3 = 7$ and $f(-4) = 2(-4) + 3 = -5$.

Consequently, $f(2) = 7$ indicates that the point $(2, 7)$ lies on the graph of the function.

Likewise $f(-4) = -5$ indicates that the point $(-4, -5)$ also lies on the graph.



Note:
- $f(x)$ is read as "f of x" and is the value of the function (or y) at any value of x.
- If (x, y) is any point on the graph then $y = f(x)$, x belongs to the domain and y belongs to the range.
- f is the function which converts x into $f(x)$, i.e., $f : x \longmapsto f(x)$.
- $y = f(x)$ is sometimes called the **image** of x.

Example 3

If $f : x \longmapsto 2x^2 - 3x$, find the value of: **a** $f(5)$ **b** $f(-4)$

$f(x) = 2x^2 - 3x$

a $f(5) = 2(5)^2 - 3(5)$ {replacing x by (5)}
$= 2 \times 25 - 15$
$= 35$

b $f(-4) = 2(-4)^2 - 3(-4)$ {replacing x by (-4)}
$= 2(16) + 12$
$= 44$

EXERCISE 1C

1 If $f : x \longmapsto 3x + 2$, find the value of:

a $f(0)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(-5)$ **e** $f(-\frac{1}{3})$

2 If $f : x \longmapsto 3x - x^2 + 2$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

Example 4

If $f(x) = 5 - x - x^2$, find in simplest form: **a** $f(-x)$ **b** $f(x + 2)$

a $f(-x) = 5 - (-x) - (-x)^2$ {replacing x by $(-x)$}
$= 5 + x - x^2$

b $f(x + 2) = 5 - (x + 2) - (x + 2)^2$ {replacing x by $(x + 2)$}
$= 5 - x - 2 - [x^2 + 4x + 4]$
$= 3 - x - x^2 - 4x - 4$
$= -x^2 - 5x - 1$

3 If $f(x) = 7 - 3x$, find in simplest form:

a $f(a)$ **b** $f(-a)$ **c** $f(a+3)$ **d** $f(b-1)$ **e** $f(x+2)$

4 If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

a $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$ **d** $F(x^2)$ **e** $F(x^2-1)$

5 If $G(x) = \dfrac{2x+3}{x-4}$:

a evaluate **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

b find a value of x where $G(x)$ does not exist

c find $G(x+2)$ in simplest form

d find x if $G(x) = -3$.

6 f represents a function. What is the difference in meaning between f and $f(x)$?

7 If the value of a photocopier t years after purchase is given by $V(t) = 9650 - 860t$ Yen:

a find $V(4)$ and state what $V(4)$ means

b find t when $V(t) = 5780$ and explain what this represents

c find the original purchase price of the photocopier.

8 On the same set of axes draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

9 Find $f(x) = ax + b$, a linear function, in which $f(2) = 1$ and $f(-3) = 11$.

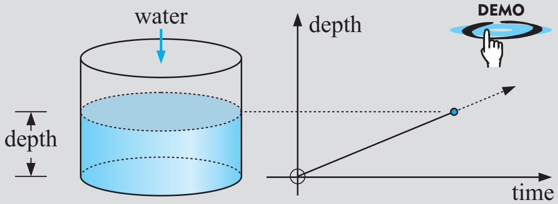
10 Given $T(x) = ax^2 + bx + c$, find a, b and c if $T(0) = -4$, $T(1) = -2$ and $T(2) = 6$.

INVESTIGATION FLUID FILLING FUNCTIONS

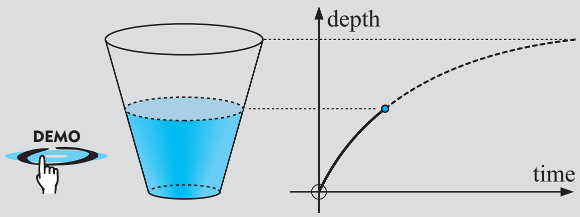
When water is added at a **constant rate** to a cylindrical container the depth of water in the container is a function of time.

This is because the volume of water added is directly proportional to the time taken to add it. If water was not added at a constant rate the direct proportionality would not exist.

The depth-time graph for the case of a cylinder would be as shown alongside:

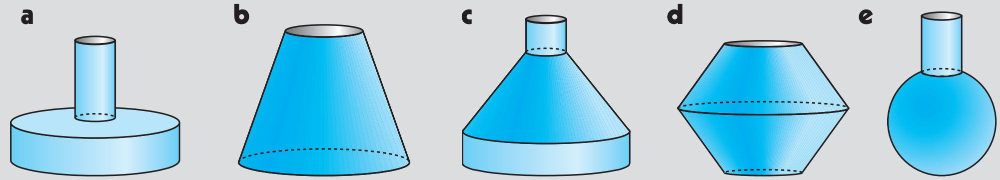


The question arises: 'What changes in appearance of the graph occur for different shaped containers?' Consider a vase of conical shape.



What to do:

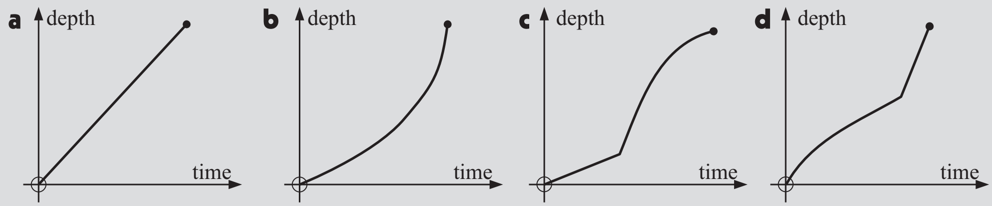
1 For each of the following containers, draw a 'depth v time' graph as water is added:



2 Use the water filling demonstration to check your answers to question **1**.

3 Write a brief report on the connection between the shape of a vessel and the corresponding shape of its depth-time graph. You may wish to discuss this in parts. For example, first examine cylindrical containers, then conical, then other shapes. Gradients of curves must be included in your report.

4 Draw possible containers as in question **1** which have the following 'depth v time' graphs:



D COMPOSITE FUNCTIONS, f ∘ g

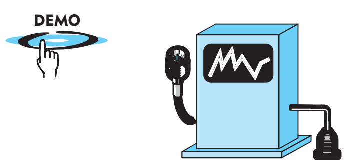
Given $f: x \longmapsto f(x)$ and $g: x \longmapsto g(x)$, then the **composite function** of f and g will convert x into $f(g(x))$.

$f \circ g$ is used to represent the composite function of f and g.

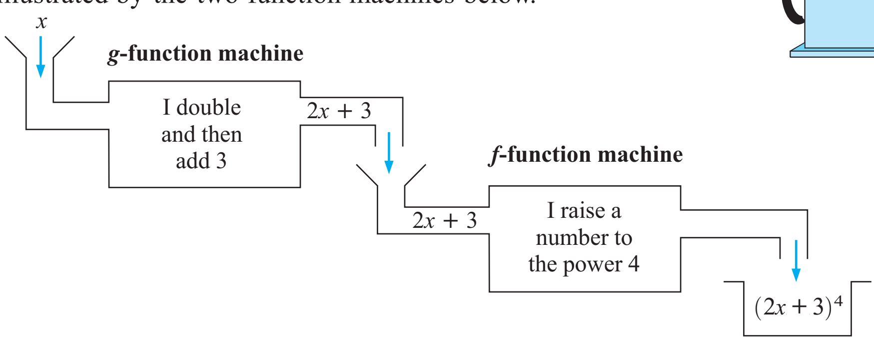
$f \circ g$ means f following g and $(f \circ g)(x) = f(g(x))$ i.e., $f \circ g: x \longmapsto f(g(x))$.

Consider $f: x \longmapsto x^4$ and $g: x \longmapsto 2x+3$.

$f \circ g$ means that g converts x to $2x+3$ and then f converts $(2x+3)$ to $(2x+3)^4$.



This is illustrated by the two function machines below.



Algebraically, if $f(x) = x^4$ and $g(x) = 2x + 3$, then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x+3) \quad \{g \text{ operates on } x \text{ first}\} \\ &= (2x+3)^4 \quad \{f \text{ operates on } g(x) \text{ next}\}\end{aligned}$$

and

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^4) \\ &= 2(x^4) + 3 \\ &= 2x^4 + 3\end{aligned}$$

So, in general, $f(g(x)) \neq g(f(x))$.

The ability to break down functions into composite functions is useful in **differential calculus**.

Example 5

Given $f : x \longmapsto 2x + 1$ and $g : x \longmapsto 3 - 4x$ find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

$f(x) = 2x + 1$ and $g(x) = 3 - 4x$

a $\therefore$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3-4x) \\ &= 2(3-4x)+1 \\ &= 6 - 8x + 1 \\ &= 7 - 8x\end{aligned}$$

b

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x+1) \\ &= 3 - 4(2x+1) \\ &= 3 - 8x - 4 \\ &= -8x - 1\end{aligned}$$

Note: If $f(x) = 2x + 1$ then

$$\begin{aligned}f(\Delta) &= 2(\Delta) + 1, \\ f(*) &= 2(*) + 1, \\ f(3x-4) &= 2(3x-4) + 1.\end{aligned}$$

EXERCISE 1D

1 Given $f : x \longmapsto 2x + 3$ and $g : x \longmapsto 1 - x$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ g)(-3)$

2 Given $f : x \longmapsto x^2$ and $g : x \longmapsto 2 - x$ find $(f \circ g)(x)$ and $(g \circ f)(x)$.
Find also the domain and range of $f \circ g$ and $g \circ f$.

3 Given $f : x \longmapsto x^2 + 1$ and $g : x \longmapsto 3 - x$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** x if $(g \circ f)(x) = f(x)$

4 **a** If $ax + b = cx + d$ for all values of x, show that $a = c$ and $b = d$.
(**Hint:** If it is true for all x, it is true for $x = 0$ and $x = 1$.)

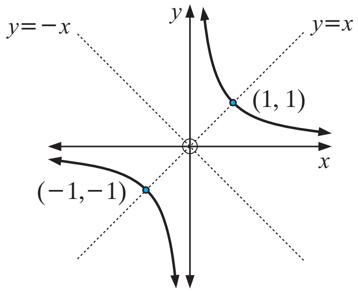
b Given $f(x) = 2x + 3$ and $g(x) = ax + b$ and that $(f \circ g)(x) = x$ for all values of x, deduce that $a = \frac{1}{2}$ and $b = -\frac{3}{2}$.

c Is the result in **b** true if $(g \circ f)(x) = x$ for all x?

E THE RECIPROCAL FUNCTION $x \mapsto \frac{1}{x}$

$x \longmapsto \dfrac{1}{x}$, i.e., $f(x) = \dfrac{1}{x}$ is defined as the **reciprocal function**.

It has graph:



- $f(x) = \dfrac{1}{x}$ is **asymptotic** to the x-axis and to the y-axis.

[The graph gets closer to the axes as it gets further from the origin.]

Notice that:

- $f(x) = \dfrac{1}{x}$ is meaningless when $x = 0$
- The graph of $f(x) = \dfrac{1}{x}$ exists in the first and third quadrants only.
- $f(x) = \dfrac{1}{x}$ is symmetric about $y = x$ and $y = -x$
- as $x \to \infty$, $f(x) \to 0$ (from above)
  as $x \to -\infty$, $f(x) \to 0$ (from below)
  as $x \to 0$ (from right), $y \to \infty$
  as $x \to 0$ (from left), $y \to -\infty$
  $\rightarrow$ reads *approaches* or *tends to*

EXERCISE 1E

1 Sketch the graph of $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{2}{x}$, $h(x) = \dfrac{4}{x}$ on the same set of axes.
Comment on any similarities and differences.

2 Sketch the graphs of $f(x) = -\dfrac{1}{x}$, $g(x) = -\dfrac{2}{x}$, $h(x) = -\dfrac{4}{x}$ on the same set of axes.
Comment on any similarities and differences.

F INVERSE FUNCTIONS

The operations of $+$ and $-$, $\times$ and $\div$, squaring and finding the square root are inverse operations as one undoes what the other does.

For example, $x + 3 - 3 = x$, $x \times 3 \div 3 = x$ and $\sqrt{8^2} = 8$.

A function $y = f(x)$ *may or may not* have an inverse function.

If $y = f(x)$ has an **inverse function**, this new function

- must indeed be a function, i.e., satisfy the vertical line test
- must be the reflection of $y = f(x)$ in the line $y = x$
- must satisfy the condition that f^{-1}: $f(x) \to x$ (i.e., the inverse).

The function $y = x$, defined as $e : x \longmapsto x$, i.e., $e(x) = x$ is the **identity function**.

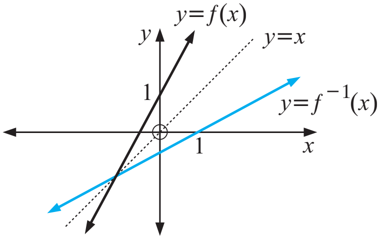
This means that, for any function f that has an inverse function f^{-1}, $f \circ f^{-1}$ and $f^{-1} \circ f$ must always equal the identity function e, i.e., $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$, i.e., the inverse function undoes the effect of the function on x.

The inverse function of $y = f(x)$ is denoted by $y = f^{-1}(x)$.

If (x, y) lies on f, then (y, x) lies on f^{-1}. So reflecting the function in $y = x$ has the algebraic effect of interchanging x and y,

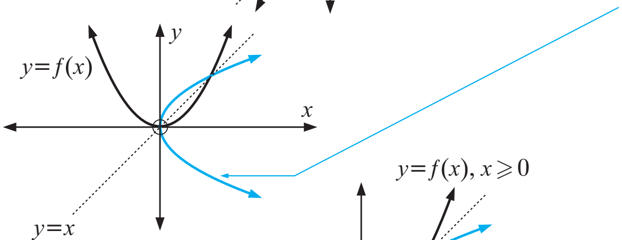
e.g., $f : y = 5x + 2$ becomes $f^{-1} : x = 5y + 2$.

For example,

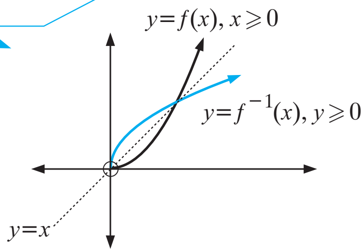


$y = f^{-1}(x)$ is the inverse of $y = f(x)$ as

- it is also a function
- it is the reflection of $y = f(x)$ in the oblique line $y = x$.



This is the reflection of $y = f(x)$ in $y = x$, but it is not the inverse function of $y = f(x)$ as it fails the vertical line test.
We say that the function $y = f(x)$ does not have an inverse.



Note: $y = f(x)$ subject to $x \geqslant 0$ does have an inverse function, drawn alongside. Also, although not drawn here, $y = f(x)$ subject to $x \leqslant 0$ does have an inverse function.

Example 6

Consider $f : x \longmapsto 2x + 3$.

a On the same axes, graph f and its inverse function f^{-1}.

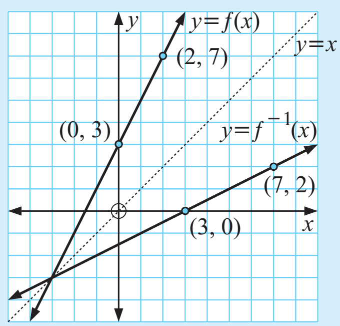
b Find $f^{-1}(x)$ using **i** coordinate geometry and the slope of $f^{-1}(x)$ from **a**

ii variable interchange.

c Check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

a $f(x) = 2x + 3$ passes through (0, 3) and (2, 7).

$\therefore$ $f^{-1}(x)$ passes through (3, 0) and (7, 2).



b **i** This line has slope $\dfrac{2-0}{7-3} = \dfrac{1}{2}$.

So, its equation is $\dfrac{y-0}{x-3} = \dfrac{1}{2}$

i.e., $y = \dfrac{x-3}{2}$

i.e., $f^{-1}(x) = \dfrac{x-3}{2}$

ii f is $y = 2x + 3$, so f^{-1} is $x = 2y + 3$

$\therefore$ $x - 3 = 2y$

$\therefore$ $\dfrac{x-3}{2} = y$ i.e., $f^{-1}(x) = \dfrac{x-3}{2}$

c $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$

$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f\left(\frac{x-3}{2}\right) \\ &= 2\left(\frac{x-3}{2}\right)+3 \\ &= x\end{aligned}$$

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}(2x+3) \\ &= \frac{(2x+3)-3}{2} \\ &= \frac{2x}{2} \\ &= x\end{aligned}$$

Note:
If f includes point (a, b) then f^{-1} includes point (b, a), i.e., the point obtained by interchanging the coordinates.

EXERCISE 1F

1 For each of the following functions f

- **i** on the same axes graph $y = x$, f and f^{-1}
- **ii** find $f^{-1}(x)$ using coordinate geometry and **i**
- **iii** find $f^{-1}(x)$ using variable interchange:

a $f : x \longmapsto 3x+1$

b $f : x \longmapsto \dfrac{x+2}{4}$.

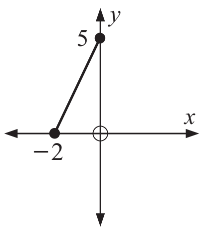
2 For each of the following functions f

- **i** find $f^{-1}(x)$ **ii** sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same axes
- **iii** show that $f^{-1} \circ f = f \circ f^{-1} = e$, the identity function:

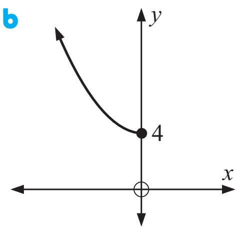
a $f : x \longmapsto 2x+5$ **b** $f : x \longmapsto \dfrac{3-2x}{4}$ **c** $f : x \longmapsto x+3$

3 Copy the graphs of the following functions and in each case include the graphs of $y = x$ and $y = f^{-1}(x)$.

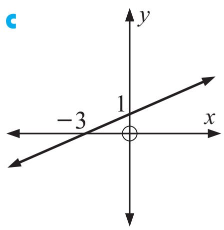
a



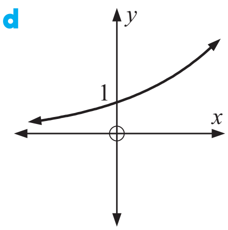
b



c



d



4 **a** Sketch the graph of $f : x \longmapsto x^2 - 4$ and reflect it in the line $y = x$.
b Does f have an inverse function?
c Does f where $x \geqslant 0$ have an inverse function?

5 Sketch the graph of $f : x \longmapsto x^3$ and its inverse function $f^{-1}(x)$.

G FUNCTIONS WHICH HAVE INVERSES

It is important to understand the distinction between one-to-one and many-to-one functions.

A **one-to-one** function is any function where
- for each x there is only one value of y and
- for each y there is only one value of x.

Functions that are **one-to-one** satisfy both the '**vertical line test**' and the '**horizontal line test**'. This means that:
- no vertical line can meet the graph more than once and
- no horizontal line can meet the graph more than once.

Functions that are not one-to-one are called **many-to-one** and whilst these functions must satisfy the '**vertical line test**' they *do not satisfy* the '**horizontal line test**', i.e., at least one y-value has more than one corresponding x-value.

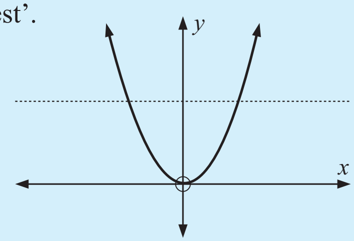
Note:
- If the function $y = f(x)$ is **one-to-one**, it will have an inverse function $y = f^{-1}(x)$.
- If a function $y = f(x)$ is **many-to-one**, it *will not* have an inverse function.
- Many-to-one functions can have inverse functions for a restricted part of the domain (see **Example 7**).

Example 7

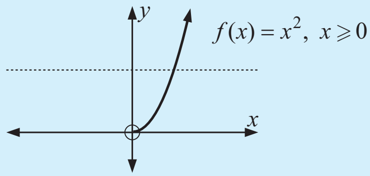
Consider $f : x \longmapsto x^2$.

a Explain why the function defined above does not have an inverse function.
b Does $f : x \longmapsto x^2$ where $x \geqslant 0$ have an inverse function?
c Find $f^{-1}(x)$ for $f : x \longmapsto x^2$, $x \geqslant 0$.
d Sketch $y = f(x)$, $y = x$ and $y = f^{-1}(x)$ for f in **b** and f^{-1} in **c**.

a $f : x \longmapsto x^2$ has domain $x \in \mathcal{R}$ and is many-to-one.
It does not pass the 'horizontal line test'.

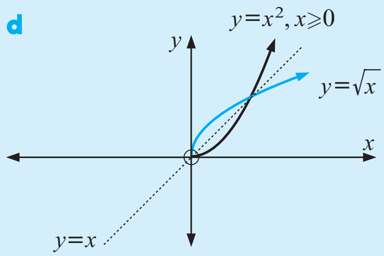


b If we restrict the domain to $x \geqslant 0$ or $x \in [0, \infty[$, or in fact any domain which makes f one-to-one, it satisfies the 'horizontal line test' and so has an inverse function.



c
f is defined by $y = x^2$, $x \geqslant 0$
$\therefore$ f^{-1} is defined by $x = y^2$, $y \geqslant 0$
$\therefore$ $y = \pm\sqrt{x}$, $y \geqslant 0$
i.e., $y = \sqrt{x}$
$\{$as $-\sqrt{x}$ is $\leqslant 0\}$
So, $f^{-1}(x) = \sqrt{x}$

d



Note: The function $f(x) = \dfrac{1}{x}$, $x \neq 0$, called the **reciprocal function**, is said to be a **self-inverse function** as $f = f^{-1}$.

This is because the graph of $y = \dfrac{1}{x}$ is symmetrical about the line $y = x$.

Any function with a graph which is symmetrical about the line $y = x$ must be a **self-inverse function**.

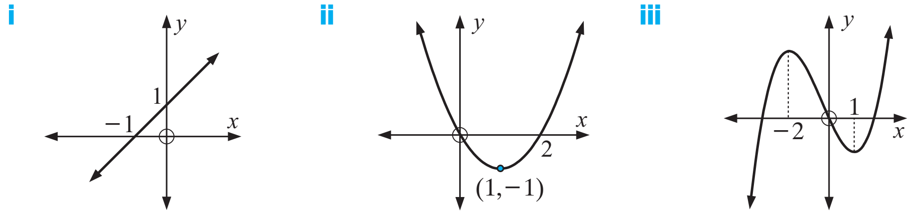
EXERCISE 1G

Note: If the domain of a function is the set of all real numbers, then the statement $x \in \mathcal{R}$ will be omitted.

1 **a** Show that $f: x \longmapsto \dfrac{1}{x}$ has an inverse function for all $x \neq 0$.
b Find f^{-1} algebraically and show that f is a self-inverse function.

2 Show that $f: x \longmapsto \dfrac{3x-8}{x-3}$, $x \neq 3$ is a self-inverse function by:
a reference to its graph **b** using algebra.

3 The '**horizontal line test**' says that:
for a function to have an inverse function, no horizontal line can cut it more than once.
a Explain why this is a valid test for the existence of an inverse function.
b Which of the following functions have an inverse function?



c For the functions in **b** which do not have an inverse, specify domains as wide as possible where each function does have an inverse.

4 Consider $f: x \longmapsto x^2$ where $x \leqslant 0$.
a Find $f^{-1}(x)$.
b Sketch $y = f(x)$, $y = x$ and $y = f^{-1}(x)$ on the same set of axes.

5 **a** Explain why $f: x \longmapsto x^2 - 4x + 3$ is a function but does not have an inverse function.
b Explain why f for $x \geqslant 2$ has an inverse function.
c Show that the inverse function of the function in **b** is $f^{-1}(x) = 2 + \sqrt{1+x}$.
d If the domain of f is restricted to $x \geqslant 2$, state the domain and range of
i f **ii** f^{-1}.
e Show that $f \circ f^{-1} = f^{-1} \circ f = e$, the identity function.

6 Given $f: x \longmapsto (x+1)^2 + 3$ where $x \geqslant -1$,
a find the defining equation of f^{-1}
b sketch, using technology, the graphs of $y = f(x)$, $y = x$ and $y = f^{-1}(x)$
c state the domain and range of **i** f **ii** f^{-1}.

7 Consider the functions $f: x \longmapsto 2x + 5$ and $g: x \longmapsto \dfrac{8-x}{2}$.
a Find $g^{-1}(-1)$. **b** Solve for x the equation $(f \circ g^{-1})(x) = 9$.

8 Given $f: x \longmapsto 5^x$ and $g: x \longmapsto \sqrt{x}$,
a find **i** $f(2)$ **ii** $g^{-1}(4)$ **b** solve the equation $(g^{-1} \circ f)(x) = 25$.

9 Given $f: x \longmapsto 2x$ and $g: x \longmapsto 4x - 3$ show that $(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$.

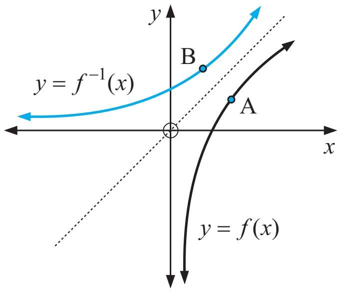
10 Which of these functions is a self inverse function, that is $f^{-1}(x) = f(x)$?

a $f(x) = 2x$ **b** $f(x) = x$ **c** $f(x) = -x$ **d** $f(x) = \frac{2}{x}$ **e** $f(x) = -\frac{6}{x}$

11 Show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$ for:

a $f(x) = 3x + 1$ **b** $f(x) = \frac{x+3}{4}$ **c** $f(x) = \sqrt{x}$

12



a B is the image of A under a reflection in the line $y = x$.
If A is $(x, f(x))$, what are the coordinates of B under the reflection?

b Substitute your result from **a** into $y = f^{-1}(x)$. What result do you obtain?

c Explain how to establish that $f(f^{-1}(x)) = x$ also.

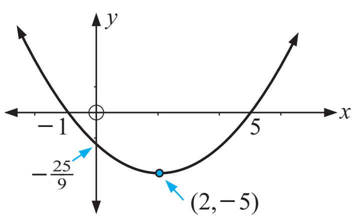
REVIEW SET 1A

1 If $f(x) = 2x - x^2$ find: **a** $f(2)$ **b** $f(-3)$ **c** $f(-\frac{1}{2})$

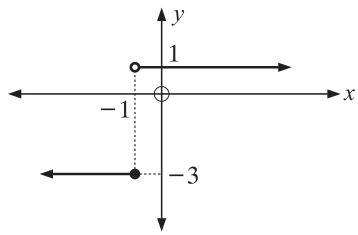
2 For the following graphs determine:

i the range and domain **ii** the x and y-intercepts **iii** whether it is a function
iv if they have an inverse function

a

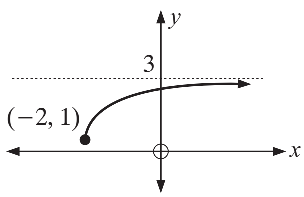


b

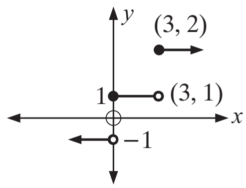


3 For each of the following graphs find the domain and range:

a



b



4 If $h(x) = 7 - 3x$:

a find in simplest form $h(2x - 1)$ **b** find x if $h(2x - 1) = -2$

5 Find a, b and c if $f(0) = 5$, $f(-2) = 21$ and $f(3) = -4$ and $f(x) = ax^2 + bx + c$.

6 Consider $f(x) = \frac{1}{x^2}$.

a For what value of x is $f(x)$ meaningless?

b Sketch the graph of this function using technology.

c State the domain and range of the function.

7 If $f(x) = 2x - 3$ and $g(x) = x^2 + 2$, find: **a** $f(g(x))$ **b** $g(f(x))$

8 If $f(x) = 1 - 2x$ and $g(x) = \sqrt{x}$:

a find in simplest form **i** $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

b What is the domain and range of $f \circ g$ and $g \circ f$?

9 Find an f and a g function given that:

a $f(g(x)) = \sqrt{1 - x^2}$ **b** $g(f(x)) = \left(\dfrac{x-2}{x+1}\right)^2$

REVIEW SET 1B

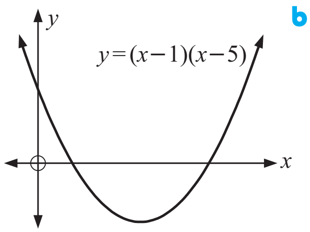
1 If $g(x) = x^2 - 3x$, find in simplest form **a** $g(x+1)$ **b** $g(x^2 - 2)$

2 For each of the following functions $f(x)$ find $f^{-1}(x)$:

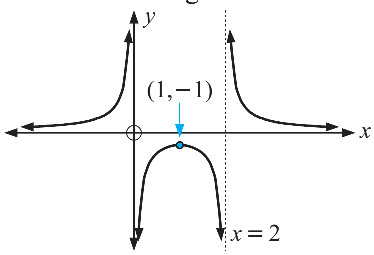
a $f(x) = 7 - 4x$ **b** $f(x) = \dfrac{3 + 2x}{5}$

3 For each of the following graphs, find the domain and range.

a

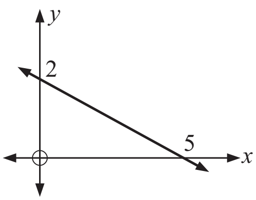


b

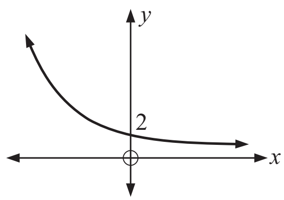


4 Copy the following graphs and draw the graph of each inverse function:

a



b



5 Find $f^{-1}(x)$ given that $f(x)$ is: **a** $4x + 2$ **b** $\dfrac{3 - 5x}{4}$

6 Consider $x \longmapsto 2x - 7$.

a On the same set of axes graph $y = x$, f and f^{-1}.

b Find $f^{-1}(x)$ using variable interchange.

c Show that $f \circ f^{-1} = f^{-1} \circ f = e$, the identity function.

7 **a** Sketch the graph of $g : x \longmapsto x^2 + 6x + 7$.

b Explain why g for $x \in \,]-\infty, -3]$ has an inverse function g^{-1}.

c Find algebraically, the equation of g^{-1}. **d** Sketch the graph of g^{-1}.

e Find the range of g and hence the domain and range of g^{-1}.

8 Given $h : x \longmapsto (x-4)^2 + 3$, $x \in [4, \infty[$

a find the defining equation of h^{-1}. **b** Show that $h \circ h^{-1} = h^{-1} \circ h = x$

9 Given $f : x \longmapsto 3x + 6$ and $h : x \longmapsto \dfrac{x}{3}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

ANSWERS

EXERCISE 1A

1 a, d, e **2** a, b, c, e, g **3** No, e.g., $x = 1$ **4** $y = \pm\sqrt{9 - x^2}$

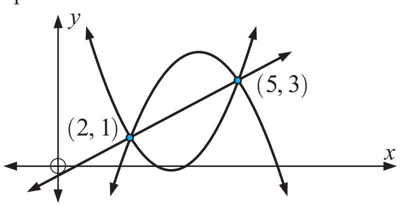
EXERCISE 1B

1 **a** Domain $\{x:\ -1 < x \leqslant 5\}$, Range $\{y:\ 1 < y \leqslant 3\}$
b Domain $\{x:\ x \neq 2\}$, Range $\{y:\ y \neq -1\}$
c Domain $\{x:\ x \in \mathcal{R}\}$, Range $\{y:\ 0 < y \leqslant 2\}$
d Domain $\{x:\ x \in \mathcal{R}\}$, Range $\{y:\ y \geqslant -1\}$
e Domain $\{x:\ x \geqslant -4\}$, Range $\{y:\ y \geqslant -3\}$
f Domain $\{x:\ x \neq \pm 2\}$, Range $\{y:\ y \leqslant -1 \text{ or } y > 0\}$

2 **a** Domain $\{x:\ x \geqslant 0\}$, Range $\{y:\ y \geqslant 0\}$
b Domain $\{x:\ x \neq 0\}$, Range $\{y:\ y > 0\}$
c Domain $\{x:\ x \leqslant 4\}$, Range $\{y:\ y \geqslant 0\}$
d Domain $\{x:\ x \in \mathcal{R}\}$, Range $\{y:\ y \geqslant -2\frac{1}{4}\}$
e Domain $\{x:\ x \in \mathcal{R}\}$, Range $\{y:\ y \leqslant 2\frac{1}{12}\}$
f Domain $\{x:\ x \neq 0\}$, Range $\{y:\ y \leqslant -2 \text{ or } y \geqslant 2\}$
g Domain $\{x:\ x \neq 2\}$, Range $\{y:\ y \neq 1\}$
h Domain $\{x:\ x \in \mathcal{R}\}$, Range $\{y:\ y \in \mathcal{R}\}$
i Domain $\{x: x \neq -1 \text{ or } 2\}$, Range $\{y: y \leqslant \frac{1}{3} \text{ or } y \geqslant 3\}$
j Domain $\{x:\ x \neq 0\}$, Range $\{y: y \geqslant 2\}$
k Domain $\{x:\ x \neq 0\}$, Range $\{y:\ y \leqslant -2 \text{ or } y \geqslant 2\}$
l Domain $\{x:\ x \in \mathcal{R}\}$, Range $\{y:\ y \geqslant -8\}$

EXERCISE 1C

1 **a** 2 **b** 8 **c** -1 **d** -13 **e** 1
2 **a** 2 **b** 2 **c** -16 **d** -68 **e** $\frac{17}{4}$
3 **a** $7-3a$ **b** $7+3a$ **c** $-3a-2$ **d** $10-3b$ **e** $1-3x$
4 **a** $2x^2+19x+43$ **b** $2x^2-11x+13$ **c** $2x^2-3x-1$
d $2x^4+3x^2-1$ **e** $2x^4-x^2-2$
5 **a** **i** $-\frac{7}{2}$ **ii** $-\frac{3}{4}$ **iii** $-\frac{4}{9}$ **b** $x=4$ **c** $\dfrac{2x+7}{x-2}$ **d** $x = \frac{9}{5}$
6 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.
7 **a** 6210 Yen value after 4 years **b** $t = 4.5$, the time for the photocopier to reach a value of 5780 Yen. **c** 9650 Yen
8



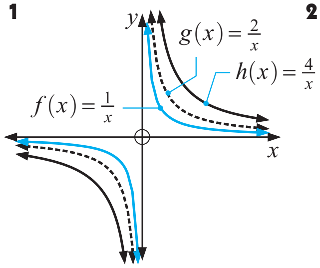
9 $f(x) = -2x + 5$
10 $a = 3,\ b = -1,\ c = -4,\ T(x) = 3x^2 - x - 4$

EXERCISE 1D

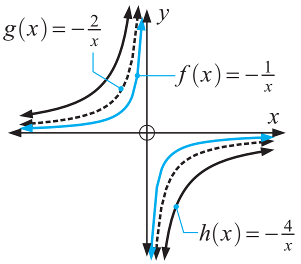
1 **a** $5-2x$ **b** $-2x-2$ **c** 11
2 $f(g(x)) = (2-x)^2$, Domain $\{x:\ x \text{ is in } \mathcal{R}\}$,
Range $\{y:\ y \geqslant 0\}$
$g(f(x)) = 2 - x^2$, Domain $\{x:\ x \text{ is in } \mathcal{R}\}$,
Range $\{y:\ y \geqslant 2\}$
3 **a** $x^2-6x+10$ **b** $2-x^2$ **c** $x = \pm\frac{1}{\sqrt{2}}$
4 **a** Let $x = 0$, $\therefore\ b = d$ and so
$ax + b = cx + b$
$\therefore\ ax = cx$ for all x
Let $x = 1$, $\therefore\ a = c$
b $(f \circ g)(x) = [2a]x + [2b+3] = 1x + 0$ for all x
$\therefore\ 2a = 1$ and $2b + 3 = 0$
$\therefore\ a = \frac{1}{2}$ and $b = -\frac{3}{2}$
c Yes, $\{(g \circ f)(x) = [2a]x + [3a + b]\}$

EXERCISE 1E

1

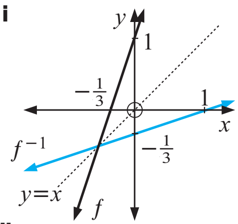


2



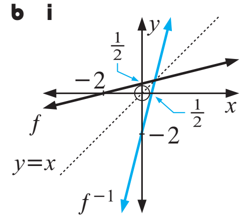
EXERCISE 1F

1 **a** **i**



ii, iii $f^{-1}(x) = \dfrac{x-1}{3}$

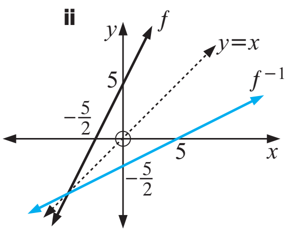
b **i**



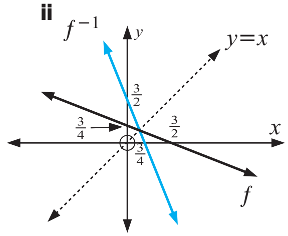
ii, iii $f^{-1}(x) = 4x - 2$

2 **a** **i** $f^{-1}(x) = \dfrac{x-5}{2}$ **b** **i** $f^{-1}(x) = -2x + \frac{3}{2}$

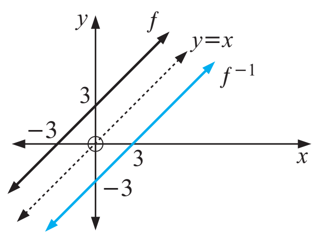
ii



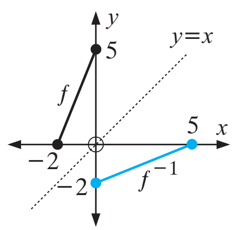
ii



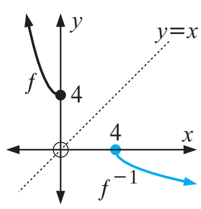
c **i** $f^{-1}(x) = x - 3$ **ii**



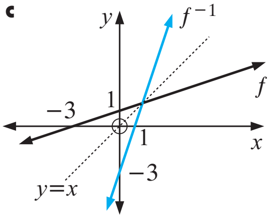
3 **a**



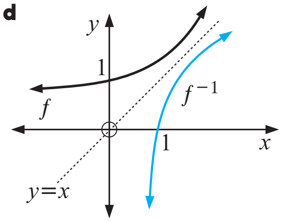
b



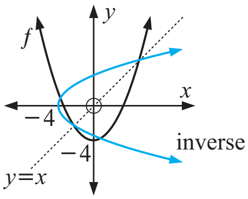
c



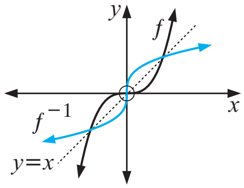
d



4 **a**



5



b No **c** Yes, it is $y = \sqrt{x+4}$

EXERCISE 1G

1 **a** $f : x \mapsto \frac{1}{x}$, $x \neq 0$ satisfies both the vertical and horizontal line tests and $\therefore$ has an inverse function.

b $f^{-1}(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x}$ i.e., $f = f^{-1}$

$\therefore$ f is a self-inverse function

2 **a** $y = \frac{3x-8}{x-3}$ is symmetrical about $y = x$,

$\therefore$ f is a self-inverse function.

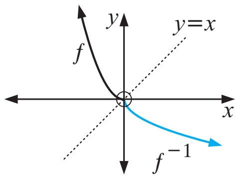
b $f^{-1}(x) = \frac{3x-8}{x-3}$ and $f(x) = \frac{3x-8}{x-3}$

i.e., $f = f^{-1}$ $\therefore$ f is a self-inverse function

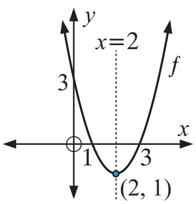
3 **b** **i** is the only one

c **ii** Domain $\{x\colon x \geqslant 1\}$ **iii** Domain $\{x\colon x \geqslant 1\}$

4 **a** $f^{-1}(x) = -\sqrt{x}$ **b**



5 **a**



A horizontal line above the vertex cuts the graph **twice**. So, it does not have an inverse.

b For $x \geqslant 2$, all horizontal lines cut 0 or once only. $\therefore$ has an inverse

c **Hint:** Inverse is $x = y^2 - 4y + 3$ for $y \geqslant 2$

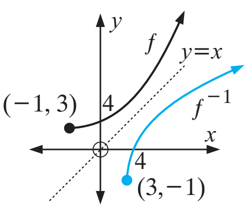
d **i** Domain is $\{x : x \geqslant 2\}$, Range is $\{y : y \geqslant -1\}$

ii Domain is $\{x : x \geqslant -1\}$, Range is $\{y : y \geqslant 2\}$

e **Hint:** Find $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$ and show that they both equal x.

6 **a** $f^{-1}(x) = \sqrt{x-3} - 1$, $x \geqslant 3$

b



c **i** Domain $\{x : x \geqslant -1\}$ Range $\{y : y \geqslant 3\}$

ii Domain $\{x : x \geqslant 3\}$ Range $\{y : y \geqslant -1\}$

7 **a** 10 **b** $x = 3$ **8** **a** **i** 25 **ii** 16 **b** $x = 1$

9 $(f^{-1} \circ g^{-1})(x) = \frac{x+3}{8}$ and $(g \circ f)^{-1}(x) = \frac{x+3}{8}$

10 **a** Is not **b** Is **c** Is **d** Is **e** Is

12 **a** B is $(f(x), x)$ **b** $x = f^{-1}(f(x)) = (f^{-1} \circ f)(x)$

c Start with B first and repeat the process used in **a** and **b**.

REVIEW SET 1A

1 **a** 0 **b** −15 **c** $-\frac{5}{4}$

2 **a** **i** Range $= \{y : y \geqslant -5\}$, Domain $= \{x : x \text{ is in } \mathcal{R}\}$

ii x-int. −1, 5; y-int. $-\frac{25}{9}$ **iii** is a function **iv** no

b **i** Range $= \{y : y = 1 \text{ or } -3\}$ Domain $= \{x : x \text{ is in } \mathcal{R}\}$

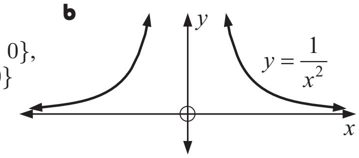
ii no x-intercepts; y-intercept 1 **iii** is a function **iv** no

3 **a** Domain $= \{x : x \geqslant -2\}$, Range $= \{y : 1 \leqslant y < 3\}$

b Domain $= \{x \text{ is in } \mathcal{R}\}$, Range $= \{y : y = -1, 1 \text{ or } 2\}$

4 **a** $10 - 6x$ **b** $x = 2$ **5** $a = 1$, $b = -6$, $c = 5$

6 **a** $x = 0$ **b**



c Domain $= \{x : x \neq 0\}$, Range $= \{y : y > 0\}$

7 **a** $2x^2 + 1$ **b** $4x^2 - 12x + 11$

8 **a** **i** $1 - 2\sqrt{x}$ **ii** $\sqrt{1-2x}$

b For $f \circ g$, Domain $\{x\colon x \geqslant 0\}$, Range $\{y\colon y \leqslant 1\}$

For $g \circ f$, Domain $\{x\colon x \leqslant \frac{1}{2}\}$, Range $\{y\colon y \geqslant 0\}$

9 **a** $f(x) = \sqrt{x}$, $g(x) = 1 - x^2$ **b** $g(x) = x^2$, $f(x) = \frac{x-2}{x+1}$

REVIEW SET 1B

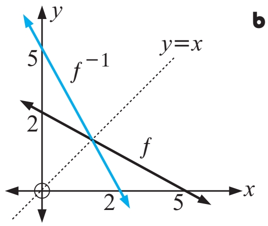
1 **a** $x^2 - x - 2$ **b** $x^4 - 7x^2 + 10$

2 **a** $f^{-1}(x) = \frac{7-x}{4}$ **b** $f^{-1}(x) = \frac{5x-3}{2}$

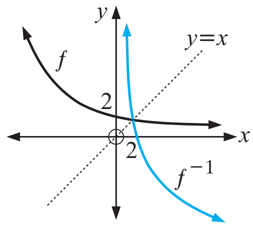
3 **a** Domain $\{x : x \in \mathcal{R}\}$, Range $\{y : y \geqslant -4\}$

b Domain $\{x : x \neq 0, 2\}$, Range $\{y : y \leqslant -1 \text{ or } y > 0\}$

4 **a**

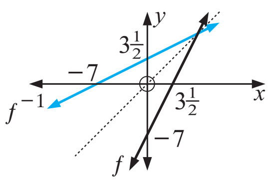


b



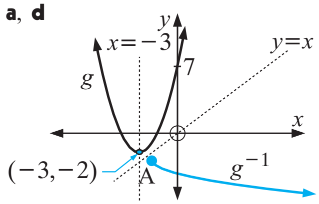
5 **a** $f^{-1}(x) = \frac{x-2}{4}$ **b** $f^{-1}(x) = \frac{3-4x}{5}$

6 **a**



b $f^{-1}(x) = \frac{x+7}{2}$

7 **a, d**



b If $x \leqslant -3$, we have the graph to the left of $x = -3$ and any horizontal line cuts it at most once.

c $y = -3 - \sqrt{x+2}$

e Range of g $\{y\colon y \geqslant -2\}$, Domain of g^{-1} $\{x\colon x \geqslant -2\}$
Range of g^{-1} $\{y\colon y \leqslant -3\}$

8 **a** $h^{-1}(x) = 4 + \sqrt{x-3}$

9 $(f^{-1} \circ h^{-1})(x) = x - 2$ and $(h \circ f)^{-1}(x) = x - 2$

EXERCISE 2A

1 **a** 4, 13, 22, 31, ... **b** 45, 39, 33, 27, ...
c 2, 6, 18, 54, ... **d** 96, 48, 24, 12, ...

2 **a** Starts at 8 and each term is 8 more than the previous term. Next two terms 40, 48.
b Starts at 2, each term is 3 more than the previous term; 14, 17.
c Starts at 36, each term is 5 less than the previous term; 16, 11.
d Starts at 96, each term is 7 less than the previous term; 68, 61.
e Starts at 1, each term is 4 times the previous term; 256, 1024.
f Starts at 2, each term is 3 times the previous term; 162, 486.
g Starts at 480, each term is half the previous term; 30, 15.
h Starts at 243, each term is $\frac{1}{3}$ of the previous term; 3, 1.
i Starts at 50 000, each term is $\frac{1}{5}$ of the previous term; 80, 16.

3 **a** Each term is the square of the number of the term; 25, 36, 49.
b Each term is the cube of the number of the term; 125, 216, 343.
c Each term is $n \times (n+1)$ where n is the number of the term; 30, 42, 56.

EXERCISE 2B

1 **a** 2, 4, 6, 8, 10 **b** 4, 6, 8, 10, 12 **c** 1, 3, 5, 7, 9
d −1, 1, 3, 5, 7 **e** 5, 7, 9, 11, 13 **f** 13, 15, 17, 19, 21
g 4, 7, 10, 13, 16 **h** 1, 5, 9, 13, 17

2 **a** 2, 4, 8, 16, 32 **b** 6, 12, 24, 48, 96
c $3, 1\frac{1}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}$ **d** −2, 4, −8, 16, −32

3 17, 11, 23, −1, 47

EXERCISE 2C

1 **a** $u_1 = 6,\ d = 11$ **b** $u_n = 11n - 5$ **c** 545
d yes, u_{30} **e** no

2 **a** $u_1 = 87,\ d = -4,$ **b** $u_n = 91 - 4n$ **c** −69 **d** no

3 **b** $u_1 = 1,\ d = 3$ **c** 169 **d** $u_{151} = 451$

4 **b** $u_1 = 32,\ d = -\frac{7}{2}$ **c** −227 **d** $n \geqslant 68$

5 **a** $k = 17\frac{1}{2}$ **b** $k = 4$ **c** $k = 3,\ k = -1$

6 **a** $u_n = 6n - 1$ **b** $u_n = -\frac{3}{2}n + \frac{11}{2}$ **c** $u_n = -5n + 36$
d $u_n = -\frac{3}{2}n + \frac{1}{2}$

7 **a** $6\frac{1}{4}, 7\frac{1}{2}, 8\frac{3}{4}$ **b** $3\frac{5}{7}, 8\frac{3}{7}, 13\frac{1}{7}, 17\frac{6}{7}, 22\frac{4}{7}, 27\frac{2}{7}$

8 **a** $u_1 = 36,\ d = -\frac{2}{3}$ **b** 100 **9** 100 006

EXERCISE 2D

1 **a** $b = 18,\ c = 54$ **b** $b = 2\frac{1}{2},\ c = 1\frac{1}{4}$ **c** $b = 3,\ c = -1\frac{1}{2}$

2 **a** $u_1 = 5,\ r = 2$ **b** $u_n = 5 \times 2^{n-1},\ u_{15} = 81\,920$

3 **a** $u_1 = 12,\ r = -\frac{1}{2}$ **b** $u_n = 12 \times (-\frac{1}{2})^{n-1},\ u_{13} = \frac{3}{1024}$

4 **a** $u_1 = 8,\ r = -\frac{3}{4},\ u_{10} = -0.600\,677\,49$

5 **a** $u_1 = 8,\ r = \frac{1}{\sqrt{2}},\ u_n = 2^{\frac{7}{2} - \frac{n}{2}}$

6 **a** $k = \pm 14$ **b** $k = 2$ **c** $k = -2$ or 4

7 **a** $u_n = 3 \times 2^{n-1}$ **b** $u_n = 32 \times (-\frac{1}{2})^{n-1}$
c $u_n = 3 \times (\sqrt{2})^{n-1}$ **d** $u_n = 10 \times (\sqrt{2})^{1-n}$

8 **a** $u_9 = 13\,122$ **b** $u_{14} = 2916\sqrt{3} \doteqdot 5050.66$
c $u_{18} \doteqdot 0.000\,091\,55$ **9** **a** \$3993.00 **b** \$993.00

10 11 470.39 Euro **11** **a** 43 923 Yen **b** 13 923 Yen

12 \$23 602.32 **13** 148 024.43 Yen **14** £51 249.06

15 \$14 976.01 **16** £11 477.02 **17** 19 712.33 Euro

18 19 522.47 Yen

19 **a** **i** 1550 ants **ii** 4820 ants **b** 12.2 weeks

20 **a** 278 animals **b** Year 2037

EXERCISE 2E.1

1 **a** **i** $S_n = 3 + 11 + 19 + 27 + ... + (8n - 5)$ **ii** 95
b **i** $S_n = 42 + 37 + 32 + ... + (47 - 5n)$ **ii** 160
c **i** $S_n = 12 + 6 + 3 + 1\frac{1}{2} + ... + 12(\frac{1}{2})^{n-1}$ **ii** $23\frac{1}{4}$
d **i** $S_n = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + ... + 2(\frac{3}{2})^{n-1}$ **ii** $26\frac{3}{8}$
e **i** $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \dfrac{1}{2^{n-1}}$ **ii** $1\frac{15}{16}$
f **i** $S_n = 1 + 8 + 27 + 64 + ... + n^3$ **ii** 225

EXERCISE 2E.2

1 **a** 820 **b** 3087.5 **c** −1460 **d** −740

2 **a** 1749 **b** 2115 **c** $1410\frac{1}{2}$ **3** 203

4 −115.5 **5** 18 **6** **a** 65 **b** 1914 **c** 47 850

7 **a** 14 025 **b** 71 071 **c** 3367

9 **a** $u_n = 2n - 1$ **c** $S_1 = 1,\ S_2 = 4,\ S_3 = 9,\ S_4 = 16$

10 56, 49 **11** 10, 4, −2 or −2, 4, 10

12 2, 5, 8, 11, 14 or 14, 11, 8, 5, 2

EXERCISE 2E.3

1 **a** $23.9766 \doteqdot 24.0$ **b** $\doteqdot 189\,134$ **c** $\doteqdot 4.000$ **d** $\doteqdot 0.5852$

2 **a** $S_n = \dfrac{3+\sqrt{3}}{2}\left((\sqrt{3})^n - 1\right)$ **b** $S_n = 24(1 - (\frac{1}{2})^n)$
c $S_n = 1 - (0.1)^n$ **d** $S_n = \frac{40}{3}(1 - (-\frac{1}{2})^n)$

3 **c** \$26 361.59

4 **a** $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$ **b** $S_n = \dfrac{2^n - 1}{2^n}$
c $1 - (\frac{1}{2})^n = \dfrac{2^n - 1}{2^n}$ **d** as $n \to \infty,\ S_n \to 1$

5 **b** $S_n = 1 + 18(1 - (0.9)^{n-1})$ **c** 19 seconds

6 **a** **i** $u_1 = \frac{3}{10}$ **ii** $r = 0.1$ **b** $S_\infty = \frac{1}{3}$

7 **a** $\frac{4}{9}$ **b** $\frac{16}{99}$ **c** $\frac{104}{333}$

EXERCISE 2F

1 **a** 10 **b** 25 **c** 168 **d** 310

2 $2 + 5 + 8 + 11 + ... + 59 = 610$

3 **a** 160 **b** −630 **c** 135

4 **a** 3069 **b** $\frac{4095}{1024} \doteqdot 3.999$ **c** −134 217 732

5 **a** 420 **b** 2231.868 211 **6** **a** $n = 37$ **b** $n = 11$

EXERCISE 2G

1 34th week (total sold = 2057)

2 After 85 months its value is \$501.88 and after 86 months its value is \$491.84, ∴ during the 86th month its value is \$500.

3 54 or $\frac{2}{3}$ **4** 70 cm

5 The 20th terms are: arithmetic 39, geometric 3^{19}
or arithmetic $7\frac{1}{3}$, geometric $(\frac{4}{3})^{19}$

Chapter 7

Quadratic equations and functions

Contents:

INTRODUCTION

Consider the functions:

$f: x \longmapsto ax + b, \quad a \neq 0$	**Linear**
$f: x \longmapsto ax^2 + bx + c, \quad a \neq 0$	**Quadratic**
$f: x \longmapsto ax^3 + bx^2 + cx + d, \quad a \neq 0$	**Cubic**
$f: x \longmapsto ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$	**Quartic**

These functions are the simplest members of the family of polynomials.

In this chapter we will examine quadratic functions in detail.

Quadratic functions arise in many situations.

You may need to review algebraic expansion and factorisation. To do this click on the 'Background Knowledge' icon on page 13.

HISTORICAL NOTE

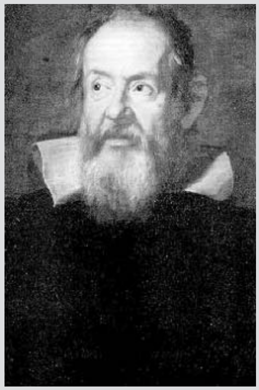
Over 400 years ago, **Galileo** (born in Pisa, Italy) conducted a series of experiments on the paths of projectiles, attempting to find a mathematical description of falling bodies.

Two of Galileo's experiments consisted of rolling a ball down a grooved ramp that was placed at a fixed height above the floor and inclined at a fixed angle to the horizontal. In one experiment the ball left the end of the ramp and descended to the floor.

In a related experiment a horizontal shelf was placed at the end of the ramp, and the ball would travel along this shelf before descending to the floor.

In each experiment Galileo altered the release height (h) of the ball and measured the distance (d) the ball travelled before landing.

The units of measurement were called 'punti'.

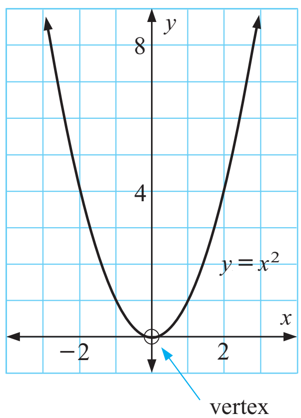


Galileo

THE SIMPLEST QUADRATIC FUNCTION

The simplest quadratic function is $y = x^2$ and its graph can be drawn from a table of values.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

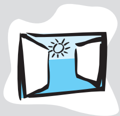


Note:

- The curve is a **parabola** and it opens upwards.
- There are no negative y values, i.e., the curve does not go below the x-axis.
- The curve is **symmetrical** about the y-axis because, for example, when $x = -3$, $y = (-3)^2$ and when $x = 3$, $y = 3^2$ have the same value.
- The curve has a **minimum turning point** or **vertex** at (0, 0).

Special note: It is essential that you can draw the graph of $y = x^2$ without having to refer to a table of values.

OPENING PROBLEM



A tennis ball is thrown vertically upwards and its height H, in m, above the ground is given at one second intervals as:

t	0	1	2	3	4	5
H	6.2	25.2	34.2	33.2	20.2	1.2

VIDEO CLIP

For you to consider:

- When the ball was released, was the thrower likely to be standing at ground level, standing on the roof of a single storey building or standing on the roof of a two storey building?
- What would the flight of the ball look like from a distance of 50 m away or from directly above the thrower?
- What is the function equation which gives the height H in terms of time t and what would its graph look like when H is plotted against t?
- What is the maximum height reached and when does this occur?
- When is the ball 30 m above the ground?

SIMULATION

A FUNCTION NOTATION $f : x \longmapsto ax^2+bx+c$

The function $f : x \longmapsto ax^2 + bx + c$ can be represented by $f(x) = ax^2 + bx + c$.

As with linear functions, for any value of x a corresponding value of y can be found by substituting into the function equation.

For example, if $y = 2x^2 - 3x + 5$, and $x = 3$, then $y = 2 \times 3^2 - 3 \times 3 + 5 = 14$
Hence, the ordered pair (3, 14) satisfies the function $y = 2x^2 - 3x + 5$.

Similarly, using function notation we could write,

if $f(x) = 2x^2 - 3x + 5$ and $x = 3$, then $f(3) = 2 \times 3^2 - 3 \times 3 + 5 = 14$

EXERCISE 7A

1 Which of the following are quadratic functions?

a $y = 3x^2 - 4x + 1$ **b** $y = 5x - 7$ **c** $y = -x^2$
d $y = \frac{2}{3}x^2 + 4$ **e** $2y + 3x^2 - 5 = 0$ **f** $y = 5x^3 + x - 6$

2 For each of the following functions, find the value of y for the given value of x:

a $y = x^2 + 5x - 4$ $\{x = 3\}$ **b** $y = 2x^2 + 9$ $\{x = -3\}$
c $y = -2x^2 + 3x - 5$ $\{x = 1\}$ **d** $y = 4x^2 - 7x + 1$ $\{x = 4\}$

3 For each of the following functions find the value of $f(x)$ given in brackets:

a $f(x) = x^2 - 2x + 3$ $\{f(2)\}$
b $f(x) = 4 - x^2$ $\{f(-3)\}$
c $f(x) = -\frac{1}{4}x^2 + 3x - 4$ $\{f(0)\}$
d $f(x) = \frac{1}{2}x^2 + 3x$ $\{f(2)\}$

Example 1

State whether the following functions are satisfied by the given ordered pairs:

a $y = 3x^2 + 2x$ (2, 16)
b $f(x) = -x^2 - 2x + 1$ (−3, 1)

a $y = 3(2)^2 + 2(2)$
$= 12 + 4$
$= 16$
i.e., when $x = 2$, $y = 16$
$\therefore$ (2, 16) does satisfy $y = 3x^2 + 2x$

b $f(-3) = -(-3)^2 - 2(-3) + 1$
$= -9 + 6 + 1$
$= -2$
i.e., $f(-3) \neq 1$
$\therefore$ (−3, 1) does not satisfy $f(x) = -x^2 - 2x + 1$

4 State whether the following quadratic functions are satisfied by the given ordered pairs:

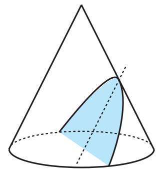
a $f(x) = 5x^2 - 10$ (0, 5)
b $y = 2x^2 + 5x - 3$ (4, 9)
c $y = -2x^2 + 3x$ $(-\frac{1}{2}, 1)$
d $y = -7x^2 + 8x + 15$ (−1, 16)

B GRAPHS OF QUADRATIC FUNCTIONS

The graphs of all quadratic functions are **parabolas**. The parabola is one of the conic sections.

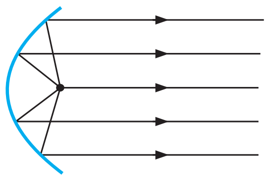
Conic sections are curves which can be obtained by cutting a cone with a plane. The Ancient Greek mathematicians were fascinated by conic sections.

You may like to find the conic sections for yourself by cutting an icecream cone. Cutting parallel to the side produces a parabola, i.e.,



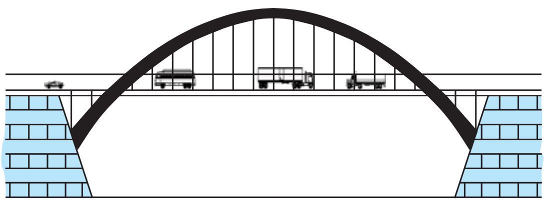
There are many examples of parabolas in every day life. The name parabola comes from the Greek word for **thrown** because when an object is thrown its path makes a parabolic shape.

Parabolic mirrors are used in car headlights, heaters, radar discs and radio telescopes because of their special geometric properties.



Alongside is a single span parabolic bridge.

Some archways also have parabolic shape.



INVESTIGATION 1 — GRAPHING $y = a(x-\alpha)(x-\beta)$

This investigation is best done using a **graphing package** or **graphics calculator**.

GRAPHING PACKAGE

TI

C

What to do:

1 **a** Use technology to assist you to draw sketch graphs of:
$y = (x-1)(x-3)$, $y = 2(x-1)(x-3)$, $y = -(x-1)(x-3)$,
$y = -3(x-1)(x-3)$ and $y = -\frac{1}{2}(x-1)(x-3)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of a in $y = a(x-1)(x-3)$?

2 **a** Use technology to assist you to draw sketch graphs of:
$y = 2(x-1)(x-4)$, $y = 2(x-3)(x-5)$, $y = 2(x+1)(x-2)$,
$y = 2x(x+5)$ and $y = 2(x+2)(x+4)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of α and β in $y = 2(x-\alpha)(x-\beta)$?

3 **a** Use technology to assist you to draw sketch graphs of:
$y = 2(x-1)^2$, $y = 2(x-3)^2$, $y = 2(x+2)^2$, $y = 2x^2$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of α in $y = 2(x-\alpha)^2$?

4 Copy and complete:

- If a quadratic has factorisation $y = a(x-\alpha)(x-\beta)$ it ...... the x-axis at ......
- If a quadratic has factorisation $y = a(x-\alpha)^2$ it ...... the x-axis at ......

INVESTIGATION 2 — GRAPHING $y = a(x-h)^2+k$

This investigation is also best done using technology.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to assist you to draw sketch graphs of:
$y = (x-3)^2+2$, $y = 2(x-3)^2+2$, $y = -2(x-3)^2+2$,
$y = -(x-3)^2+2$ and $y = -\frac{1}{3}(x-3)^2+2$

b Find the coordinates of the vertex for each function in **a**.

c What is the geometrical significance of a in $y = a(x-3)^2+2$?

TI

C

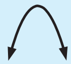
2 **a** Use technology to assist you to to draw sketch graphs of:
$y = 2(x-1)^2+3$, $y = 2(x-2)^2+4$, $y = 2(x-3)^2+1$,
$y = 2(x+1)^2+4$, $y = 2(x+2)^2-5$ and $y = 2(x+3)^2-2$

b Find the coordinates of the vertex for each function in **a**.

c What is the geometrical significance of h and k in $y = 2(x-h)^2+k$?

3 Copy and complete:
If a quadratic is in the form $y = a(x-h)^2+k$ then its vertex has coordinates ....

From **Investigations 1** and **2** you should have discovered that:

- The coefficient of x^2 (which is a) controls the degree of width of the graph and whether it opens upwards or downwards.
  - $a > 0$  (concave up) whereas $a < 0$ produces  (concave down)
  - If $-1 < a < 1$, $a \neq 0$ the graph is wider than $y = x^2$.
    If $a < -1$ or $a > 1$ the graph is narrower than $y = x^2$.
- In the form $y = a(x-\alpha)(x-\beta)$ the graph **cuts** the x-axis at α and β.
- In the form $y = a(x-\alpha)^2$ the graph **touches** the x-axis at α.
- In the form $y = a(x-h)^2 + k$ the graph has vertex (h, k) and axis of symmetry $x = h$.

THE FORM $y = a(x-\alpha)(x-\beta)$

If we are given an equation of the form $y = a(x-\alpha)(x-\beta)$ we can easily graph it using

- the x-intercepts (α and β)
- the axis of symmetry $\left(x = \frac{\alpha+\beta}{2}\right)$
- the coordinates of its vertex $\left(\frac{\alpha+\beta}{2}, f\left(\frac{\alpha+\beta}{2}\right)\right)$
- the y-intercept (let $x = 0$).

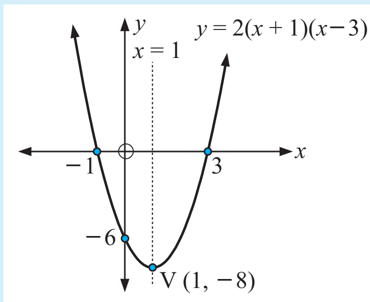
Example 2

Using axis intercepts only, sketch the graph of $y = 2(x+1)(x-3)$.

$y = 2(x+1)(x-3)$
has x-intercepts -1 and 3, $\therefore$ the axis of symmetry is midway between the x-intercepts i.e., $x = 1$

when $x = 1$, $y = 2(2)(-2) = -8$
$\therefore$ the vertex is $(1, -8)$

when $x = 0$, $y = 2(1)(-3) = -6$
$\therefore$ the y-intercept is -6



EXERCISE 7B.1

1 For each of the following functions:

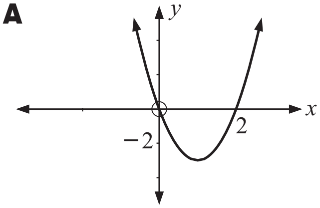
i state the x-intercepts
ii state the equation of the axis of symmetry
iii find the coordinates of the vertex
iv find the y-intercept
v sketch the graph of the function
vi use technology to check your answers.

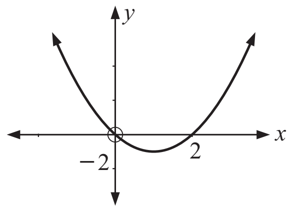
a $y = (x+2)(x-2)$
b $y = 2(x-1)(x-3)$
c $y = 3(x-1)(x-2)$
d $y = \frac{1}{2}x(x-4)$
e $y = -2x(x+3)$
f $y = -\frac{1}{2}(x+2)(x+3)$

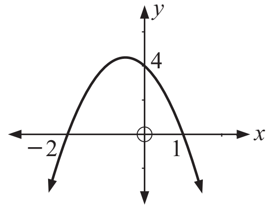
2 Match each function with its corresponding graph:

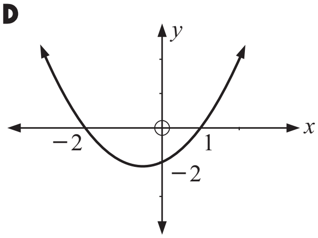
a $y = x(x-2)$ **b** $y = 3x(x-2)$ **c** $y = -x(x-2)$

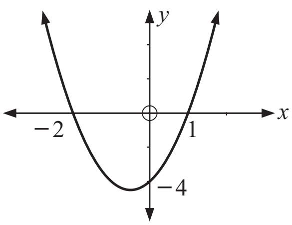
d $y = (x+2)(x-1)$ **e** $y = 2(x+2)(x-1)$ **f** $y = -2(x+2)(x-1)$

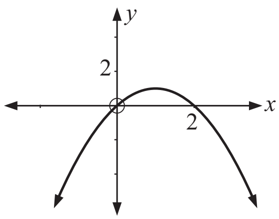
A


B


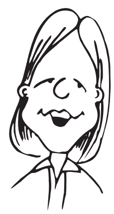
C


D


E


F


THE FORM $y = a(x-h)^2+k$



If we are given an equation of the form $y = a(x-h)^2 + k$ we can easily graph it using:

- the axis of symmetry $(x = h)$
- the coordinates of the vertex $(h,\ k)$
- the y-intercept. (let $x = 0$)

In this form the axis of symmetry and the coordinates of the vertex are easy to read off.

Example 3

Use the vertex, axis of symmetry and y-intercept to sketch the graph of:

a $y = -2(x-2)^2 - 1$ **b** $y = \frac{1}{2}(x+3)^2$

a $y = -2(x-2)^2 - 1$

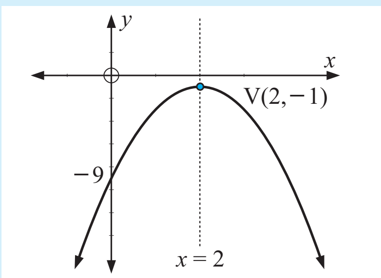
has axis of symmetry $x = 2$

and vertex $(2,\ -1)$

when $x = 0$, $y = -2(-2)^2 - 1 = -9$

$\therefore$ y-intercept is -9

$a < 0$, $\therefore$ the shape is 



b $y = \frac{1}{2}(x+3)^2$

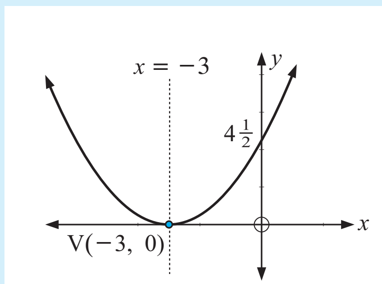
has axis of symmetry $x = -3$

and vertex $(-3,\ 0)$

when $x = 0$, $y = \frac{1}{2}(3)^2 = 4\frac{1}{2}$

$\therefore$ y-intercept is $4\frac{1}{2}$

$a > 0$, $\therefore$ the shape is 



EXERCISE 7B.2

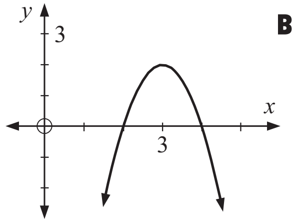
1 For each of the following functions:

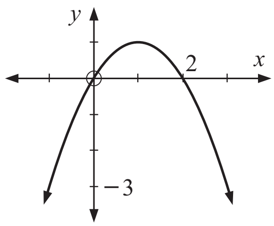
i state the equation of the axis of symmetry
ii find the coordinates of the vertex
iii find the y-intercept
iv sketch the graph of the function
v use technology to check your answers.

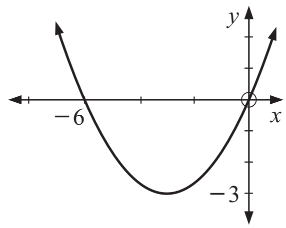
a $y = (x-4)^2 + 3$
b $y = 2(x+1)^2$
c $y = -(x+3)^2 + 2$
d $y = 3(x+2)^2 - 4$
e $y = \frac{1}{2}(x-2)^2$
f $y = -\frac{3}{2}(x+2)^2 - 4$

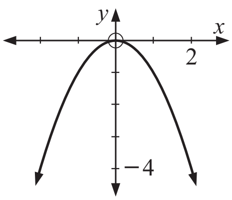
2 Match each quadratic function with its corresponding graph:

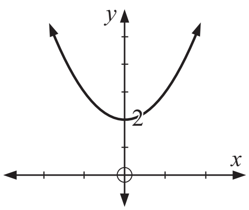
a $y = -(x+1)^2 + 3$
b $y = -2(x-3)^2 + 2$
c $y = x^2 + 2$
d $y = -(x-1)^2 + 1$
e $y = (x-2)^2 - 2$
f $y = \frac{1}{3}(x+3)^2 - 3$
g $y = -x^2$
h $y = -\frac{1}{2}(x-1)^2 + 1$
i $y = 2(x+2)^2 - 1$

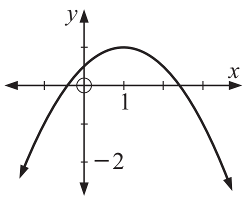
A


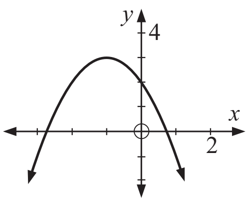
B


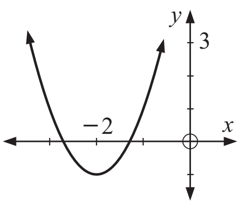
C


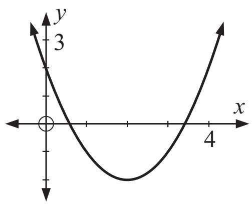
D


E


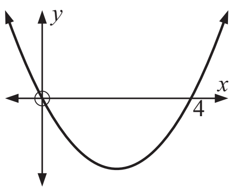
F


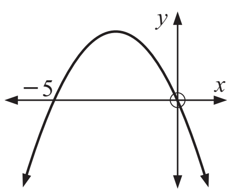
G


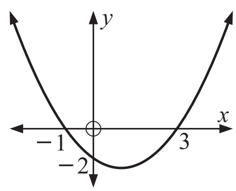
H


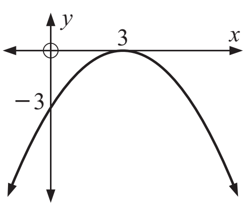
I


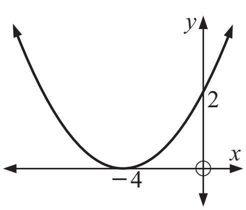
3 For each of the following find the equation of the axis of symmetry:

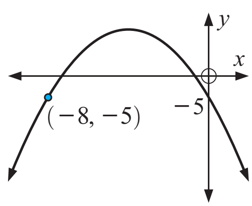
a


b


c


d


e


f


4 For each of the following quadratic functions:

i sketch the graph using axes intercepts and hence find

ii the equation of the axis of symmetry **iii** the coordinates of the vertex.

a $y = x^2 + 4x$ **b** $y = x(x-4)$ **c** $y = 3(x-2)^2$

d $y = 2(x-1)(x+3)$ **e** $y = -2(x-1)^2$ **f** $y = -3(x+2)(x-2)$

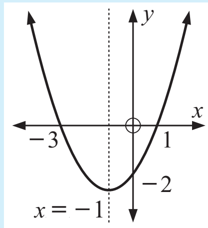
Example 4

Sketch the parabola which has x-intercepts -3 and 1, and y-intercept -2. Find the equation of the axis of symmetry.

The axis of symmetry lies halfway between the x-intercepts $\therefore$ axis of symmetry is $x = -1$.

$\{\dfrac{-3+1}{2} = -1\}$

Note: The graph is concave up. Can you see why?



5 For each of the following:

i sketch the parabola **ii** find the equation of the axis of symmetry.

a x-intercepts 3 and -1, y-int. -4 **b** x-intercepts 2 and -2, y-int. 4

c x-intercept -3 (touching), y-int. 6 **d** x-intercept 1 (touching), y-int. -4

6 Find all x-intercepts of the following graphs of quadratic functions:

a cuts the x-axis at -1, axis of symmetry $x = -3$

b touches the x-axis at 3.

C COMPLETING THE SQUARE

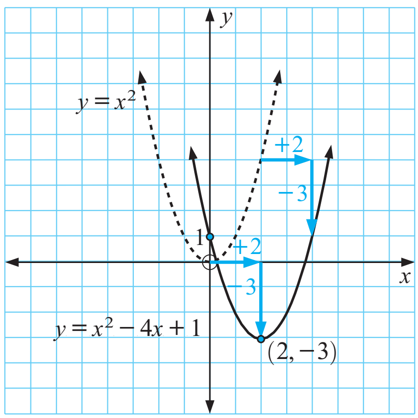
If we wish to find the vertex of a quadratic given in general form $y = ax^2 + bx + c$ then one approach is to convert it to the form $y = a(x-h)^2 + k$ where we can read off the vertex (h, k). To do this we may choose to '**complete the square**'.

Consider a case where $a = 1$; $y = x^2 - 4x + 1$.

$$y = x^2 - 4x + 1$$
$$\therefore \quad y = x^2 - 4x + 2^2 + 1 - 2^2$$
$$\therefore \quad y = x^2 - 4x + 2^2 \quad - \quad 3$$
$$\therefore \quad y = (x-2)^2 - 3$$

So, $y = x^2 - 4x + 1$ is really $y = (x-2)^2 - 3$

and therefore the graph of $y = x^2 - 4x + 1$ can be considered as the graph of $y = x^2$ after it has been translated 2 units to the right and 3 units down, i.e., $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.



Example 5

Write $y = x^2 + 4x + 3$ in the form $y = (x - h)^2 + k$ using completing the square and hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

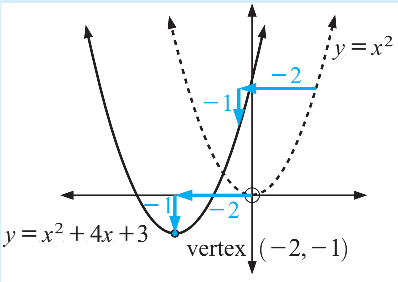
$y = x^2 + 4x + 3$

$\therefore \quad y = x^2 + 4x + 2^2 + 3 - 2^2$

$\therefore \quad y = (x + 2)^2 - 1$

shift 2 units left; shift 1 unit down

Vertex is $(-2, -1)$ and the y-intercept is 3



EXERCISE 7C

1 Write the following quadratics in the form $y = (x - h)^2 + k$ using 'completing the square' and hence sketch each function, stating the vertex:

a $y = x^2 - 2x + 3$ **b** $y = x^2 + 4x - 2$ **c** $y = x^2 - 4x$

d $y = x^2 + 3x$ **e** $y = x^2 + 5x - 2$ **f** $y = x^2 - 3x + 2$

g $y = x^2 - 6x + 5$ **h** $y = x^2 + 8x - 2$ **i** $y = x^2 - 5x + 1$

Example 6

Convert $y = 3x^2 - 4x + 1$ into the form $y = a(x - h)^2 + k$ by 'completing the square'. Hence, write down the coordinates of its vertex and sketch the graph of the function.

$y = 3x^2 - 4x + 1$

$= 3[x^2 - \frac{4}{3}x + \frac{1}{3}]$ {take out a factor of 3}

$= 3[x^2 - 2(\frac{2}{3})x + (\frac{2}{3})^2 - (\frac{2}{3})^2 + \frac{1}{3}]$ {complete the square}

$= 3[(x - \frac{2}{3})^2 - \frac{4}{9} + \frac{1}{3}]$ {write as a perfect square}

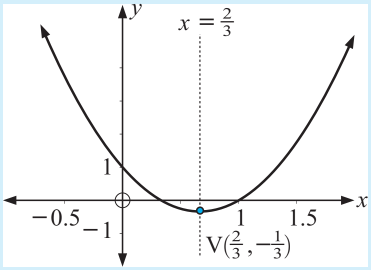
$= 3[(x - \frac{2}{3})^2 - \frac{4}{9} + \frac{3}{9}]$ {get common denominator}

$= 3[(x - \frac{2}{3})^2 - \frac{1}{9}]$ {add fractions}

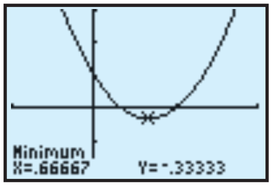
$= 3(x - \frac{2}{3})^2 - \frac{1}{3}$ {expand to put into desired form}

So the vertex is $(\frac{2}{3}, -\frac{1}{3})$

The y-intercept is 1.



We can use technology to confirm this. For example:



2 For each of the following quadratics:

- **i** convert into the form $y = a(x-h)^2 + k$ by 'completing the square'
- **ii** state the coordinates of the vertex
- **iii** find the y-intercept.
- **iv** Hence, sketch the graph of the quadratic.
- **v** Use technology to check your answer.

a $y = 2x^2 + 4x + 5$ **b** $y = 2x^2 - 8x + 3$

c $y = 2x^2 - 6x + 1$ **d** $y = 3x^2 - 6x + 5$

e $y = -x^2 + 4x + 2$ **f** $y = -2x^2 - 5x + 3$

3 By using your **graphing package** or **graphics calculator**, graph each of the following functions, and hence write each function in the form $y = a(x-h)^2 + k$:

a $y = x^2 - 4x + 7$ **b** $y = x^2 + 6x + 3$ **c** $y = -x^2 + 4x + 5$

d $y = 2x^2 + 6x - 4$ **e** $y = -2x^2 - 10x + 1$ **f** $y = 3x^2 - 9x - 5$

D QUADRATIC EQUATIONS

Apex Leather Jacket Co. makes and sells x leather jackets each day and their revenue function is given by $R = 12.5x^2 - 550x + 8125$ dollars.

How many jackets must be made and sold each week in order to obtain income of \$3000 each week?

Clearly we need to solve the equation:

$$12.5x^2 - 550x + 8125 = 3000$$
$$\text{i.e.,} \quad 12.5x^2 - 550x + 5125 = 0$$

This equation, which is of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

> A **quadratic equation**, with variable x, is an equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$.

To solve quadratic equations we can:

- **factorise** the quadratic and use the **Null Factor law**: "if $ab = 0$ then $a = 0$ or $b = 0$"
- **complete the square**
- use the **quadratic formula**
- use **technology**.

Definition:

The **roots** (or **solutions**) of $ax^2 + bx + c = 0$ are all the values of x which satisfy the equation (i.e., make it true).

For example, $x = 2$ is a root of $x^2 - 3x + 2 = 0$ since, when $x = 2$

$$x^2 - 3x + 2 = (2)^2 - 3(2) + 2 = 4 - 6 + 2 = 0 \quad \checkmark$$

SOLVING USING FACTORISATION

Step 1: Make one side of the equation 0 by transferring all terms to one side.
Step 2: Fully factorise the other side.
Step 3: Use the 'Null Factor law': "if $ab = 0$ then $a = 0$ or $b = 0$".
Step 4: Solve the resulting elementary linear equations.

Example 7

Solve for x: **a** $3x^2 + 5x = 0$ **b** $x^2 = 5x + 6$

a
$$3x^2 + 5x = 0$$
$$\therefore\ x(3x+5) = 0$$
$$\therefore\ x = 0 \text{ or } 3x + 5 = 0$$
$$\therefore\ x = 0 \text{ or } x = -\tfrac{5}{3}$$

b
$$x^2 = 5x + 6$$
$$\therefore\ x^2 - 5x - 6 = 0$$
$$\therefore\ (x-6)(x+1) = 0$$
$$\therefore\ x - 6 = 0 \text{ or } x + 1 = 0$$
$$\therefore\ x = 6 \text{ or } -1$$

EXERCISE 7D.1

In each of the following, check answers with technology.

1 Solve the following using 'factorisation':

a $4x^2 + 7x = 0$ **b** $6x^2 + 2x = 0$ **c** $3x^2 - 7x = 0$
d $2x^2 - 11x = 0$ **e** $3x^2 = 8x$ **f** $9x = 6x^2$
g $x^2 - 5x + 6 = 0$ **h** $x^2 = 2x + 8$ **i** $x^2 + 21 = 10x$
j $9 + x^2 = 6x$ **k** $x^2 + x = 12$ **l** $x^2 + 8x = 33$

Example 8

Solve for x: **a** $4x^2 + 1 = 4x$ **b** $6x^2 = 11x + 10$

a
$$4x^2 + 1 = 4x$$
$$\therefore\ 4x^2 - 4x + 1 = 0$$
$$\therefore\ (2x-1)^2 = 0$$
$$\therefore\ x = \tfrac{1}{2}$$

b
$$6x^2 = 11x + 10$$
$$\therefore\ 6x^2 - 11x - 10 = 0$$
$$\therefore\ (2x-5)(3x+2) = 0$$
{using a factorisation technique}
$$\therefore\ x = \tfrac{5}{2} \text{ or } -\tfrac{2}{3}$$

2 Solve the following using factorisation:

a $9x^2 - 12x + 4 = 0$	**b** $2x^2 - 13x - 7 = 0$	**c** $3x^2 = 16x + 12$
d $3x^2 + 5x = 2$	**e** $2x^2 + 3 = 5x$	**f** $3x^2 = 4x + 4$
g $3x^2 = 10x + 8$	**h** $4x^2 + 4x = 3$	**i** $4x^2 = 11x + 3$
j $12x^2 = 11x + 15$	**k** $7x^2 + 6x = 1$	**l** $15x^2 + 2x = 56$

Example 9

Solve for x:

$3x + \frac{2}{x} = -7$

$$3x + \frac{2}{x} = -7$$

$\therefore \quad x(3x + \frac{2}{x}) = -7x$ {multiply both sides by x to eliminate the fraction}

$\therefore \quad 3x^2 + 2 = -7x$ {clear the bracket}

$\therefore \quad 3x^2 + 7x + 2 = 0$ {equate to 0}

$\therefore \quad (x+2)(3x+1) = 0$ {on factorising}

$\therefore \quad x = -2 \text{ or } -\frac{1}{3}$

3 Solve for x:

a $(x+1)^2 = 2x^2 - 5x + 11$	**b** $(x+2)(1-x) = -4$
c $5 - 4x^2 = 3(2x+1) + 2$	**d** $x + \frac{2}{x} = 3$
e $2x - \frac{1}{x} = -1$	**f** $\frac{x+3}{1-x} = -\frac{9}{x}$

FINDING x GIVEN y IN $y = ax^2 + bx + c$

It is also possible to substitute a value for y to find a corresponding value for x. However, unlike linear functions, with quadratic functions there may be 0, 1 or 2 possible values for x for any one value of y.

Example 10

If $y = x^2 - 6x + 8$ find the value(s) of x when: **a** $y = 15$ **b** $y = -1$

a If $y = 15$,

$x^2 - 6x + 8 = 15$

$\therefore \quad x^2 - 6x - 7 = 0$

$\therefore \quad (x+1)(x-7) = 0$

$\therefore \quad x = -1 \quad \text{or} \quad x = 7$

i.e., 2 solutions.

b If $y = -1$,

$x^2 - 6x + 8 = -1$

$\therefore \quad x^2 - 6x + 9 = 0$

$\therefore \quad (x-3)^2 = 0$

$\therefore \quad x = 3$

i.e., only one solution

4 Find the value(s) of x for the given value of y for each of the following quadratic functions:

a $y = x^2 + 6x + 10$ $\{y = 1\}$ **b** $y = x^2 + 5x + 8$ $\{y = 2\}$

c $y = x^2 - 5x + 1$ $\{y = -3\}$ **d** $y = 3x^2$ $\{y = -3\}$

Example 11

If $f(x) = x^2 + 4x + 11$ find x when **a** $f(x) = 23$ **b** $f(x) = 7$

a If $f(x) = 23$

$\therefore\ x^2 + 4x + 11 = 23$

$\therefore\ x^2 + 4x - 12 = 0$

$\therefore\ (x+6)(x-2) = 0$ {factorising}

$\therefore\ x = -6$ or 2

i.e., 2 solutions.

b If $f(x) = 7$

$\therefore\ x^2 + 4x + 11 = 7$

$\therefore\ x^2 + 4x + 4 = 0$

$\therefore\ (x+2)^2 = 0$ {factorising}

$\therefore\ x = -2$

i.e., one solution only.

5 Find the value(s) of x given that:

a $f(x) = 3x^2 - 2x + 5$ and $f(x) = 5$

b $f(x) = x^2 - x - 5$ and $f(x) = 1$

c $f(x) = -2x^2 - 13x + 3$ and $f(x) = -4$

d $f(x) = 2x^2 - 12x + 1$ and $f(x) = -17$

Example 12

A stone is thrown into the air and its height in metres above the ground is given by the function $h(t) = -5t^2 + 30t + 2$ where t is the time (in seconds) from when the stone is thrown.

a How high above the ground is the stone at time $t = 3$ seconds?

b How high above the ground was the stone released?

c At what time was the stone's height above the ground 27 m?

a $h(3) = -5(3)^2 + 30(3) + 2$

$= -45 + 90 + 2$

$= 47$

i.e., 47 m above ground.

b The stone is released when $t = 0$ sec

$\therefore\ h(0) = -5(0)^2 + 30(0) + 2 = 2$

$\therefore$ released 2 m above ground level.

c When $h(t) = 27$

$-5t^2 + 30t + 2 = 27$

$\therefore\ -5t^2 + 30t - 25 = 0$

$\therefore\ t^2 - 6t + 5 = 0$ {dividing each term by -5}

$\therefore\ (t-1)(t-5) = 0$ {factorising}

$\therefore\ t = 1$ or 5

i.e., after 1 sec and after 5 sec. Can you explain the two answers?

6 An object is projected into the air with a velocity of 30 m/s. Its height in metres, after t seconds, is given by the function $h(t) = 30t - 5t^2$.

- **a** Calculate the height after: **i** 1 second **ii** 5 seconds **iii** 3 seconds.
- **b** Calculate the time(s) at which the height is: **i** 40 m **ii** 0 m.
- **c** Explain your answers in part **b**.

7 A cake manufacturer finds that the profit in dollars, from making x cakes per day, is given by the function $P(x) = -\frac{1}{4}x^2 + 16x - 30$.

- **a** Calculate the profit if: **i** 0 cakes **ii** 10 cakes are made per day.
- **b** How many cakes per day are made if the profit is \$57?

SOLVING USING 'COMPLETING THE SQUARE'

As you would be aware by now, not all quadratics factorise easily. In fact, $x^2 + 4x + 1$ cannot be factorised by using a simple factorisation approach.

This means that we need a different approach in order to solve $x^2 + 4x + 1 = 0$.

One way is to use the 'completing the square' technique.

So, equations of the form $ax^2 + bx + c = 0$ can be converted to the form $(x + p)^2 = q$ from which the solutions are easy to obtain.

Notice that if $X^2 = a$, then $X = \pm\sqrt{a}$ is used.

Example 13

Solve exactly for x: **a** $(x+2)^2 = 7$ **b** $(x-1)^2 = -5$

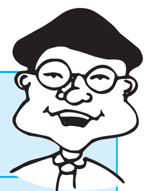
a
$$(x+2)^2 = 7$$
$$\therefore\quad x + 2 = \pm\sqrt{7}$$
$$\therefore\quad x = -2 \pm \sqrt{7}$$

b $(x-1)^2 = -5$
has no real solutions
{the perfect square, $(x-1)^2$ cannot be negative}

EXERCISE 7D.2

1 Solve for exact values of x:

a $(x+5)^2 = 2$	**b** $(x+6)^2 = 11$	**c** $(x-4)^2 = 8$
d $(x-8)^2 = 7$	**e** $2(x+3)^2 = 10$	**f** $3(x-2)^2 = 18$
g $(x+1)^2 + 1 = 11$	**h** $(2x+1)^2 = 3$	



Example 14

Solve for exact values of x: $x^2 + 4x + 1 = 0$

$$x^2 + 4x + 1 = 0$$
$$\therefore\quad x^2 + 4x = -1 \qquad \text{\{put the constant on the RHS\}}$$
$$\therefore\quad x^2 + 4x + 2^2 = -1 + 2^2 \qquad \text{\{completing the square\}}$$
$$\therefore\quad (x+2)^2 = 3 \qquad \text{\{factorising\}}$$
$$\therefore\quad x + 2 = \pm\sqrt{3} \qquad \text{\{solving\}}$$
$$\therefore\quad x = -2 \pm \sqrt{3}$$

The squared number we add to both sides is $\left(\dfrac{\text{coefficient of } x}{2}\right)^2$

2 Solve for exact values of x by completing the square:

a $x^2 - 4x + 1 = 0$ **b** $x^2 + 6x + 2 = 0$ **c** $x^2 - 14x + 46 = 0$
d $x^2 = 4x + 3$ **e** $x^2 + 6x + 7 = 0$ **f** $x^2 = 2x + 6$
g $x^2 + 6x = 2$ **h** $x^2 + 10 = 8x$ **i** $x^2 + 6x = -11$

3 If the coefficient of x^2 is not 1, we first divide throughout to make it 1.

For example, $2x^2 + 10x + 3 = 0$ becomes $x^2 + 5x + \frac{3}{2} = 0$
$-3x^2 + 12x + 5 = 0$ becomes $x^2 - 4x - \frac{5}{3} = 0$

Solve for exact values of x by completing the square:

a $2x^2 + 4x + 1 = 0$ **b** $2x^2 - 10x + 3 = 0$ **c** $3x^2 + 12x + 5 = 0$
d $3x^2 = 6x + 4$ **e** $5x^2 - 15x + 2 = 0$ **f** $4x^2 + 4x = 5$

E THE QUADRATIC FORMULA

Many quadratic equations cannot be solved by factorising, and completing the square is rather tedious. Consequently, the **quadratic formula** has been developed. This formula is:

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Consider the Apex Leather Jacket Co. equation from page **131**. We need to solve:

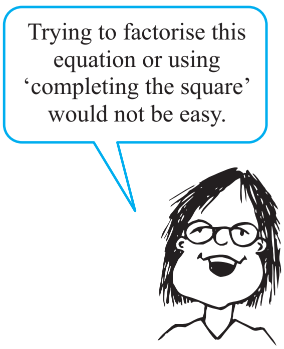
$$12.5x^2 - 550x + 5125 = 0$$

Here we have $a = 12.5$, $b = -550$, $c = 5125$

$$\therefore \quad x = \frac{550 \pm \sqrt{(-550)^2 - 4(12.5)(5125)}}{2(12.5)}$$

$$= \frac{550 \pm \sqrt{46\,250}}{25}$$

$$\doteqdot 30.60 \quad \text{or} \quad 13.40$$



But as x needs to be a whole number, $x = 13$ or 31 would produce income of around \$3000 each week.

The following proof of the quadratic formula is worth careful examination.

Proof: If $ax^2 + bx + c = 0$,

then $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ {dividing each term by a, as $a \neq 0$}

$\therefore \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$

$\therefore \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ {completing the square on LHS}

$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a}\left(\frac{4a}{4a}\right) + \frac{b^2}{4a^2}$

$$\therefore \quad \left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$$\therefore \quad x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$\therefore \quad x = -\frac{b}{2a} \pm \sqrt{\frac{b^2-4ac}{4a^2}}$$

$$\text{i.e.,} \quad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Example 15

Solve for x: **a** $x^2-2x-2=0$ **b** $2x^2+3x-4=0$

a $x^2-2x-2=0$ has $a=1,\ b=-2,\ c=-2$

$$\therefore \quad x = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-2)}}{2(1)}$$

$$\therefore \quad x = \frac{2\pm\sqrt{4+8}}{2}$$

$$\therefore \quad x = \frac{2\pm\sqrt{12}}{2}$$

$$\therefore \quad x = \frac{2\pm2\sqrt{3}}{2}$$

$$\therefore \quad x = 1\pm\sqrt{3}$$

Solutions are: $1+\sqrt{3}$ and $1-\sqrt{3}$.

b $2x^2+3x-4=0$ has $a=2,\ b=3,\ c=-4$

$$\therefore \quad x = \frac{-3\pm\sqrt{3^2-4(2)(-4)}}{2(2)}$$

$$\therefore \quad x = \frac{-3\pm\sqrt{9+32}}{4}$$

$$\therefore \quad x = \frac{-3\pm\sqrt{41}}{4}$$

Solutions are:

$\dfrac{-3+\sqrt{41}}{4}$ and $\dfrac{-3-\sqrt{41}}{4}$.

EXERCISE 7E

1 Use the quadratic formula to solve for exact values of x:

a $x^2-4x-3=0$ **b** $x^2+6x+7=0$ **c** $x^2+1=4x$

d $x^2+4x=1$ **e** $x^2-4x+2=0$ **f** $2x^2-2x-3=0$

g $x^2-2\sqrt{2}x+2=0$ **h** $(3x+1)^2=-2x$ **i** $(x+3)(2x+1)=9$

2 Use the quadratic formula to solve for exact values of x:

a $(x+2)(x-1)=2-3x$ **b** $(2x+1)^2=3-x$ **c** $(x-2)^2=1+x$

d $\dfrac{x-1}{2-x}=2x+1$ **e** $x-\dfrac{1}{x}=1$ **f** $2x-\dfrac{1}{x}=3$

Note: If asked to solve quadratic equations for **exact values** of x, use the quadratic formula. Use **completing the square** only when asked to do so.

F SOLVING QUADRATIC EQUATIONS WITH TECHNOLOGY

A **graphics calculator** or **graphing package** could be used to solve quadratic equations.

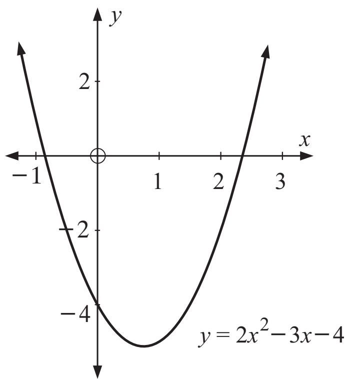
However, exact solutions in square root form would be lost in most cases. Approximate decimal solutions are usually generated. At this stage we will find solutions using **graphs** of quadratics and examine intersections with the x-axis (to get zeros) or the intersection of two functions to find the x-coordinates of the points where they meet.

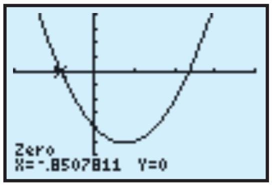
We have chosen to use this approach, even though it may not be the quickest, so that an understanding of the link between the algebra and the graphics is fully appreciated.

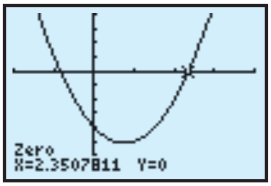
Consider the equation $2x^2 - 3x - 4 = 0$.

Our approach will be:

- draw the graph of $y = 2x^2 - 3x - 4$
- now $2x^2 - 3x - 4 = 0$ when $y = 0$ and this occurs at the x-intercepts of the graph.







The solutions are: $x \doteqdot -0.8508$ or 2.351

Click on the appropriate icon for helpful instructions if using a **graphics calculator** and/or **graphing package**.

GRAPHING PACKAGE TI C

EXERCISE 7F

1 Use technology to solve:

a $x^2 + 4x + 2 = 0$	**b** $x^2 + 6x - 2 = 0$	**c** $2x^2 - 3x - 7 = 0$
d $3x^2 - 7x - 11 = 0$	**e** $4x^2 - 11x - 13 = 0$	**f** $5x^2 + 6x - 17 = 0$

To solve a more complicated equation like $(x-2)(x+1) = 2 + 3x$ we could:

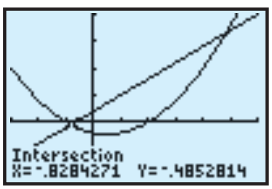
- make the RHS zero i.e., $(x-2)(x+1) - 2 - 3x = 0$.
  Plot $y = (x-2)(x+1) - 2 - 3x$ and find the x-intercepts.
- plot $y = (x-2)(x+1)$ and $y = 2 + 3x$ on the same axes and find the x-coordinates where the two graphs meet.

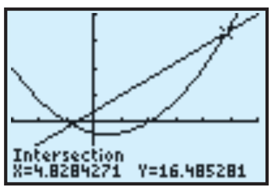
GRAPHING PACKAGE TI C

If using a graphics calculator with

$Y_1 = (x-2)(x+1)$ and

$Y_2 = 2 + 3x$ we get





So, the solutions are

$x \doteqdot -0.8284$ or 4.8284

2 Use technology to solve:

a $(x+2)(x-1) = 2-3x$ **b** $(2x+1)^2 = 3-x$ **c** $(x-2)^2 = 1+x$

d $\dfrac{x-1}{2-x} = 2x+1$ **e** $x - \dfrac{1}{x} = 1$ **f** $2x - \dfrac{1}{x} = 3$

G PROBLEM SOLVING WITH QUADRATICS

When solving some problems algebraically, a quadratic equation results. Consequently, we are only interested in any **real solutions** which result as we are looking for real answers.

If the resulting quadratic equation has no real roots then the problem has no real solution.

Also, any answer must be checked to see if it is reasonable. For example:

- if you are finding a length then it must be positive, so reject any negative solutions
- if you are finding 'how many people present' then clearly a fractional answer would be unacceptable.

General problem solving method:

Step 1: If the information is given in words, translate it into algebra using x for the unknown, say. An equation results.
Step 2: Solve the equation by a suitable method.
Step 3: Examine the solutions carefully to see if they are acceptable.
Step 4: Give your answer in a sentence.

Example 16

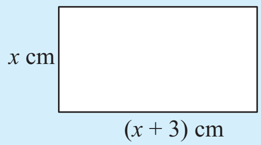
A rectangle has length 3 cm longer than its width. Its area is 42 cm^2. Find its width.

If the width is x cm, then the length is $(x+3)$ cm.

Therefore $x(x+3) = 42$ {equating areas}

$\therefore$ $x^2 + 3x - 42 = 0$

$\therefore$ $x \doteqdot -8.15$ or 5.15 {using technology}



We reject the negative solution as lengths are positive $\therefore$ width $\doteqdot 5.15$ cm.

EXERCISE 7G

1 Two integers differ by 12 and the sum of their squares is 74. Find the integers.

2 The sum of a number and its reciprocal is $5\frac{1}{5}$. Find the number.

3 The sum of a natural number and its square is 210. Find the number.

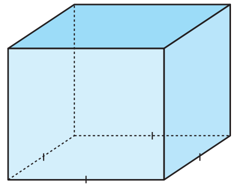
4 The product of two consecutive even numbers is 360. Find the numbers.

5 The product of two consecutive odd numbers is 255. Find the numbers.

6 The number of diagonals of an n-sided polygon is given by the formula $D = \dfrac{n}{2}(n-3)$. A polygon has 90 diagonals. How many sides does it have?

7 The length of a rectangle is 4 cm longer than its width. Find its width given that its area is 26 cm^2.

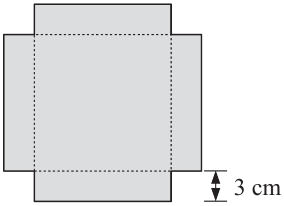
8 A rectangular box has a square base and its height is 1 cm longer than the length of one side of its base.



a If x cm is the length of one side of its base, show that its total surface area, A, is given by $A = 6x^2 + 4x$ cm^2.

b If the total surface area is 240 cm^2, find the dimensions of the box.

9

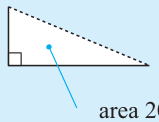


An open box contains 80 cm^3 and is made from a square piece of tinplate with 3 cm squares cut from each of its 4 corners. Find the dimensions of the original piece of tinplate.

Example 17

Is it possible to bend a 12 cm length of wire to form the legs of a right angled triangle with area 20 cm^2?

i.e., becomes

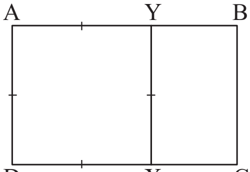


$$\text{Area,} \quad A = \tfrac{1}{2}(12 - x)x$$
$$\therefore \quad \tfrac{1}{2}x(12 - x) = 20$$
$$\therefore \quad x(12 - x) = 40$$
$$\therefore \quad 12x - x^2 - 40 = 0$$
$$\therefore \quad x^2 - 12x + 40 = 0 \quad \text{which becomes} \quad x = \frac{12 \pm \sqrt{-16}}{2}$$

Thus there are no real solutions, indicating the **impossibility**.

10 Is it possible to bend a 20 cm length of wire into the shape of a rectangle which has an area of 30 cm^2?

11 The *golden rectangle* is the rectangle defined by the following statement:

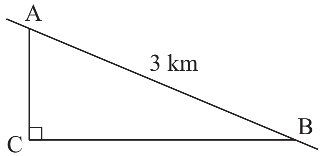


The golden rectangle can be divided into a square and a smaller rectangle by a line which is parallel to its shorter sides, and the smaller rectangle is **similar** to the original rectangle.

Thus, if ABCD is the golden rectangle, ADXY is a square and BCXY is similar to ABCD, (i.e., BCXY is a reduction of ABCD).

The ratio of $\dfrac{\text{AB}}{\text{AD}}$ for the golden rectangle is called the **golden ratio**.

Show that the golden ratio is $\dfrac{1 + \sqrt{5}}{2}$. (**Hint:** Let AB $= x$ units and BC $= 1$ unit.)

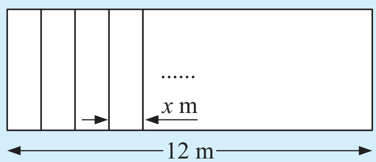
12 

A triangular paddock has a road AB forming its hypotenuse. AB is 3 km long. The fences AC and CB are at right angles. If BC is 400 m longer than AC, find the area of the paddock in hectares.

13 Find the width of a uniform concrete path placed around a 30 m by 40 m rectangular lawn given that the concrete has area one quarter of the lawn.

Example 18

A wall is 12 m long and is timber panelled using vertical sheets of panelling of equal width. If the sheets had been 0.2 m wider, 2 less sheets would have been required. What is the width of the timber panelling used?



Let x m be the width of each panel used.

$\therefore$ $\dfrac{12}{x}$ is the number of sheets needed.

Now if the sheets are $\left(x + \frac{1}{5}\right)$ m in width

$\left(\dfrac{12}{x} - 2\right)$ sheets are needed.

So, $\left(x + \frac{1}{5}\right)\left(\dfrac{12}{x} - 2\right) = 12$ {length of wall}

$\therefore \quad 12 - 2x + \dfrac{12}{5x} - \frac{2}{5} = 12$ {expanding LHS}

$\therefore \quad -2x + \dfrac{12}{5x} - \frac{2}{5} = 0$

$\therefore \quad -10x^2 + 12 - 2x = 0$ {$\times$ each term by $5x$}

$\therefore \quad 5x^2 + x - 6 = 0$ {$\div$ each term by -2}

$\therefore \quad (5x+6)(x-1) = 0$

$\therefore \quad x = -\frac{6}{5}$ or 1 where $x > 0$

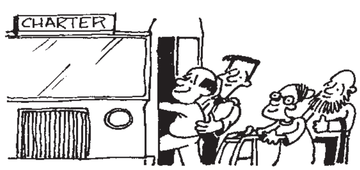
$\therefore$ each sheet is 1 m wide.

14 Chuong and Hassan both drive 40 km from home to work each day. One day Chuong said to Hassan, "If you drive home at your usual speed, I will average 40 kmph faster than you and arrive home in 20 minutes less time." Find Hassan's speed.

15 If the average speed of an aeroplane had been 120 kmph less, it would have taken a half an hour longer to fly 1000 km. Find the speed of the plane.

16 Two trains travel a 105 km track each day. The express travels 10 kmph faster and takes 30 minutes less than the normal train. Find the speed of the express.

17 A group of elderly citizens chartered a bus for \$160. However, at the last minute, due to illness, 8 of them had to miss the trip. Consequently the other citizens had to pay an extra \$1 each. How many elderly citizens went on the trip?



H QUADRATIC GRAPHS (REVIEW)

REVIEW OF TERMINOLOGY

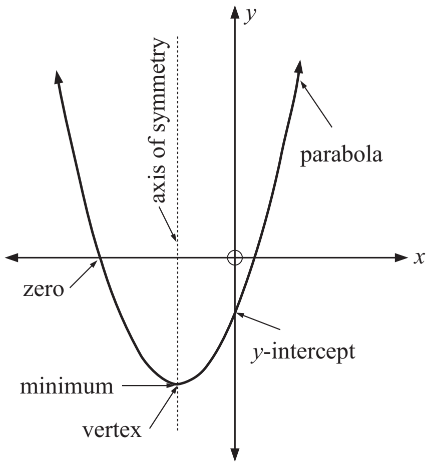
The equation of a **quadratic function** is given by $y = ax^2 + bx + c$, where $a \neq 0$.

The graph of a quadratic function is called a **parabola**. The point where the graph 'turns' is called the **vertex**.

If the graph opens upward, the y coordinate of the vertex is the **minimum** (concave up), while if the graph opens downward, the y-coordinate of the vertex is the **maximum** (concave down).

The vertical line that passes through the vertex is called the **axis of symmetry**. All parabolas are symmetrical about the axis of symmetry.

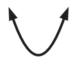
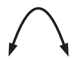
The point where the graph crosses the y-axis is the y-**intercept**.

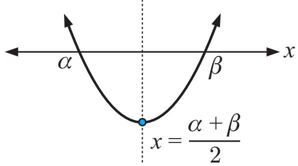
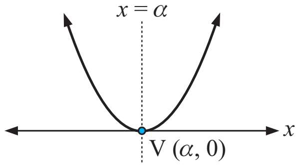
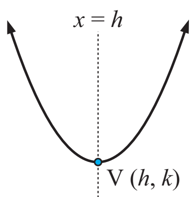


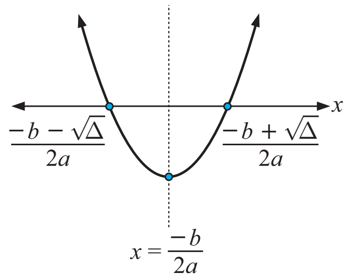
The points (if they exist) where the graph crosses the x-axis should be called the x-**intercepts**, but more commonly are called the **zeros** of the function.

OPENING UPWARDS OR DOWNWARDS?

If the coefficient of x^2:

- is positive, the graph is concave up  $a > 0$
- is negative, the graph is concave down  $a < 0$.

Quadratic form, $a \neq 0$	*Graph*	*Facts*
• $y = a(x-\alpha)(x-\beta)$ α, β are real		x-intercepts are α and β axis of symmetry is $x = \frac{\alpha+\beta}{2}$
• $y = a(x-\alpha)^2$ α is real		touches x-axis at α vertex is $(\alpha, 0)$ axis of symmetry is $x = \alpha$
• $y = a(x-h)^2 + k$		vertex is (h, k) axis of symmetry is $x = h$

Quadratic form, $a \neq 0$	*Graph*	*Facts*
• $y = ax^2 + bx + c$ (general quadratic form)		axis of symmetry is $x = \dfrac{-b}{2a}$ x-intercepts for $\Delta \geqslant 0$ are $\dfrac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$

Notice that the axis of symmetry is always easily found.

Note: $-\dfrac{b}{2a}$ is the **average** of $\dfrac{-b-\sqrt{\Delta}}{2a}$ and $\dfrac{-b+\sqrt{\Delta}}{2a}$.

as the sum equals $\dfrac{-b-\sqrt{\Delta}}{2a} + \dfrac{-b+\sqrt{\Delta}}{2a} = \dfrac{-2b}{2a} = \dfrac{-b}{a}$

and so the average is $\dfrac{\text{the sum}}{2} = \dfrac{-b}{2a}$.

SKETCHING GRAPHS USING KEY FACTS

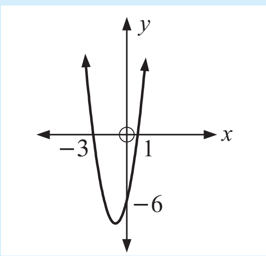
Example 19

Using axis intercepts only, sketch the graphs of:

a $y = 2(x+3)(x-1)$ **b** $y = -2(x-1)(x-2)$ **c** $y = \frac{1}{2}(x+2)^2$

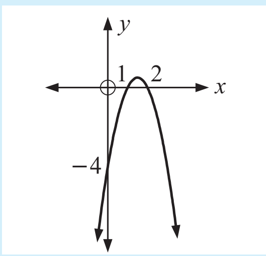
a $y = 2(x+3)(x-1)$
has x-intercepts -3, 1

when $x = 0$,
$y = 2(3)(-1)$
$= -6$
y-intercept is -6



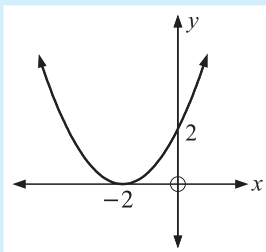
b $y = -2(x-1)(x-2)$
has x-intercepts 1, 2

when $x = 0$,
$y = -2(-1)(-2)$
$= -4$
y-intercept -4



c $y = \frac{1}{2}(x+2)^2$
touches x-axis at -2

when $x = 0$,
$y = \frac{1}{2}(2)^2$
$= 2$
has y-intercept 2



EXERCISE 7H

1 Using axis intercepts only, sketch the graphs of:

a $y = (x-4)(x+2)$ **b** $y = -(x-4)(x+2)$ **c** $y = 2(x+3)(x+5)$

d $y = -3x(x+4)$ **e** $y = 2(x+3)^2$ **f** $y = -\frac{1}{4}(x+2)^2$

2 What is the axis of symmetry of each graph in question **1**?

Example 20

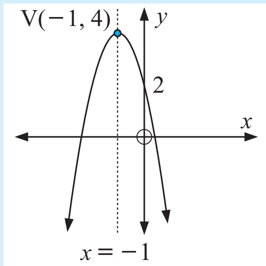
Use the vertex, axis of symmetry and y-intercept to graph $y = -2(x+1)^2 + 4$.

The vertex is $(-1, 4)$.

The axis of symmetry is $x = -1$.

When $x = 0$, $y = -2(1)^2 + 4$
$= 2$

$a < 0$ $\therefore$ $\cap$ shape



3 Use the vertex, axis of symmetry and y-intercept to graph:

a $y = (x-1)^2 + 3$ **b** $y = 2(x+2)^2 + 1$ **c** $y = -2(x-1)^2 - 3$

d $y = \frac{1}{2}(x-3)^2 + 2$ **e** $y = -\frac{1}{3}(x-1)^2 + 4$ **f** $y = -\frac{1}{10}(x+2)^2 - 3$

Example 21

For the quadratic $y = 2x^2 + 6x - 3$, find:

a the equation of the axis of symmetry **b** the coordinates of the vertex

c the axes intercepts. **d** Hence, sketch the graph.

For $y = 2x^2 + 6x - 3$, $a = 2$, $b = 6$, $c = -3$ $a > 0$ $\therefore$

a $\frac{-b}{2a} = \frac{-6}{4} = -\frac{3}{2}$

$\therefore$ axis of symmetry is $x = -\frac{3}{2}$.

b When $x = -\frac{3}{2}$,

$y = 2(-\frac{3}{2})^2 + 6(-\frac{3}{2}) - 3$

$= -7.5$ {simplifying}

$\therefore$ vertex is $(-\frac{3}{2}, -7\frac{1}{2})$.

c When $x = 0$, $y = -3$

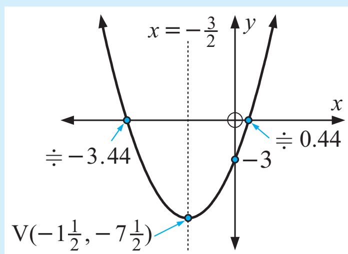
$\therefore$ y-intercept is -3.

When $y = 0$, $2x^2 + 6x - 3 = 0$

$\therefore$ $x = \dfrac{-6 \pm \sqrt{36 - 4(2)(-3)}}{4}$

$\therefore$ $x \doteqdot -3.44$ or 0.44

d



Example 22

Determine the coordinates of the vertex of $y = 2x^2 - 8x + 1$.

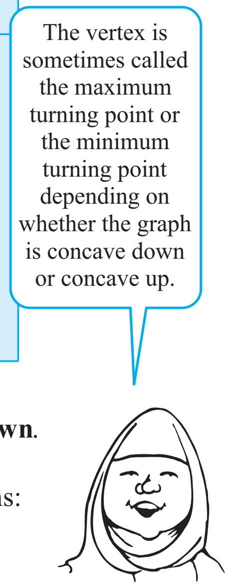
$y = 2x^2 - 8x + 1$ has $a = 2$, $b = -8$, $c = 1$

and so $\dfrac{-b}{2a} = \dfrac{-(-8)}{2 \times 2} = 2$

$\therefore$ equation of axis of symmetry is $x = 2$

and when $x = 2$, $y = 2(2)^2 - 8(2) + 1$
$= -7$

$\therefore$ the vertex has coordinates $(2, -7)$.



Note: A parabola opening upwards has a shape called **concave up**.
A parabola opening downwards has a shape called **concave down**.

4 Find the turning point (vertex) for the following quadratic functions:

- **a** $y = x^2 - 4x + 2$
- **b** $y = x^2 + 2x - 3$
- **c** $y = 2x^2 + 4$
- **d** $y = -3x^2 + 1$
- **e** $y = 2x^2 + 8x - 7$
- **f** $y = -x^2 - 4x - 9$
- **g** $y = 2x^2 + 6x - 1$
- **h** $y = 2x^2 - 10x + 3$
- **i** $y = -\frac{1}{2}x^2 + x - 5$
- **j** $y = -2x^2 + 8x - 2$

5 Find the x-intercepts for:

- **a** $y = x^2 - 9$
- **b** $y = 2x^2 - 6$
- **c** $y = x^2 + 7x + 10$
- **d** $y = x^2 + x - 12$
- **e** $y = 4x - x^2$
- **f** $y = -x^2 - 6x - 8$
- **g** $y = -2x^2 - 4x - 2$
- **h** $y = 4x^2 - 24x + 36$
- **i** $y = x^2 - 4x + 1$
- **j** $y = x^2 + 4x - 3$
- **k** $y = x^2 - 6x - 2$
- **l** $y = x^2 + 8x + 11$

6 For the following quadratics, find:

- **i** the equation of the axis of symmetry
- **ii** the coordinates of the vertex
- **iii** the axes intercepts, if they exist
- **iv** Hence, sketch the graph.

- **a** $y = x^2 - 2x + 5$
- **b** $y = x^2 + 4x - 1$
- **c** $y = 2x^2 - 5x + 2$
- **d** $y = -x^2 + 3x - 2$
- **e** $y = -3x^2 + 4x - 1$
- **f** $y = -2x^2 + x + 1$
- **g** $y = 6x - x^2$
- **h** $y = -x^2 - 6x - 8$
- **i** $y = -\frac{1}{4}x^2 + 2x + 1$

I THE DISCRIMINANT, Δ

In the quadratic formula, $b^2 - 4ac$, which is under the square root sign, is called **the discriminant**.

The symbol **delta** Δ, is used to represent the discriminant, i.e., $\Delta = b^2 - 4ac$.

The quadratic formula becomes $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ if Δ replaces $b^2 - 4ac$.

Note:
- if $\Delta = 0$, $x = \dfrac{-b}{2a}$ is the **only solution** (a **repeated** or **double root**)
- if $\Delta > 0$, $\sqrt{\Delta}$ is a positive real number and so there are **two distinct real roots**, $\dfrac{-b+\sqrt{\Delta}}{2a}$ and $\dfrac{-b-\sqrt{\Delta}}{2a}$
- if $\Delta < 0$, $\sqrt{\Delta}$ is not a real number and so there are **no real roots**.

Note: If a, b and c are rational and Δ is a **perfect square** then the equation has two rational roots which can be found by factorisation.

Example 23

Use the discriminant to determine the nature of the roots of:

a $2x^2 - 3x + 4 = 0$ **b** $4x^2 - 4x - 1 = 0$

a
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-3)^2 - 4(2)(4)\\ &= -23 \text{ which is } < 0\end{aligned}$$
$\therefore$ no real roots

b
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-4)^2 - 4(4)(-1)\\ &= 32 \text{ which is } > 0\end{aligned}$$
$\therefore$ has 2 distinct irrational roots

EXERCISE 7I.1

1 By using the discriminant only, state the nature of the solutions of:

a $x^2 + 7x - 2 = 0$ **b** $x^2 + 4\sqrt{2}x + 8 = 0$ **c** $2x^2 + 3x - 1 = 0$

d $6x^2 + 5x - 4 = 0$ **e** $x^2 + x + 6 = 0$ **f** $9x^2 + 6x + 1 = 0$

2 By using the discriminant only, determine which of the following quadratic equations have rational roots which can be found by factorisation.

a $2x^2 + 7x - 4 = 0$ **b** $3x^2 - 7x - 6 = 0$ **c** $2x^2 + 6x + 1 = 0$

d $6x^2 + 19x + 10 = 0$ **e** $4x^2 - 3x + 3 = 0$ **f** $8x^2 - 10x - 3 = 0$

Example 24

For $x^2 - 2x + m = 0$, find Δ and hence find the values of m for which the equation has: **a** a repeated root **b** 2 distinct real roots **c** no real roots.

$x^2 - 2x + m = 0$ has $a = 1$, $b = -2$ and $c = m$

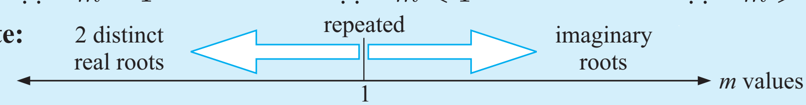
$\therefore \Delta = b^2 - 4ac = (-2)^2 - 4(1)(m) = 4 - 4m$

a For a repeated root
$$\begin{aligned}\Delta &= 0\\ \therefore 4 - 4m &= 0\\ \therefore 4 &= 4m\\ \therefore m &= 1\end{aligned}$$

b For 2 distinct real roots
$$\begin{aligned}\Delta &> 0\\ \therefore 4 - 4m &> 0\\ \therefore -4m &> -4\\ \therefore m &< 1\end{aligned}$$

c For no real roots
$$\begin{aligned}\Delta &< 0\\ \therefore 4 - 4m &< 0\\ \therefore -4m &< -4\\ \therefore m &> 1\end{aligned}$$

Note:



3 For the following quadratic equations, determine the discriminant in simplest form and hence find the values of m for which the equation has:

i a repeated root **ii** two distinct roots **iii** no real roots.

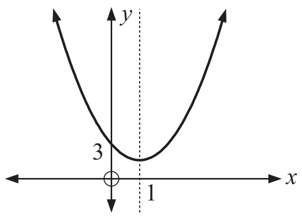
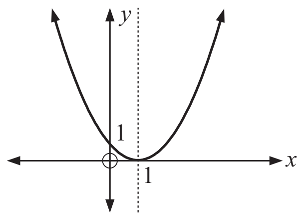
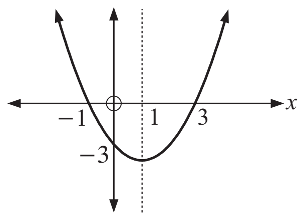
a $x^2+3x+m=0$ **b** $x^2-5x+m=0$ **c** $mx^2-x+1=0$

d $mx^2+2x+3=0$ **e** $2x^2+7x+m=0$ **f** $mx^2-5x+4=0$

THE DISCRIMINANT AND THE QUADRATIC GRAPH

Consider the graphs of: $y=x^2-2x+3$, $y=x^2-2x+1$, $y=x^2-2x-3$.

All of these curves have axis of symmetry with equation $x=1$.

$y=x^2-2x+3$	$y=x^2-2x+1$	$y=x^2-2x-3$
		
$\Delta = b^2-4ac$ $= (-2)^2-4(1)(3)$ $= -8$	$\Delta = b^2-4ac$ $= (-2)^2-4(1)(1)$ $= 0$	$\Delta = b^2-4ac$ $= (-2)^2-4(1)(-3)$ $= 16$
$\Delta<0$	$\Delta=0$	$\Delta>0$
does not cut the x-axis	touches the x-axis	cuts the x-axis twice

Thus, the **discriminant** Δ, helps us to decide between the possibilities of:

- not cutting the x-axis $(\Delta<0)$
- touching the x-axis $(\Delta=0)$
- cutting the x-axis twice $(\Delta>0)$.

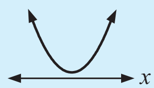
Example 25

Use the discriminant to determine the relationship between the graph and the x-axis of: **a** $y=x^2+3x+4$ **b** $y=-2x^2+5x+1$

a $a=1,\quad b=3,\quad c=4$

$\therefore\ \Delta = b^2-4ac$
$= 9-4(1)(4)$
$= -7$ which is <0

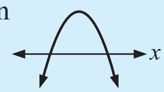
$a>0$ $\therefore$ concave up



$\therefore$ graph does not cut the x-axis, i.e., lies entirely above the x-axis.

b $a=-2,\quad b=5,\quad c=1$

$a<0$ $\therefore$ concave down



$\therefore\ \Delta = b^2-4ac$
$= 25-4(-2)(1)$
$= 33$ which is >0

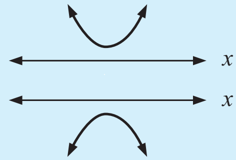
$\therefore$ graph cuts the x-axis twice.

POSITIVE DEFINITE AND NEGATIVE DEFINITE QUADRATICS

Definitions:

Positive definite quadratics are quadratics which are definitely positive for all values of x, i.e., $ax^2 + bx + c > 0$ for all x.

Negative definite quadratics are quadratics which are definitely negative for all values of x, i.e., $ax^2 + bx + c < 0$ for all x.



TESTS

- A quadratic is **positive definite** if $a > 0$ and $\Delta < 0$, i.e., $y > 0$ for all $x \in \mathcal{R}$.
- A quadratic is **negative definite** if $a < 0$ and $\Delta < 0$, i.e., $y < 0$ for all $x \in \mathcal{R}$.

EXERCISE 7I.2

1 Use the discriminant to determine the relationship between the graph and x-axis for:

a $y = x^2 + 7x - 2$ **b** $y = x^2 + 4\sqrt{2}x + 8$ **c** $y = -2x^2 + 3x + 1$

d $y = 6x^2 + 5x - 4$ **e** $y = -x^2 + x + 6$ **f** $y = 9x^2 + 6x + 1$

2 Show that:

a $x^2 - 3x + 6 > 0$ for all x **b** $4x - x^2 - 6 < 0$ for all x

c $2x^2 - 4x + 7$ is positive definite **d** $-2x^2 + 3x - 4$ is negative definite

3 Explain why $3x^2 + kx - 1$ is never always positive for any value of k.

4 Under what conditions is $2x^2 + kx + 2$ positive definite?

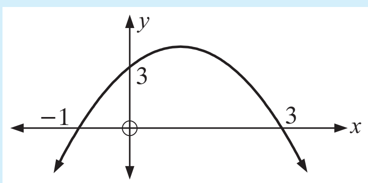
J DETERMINING THE QUADRATIC FROM A GRAPH

If we are given sufficient information on or about a graph we can determine the quadratic function in whatever form is required.

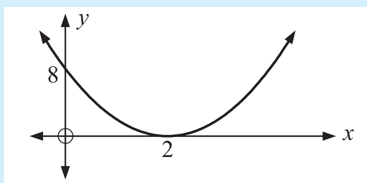
Example 26

Find the equation of the quadratic with graph:

a



b



a Since the x-intercepts are -1 and 3 then $y = a(x+1)(x-3)$, $a < 0$.

But when $x = 0$, $y = 3$

$\therefore$ $3 = a(1)(-3)$

$\therefore$ $a = -1$

So, $y = -(x+1)(x-3)$.

b Since it touches at 2, then $y = a(x-2)^2$, $a > 0$.

But when $x = 0$, $y = 8$

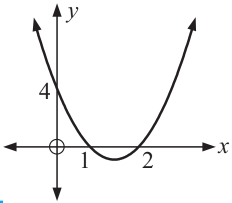
$\therefore$ $8 = a(-2)^2$

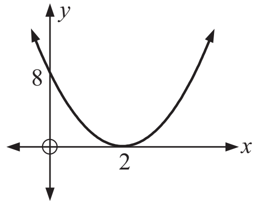
$\therefore$ $a = 2$

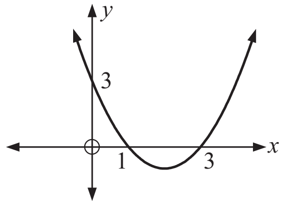
So, $y = 2(x-2)^2$.

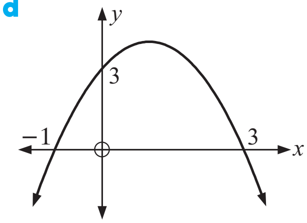
EXERCISE 7J

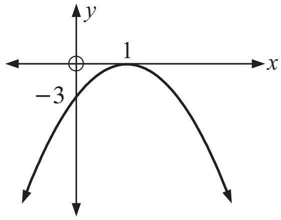
1 Find the equation of the quadratic with graph:

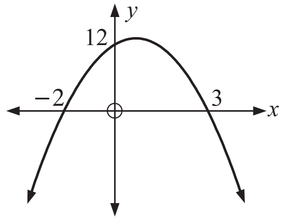
a


b


c


d


e


f


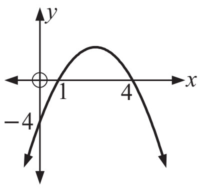
2 Match the given graphs to the possible formulae stated:

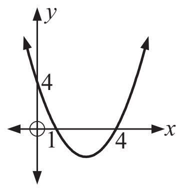
a $y = 2(x-1)(x-4)$ **b** $y = -(x+1)(x-4)$

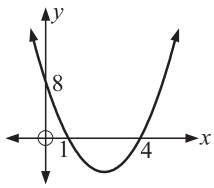
c $y = (x-1)(x-4)$ **d** $y = (x+1)(x-4)$

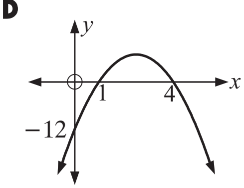
e $y = 2(x+4)(x-1)$ **f** $y = -3(x+4)(x-1)$

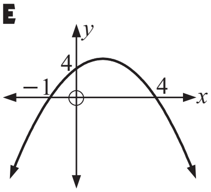
g $y = -(x-1)(x-4)$ **h** $y = -3(x-1)(x-4)$

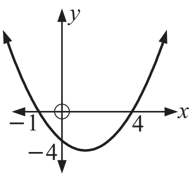
A


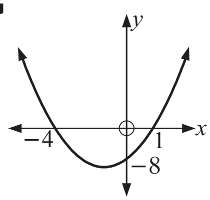
B


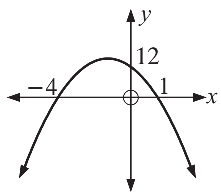
C


D


E


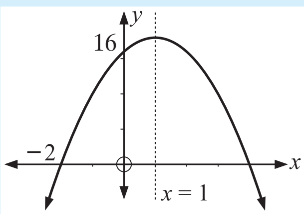
F


G


H


Example 27

Find the equation of the quadratic with graph:



As the axis of symmetry is $x = 1$,
the other x-intercept is 4

$\therefore \quad y = a(x+2)(x-4)$

But when $x = 0$, $y = 16$

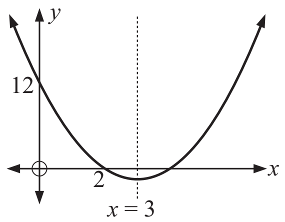
$\therefore \quad 16 = a(2)(-4)$

$\therefore \quad a = -2$

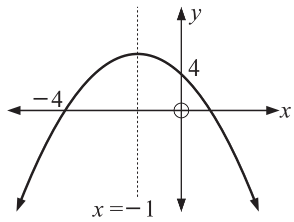
$\therefore$ quadratic is $y = -2(x+2)(x-4)$

3 Find the quadratic with graph:

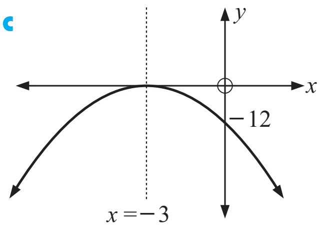
a



b



c



Example 28

Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph cuts the x-axis at 4 and -3 and passes through the point $(2, -20)$.

Since the x-intercepts are 4 and -3, the equation is

$$y = a(x-4)(x+3) \quad \text{where} \quad a \neq 0.$$

But when $x = 2$, $y = -20$ $\quad \therefore \quad -20 = a(2-4)(2+3)$

$$\therefore \quad -20 = a(-2)(5)$$

$$\therefore \quad a = 2$$

$\therefore$ equation is $y = 2(x-4)(x+3)$ i.e., $y = 2x^2 - 2x - 24$

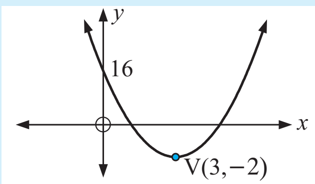
4 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

- **a** cuts the x-axis at 5 and 1, and passes through $(2, -9)$
- **b** cuts the x-axis at 2 and $-\frac{1}{2}$, and passes through $(3, -14)$
- **c** touches the x-axis at 3 and passes through $(-2, -25)$
- **d** touches the x-axis at -2 and passes through $(-1, 4)$
- **e** cuts the x-axis at 3, passes through $(5, 12)$ and has axis of symmetry $x = 2$
- **f** cuts the x-axis at 5, passes through $(2, 5)$ and has axis of symmetry $x = 1$.

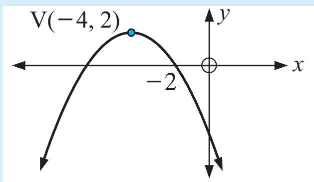
Example 29

Find the equation of the quadratic given its graph is:

a



b



a For vertex $(3, -2)$
the quadratic has form
$y = a(x-3)^2 - 2$

But when $x = 0$, $y = 16$

$$\therefore \quad 16 = a(-3)^2 - 2$$
$$\therefore \quad 16 = 9a - 2$$
$$\therefore \quad 9a = 18$$
$$\therefore \quad a = 2$$

i.e., $y = 2(x-3)^2 - 2$

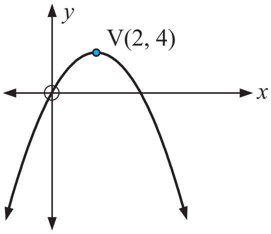
b For vertex $(-4, 2)$
the quadratic has form
$y = a(x+4)^2 + 2$

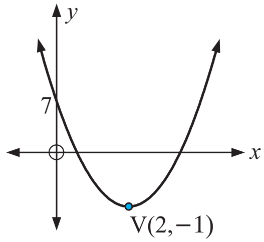
But when $x = -2$, $y = 0$

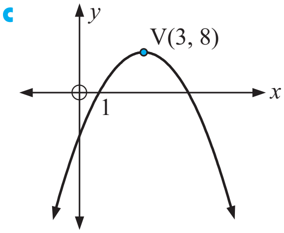
$$\therefore \quad 0 = a(2)^2 + 2$$
$$\therefore \quad 4a = -2$$
$$\therefore \quad a = -\tfrac{1}{2}$$

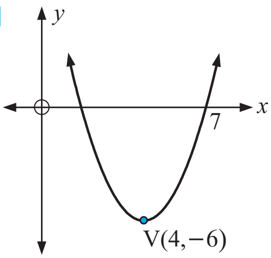
i.e., $y = -\frac{1}{2}(x+4)^2 + 2$

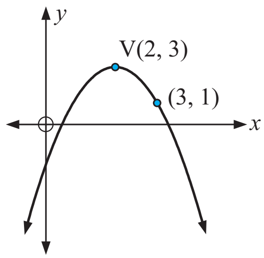
5 Find the equation of the quadratic given its graph is: (V is the vertex.)

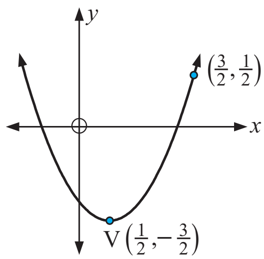
a


b


c


d


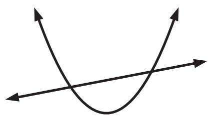
e


f


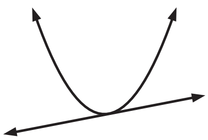
K WHERE FUNCTIONS MEET

Consider the graphs of a quadratic function and a linear function on the same set of axes.

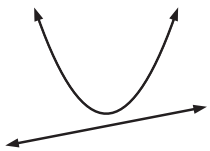
Notice that we could have:



cutting
(2 points of intersection)



touching
(1 point of intersection)



missing
(no points of intersection)

The graphs could meet and the coordinates of the points of intersection of the graphs of the two functions can be found by *solving the two equations simultaneously*.

Example 30

Find the coordinates of the points of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$

$$\therefore \quad x^2 - 2x - 15 = 0 \qquad \{\text{RHS} = 0\}$$

$$\therefore \quad (x-5)(x+3) = 0 \qquad \{\text{factorising}\}$$

$$\therefore \quad x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

$\therefore$ graphs meet at (5, 2) and (−3, −6).

EXERCISE 7K

1 Find the coordinates of the point(s) of intersection of the graphs with equations:

a $y = x^2 - 2x + 8$ and $y = x + 6$
b $y = -x^2 + 3x + 9$ and $y = 2x - 3$
c $y = x^2 - 4x + 3$ and $y = 2x - 6$
d $y = -x^2 + 4x - 7$ and $y = 5x - 4$

2 Use a **graphing package** or a **graphics calculator** to find the coordinates of the points of intersection (to two decimal places) of the graphs with equations:

a $y = x^2 - 3x + 7$ and $y = x + 5$
b $y = x^2 - 5x + 2$ and $y = x - 7$
c $y = -x^2 - 2x + 4$ and $y = x + 8$
d $y = -x^2 + 4x - 2$ and $y = 5x - 6$

GRAPHING PACKAGE TI C

3 Find, by algebraic means, the points of intersection of the graphs with equations:

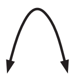
a $y = x^2$ and $y = x + 2$
b $y = x^2 + 2x - 3$ and $y = x - 1$
c $y = 2x^2 - x + 3$ and $y = 2 + x + x^2$
d $xy = 4$ and $y = x + 3$

4 Use technology to check your solutions to the questions in **3**.

GRAPHING PACKAGE

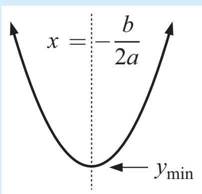
L QUADRATIC MODELLING

There are many situations in the real world where the relationship between two variables is a quadratic function.

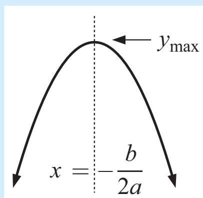
This means that the graph of such a relationship will be either  or  and the function will have a minimum or maximum value.

For $y = ax^2 + bx + c$:

- if $a > 0$, the **minimum** value of y occurs at $x = -\dfrac{b}{2a}$



- if $a < 0$, the **maximum** value of y occurs at $x = -\dfrac{b}{2a}$.



Note: we are optimising y, not x. $x = -\dfrac{b}{2a}$ helps us calculate this y-value.

The process of finding the maximum or minimum value of a function is called **optimisation**.

Optimisation is a very useful tool when looking at such issues as:

- maximising profits
- minimising costs
- maximising heights reached etc.

Example 31

The height H metres, of a rocket t seconds after it is fired vertically upwards is given by $H(t) = 80t - 5t^2, \quad t \geqslant 0$.

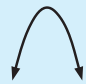
a How long does it take for the rocket to reach its maximum height?

b What is the maximum height reached by the rocket?

c How long does it take for the rocket to fall back to earth?

a $$H(t) = 80t - 5t^2$$

$$\therefore \quad H(t) = -5t^2 + 80t \quad \text{where} \quad a = -5 \quad \therefore$$

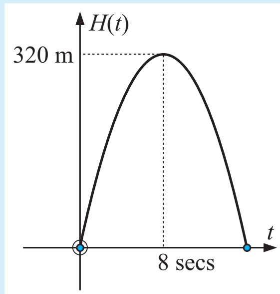


The maximum height reached occurs when $t = \dfrac{-b}{2a} = \dfrac{-80}{2(-5)} = 8$

i.e., the maximum height is reached after 8 seconds.

b $$H(8) = 80 \times 8 - 5 \times 8^2$$
$$= 640 - 320$$
$$= 320$$

i.e., the maximum height reached is 320 m.



c The rocket falls back to earth when $H(t) = 0$

$$\therefore \quad 0 = 80t - 5t^2$$
$$\therefore \quad 5t^2 - 80t = 0$$
$$\therefore \quad 5t(t - 16) = 0 \qquad \{\text{factorising}\}$$
$$\therefore \quad t = 0 \text{ or } t = 16$$

i.e., the rocket falls back to earth after 16 seconds.

EXERCISE 7L

1 The height H metres, of a ball hit vertically upwards, t seconds after it is hit is given by $H(t) = 36t - 2t^2$.

a How long does it take for the ball to reach its maximum height?

b What is the maximum height of the ball?

c How long does it take for the ball to hit the ground?

2 A skateboard manufacturer finds that the cost $\$C$ of making x skateboards per day is given by $C(x) = x^2 - 24x + 244$.

a How many skateboards should be made per day to minimise the cost of production?

b What is the minimum cost?

c What is the cost if no skateboards are made in a day?

3 The driver of a car travelling downhill on a road applied the brakes. The velocity (v) of the car in m/s, t seconds after the brakes were applied was given by $v(t) = -\frac{1}{2}t^2 + \frac{1}{2}t + 15$ m/s.

a How fast was the car travelling when the driver applied the brakes?

b After how many seconds did the car reach its maximum velocity? Explain why this may have happened.

c What was the maximum velocity reached?

d How long does it take for the car to stop?

4 The hourly profit ($\$P$) obtained from operating a fleet of n taxis is given by $P(n) = 84n - 45 - 2n^2$.

a What number of taxis gives the maximum hourly profit?

b What is the maximum hourly profit?

c How much money is lost per hour if no taxis are on the road?

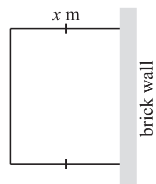
5 The temperature T^o Celsius in a greenhouse t hours after dusk (7.00 pm) is given by $T(t) = \frac{1}{4}t^2 - 5t + 30, \quad (t \leqslant 20)$.

a What was the temperature in the greenhouse at dusk?

b At what time was the temperature at a minimum?

c What was the minimum temperature?

6 A vegetable gardener has 40 m of fencing to enclose a rectangular garden plot where one side is an existing brick wall. If the width is x m as shown:

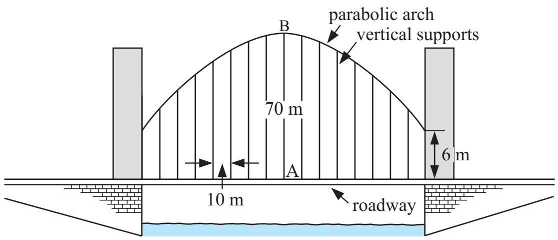


a Show that the area (A) enclosed is given by $A = -2x^2 + 40x$ m^2.

b Find x such that the vegetable garden has maximum area.

c What is the maximum area?

7 Consider the following diagram of a bridge:



AB is the longest vertical support of a bridge which contains a parabolic arch. The vertical supports are 10 m apart. The arch meets the vertical end supports 6 m above the road.

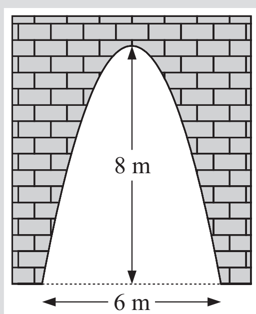
a If axes are drawn on the diagram of the bridge above, with x-axis the road and y-axis on AB, find the equation of the parabolic arch in the form $y = ax^2 + c$.

b Hence, determine the lengths of all other vertical supports.

GRAPHICS CALCULATOR INVESTIGATION TUNNELS AND TRUCKS

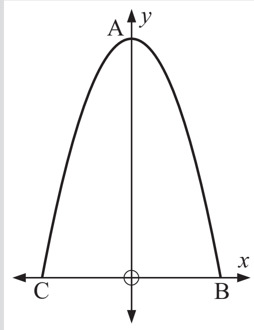
A tunnel is parabolic in shape with dimensions shown:

A truck carrying a wide load is 4.8 m high and 3.9 m wide and needs to pass through the tunnel. Your task is to determine if the truck will fit through the tunnel.



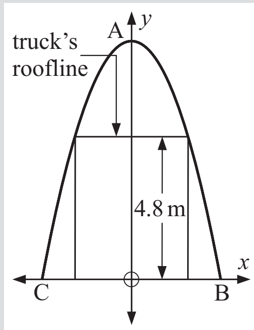
What to do:

1. If a set of axes is fitted to the parabolic tunnel as shown, state the coordinates of points A, B and C.



2. Using a **graphics calculator**:
   a. enter the x-coordinates of A, B and C into **List 1**
   b. enter the y-coordinates of A, B and C into **List 2**.
3. Draw a **scatterplot** of points A, B and C.
4. Set your calculator to display 4 decimal places and determine the equation of the parabolic boundary of the tunnel in the form $y = ax^2 + bx + c$, by fitting a **quadratic model** to the data.
5. Place the end view of the truck on the same set of axes as above.

   What is the equation of the truck's roofline?



6. You should have found that the equation of the parabolic boundary of the tunnel is $y = -0.8889x^2 + 8$ and the equation of the truck's roofline is $y = 4.8$.

   Graph these equations on the same set of axes. Calculate the **points of intersection** of the graphs of these functions.
7. Using the points of intersection found in **6**, will the truck pass through the tunnel? What is the maximum width of a truck that is 4.8 m high if it is to pass through the tunnel?
8. Investigate the maximum width of a truck that is 3.7 m high if it is to pass through the tunnel.
9. What is the maximum width of a 4.1 m high truck if it is to pass through a parabolic tunnel 6.5 m high and 5 m wide?

REVIEW SET 7A

1. For $y = -2(x+2)(x-1)$:
   a. state the x-intercepts
   b. state the equation of the axis of symmetry
   c. find the coordinates of the vertex
   d. find the y-intercept
   e. sketch the graph of the function
   f. use technology to check your answers.

2. For $y = \frac{1}{2}(x-2)^2 - 4$:
   a. state the equation of the axis of symmetry
   b. find the coordinates of the vertex
   c. find the y-intercept
   d. sketch the graph of the function
   e. use technology to check your answers.

3 For $y = x^2 - 4x - 1$:

a convert into the form $y = (x-h)^2 + k$ by 'completing the square'

b state the coordinates of the vertex

c find the y-intercept.

d Hence sketch the graph.

e Use technology to check your answer.

4 For $y = 2x^2 + 6x - 3$:

a convert into the form $y = (x-h)^2 + k$ by 'completing the square'

b state the coordinates of the vertex

c find the y-intercept.

d Hence sketch the graph.

e Use technology to check your answer.

5 Solve the following equations, giving exact answers:

a $x^2 - 11x = 60$ **b** $3x^2 - x - 10 = 0$ **c** $3x^2 - 12x = 0$

6 Solve the following equations:

a $x^2 + 10 = 7x$ **b** $x + \dfrac{12}{x} = 7$ **c** $2x^2 - 7x + 3 = 0$

7 Solve the following equation by completing the square: $x^2 + 7x - 4 = 0$

8 Solve the following equation by completing the square: $x^2 + 4x + 1 = 0$

9 Solve the following using the quadratic formula:

a $x^2 - 7x + 3 = 0$ **b** $2x^2 - 5x + 4 = 0$

REVIEW SET 7B

1 Draw the graph of $y = -x^2 + 2x$.

2 Find the equation of the axis of symmetry and the vertex of $y = -3x^2 + 8x + 7$.

3 Find the equation of the axis of symmetry and the vertex of $y = 2x^2 + 4x - 3$.

4 Use the discriminant only to determine the number of solutions to:

a $3x^2 - 5x + 7 = 0$ **b** $-2x^2 - 4x + 3 = 0$

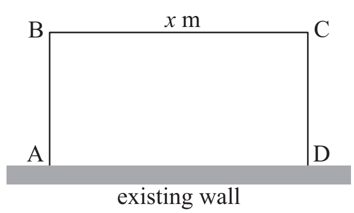
5 Show that $5 + 7x + 3x^2$ is positive definite.

6 Find the maximum or minimum value of the relation $y = -2x^2 + 4x + 3$ and the value of x for which the maximum or minimum occurs.

7 Find the points of intersection of $y = x^2 - 3x$ and $y = 3x^2 - 5x - 24$.

8 For what values of k does the graph of $y = -2x^2 + 5x + k$ not cut the x-axis?

9 60 m of chicken wire is available for constructing a chicken enclosure against an existing wall. The enclosure is to be rectangular.



a If BC $= x$ m, show that the area of rectangle ABCD is given by $A = (30x - \frac{1}{2}x^2)$ m^2.

b Find the dimensions of the enclosure which will maximise the area enclosed.

REVIEW SET 7C, 7D, 7E

Click on the icon to obtain printable review sets and answers

REVIEW SET 7C REVIEW SET 7D REVIEW SET 7E

EXERCISE 7A

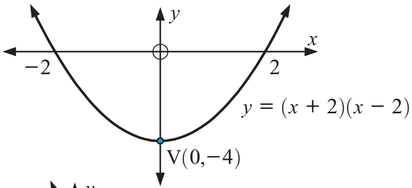
1 **a**, **c**, **d**, **e** **2** **a** $y=20$ **b** $y=27$ **c** $y=-4$ **d** $y=37$

3 **a** 3 **b** -5 **c** -4 **d** 8

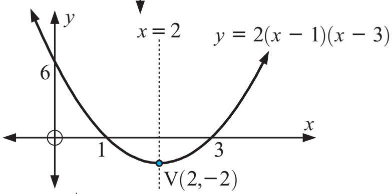
4 **a** no **b** no **c** no **d** no

EXERCISE 7B.1

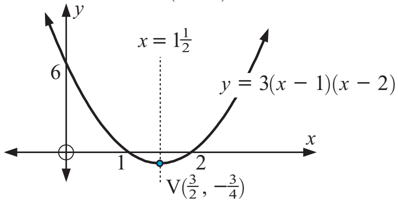
1 **a** **i** ± 2 **ii** $x=0$ **iii** $(0,\,-4)$ **iv** -4



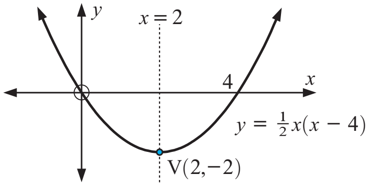
b **i** 1, 3 **ii** $x=2$ **iii** $(2,\,-2)$ **iv** 6



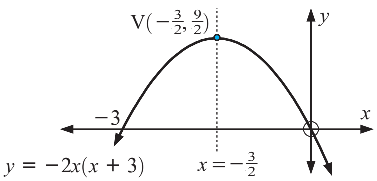
c **i** 1, 2 **ii** $x=\frac{3}{2}$ **iii** $(\frac{3}{2},\,-\frac{3}{4})$ **iv** 6



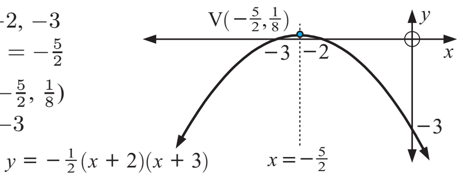
d **i** 0, 4 **ii** $x=2$ **iii** $(2,\,-2)$ **iv** 0



e **i** 0, -3 **ii** $x=-\frac{3}{2}$ **iii** $(-\frac{3}{2},\,\frac{9}{2})$ **iv** 0



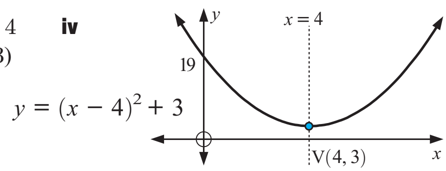
f **i** -2, -3 **ii** $x=-\frac{5}{2}$ **iii** $(-\frac{5}{2},\,\frac{1}{8})$ **iv** -3



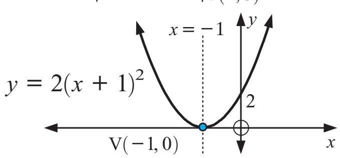
2 **a** B **b** A **c** F **d** D **e** E **f** C

EXERCISE 7B.2

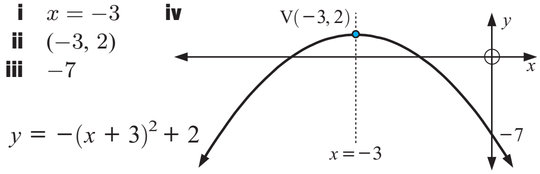
1 **a** **i** $x=4$ **ii** $(4,\,3)$ **iii** 19 **iv**



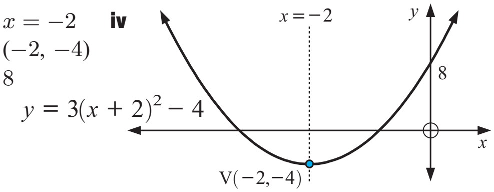
b **i** $x=-1$ **ii** $(-1,\,0)$ **iii** 2 **iv**



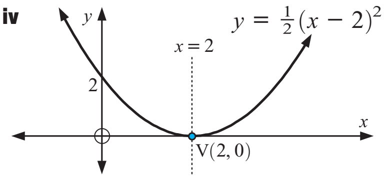
c **i** $x=-3$ **ii** $(-3,\,2)$ **iii** -7 **iv**



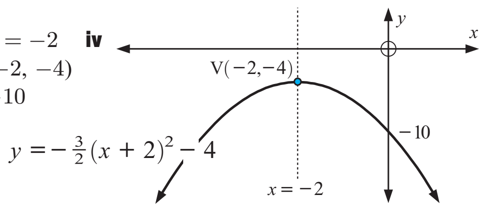
d **i** $x=-2$ **ii** $(-2,\,-4)$ **iii** 8 **iv**



e **i** $x=2$ **ii** $(2,\,0)$ **iii** 2 **iv**



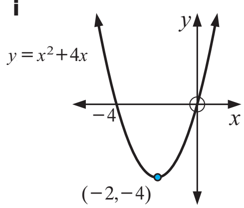
f **i** $x=-2$ **ii** $(-2,\,-4)$ **iii** -10 **iv**



2 **a** G **b** A **c** E **d** B **e** I **f** C **g** D **h** F **i** H

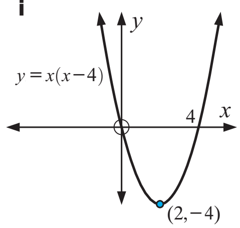
3 **a** $x=2$ **b** $x=-\frac{5}{2}$ **c** $x=1$ **d** $x=3$ **e** $x=-4$ **f** $x=-4$

4 **a** **i**



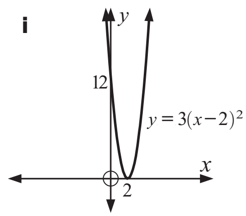
ii $x=-2$ **iii** $(-2,\,-4)$

b **i**



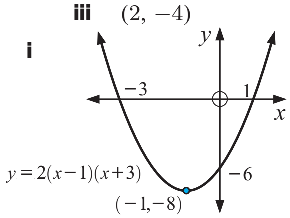
ii $x=2$ **iii** $(2,\,-4)$

c **i**



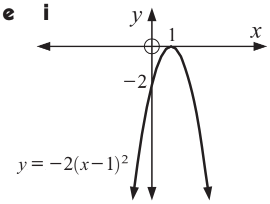
ii $x=2$ **iii** $(2,\,0)$

d **i**



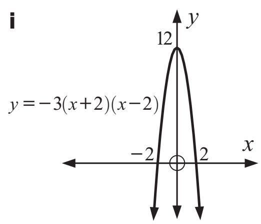
ii $x=-1$ **iii** $(-1,\,-8)$

e **i**



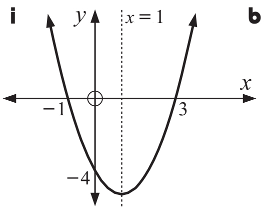
ii $x=1$ **iii** $(1,\,0)$

f **i**



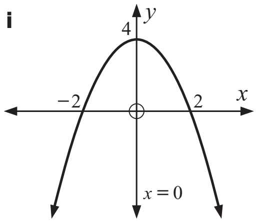
ii $x=0$ **iii** $(0,\,12)$

5 **a** **i**



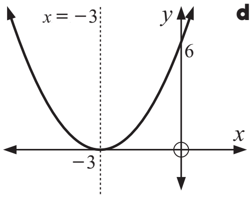
ii axis of sym. $x = 1$

b **i**



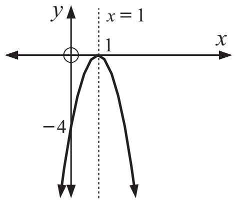
ii axis of sym. $x = 0$

c **i**



ii axis of sym. $x = -3$

d **i**

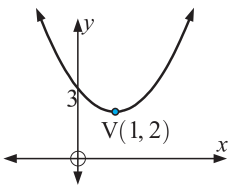


ii axis of sym. $x = 1$

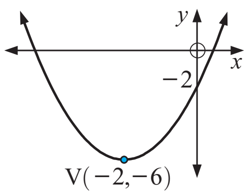
6 **a** -1 and -5 **b** 3 (touching)

EXERCISE 7C

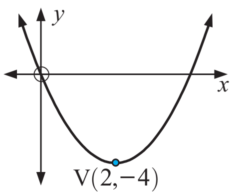
1 **a** $y = (x-1)^2 + 2$



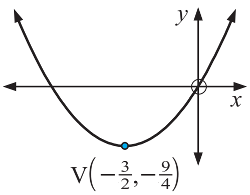
b $y = (x+2)^2 - 6$



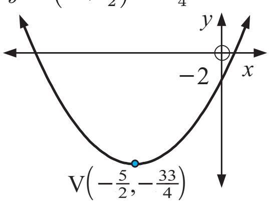
c $y = (x-2)^2 - 4$



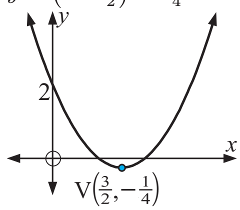
d $y = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$



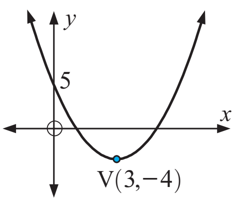
e $y = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$



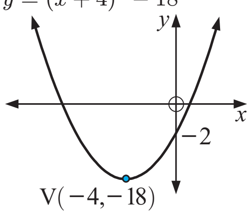
f $y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$



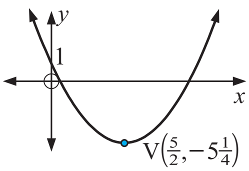
g $y = (x-3)^2 - 4$



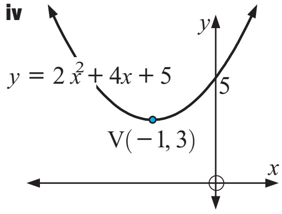
h $y = (x+4)^2 - 18$



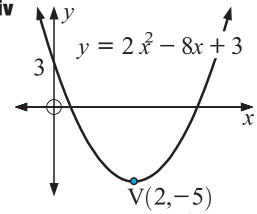
i $y = \left(x - \frac{5}{2}\right)^2 - 5\frac{1}{4}$



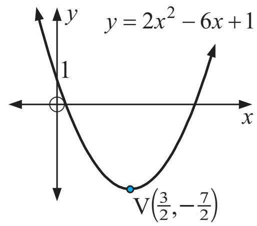
2 **a** **i** $y = 2(x+1)^2 + 3$ **ii** $(-1, 3)$ **iii** 5 **iv**



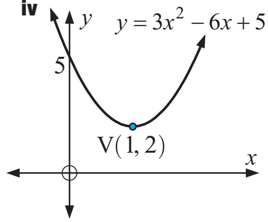
b **i** $y = 2(x-2)^2 - 5$ **ii** $(2, -5)$ **iii** 3 **iv**



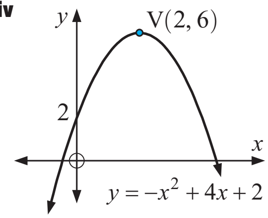
c **i** $y = 2(x - \frac{3}{2})^2 - \frac{7}{2}$ **ii** $(\frac{3}{2}, -\frac{7}{2})$ **iii** 1 **iv**



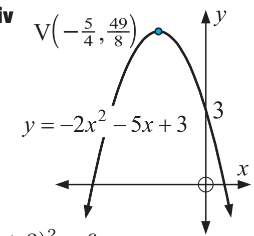
d **i** $y = 3(x-1)^2 + 2$ **ii** $(1, 2)$ **iii** 5 **iv**



e **i** $y = -(x-2)^2 + 6$ **ii** $(2, 6)$ **iii** 2 **iv**



f **i** $y = -2(x + \frac{5}{4})^2 + \frac{49}{8}$ **ii** $(-\frac{5}{4}, \frac{49}{8})$ **iii** 3 **iv**



3 **a** $y = (x-2)^2 + 3$ **b** $y = (x+3)^2 - 6$
c $y = -(x-2)^2 + 9$ **d** $y = 2\left(x + \frac{3}{2}\right)^2 - \frac{17}{2}$
e $y = -2\left(x + \frac{5}{2}\right)^2 + \frac{27}{2}$ **f** $y = 3\left(x - \frac{3}{2}\right)^2 - \frac{47}{4}$

EXERCISE 7D.1

1 **a** $x = 0, -\frac{7}{4}$ **b** $x = 0, -\frac{1}{3}$ **c** $x = 0, \frac{7}{3}$ **d** $x = 0, \frac{11}{2}$
e $x = 0, \frac{8}{3}$ **f** $x = 0, \frac{3}{2}$ **g** $x = 3, 2$ **h** $x = 4, -2$
i $x = 3, 7$ **j** $x = 3$ **k** $x = -4, 3$ **l** $x = -11, 3$

2 **a** $x = \frac{2}{3}$ **b** $x = -\frac{1}{2}, 7$ **c** $x = -\frac{2}{3}, 6$ **d** $x = \frac{1}{3}, -2$
e $x = \frac{3}{2}, 1$ **f** $x = -\frac{2}{3}, 2$ **g** $x = -\frac{2}{3}, 4$ **h** $x = \frac{1}{2}, -\frac{3}{2}$
i $x = -\frac{1}{4}, 3$ **j** $x = -\frac{3}{4}, \frac{5}{3}$ **k** $x = \frac{1}{7}, -1$ **l** $x = -2, \frac{28}{15}$

3 **a** $x = 2, 5$ **b** $x = -3, 2$ **c** $x = 0, -\frac{3}{2}$ **d** $x = 1, 2$
e $x = \frac{1}{2}, -1$ **f** $x = 3$

4 **a** $x = -3$ **b** $x = -3$ or -2 **c** $x = 1$ or 4 **d** no solution

5 **a** $x = 0$ or $\frac{2}{3}$ **b** $x = 3$ or -2 **c** $x = \frac{1}{2}$ or -7 **d** $x = 3$

6 **a** **i** 25 m **ii** 25 m **iii** 45 m
b **i** 2 secs and 4 secs **ii** 0 secs and 6 secs
c once going up and once coming down

7 **a** **i** −$30 **ii** $105 **b** 6 or 58 cakes

EXERCISE 7D.2

1 **a** $x = -5 \pm \sqrt{2}$ **b** $x = -6 \pm \sqrt{11}$ **c** $x = 4 \pm 2\sqrt{2}$
d $x = 8 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{5}$ **f** $x = 2 \pm \sqrt{6}$
g $x = -1 \pm \sqrt{10}$ **h** $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$

2 **a** $x = 2 \pm \sqrt{3}$ **b** $x = -3 \pm \sqrt{7}$ **c** $x = 7 \pm \sqrt{3}$

d $x=2\pm\sqrt{7}$ **e** $x=-3\pm\sqrt{2}$ **f** $x=1\pm\sqrt{7}$
g $x=-3\pm\sqrt{11}$ **h** $x=4\pm\sqrt{6}$ **i** no real solns.

3 **a** $x=-1\pm\frac{1}{\sqrt{2}}$ **b** $x=\frac{5}{2}\pm\frac{\sqrt{19}}{2}$ **c** $x=-2\pm\sqrt{\frac{7}{3}}$
d $x=1\pm\sqrt{\frac{7}{3}}$ **e** $x=\frac{3}{2}\pm\sqrt{\frac{37}{20}}$ **f** $x=-\frac{1}{2}\pm\frac{\sqrt{6}}{2}$

EXERCISE 7E

1 **a** $x=2\pm\sqrt{7}$ **b** $x=-3\pm\sqrt{2}$ **c** $x=2\pm\sqrt{3}$
d $x=-2\pm\sqrt{5}$ **e** $x=2\pm\sqrt{2}$ **f** $x=\frac{1}{2}\pm\frac{1}{2}\sqrt{7}$
g $x=\sqrt{2}$ **h** $x=-\frac{4}{9}\pm\frac{\sqrt{7}}{9}$ **i** $x=-\frac{7}{4}\pm\frac{\sqrt{97}}{4}$

2 **a** $x=-2\pm2\sqrt{2}$ **b** $x=-\frac{5}{8}\pm\frac{\sqrt{57}}{8}$ **c** $x=\frac{5}{2}\pm\frac{\sqrt{13}}{2}$
d $x=\frac{1}{2}\pm\frac{1}{2}\sqrt{7}$ **e** $x=\frac{1}{2}\pm\frac{\sqrt{5}}{2}$ **f** $x=\frac{3}{4}\pm\frac{\sqrt{17}}{4}$

EXERCISE 7F

1 **a** $x=-3.414$ or -0.586 **b** $x=0.317$ or -6.317
c $x=2.77$ or -1.27 **d** $x=-1.08$ or 3.41
e $x=-0.892$ or 3.64 **f** $x=1.34$ or -2.54

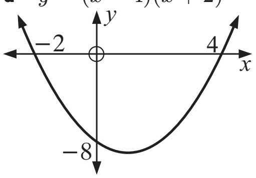
2 **a** $x=-4.83$ or 0.828 **b** $x=-1.57$ or 0.319
c $x=0.697$ or 4.30 **d** $x=-0.823$ or 1.82
e $x=-0.618$ or 1.62 **f** $x=-0.281$ or 1.78

EXERCISE 7G

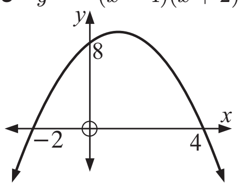
1 7 and -5 or -7 and 5 **2** 5 or $\frac{1}{5}$ **3** 14
4 18 and 20 or -18 and -20 **5** 15 and 17 or -15 and -17
6 15 sides **7** 3.48 cm **8** **b** 6 cm by 6 cm by 7 cm
9 11.2 cm square **10** no **12** 221 ha **13** 2.03 m
14 52.1 km/h **15** 553 km/h **16** 51.1 km/h **17** 32

EXERCISE 7H

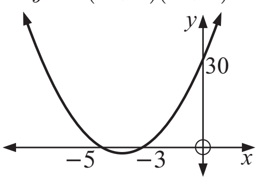
1 **a** $y=(x-4)(x+2)$



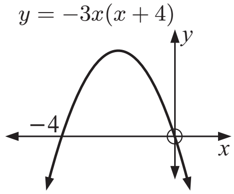
b $y=-(x-4)(x+2)$



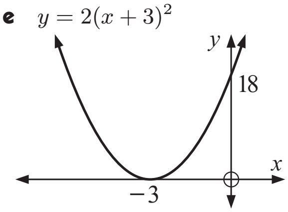
c $y=2(x+3)(x+5)$



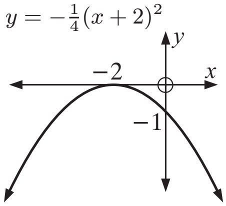
d $y=-3x(x+4)$



e $y=2(x+3)^2$

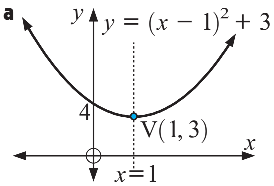


f $y=-\frac{1}{4}(x+2)^2$

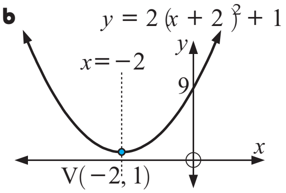


2 **a** $x=1$ **b** $x=1$ **c** $x=-4$ **d** $x=-2$ **e** $x=-3$ **f** $x=-2$

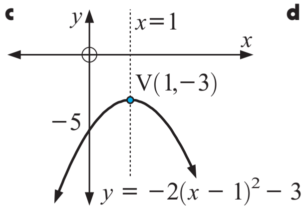
3 **a**



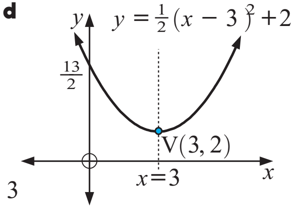
b



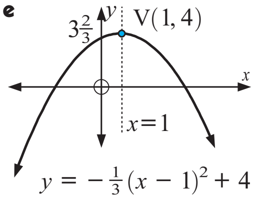
c



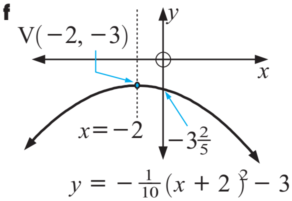
d



e



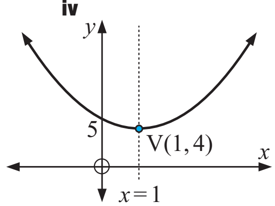
f



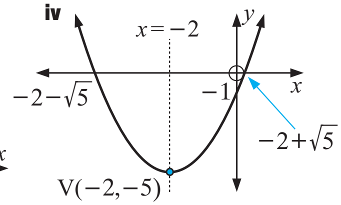
4 **a** $(2,-2)$ **b** $(-1,-4)$ **c** $(0,4)$ **d** $(0,1)$ **e** $(-2,-15)$
f $(-2,-5)$ **g** $(-\frac{3}{2},-\frac{11}{2})$ **h** $(\frac{5}{2},-\frac{19}{2})$ **i** $(1,-\frac{9}{2})$ **j** $(2,6)$

5 **a** ±3 **b** $\pm\sqrt{3}$ **c** -5 and -2 **d** 3 and -4 **e** 0 and 4
f -4 and -2 **g** -1 (touching) **h** 3 (touching)
i $2\pm\sqrt{3}$ **j** $-2\pm\sqrt{7}$ **k** $3\pm\sqrt{11}$ **l** $-4\pm\sqrt{5}$

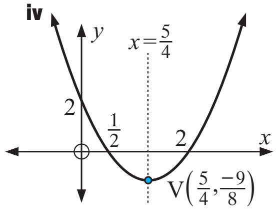
6 **a** **i** $x=1$
ii $(1,4)$
iii no x-intercept, y-intercept 5
iv



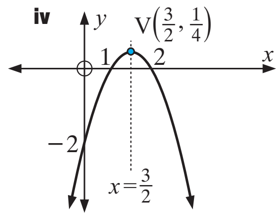
b **i** $x=-2$
ii $(-2,-5)$
iii x-int. $-2\pm\sqrt{5}$, y-intercept -1
iv



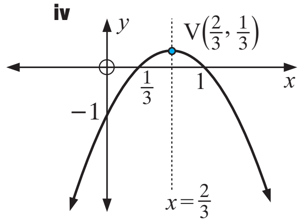
c **i** $x=\frac{5}{4}$
ii $(\frac{5}{4},-\frac{9}{8})$
iii x-intercepts $\frac{1}{2}$, 2, y-intercept 2
iv



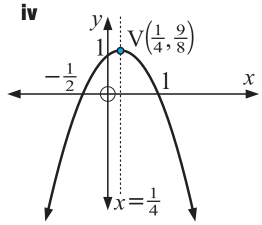
d **i** $x=\frac{3}{2}$
ii $(\frac{3}{2},\frac{1}{4})$
iii x-intercepts 1, 2, y-intercept -2
iv



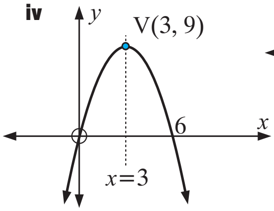
e **i** $x=\frac{2}{3}$
ii $(\frac{2}{3},\frac{1}{3})$
iii x-intercepts $\frac{1}{3}$, 1, y-intercept -1
iv



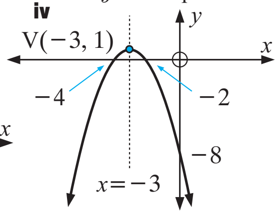
f **i** $x=\frac{1}{4}$
ii $(\frac{1}{4},\frac{9}{8})$
iii x-intercepts $-\frac{1}{2}$, 1, y-intercept 1
iv



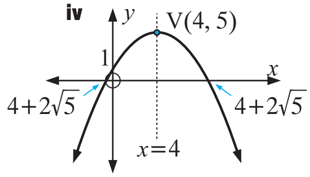
g **i** $x = 3$
ii $(3, 9)$
iii x-intercepts 0, 6, y-intercept 0
iv



h **i** $x = -3$
ii $(-3, 1)$
iii x-int. $-2, -4$, y-intercept -8
iv



i **i** $x = 4$
ii $(4, 5)$
iii x-int. $4 \pm 2\sqrt{5}$, y-intercept 1
iv



EXERCISE 7I.1

1 **a** 2 real distinct roots **b** a repeated root
c 2 real distinct roots **d** 2 real distinct roots
e no real roots **f** a repeated root

2 **a**, **b**, **d**, **f**

3 **a** $\Delta = 9 - 4m$ **i** $m = \frac{9}{4}$ **ii** $m < \frac{9}{4}$ **iii** $m > \frac{9}{4}$
b $\Delta = 25 - 4m$ **i** $m = \frac{25}{4}$ **ii** $m < \frac{25}{4}$ **iii** $m > \frac{25}{4}$
c $\Delta = 1 - 4m$ **i** $m = \frac{1}{4}$ **ii** $m < \frac{1}{4}$ **iii** $m > \frac{1}{4}$
d $\Delta = 4 - 12m$ **i** $m = \frac{1}{3}$ **ii** $m < \frac{1}{3}$ **iii** $m > \frac{1}{3}$
e $\Delta = 49 - 8m$ **i** $m = \frac{49}{8}$ **ii** $m < \frac{49}{8}$ **iii** $m > \frac{49}{8}$
f $\Delta = 25 - 16m$ **i** $m = \frac{25}{16}$ **ii** $m < \frac{25}{16}$ **iii** $m > \frac{25}{16}$

EXERCISE 7I.2

1 **a** cuts x-axis twice **b** touches x-axis **c** cuts x-axis twice
d cuts x-axis twice **e** cuts x-axis twice **f** touches x-axis

2 **a** $a = 1$ which is > 0 and $\Delta = -15$ which is < 0
b $a = -1$ which is < 0 and $\Delta = -8$ which is < 0
c $a = 2$ which is > 0 and $\Delta = -40$ which is < 0
d $a = -2$ which is < 0 and $\Delta = -23$ which is < 0

3 $a = 3$ which is > 0 and $\Delta = k^2 + 12$ which is always > 0 {as $k^2 > 0$ for all k}

4 $a = 2$ which is > 0 and $\Delta = k^2 - 16$ $\therefore$ positive definite when $k^2 < 16$ i.e., $-4 < k < 4$

EXERCISE 7J

1 **a** $y = 2(x-1)(x-2)$ **b** $y = 2(x-2)^2$
c $y = (x-1)(x-3)$ **d** $y = -(x-3)(x+1)$
e $y = -3(x-1)^2$ **f** $y = -2(x+2)(x-3)$

2 **a** C **b** E **c** B **d** F **e** G **f** H **g** A **h** D

3 **a** $y = \frac{3}{2}(x-2)(x-4)$ **b** $y = -\frac{1}{2}(x+4)(x-2)$
c $y = -\frac{4}{3}(x+3)^2$

4 **a** $y = 3x^2 - 18x + 15$ **b** $y = -4x^2 + 6x + 4$
c $y = -x^2 + 6x - 9$ **d** $y = 4x^2 + 16x + 16$
e $y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$ **f** $y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$

5 **a** $y = -(x-2)^2 + 4$ **b** $y = 2(x-2)^2 - 1$
c $y = -2(x-3)^2 + 8$ **d** $y = \frac{2}{3}(x-4)^2 - 6$
e $y = -2(x-2)^2 + 3$ **f** $y = 2(x - \frac{1}{2})^2 - \frac{3}{2}$

EXERCISE 7K

1 **a** $(1, 7)$ and $(2, 8)$ **b** $(4, 5)$ and $(-3, -9)$
c $(3, 0)$ (touching) **d** graphs do not meet

2 **a** $(0.59, 5.59)$ and $(3.41, 8.41)$ **b** $(3, -4)$ touching
c graphs do not meet **d** $(-2.56, -18.81)$ and $(1.56, 1.81)$

3 **a** $(2, 4), (-1, 1)$ **b** $(1, 0), (-2, -3)$ **c** $(1, 4)$
d $(1, 4), (-4, -1)$

EXERCISE 7L

1 **a** 9 seconds **b** 162 m **c** 18 sec

2 **a** 12 **b** \$100 **c** \$244

3 **a** 15 m/s **b** $\frac{1}{2}$ sec; since the car was travelling downhill, it was accelerating. $\therefore$ when the brake was applied, the speed of the vehicle still increased for a short time.
c $15\frac{1}{8}$ m/s **d** 6 seconds

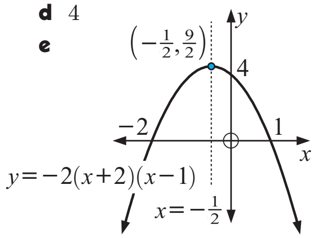
4 **a** 21 **b** \$837 **c** \$45 **5** **a** 30°C **b** 5.00 am **c** 5°C

6 **b** $x = 10$ **c** 200 m^2

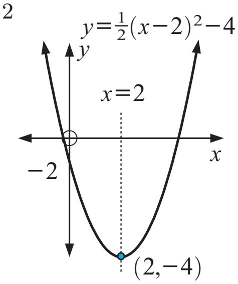
7 **a** $y = -\frac{1}{100}x^2 + 70$ **b** supports are 21 m, 34 m, 45 m, 54 m, 61 m, 66 m, 69 m

REVIEW SET 7A

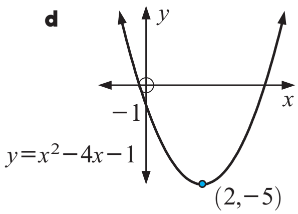
1 **a** $-2, 1$
b $x = -\frac{1}{2}$
c $(-\frac{1}{2}, \frac{9}{2})$
d 4
e



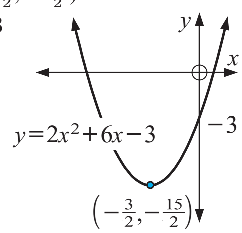
2 **a** $x = 2$
b $(2, -4)$
c -2
d



3 **a** $y = (x-2)^2 - 5$
b $(2, -5)$
c -1
d



4 **a** $y = 2\left(x + \frac{3}{2}\right)^2 - \frac{15}{2}$
b $(-\frac{3}{2}, -\frac{15}{2})$
c -3
d



5 **a** $x = 15$ or -4 **b** $x = -\frac{5}{3}$ or 2 **c** $x = 0$ or 4

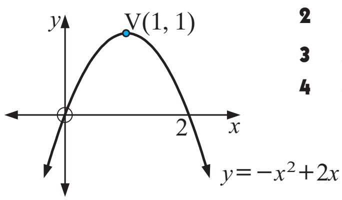
6 **a** $x = 5$ or 2 **b** $x = 3$ or 4 **c** $x = \frac{1}{2}$ or 3

7 $x = -\frac{7}{2} \pm \frac{\sqrt{65}}{2}$ **8** $x = -2 \pm \sqrt{3}$

9 **a** $x = \frac{7}{2} \pm \frac{\sqrt{37}}{2}$ **b** no real roots

REVIEW SET 7B

1



2 $x = \frac{4}{3}$, $V(\frac{4}{3}, 12\frac{1}{3})$

3 $x = -1$, $V(-1, -5)$

4 **a** no real solutions
b two real distinct solutions

5 $a = 3$ which is > 0 and $\Delta = -11$ which is < 0

6 $a = -2$ which is < 0 $\therefore$  a max.
max. $= 5$ when $x = 1$

7 $(4, 4)$ and $(-3, 18)$ **8** $k < -3\frac{1}{8}$ **9** **b** 15 m by 30 m