Minnesota State High School Mathematics League

2020-21 State Tournament, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have **15** minutes for this Tournament event.

NO CALCULATORS are allowed on this event.

$a+b=$ ______________

1. The origin and the x- and y-intercepts of a line with a slope of $-\frac{1}{4}$ determine a triangle in Quadrant I whose area is 20. The **sum** of the x- and y-intercepts can be written as $a\sqrt{b}$, where b is square-free. Determine the value of $a+b$.

A = ______________

2. Determine the value of the digit, A, if $\underline{A}\underline{A}\underline{A}_9 = \left(9^3-1\right)_{10}$.

\$ ______________

3. While in Oregon, a state with no sales tax, I purchased a pair of earphones and a charger and spent more than \$60. If the charger had cost \$6 more, the cost of the earphones would have been 25% of the total cost. If the earphones had cost \$6 less, the cost of the charger would have been 80% of the total cost. Determine exactly the price of the charger.

$m+n=$ ______________

4. $(m,\ n)$ is an ordered pair that satisfies this system of equations:

$$2|x-1| + 3|y+2| = 16$$
$$2x+5y=-24$$

If $m>0$, determine the value of $m+n$.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2020-21 State Tournament, Individual Event A

SOLUTIONS

15

1. The origin and the x- and y-intercepts of a line with a slope of $-\frac{1}{4}$ determine a triangle in Quadrant I whose area is 20. The **sum** of the x- and y-intercepts can be written as $a\sqrt{b}$, where b is square-free. Determine the value of $a+b$.

See Figure 1.1. $\frac{1}{2}(p)(4p)=20 \Rightarrow 2p^2=20 \Rightarrow p=\sqrt{10}$. $p+4p=5p=5\sqrt{10}$.

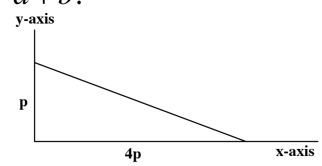

Figure 1.1

8

2. Determine the value of the digit, A, if $\underline{A}\underline{A}\underline{A}_9=\left(9^3-1\right)_{10}$.

$\underline{A}\underline{A}\underline{A}_9 = A\left(9^2\right)+A(9)+A = A\left(9^2+9+1\right)$. $9^3-1=(9-1)\left(9^2+9+1\right)$. *Therefore,* $A=9-1$.

48

3. While in Oregon, a state with no sales tax, I purchased a pair of earphones and a charger and spent more than \$60. If the charger had cost \$6 more, the cost of the earphones would have been 25% of the total cost. If the earphones had cost \$6 less, the cost of the charger would have been 80% of the total cost. Determine exactly the price of the charger.

$E=\frac{1}{4}(E+C+6) \Rightarrow 4E=E+C+6 \Rightarrow C-3E=-6$. $C=\frac{4}{5}(C+E-6) \Rightarrow$
$5C=4C+4E-24 \Rightarrow C-4E=-24$. *Therefore,* $4C-12E=-24$ *and* $-3C+12E=72$.
Adding these two equations, yields $C=48$.

-3

4. $(m,\ n)$ is an ordered pair that satisfies this system of equations:

$$2|\,x-1\,| \; + \; 3|\,y+2\,| \; = 16$$
$$2x+5y=-24$$

If $m>0$, determine the value of $m+n$.

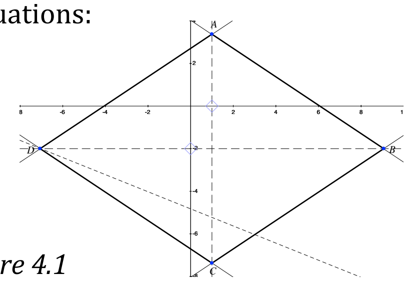

Figure 4.1

Case 1: $x \ge 1$ *and* $y \ge -2$ *: a line with a slope of* $-\frac{2}{3}$ *and an x-intercept of* 6. *The solution for this case is* $\overline{AB}$ *in Figure 4.1. Case 2:* $x \ge 1$ *and* $y < -2: 2(x-1)+3(-y-2)=16 \Rightarrow 2x-3y=24$, $\overleftrightarrow{AB}$ *reflected through* $y=-2$. *Similarly for case 3:* $x<1$ *and* $y<-2$, $2x+3y=-20$, *and case 4:* $x<1$ *and* $y \ge -2$, $2x-3y=-8$. *Because of the symmetry of the absolute values about the lines* $x=1$ *and* $y=-2$, *rhombus* $ABCD$ *is the graph of the first equation. The graph of the second equation is a line with a slope* $-\frac{2}{5}$ *and an x-intercept of* -12. *Solving systems with lines* $\overleftrightarrow{DA}$ *and* $\overleftrightarrow{CB}$ *shows it intersects the rhombus at* $(-7,-2)$ *and* $(3,-6)$. *Since* $m>0$, *the second solution is* $(m,\ n)$.

Minnesota State High School Mathematics League

2020-21 State Tournament, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have **15** minutes for this Tournament event.

NO CALCULATORS are allowed on this event.

______ $cm.$

1. On the screen of my graphing calculator point $A\left(1,\ \frac{1}{2}\right)$ is $0.2\ cm.$ from the origin. In centimeters how far from the origin would point $B\ (5,\ 10)$ be on this calculator?

$p+q=$ ______

2. In $\triangle ABC$, $AB=BC=8$. Point D is on $\overline{AC}$, such that $BD=CD=5$. If $AD=\frac{p}{q}$, where p and q are relatively prime integers, determine the value of $p+q$.

$DM=$ ______

3. In $\triangle ABC$, $AB=10$, $BC=17$, and $AC=21$. Cevian $\overline{BD}$ has a length of 8. Let M be the midpoint of $\overline{AB}$. Determine the length of $\overline{DM}$.

$k=$ ______

4. In Figure 4, OAB is a quarter circle with $AO\perp OB$. $\overline{CD}$ is drawn parallel to $\overline{OB}$, with C on $\overline{OA}$ and D on $\overset{\frown}{AB}$. A circle is drawn tangent to $\overline{CD}$ and $\overline{OB}$. If $CD=6$, the area of the shaded region is $k\pi$. Determine the value of k.

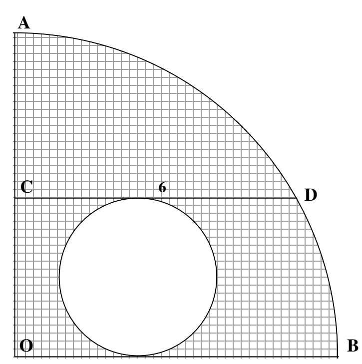

Figure 4

Name: ______ Team: ______

Minnesota State High School Mathematics League

2020-21 State Tournament, Individual Event B

SOLUTIONS

2 cm.

1. On the screen of my graphing calculator point $A\left(1,\ \frac{1}{2}\right)$ is $0.2\ cm.$ from the origin. In centimeters how far from the origin would point $B\ (5,\ 10)$ be on this calculator?

Let O be the origin. Then $OA=\sqrt{1^2+\left(\frac{1}{2}\right)^2}=\frac{\sqrt{5}}{2}$. $OB=\sqrt{5^2+10^2}=5\sqrt{5}$, *or 10 times longer than* OA.

44

2. In $\triangle ABC$, $AB=BC=8$. Point D is on $\overline{AC}$, such that $BD=CD=5$. If $AD=\frac{p}{q}$, where p and q are relatively prime integers, determine the value of $p+q$.

See Figure 2.1 and note $\triangle BDC \sim \triangle ABC$. $\frac{5}{8}=\frac{8}{5+x} \Rightarrow x=\frac{39}{5}$. *Or use Stewart's Formula:* $8^2(5)+8^2(x)=5^2(5+x)+5(x)(5+x) \Rightarrow$

$8^2(5+x)=\left(5^2+5x\right)(5+x) \Rightarrow 64=25+5x \Rightarrow x=\frac{39}{5}$.

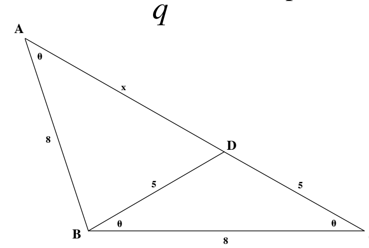

Figure 2.1

5

3. In $\triangle ABC$, $AB=10$, $BC=17$, and $AC=21$. Cevian $\overline{BD}$ has a length of 8. Let M be the midpoint of $\overline{AB}$. Determine the length of $\overline{DM}$.

By Hero's Formula, the area of $\triangle ABC=\sqrt{24(14)(7)(3)}=\sqrt{\left(2^3\cdot 3\right)(2\cdot 7)(7)(3)}=2^2\cdot 7\cdot 3=84$. *Also the area of* $\triangle ABC=84=\frac{1}{2}(21)h \Rightarrow h=8$. *Therefore, cevian* $\overline{BD}$ *is an altitude and* $\triangle ABD$ *is a right triangle.* DM *= half the hypotenuse* $=\frac{10}{2}=5$.

9

4. In Figure 4, OAB is a quarter circle with $AO \perp OB$. $\overline{CD}$ is drawn parallel to $\overline{OB}$, with C on $\overline{OA}$ and D on $\overset{\frown}{AB}$. A circle is drawn tangent to $\overline{CD}$ and $\overline{OB}$. If $CD=6$, the area of the shaded region is $k\pi$. Determine the value of k.

Let the radius of the quarter circle be R and the radius of the small circle be r. The the area of the shaded region is $\frac{\pi R^2}{4}-\pi r^2=\frac{\left(R^2-4r^2\right)\pi}{4}$.

Triangle OCD is a right triangle with $OD=R$, $OC=2r$ *and* $CD=6$.

Therefore, $(2r)^2+6^2=R^2 \Rightarrow 36=R^2-4r^2$. *Substituting, the area of the shaded region is* 9π.

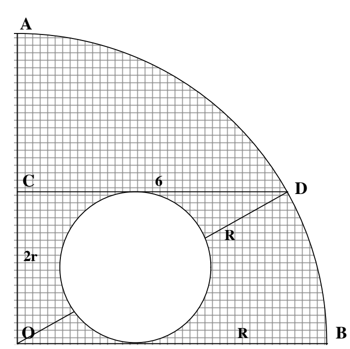

Figure 4.1

Minnesota State High School Mathematics League

2020-21 State Tournament, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have **15** minutes for this Tournament event.

NO CALCULATORS are allowed on this event.

$m =$ ____________

1. When $\left(2+\frac{1}{\sqrt{x}}\right)^5$ is expanded, one of the terms is $\frac{m}{x}$ for some integer m. Determine the value of m.

$p+q =$ ____________

2. In $\triangle ABC$, $\sin A = \frac{3}{4}$, $AB = 8$, and $BC = 10$. If $\cos C = \frac{p}{q}$, where p and q are relatively prime integers, determine the value of $p+q$.

$x =$ ____________

3. If $\cot\left(\frac{x}{2}\right) - \tan\left(\frac{x}{2}\right) = 2\tan\left(60^\circ\right)$, what is the smallest possible positive value for x in degrees?

$p+q =$ ____________

4. Five people get on the elevator on the first floor of an 8-floor apartment building. Each person upon entering the elevator, independently and with equal probability, may choose any floor, starting with the second, to get off the elevator. The probability that all five people will get off on different floors can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2020-21 State Tournament, Individual Event C

SOLUTIONS

80

1. When $\left(2+\frac{1}{\sqrt{x}}\right)^5$ is expanded, one of the terms is $\frac{m}{x}$ for some integer m. Determine the value of m.

Each term of the expansion is $\binom{5}{k}2^{5-k}\left(x^{-\frac{1}{2}}\right)^k$. *So* $\left(-\frac{1}{2}\right)\cdot k=-1 \Rightarrow k=2$. $\binom{5}{2}\cdot 2^3=10\cdot 8=80$.

9

2. In $\triangle ABC$, $\sin A=\frac{3}{4}$, $AB=8$, and $BC=10$. If $\cos C=\frac{p}{q}$, where p and q are relatively prime integers, determine the value of $p+q$.

$\frac{\sin A}{10}=\frac{\sin C}{8} \Rightarrow \sin C=\frac{3}{5}$. *Therefore,* $\cos C=\frac{4}{5}$.

30

3. If $\cot\left(\frac{x}{2}\right)-\tan\left(\frac{x}{2}\right)=2\tan\left(60^\circ\right)$, what is the smallest possible positive value for x in degrees?

$$\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}-\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}=\frac{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}=\frac{\cos x}{\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}=\frac{2\cos x}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}=\frac{2\cos x}{\sin x}=2\cot x.$$

Therefore, $\cot x=\tan\left(60^\circ\right) \Rightarrow x=30^\circ$.

2761

4. Five people get on the elevator on the first floor of an 8-floor apartment building. Each person upon entering the elevator, independently and with equal probability, may choose any floor, starting with the second, to get off the elevator. The probability that all five people will get off on different floors can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

Each person can leave the elevator on seven different floors. So the total number of possibilities is 7^5. *In counting the successes, i.e, each person getting off on a different floor, the first person has seven choices on which floor to get off, the second person six choices, the third five choices, the fourth four choices and the last only three choices. Therefore, the desired probability is*

$$\frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{7^5}=\frac{30\cdot 12}{49\cdot 49}=\frac{360}{2401}.$$

Minnesota State High School Mathematics League

2020-21 State Tournament, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have **15** minutes for this Tournament event.

NO CALCULATORS are allowed on this event.

____________ 1. Let $g(t) = 4 + 5t$ and $h(t) = \left(g(t)\right)^2$. Determine exactly the value of $h\left(g^{-1}(12)\right)$.

$m+n=$ ____________ 2. Lines $ax + (a+k)y = a + 2k$ and $bx + (b+j)y = b + 2j$ intersect at a single point $(m,\ n)$. Determine the value of $m+n$.

$b-a=$ ____________ 3. Let $\lfloor A \rfloor$ denote the greatest integer less than or equal to A. All values of x such that $\left\lfloor 10^{\log_3 \sqrt{x}} \right\rfloor$ has three digits, lie in the interval $[a,\ b)$. Determine the minimum value of $b-a$.

$p+q+r+s=$ ____________ 4. In a hyperbola, c is the distance from the center to one of the foci. In the hyperbola $\frac{(x+2)^2}{a^2} - \frac{(y-3)^2}{b^2} = 1$, the terms $a,\ b,\ c$ form an arithmetic progression, with $a < b < c$. If $b = 10$, the foci are $(p,\ q)$ and $(r,\ s)$. Determine the value of $p+q+r+s$.

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2020-21 State Tournament, Individual Event D

SOLUTIONS

144

1. Let $g(t)=4+5t$ and $h(t)=(g(t))^2$. Determine exactly the value of $h\left(g^{-1}(12)\right)$.

Since $h(t)=(g(t))^2$, *we don't have to solve for t:* $h\left(g^{-1}(12)\right)=\left(g\left(g^{-1}(12)\right)\right)^2=12^2=144$.

1

2. Lines $ax+(a+k)y=a+2k$ and $bx+(b+j)y=b+2j$ intersect at a single point $(m,\ n)$. Determine the value of $m+n$.

$b\left(ax+(a+k)y=a+2k\right)\Rightarrow abx+aby+bky=ab+2bk$.
$-a\left(bx+(b+j)y=b+2j\right)\Rightarrow -abx-aby-ajy=-ab-2aj$. *Adding these two equations yields:* $(bk-aj)y=2(bk-aj)\Rightarrow y=2$. *Substituting, yields:* $ax+2(a+k)=a+2k\Rightarrow x=-1$. *So* $(m,\ n)=(-1,\ 2)$.

648

3. Let $\lfloor A\rfloor$ denote the greatest integer less than or equal to A. All values of x such that $\left\lfloor 10^{\log_3\sqrt{x}}\right\rfloor$ has three digits, lie in the interval $[a,\ b)$. Determine the minimum value of $b-a$.

$100\le 10^{\log_3\sqrt{x}}<1000\Rightarrow 2\le\log_3\sqrt{x}<3\Rightarrow 3^2\le\sqrt{x}<3^3\Rightarrow 81\le x<729$. $729-81=648$.

2

4. In a hyperbola, c is the distance from the center to one of the foci. In the hyperbola $\dfrac{(x+2)^2}{a^2}-\dfrac{(y-3)^2}{b^2}=1$, the terms $a,\ b,\ c$ form an arithmetic progression, with $a<b<c$. If $b=10$, the foci are $(p,\ q)$ and $(r,\ s)$. Determine the value of $p+q+r+s$.

In a hyperbola $a^2+b^2=c^2$. *Therefore,* $(10-d)^2+(10)^2=(10+d)^2\Rightarrow$
$100-20d+d^2+100=100+20d+d^2\Rightarrow d=2.5$. *Therefore,* $c=12.5$. *The center is* $(-2,\ 3)$ *and the foci are at* $(-14.5,\ 3)$ *and* $(10.5,\ 3)$. $16.5-14.5=2$.

Minnesota State High School Mathematics League

2020-21 State Tournament, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of **30** minutes, submit only one set of answers. Place your answer to each question on the line provided.

$p+q=$ ________________

1. *Figure 1:* $\triangle ABC$ with $AB=4$, $BC=6$, and $AC=8$. Side $\overline{BA}$ is extended to F, so that $BF=6$. Side $\overline{BC}$ is extended to D, so that $BD=8$. Draw $\overline{AD}$ and $\overline{CF}$ and mark their intersection X. Draw $\overline{BX}$ and let the intersection of $\overline{BX}$ and $\overline{AC}$ be E. The length of $\overline{CE}$ can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

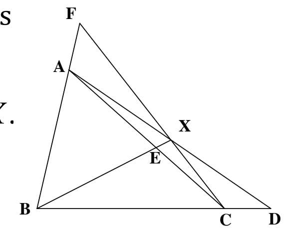

Figure 1

$b=$ ________________

2. In a circle centered at O, $\overline{AB}$ and $\overline{CD}$ are two perpendicular chords. If $AC=6$ and $BD=10$, the radius of the circle can be written as $\sqrt{b}$, where b is square-free. Determine the value of b.

$p+q=$ ________________

3. If x is measured in radians, the greatest **negative** solution to $\tan 2x = \frac{\cos x - \sin x}{\cos x + \sin x}$ can be written as $\frac{p\pi}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$p+q=$ ________________

4. $\overline{AD}$ is a median in $\triangle ABC$, with $AC=10$ and $BC=12$. Point M is the midpoint of $\overline{AD}$. Ray $\overrightarrow{BM}$ intersects $\overline{AC}$ at E. If CE can be written as $\frac{p}{q}$, where p and q are relatively prime integers, determine the value of $p+q$.

$a+b=$ ________________

5. Find the **sum** of all values for a and b which make the following two systems equivalent, i.e. have the same solution set.

$$\left\{ \begin{array}{c} ax+2y=b+1 \\ x+y=3 \end{array} \right\} \qquad \left\{ \begin{array}{c} 2x+y=a^2+2 \\ x+3y=3 \end{array} \right\}$$

$n=$ ________________

6. Let $f(x)=\left(\frac{n}{4}\right)x^2+(n+1)x+(n-2)$. Find the least integer n greater than 2021, such that $f(x)$ has rational roots.

Team: ________________

Minnesota State High School Mathematics League

2020-21 State Tournament, Team Event

SOLUTIONS (page 1)

31

1. *Figure 1:* $\triangle ABC$ with $AB=4$, $BC=6$, and $AC=8$. Side $\overline{BA}$ is extended to F, so that $BF=6$. Side $\overline{BC}$ is extended to D, so that $BD=8$. Draw $\overline{AD}$ and $\overline{CF}$ and mark their intersection X. Draw $\overline{BX}$ and let the intersection of $\overline{BX}$ and $\overline{AC}$ be E. The length of $\overline{CE}$ can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

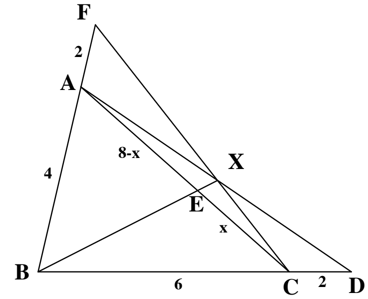

Figure 1.1

34

2. In a circle centered at O, $\overline{AB}$ and $\overline{CD}$ are two perpendicular chords. If $AC=6$ and $BD=10$, the radius of the circle can be written as $\sqrt{b}$, where b is square-free. Determine the value of b.

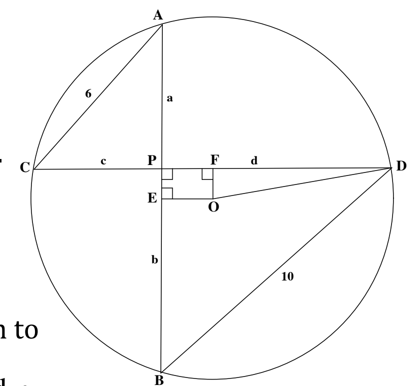

Figure 2.1

5

3. If x is measured in radians, the greatest **negative** solution to $\tan 2x = \dfrac{\cos x - \sin x}{\cos x + \sin x}$ can be written as $\dfrac{p\pi}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

23

4. $\overline{AD}$ is a median in $\triangle ABC$, with $AC=10$ and $BC=12$. Point M is the midpoint of $\overline{AD}$. Ray $\overrightarrow{BM}$ intersects $\overline{AC}$ at E. If CE can be written as $\frac{p}{q}$, where p and q are relatively prime integers, determine the value of $p+q$.

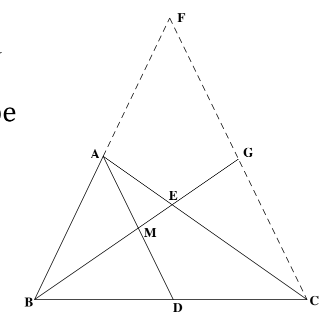

Figure 4.1

-9

5. Find the **sum** of all values for a and b which make the following two systems equivalent, i.e. have the same solution set.

$$\left\{\begin{matrix} ax+2y=b+1 \\ x+y=3 \end{matrix}\right\} \qquad \left\{\begin{matrix} 2x+y=a^2+2 \\ x+3y=3 \end{matrix}\right\}$$

2070

6. Let $f(x)=\left(\dfrac{n}{4}\right)x^2+(n+1)x+(n-2)$. Find the least integer n greater than 2021, such that $f(x)$ has rational roots.

Minnesota State High School Mathematics League

2020-21 State Tournament, Team Event

SOLUTIONS (page 2)

1. *See Figure1.1. Since cevians* $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ *intersect at* X, *Ceva's Theorem is true even if the cevians meet outside the triangle:* $\left(\frac{AF}{FB}\right)\cdot\left(\frac{BD}{DC}\right)\cdot\left(\frac{CE}{EA}\right)=1 \Rightarrow \left(\frac{2}{6}\right)\left(\frac{8}{2}\right)\left(\frac{x}{8-x}\right)=1 \Rightarrow 4x=24-3x \Rightarrow x=\frac{24}{7}$.

2. *Let* $\overline{AB}$ *and* $\overline{CD}$ *intersect at* P. *Draw* $\overline{OE}$ *and* $\overline{OF}$ *perpendicular to* $\overline{AB}$ *and* $\overline{CD}$ *and label as shown in Figure 2.1.* $a^2+c^2=36$, $b^2+d^2=100$ *and* $ab=cd$ *(Power of a Point).* $BA=a+b \Rightarrow EA=\frac{a+b}{2} \Rightarrow EP=OF=\frac{a+b}{2}-a=\frac{b-a}{2}$. $CD=c+d \Rightarrow FD=\frac{c+d}{2}$. $OD^2=OF^2+FD^2=\left(\frac{a-b}{2}\right)^2+\left(\frac{c+d}{2}\right)^2=\frac{a^2-2ab+b^2}{4}+\frac{c^2+2cd+d^2}{4}=\frac{a^2+b^2+c^2+d^2}{4}=\frac{36+100}{4}=34$. *Therefore,* $OD=\sqrt{34}$.

3. $\frac{\cos x-\sin x}{\cos x+\sin x}\cdot\frac{\cos x-\sin x}{\cos x-\sin x}=\frac{1-2\sin x\cos x}{\cos^2 x-\sin^2 x} \Rightarrow \tan 2x=\frac{\sin 2x}{\cos 2x}=\frac{1-\sin 2x}{\cos 2x}$. *Therefore,* $\sin 2x=1-\sin 2x \Rightarrow 2\sin 2x=1 \Rightarrow \sin 2x=\frac{1}{2}$. *Therefore,* $2x=\frac{\pi}{6}+2k\pi$ *or* $\frac{5\pi}{6}+2k\pi$. *So* $x=\frac{\pi}{12}+k\pi$ *or* $\frac{5\pi}{12}+k\pi$. *The greatest negative solution occurs when* $k=-1$. *Of these two values, the greatest negative is* $\frac{5\pi}{12}-\pi=\frac{-7\pi}{12}$.

4. *See Figure 4.1. Construct a line parallel to* $\overline{AD}$ *through* C. *Extend* $\overrightarrow{BA}$ *to intersect this line at* F. $\triangle BAD \sim \triangle BFC$. *Therefore,* A *is the midpoint of* $\overline{BF}$ *and* G *is the midpoint of* $\overline{FC}$. *Then* $\overline{BG}$ *and* $\overline{CA}$ *are medians of* $\triangle BFC$ *and* E, *therefore, is the centroid of* $\triangle BFC$ *and* $CE=\frac{2}{3}(CA)=\frac{20}{3}=\frac{p}{q}$.

5. *In the second system, the slopes of the lines are* $\frac{-2}{1}$ *and* $\frac{-1}{3}$, *and therefore, intersect at a single point. Therefore, the lines in the first system must also have different slopes.* $\frac{-a}{2}\neq\frac{-1}{1} \Rightarrow a\neq 2$. *If* $(m,\ n)$ *is a solution to both systems, then* $m+n=3$ *and* $m+3n=3$. *Therefore,* $(3,\ 0)$ *is a solution to both systems. Since* $(3,\ 0)$ *is the common solution,* $3a=b+1$ *and* $6=a^2+2$. *Therefore,* $a^2=4 \Rightarrow a=-2$. *Then* $b=3(-2)-1=-7$.

6. *The roots will be rational if and only if the discriminant of* $f(x)$ *is a perfect square (of a rational number). That discriminant is* $(n+1)^2-4\left(\frac{n}{4}\right)(n-2)=4n+1$. *Since* $n>2021$, $4n+1>8085$. *The smallest square greater than* 8085 *is* $8100=90^2$. *But* $4n+1$ *is odd, so* $4n+1=91^2=8281 \Rightarrow n=2070$.

Minnesota State High School Mathematics League

2020-21 State Tournament, Invitational Event

(35 minutes)

NO CALCULATORS are allowed on this event.

Quickies (8 points)

Questions in this section are intended to require very little computation and should be answered very quickly.
Each question is worth 1 point. Place your answer to each question on the line provided.

______________ 1. What is the value of $x+y+z$, if $x+2y-4z=5$, $y+2z-4x=-8$ and $z+2x-4y=-3$?

______________ % 2. Carlos took 60% of a whole pie. Maria took one fourth of the remainder. What per cent of the pie was left?

______________ 3. Determine the coefficient of x^5 when $\left(2x^2-x\right)^4$ is expanded.

$p+q=$ ______________ 4. $\frac{w}{x}=\frac{4}{3}$, $\frac{y}{z}=\frac{3}{2}$, $\frac{z}{x}=\frac{1}{6}$, $\frac{w}{y}=\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

______________ 5. a, b, c, d, and e are five distinct integers. If $a+b+|a-b|=2a$, $b+c+|b-c|=2b$, $c+d-|c-d|=2c$, and $d+e-|d-e|=2d$, which of the five integers is the least?

$A=$ ______________ 6. The equation $x^2-9x+A=0$ has one root which is twice the other root. Determine the value of A.

$p+q=$ ______________ 7. If $\tan x=2\sqrt{2}$, the value of $\cos(2x)$ can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

______________ ° 8. In $\triangle ABC$, $a^2+b^2+c^2=ab+bc+ac$. Determine the value of $m\angle A+m\angle B$, in degrees.

Name: ______________________ Team: ______________________

Problems (8 points)

Questions in this section are meant to be comparable in difficulty to those you encountered in individual events during the season. Each question is worth 2 points. Place your answer to each question on the line provided.

__________ 9. The length of one leg of a right triangle, with relatively prime integer sides, is 21. Find the **sum** of all possible lengths of the hypotenuse of this triangle.

$a+b=$ __________ 10. $ABCDEF$ is a regular hexagon with a side length of 12. $\overline{BD}$ and $\overline{EC}$ intersect at P. The distance from P to $\overline{AF}$ can be written as $a\sqrt{b}$, where b is square-free. Determine the value of $a+b$.

__________ 11. Determine the **sum** of the values of x, such that $f(x)=\frac{4}{(x-6)^2}+\frac{(x-6)^2}{4}$ takes on a minimum value.

$p+q=$ __________ 12. OAB is a quarter circle with radius R. Arc OC is centered at B, also with a radius of R. A circle, of radius r, is inscribed in curvilinear triangle OAC. The ratio of $r:R=p:q$, where p and q are relatively prime integers. Determine the value of $p+q$.

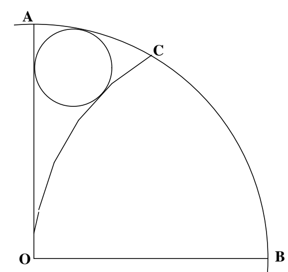

Figure 12

Challenges (8 points)

Questions in this section are meant, as the heading implies, to be challenges, and should probably only be tried after you have done as well as you can on the other sections. Each question is worth 4 points. Place your answer to each question on the line provided.

__________ 13. In right $\triangle ABC$, the length of the hypotenuse $AB=13$ and leg $AC=5$. An **excircle** of $\triangle ABC$ is drawn tangent to $\overline{CB}$ and the **extended** sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Determine the radius of this excircle.

$p+q=$ __________ 14. Let a, b, c, d be **rational** numbers such that $a+b=-1$, $ab+bc+ca=-6$, $abc+bcd+cda+dab=-5$ and $abcd=4$. The sum $a+b+c+d$ can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

Name: ________________ Team: ________________

Minnesota State High School Mathematics League

2020-21 State Tournament, Invitational Event

Quickies (8 points)

SOLUTIONS (page 1 of 3)

6 1. What is the value of $x+y+z$, if $x+2y-4z=5$, $y+2z-4x=-8$ and $z+2x-4y=-3$?

Add the three equations: $-x-y-z=-6$.

30 2. Carlos took 60% of a whole pie. Maria took one fourth of the remainder. What per cent of the pie was left?

$\frac{3}{4}(40\%)=30\%$.

−8 3. Determine the coefficient of x^5 when $\left(2x^2-x\right)^4$ is expanded.

$\binom{4}{k}\left(2x^2\right)^{4-k}(-1x)^k=\binom{4}{k}\left(2^{4-k}\right)(-1)^k x^{8-2k}x^k \Rightarrow 8-k=5 \Rightarrow k=3.\quad \binom{4}{3}\cdot 2(-1)^3=-8.$

19 4. $\frac{w}{x}=\frac{4}{3}$, $\frac{y}{z}=\frac{3}{2}$, $\frac{z}{x}=\frac{1}{6}$. $\frac{w}{y}=\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$\frac{w}{y}=\left(\frac{w}{x}\right)\left(\frac{x}{z}\right)\left(\frac{z}{y}\right)=\frac{4}{3}\cdot\frac{6}{1}\cdot\frac{2}{3}=\frac{16}{3}$.

c 5. a, b, c, d, and e are five distinct integers. If $a+b+|a-b|=2a$, $b+c+|b-c|=2b$, $c+d-|c-d|=2c$, and $d+e-|d-e|=2d$, which of the five integers is the least?

The first equation implies $a>b$; *the second,* $b>c$; *the third,* $d>c$, *and the fourth,* $e>d$. *Therefore, c is the least integer.*

18 6. The equation $x^2-9x+A=0$ has one root which is twice the other root. Determine the value of A.

$r+2r=9 \Rightarrow r=3.\ A=2r^2=18.$

2 7. If $\tan x=2\sqrt{2}$, the value of $\cos(2x)$ can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$\frac{\sin x}{\cos x}=2\sqrt{2} \Rightarrow \frac{\sin^2 x}{\cos^2 x}=8 \Rightarrow 1-\cos^2 x=8\cos^2 x \Rightarrow \cos^2 x=\frac{1}{9}.\ \cos(2x)=2\cos^2 x-1=\frac{-7}{9}.$

120 8. In $\triangle ABC$, $a^2+b^2+c^2=ab+bc+ac$. Determine the value of $m\angle A+m\angle B$, in degrees.

Because of the symmetry of the equation, $a=b=c$. *So* $m\angle A=m\angle B=60^\circ$. *Proof on Page 3.*

Minnesota State High School Mathematics League

2020-21 State Tournament, Invitational Event

SOLUTIONS (page 2 of 3)

Problems (8 points)

250

9. The length of one leg of a right triangle, with relatively prime integer sides, is 21. Find the **sum** of all possible lengths of the hypotenuse of this triangle.

13

10. $ABCDEF$ is a regular hexagon with a side length of 12. $\overline{BD}$ and $\overline{EC}$ intersect at P. The distance from P to $\overline{AF}$ can be written as $a\sqrt{b}$, where b is square-free. Determine the value of $a+b$.

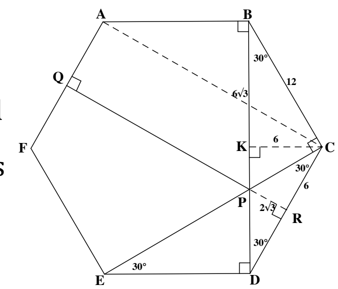

Figure 10.1

12

11. Determine the **sum** of the values of x, such that $f(x)=\dfrac{4}{(x-6)^2}+\dfrac{(x-6)^2}{4}$ takes on a minimum value.

7

12. OAB is a quarter circle with radius R. Arc OC is centered at B, also with a radius of R. A circle, of radius r, is inscribed in curvilinear triangle OAC. The ratio of $r:R=p:q$, where p and q are relatively prime integers. Determine the value of $p+q$.

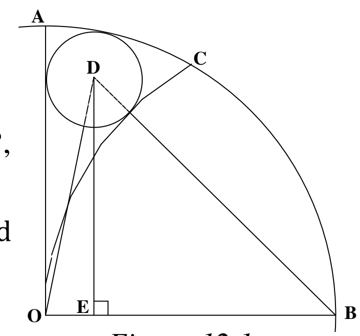

Figure 12.1

Challenges (8 points)

10

13. In right $\triangle ABC$, the length of the hypotenuse $AB=13$ and leg $AC=5$. An **excircle** of $\triangle ABC$ is drawn tangent to $\overline{CB}$ and the **extended** sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Determine the radius of this excircle.

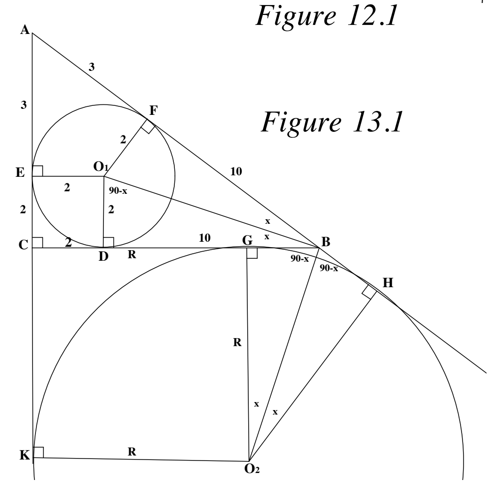

Figure 13.1

7

14. Let a, b, c, d be **rational** numbers such that $a+b=-1$, $ab+bc+ca=-6$, $abc+bcd+cda+dab=-5$ and $abcd=4$. The sum $a+b+c+d$ can be written as $\dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

Minnesota State High School Mathematics League

2020-21 State Tournament, Invitational Event

SOLUTIONS (page 3 of 3)

8. *Symmetry Proof:* $a^2+b^2+c^2=ab+bc+ac \Rightarrow 2a^2+2b^2+2c^2=2ab+2bc+2ac \Rightarrow$
$a^2-2ab+b^2+b^2-2bc+c^2+a^2-2ac+c^2=0 \Rightarrow (a-b)^2+(b-c)^2+(a-c)^2=0 \Rightarrow a=b=c.$

9. *For relatively prime integers,* m *and* n, *the legs of a right triangle are* $2mn$ *and* m^2-n^2. *So* $21=m^2-n^2=(m-n)(m+n)$. *So either* $m+n=21$ *and* $m-n=1$ *or* $m+n=7$ *and* $m-n=3$. *Solving the first system yields* $m=11,\ n=10$ *and solving the second system, yields* $m=5,\ n=2$. *The length of the hypotenuse is* m^2+n^2. *So* $11^2+10^2=221$ *and* $5^2+2^2=29$, *and* $221+29=250$.

10. *See Figure 10.1. The interior angles of a regular hexagon are* 120° $\triangle BCD$ *and* $\triangle CDE$ *are isosceles. Draw* $\overline{CK}\perp\overline{BD}$ *and* $\overline{PR}\perp\overline{CD}$, *forming 30-60-90 triangles.* $BC=12,\ CK=6,\ BK=6\sqrt{3},\ BD=12\sqrt{3}.\ CR=6$ *and* $PR=2\sqrt{3}.$ $QR=AC=BD=12\sqrt{3}.\ QP=QR-PR=12\sqrt{3}-2\sqrt{3}=10\sqrt{3}.$

11. $f(x)=K^2+\dfrac{1}{K^2}$ *takes on a minimum value of 2, whenever* $K^2=1 \Rightarrow K=\pm1$. *Therefore,* $\dfrac{x-6}{2}=\pm1 \Rightarrow$ $x-6=2$ *or* $x-6=-2 \Rightarrow x=8$ *or* $x=4$.

12. *See Figure 12.1. Let D be the center of the inscribed circle. Construct DE perpendicular to* $\overline{OB}$. *Draw OD and DB. In* $\triangle OED$, $r^2+(ED)^2=(R-r)^2$, *and in* $\triangle DEB$, $(R-r)^2+(ED)^2=BD^2=(R+r)^2$. *Therefore,* $(R-r)^2-r^2=(R+r)^2-(R-r)^2 \Rightarrow R^2=6Rr \Rightarrow \dfrac{r}{R}=\dfrac{1}{6}.$

13. *Label the diagram as shown in Figure 13.1. The radius of the incircle of a right triangle is* $\dfrac{a+b-c}{2}=\dfrac{5+12-13}{2}=2$. *Let* $CG=R$ *and so* $GB=12-R$. *Let* $x=m\angle DBO_1=m\angle FBO_1$, *then* $m\angle DBH=180-2x$. *So* $m\angle DBO_2=90-x$. *Therefore,* $\triangle O_1DB \sim \triangle BGO_2$. *Then* $\dfrac{O_1D}{BG}=\dfrac{DB}{GO_2} \Rightarrow \dfrac{2}{12-R}=\dfrac{10}{R} \Rightarrow 2R=120-10R \Rightarrow 12R=120 \Rightarrow R=10.$

14. $abc+d(bc+ca+ab)=-5 \Rightarrow abc+(-6d)=-5$. *But* $abc(-6d)=-24$. *Therefore,* abc *and* $-6d$ *are zeros of* $x^2+5x-24=0=(x+8)(x-3)$. *So* $\{abc,\ -6d\}=\{-8,\ 3\}$. $ab+bc+ca=ab+c(a+b)=ab+(-c)=-6$. *But* $ab(-c)=-abc=8$ *or* -3. *Therefore,* ab *and* $-c$ *are zeros of* $x^2+6x+8=0$ *or* $x^2+6x-3=0$. *The second equation has irrational zeros but the zeros of the first equation are* -2 *and* -4. *Therefore,* $\{ab,\ -c\}=\{-2,\ -4\}$. *If* $ab=-4$ *and* $a+b=-1$, *then* a *and* b *are zeros of* $x^2+x-4=0$. *But these zeros are irrational. If* $ab=-2$ *and* $a+b=-1$, *then* a *and* b *are zeros of* $x^2+x-2=0 \Rightarrow \{a,b\}=\{1,\ -2\}$. *Then* $ab=-2$. *Therefore,* $-c=-4 \Rightarrow c=4$. *Then* $abc=-8$, *so* $-6d=3 \Rightarrow d=\dfrac{-1}{2}$. *Therefore,* $a+b+c+d=2\tfrac{1}{2}=\dfrac{5}{2}.$