Minnesota State High School Mathematics League

2020-21 Meet 5, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have **20 minutes** for this event.

NO CALCULATORS are allowed on this event.

________________ 1. If $12102_b + 1_b = 12220_b$, what base-10 number is equivalent in value to 12220_b?
(Hint: b may be a negative integer.)

________________ 2. Summing the powers of 3 produces an interesting pattern that allows you to find the sum without adding. Discover the pattern and use it to find the the sum of: $1+3+9+27+\ldots+531441$.

$$\begin{array}{rcl} 1 & = & 1 \\ 1+3 & = & 4 \\ 1+3+9 & = & 13 \\ 1+3+9+27 & = & 40 \\ 1+3+9+27+81 & = & \_\_\_ \\ 1+3+9+27+81+243 & = & \_\_\_ \end{array}$$

________________ 3. I have \$3.00 in change, all in three types of coins. I have the same number of dimes as quarters and the mean coin value is \$0.15. How many quarters do I have?

$m+n=$ ________________ 4. Abdul, Briana, Carmine, and Diego are four members of the Student Council. Every pair of members on the Student Council is together on one and only one committee. Each committee has exactly three members. If m is the smallest possible total number of members on the Student Council and n is the number of committees on the Student Council, determine the value of $m+n$.

Name: ____________________________ Team: ____________________________

Minnesota State High School Mathematics League

2020-21 Meet 5, Individual Event A

SOLUTIONS

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1. If $12102_b + 1_b = 12220_b$, what base-10 number is equivalent in value to 12220_b? (Hint: b may be a negative integer.)

Subtract 12000_b *from both sides, yielding* $102_b + 1_b = 220_b$. *Then* $b^2 + 2 + 1 = 2b^2 + 2b \Rightarrow 0 = b^2 + 2b - 3 \Rightarrow 0 = (b+3)(b-1) \Rightarrow b = -3$ *or* $b = 1$. *Since 2 is a digit, the base cannot be 1, so b must be* -3, *implying* $12102_{-3} = 1(81) + 2(-27) + 1(9) + 0(-3) + 2 = 38$ *and* $12220_{-3} = 1(81) + 2(-27) + 2(9) + 2(-3) + 0 = 39$.

797161

2. Summing the powers of 3 produces an interesting pattern that allows you to find the sum without adding. Discover the pattern and use it to find the sum of: $1 + 3 + 9 + 27 + \ldots + 531441$.

1	$=$	1
$1+3$	$=$	4
$1+3+9$	$=$	13
$1+3+9+27$	$=$	40
$1+3+9+27+81$	$=$	___
$1+3+9+27+81+243$	$=$	___

Take the last power of three, multiply by 3, subtract 1, and divide by 2. $531441 \cdot 3 = 1594323 \Rightarrow 1594322 \div 2 = 797161$.

8

3. I have \$3.00 in change, all in three types of coins. I have the same number of dimes as quarters and the mean coin value is \$0.15. How many quarters do I have?

I have x dimes and x quarters and y c-cent coins. $25x + 10x + c \cdot y = 300$. *Also*

$$\frac{3.00}{2x+y} = .15 \Rightarrow 3 = .3x + .15y \Rightarrow y = \frac{3 - .3x}{.15} = 20 - 2x.$$

Substituting, $35x + c(20 - 2x) = 300$. *If c is a penny, then* $35x + 20 - 2x = 300 \Rightarrow x = 8.48$, *impossible. If c is a nickel, then* $35x + 100 - 10x = 300 \Rightarrow x = 8$, *a solution! If c is a half dollar, then* $35x + 1000 - 100x = 300 \Rightarrow x = 10.7$, *impossible again. If c is a dollar coin, then* $35x + 2000 - 200x = 300 \Rightarrow x = 11.35$, *impossible again. So I have 8 quarters, 8 dimes, and 4 nickels*

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4. Abdul, Briana, Carmine, and Diego are four members of the Student Council. Every pair of members on the Student Council is together on one and only one committee. Each committee has exactly three members. If m is the smallest possible total number of members on the Student Council and n is the number of committees on the Student Council, determine the value of $m + n$.

Just try to build three-member committees: Let ABC be on a committee. A second committee could be ADX. A third committee could be BDY. A fourth committee could be AYZ. A fifth committee could be BXZ. A sixth committee could be CDZ. A seventh committee could be CXY. This fulfills the two requirements and members X, Y, and Z had to be added to fulfill them. So $m = 7$ *and* $n = 7$. *The Fano Plane in Figure 4.1 illustrates this solution.*

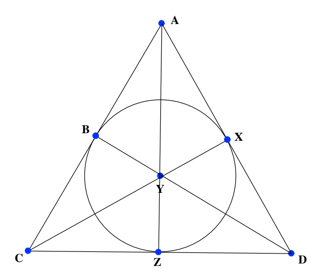

Figure 4.1

Minnesota State High School Mathematics League

2020-21 Meet 5, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

$AB =$ ______________

1. In right triangle ABC with hypotenuse $\overline{AC}$, $\sin A = \dfrac{1}{3}$ and $BC = 2\sqrt{2}$. What is the length of $\overline{AB}$?

$p+q =$ ______________

2. In *Figure 2,* A, B, and E are collinear, as are A, C, and D. $AC = 5$, $CD = 3$, $\overline{BC} \parallel \overline{ED}$, the area of $\triangle BCD = 10$, and the area of $\triangle BDE = 16$. The area of $\triangle ABC$ can be written as $\dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

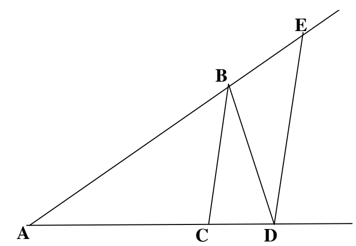

Figure 2

$p+q =$ ______________

3. In *Figure 3,* $ABCD$ is a square with side length 10. P is a point in the interior of $ABCD$ such that APD is a right triangle with $PD = 8$. $\dfrac{\textit{The area of } \triangle PCD}{\textit{The area of } \triangle PCB} = \dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

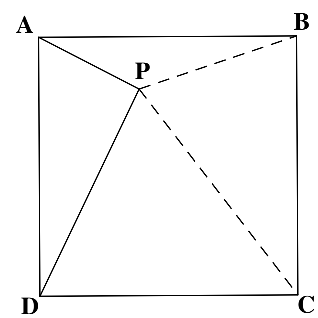

Figure 3

$a+b =$ ______________

4. In *Figure 4,* in $\triangle ABC$, $AB = AC$. Angle bisectors $\overline{AD}$ and $\overline{BE}$ are drawn, intersecting at F. $\textit{Area of } \triangle BFD : \textit{Area of } \triangle AFE = 3:8$. If $AB : BC = a : b$, where a and b are relatively prime integers, determine the value of $a+b$.

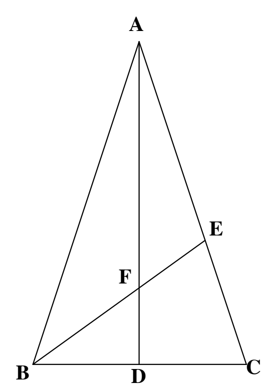

Figure 4

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2020-21 Meet 5, Individual Event B

SOLUTIONS

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1. In right triangle ABC with hypotenuse $\overline{AC}$, $\sin A = \dfrac{1}{3}$ and $BC = 2\sqrt{2}$. What is the length of $\overline{AB}$?

$$\sin A = \frac{BC}{AC} = \frac{2\sqrt{2}}{AC} = \frac{1}{3} \Rightarrow AC = 6\sqrt{2}.\ AB^2 = AC^2 - BC^2 = \left(6\sqrt{2}\right)^2 - \left(2\sqrt{2}\right)^2 = 72 - 8 = 64.$$

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2. In *Figure 2*, A, B, and E are collinear, as are A, C, and D. $AC = 5$, $CD = 3$, $\overline{BC} \parallel \overline{ED}$, the area of $\triangle BCD = 10$, and the area of $\triangle BDE = 16$. The area of $\triangle ABC$ can be written as $\dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

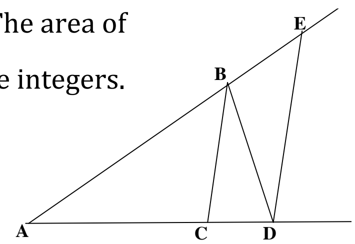

Figure 2

$$\frac{[\triangle ABC]}{[\triangle AED]} = \frac{(5)^2}{(5+3)^2} = \frac{25}{64} = \frac{x}{x+26} \Rightarrow 25x + 650 = 64x \Rightarrow 39x = 650 \Rightarrow x = \frac{650}{39} = \frac{50}{3}.$$

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3. In *Figure 3*, $ABCD$ is a square with side length 10. P is a point in the interior of $ABCD$ such that APD is a right triangle with $PD = 8$. $\dfrac{\textit{The area of } \triangle PCD}{\textit{The area of } \triangle PCB} = \dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

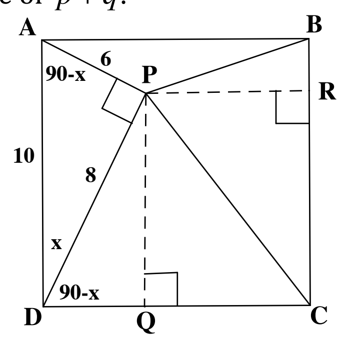

Figure 3.1

As shown in Figure 3.1, construct perpendiculars from P to the sides. $AP = 6$. By AA, $\triangle ADP \sim \triangle DPQ$. Therefore, $\dfrac{AD}{DP} = \dfrac{DP}{PQ} = \dfrac{AP}{DQ} \Rightarrow \dfrac{10}{8} = \dfrac{8}{PQ} = \dfrac{6}{DQ} \Rightarrow$ $PQ = 6.4$ and $DQ = 4.8$. $QC = 5.2 = PR$. $\dfrac{[PCD]}{[PCB]} = \dfrac{\frac{1}{2}(10)(PQ)}{\frac{1}{2}(10)(PR)} = \dfrac{32}{26} = \dfrac{16}{13}$.

3

4. In *Figure 4*, in $\triangle ABC$, $AB = AC$. Angle bisectors $\overline{AD}$ and $\overline{BE}$ are drawn, intersecting at F. *Area of* $\triangle BFD$: *Area of* $\triangle AFE = 3:8$. If $AB : BC = a : b$, where a and b are relatively prime integers, determine the value of $a+b$.

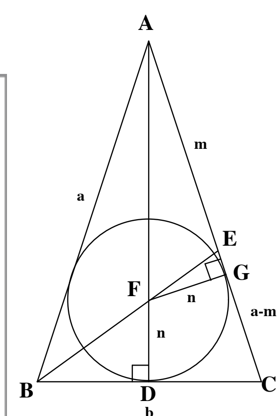

Figure 4.1

Let $AB = AC = a$, $BC = b$, $AE = m$, and $FD = n$. *Then* $\dfrac{a}{m} = \dfrac{b}{a-m} \Rightarrow m = \dfrac{a^2}{a+b}$. F *is the incenter, with* $\triangle AFG \sim \triangle ACD \Rightarrow \dfrac{AF}{a} = \dfrac{n}{\frac{b}{2}} \Rightarrow AF = \dfrac{2an}{b}$. $\dfrac{[BFD]}{[AFE]} = \dfrac{\frac{1}{2}(BD)(FD)}{\frac{1}{2}(AF)(AE)\sin\angle DAC} =$

$$\frac{\frac{1}{2}\left(\frac{b}{2}\right)n}{\frac{1}{2}\left(\frac{2an}{b}\right)\left(\frac{a^2}{a+b}\right)\left(\frac{b}{2a}\right)} = \frac{ab+b^2}{2a^2} = \frac{3}{8} \Rightarrow 8ab + 8b^2 = 6a^2 \Rightarrow 3\left(\frac{a}{b}\right)^2 - 4\left(\frac{a}{b}\right) - 4 = 0 \Rightarrow \frac{a}{b} = \frac{2}{1} \Rightarrow$$

$a : b = 2 : 1$.

Minnesota State High School Mathematics League

2020-21 Meet 5, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

______________ 1. How many ways are there to pick two distinct (different) numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that the sum of the two numbers is even?

$p+q=$ ______________ 2. From the set $\{2, 4, 6, 7, 8, 11, 12, 13\}$ a number is chosen at random to be the numerator of a fraction F and then, from the remaining numbers, a second number is chosen to be the denominator of fraction F. The probability that F is in lowest terms (reduced) can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$b-a=$ ______________ 3. In a group of 62 students, 44 are taking physics and 52 are taking mathematics. If x is the number of students taking both physics and mathematics, then $a \le x \le b$. Determine the minimum possible value of $b-a$.

______________ 4. Fourteen players enter a ping-pong tournament. The first round consists of seven matches, with each player in a match. How many ways can the players be matched against one another?

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2020-21 Meet 5, Individual Event C

SOLUTIONS

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1. How many ways are there to pick two distinct (different) numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that the sum of the two numbers is even?

Pick two odd numbers $= \binom{5}{2} = 10$ ways. Pick two even numbers $= \binom{4}{2} = 6$ ways.

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2. From the set $\{2, 4, 6, 7, 8, 11, 12, 13\}$ a number is chosen at random to be the numerator of a fraction F and then, from the remaining numbers, a second number is chosen to be the denominator of fraction F. The probability that F is in lowest terms (reduced) can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

There are $_8\mathrm{P}_2 = 56$ *possible fractions. The following pairs are relatively prime:* $(2,7)$, $(2,11)$, $(2,13)$, $(4,7)$, $(4,11)$, $(4,13)$, $(6,7)$, $(6,11)$, $(6,13)$, $(7,8)$, $(7,11)$, $(7,12)$, $(7,13)$, $(8,11)$, $(8,13)$, $(11,12)$, $(11,13)$, *and* $(12,13)$ *So each of these pairs and its reverse could produce a reduced fraction for a total of 36 fractions.* $\frac{36}{56} = \frac{9}{14}$.

10

3. In a group of 62 students, 44 are taking physics and 52 are taking mathematics. If x is the number of students taking both physics and mathematics, then $a \le x \le b$. Determine the minimum possible value of $b-a$.

If all physics students take mathematics, then $x = 44$. (This would correspond to the greatest number of students not taking math or physics.) If all 62 students take either physics or mathematics (or both), then $62 = 44 + 52 - x \Rightarrow x = 34$. Therefore, $34 \le x \le 44$.

135135

4. Fourteen players enter a ping-pong tournament. The first round consists of seven matches, with each player in a match. How many ways can the players be matched against one another?

Choose any player. There are 13 possible opponents for this player in the first round. Choose any unmatched player. There are 11 possible opponents for this player. Choose a another unmatched player. There are 9 possible opponents for this player. Continue this process until two players remain. Number of arrangements $= 13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 135135$.

Minnesota State High School Mathematics League

2010-21 Meet 5, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

_______________ 1. Simplify:

$$\sqrt{1}-\sqrt{1+3}+\sqrt{1+3+5}-\sqrt{1+3+5+7}+\ldots+\sqrt{1+3+5+\ldots+17}-\sqrt{1+3+5+\ldots+17+19}.$$

_______________ 2. **How many** ordered pairs of integers (x,y) satisfy the equation: $x^4+y^2=4y$?

$n=$ _______________ 3. If n is a positive integer such that $\log_4(\log_{16} n)=\log_2(\log_4 n)$, determine the value of n.

$a+b+c=$ _______________ 4. There are integers a, b, and c, each greater than 1, such that for all $N>1$,

$\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}}=\sqrt[48]{N^{25}}$. Determine the value of $a+b+c$.

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2020-21 Meet 5, Individual Event D

SOLUTIONS

$\boxed{-5}$

1. Simplify:
$\sqrt{1}-\sqrt{1+3}+\sqrt{1+3+5}-\sqrt{1+3+5+7}+\ldots+\sqrt{1+3+5+\ldots+17}-\sqrt{1+3+5+\ldots+17+19}$.

[2020 AMC 12B, problem #1]

$(1-2)+(3-4)+\ldots+(9-10)=5(-1)=-5.$

$\boxed{2}$

2. **How many** ordered pairs of integers (x,y) satisfy the equation: $x^4+y^2=4y$?

[2020 AMC 12B, problem #8]

$x^4+y^2-4y+4=4 \Rightarrow x^4+(y-2)^2=4$. *If* $x=0$, *then* $(y-2)^2=4 \Rightarrow y-2=\pm 2 \Rightarrow$ $y=2\pm 2 \Rightarrow y=0$ *or* $y=4$. *For any other integer value of* x, y *is not an integer.*

$\boxed{2}$

3. If n is a positive integer such that $\log_4(\log_{16} n)=\log_2(\log_4 n)$, determine the value of n.

[2020 AMC 12A, problem #10]

Let $x=\log_4(\log_{16} n)=\log_2(\log_4 n)$, *then* $4^x=\log_{16} n$ *and* $2^x=\log_4 n$. $4^x=\dfrac{\log_4 n}{\log_4 16}=\dfrac{\log_4 n}{2}\Rightarrow$ $2\cdot 4^x=\log_4 n$. *Substituting,* $2\cdot 4^x=2^x \Rightarrow 2=\dfrac{2^x}{4^x}=\left(\dfrac{1}{2}\right)^x \Rightarrow x=-1$. *So* $\dfrac{1}{2}=\log_4 n \Rightarrow n=4^{\frac{1}{2}}=2$.

$\boxed{13}$

4. There are integers a, b, and c, each greater than 1, such that for all $N>1$, $\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}}=\sqrt[48]{N^{25}}$. Determine the value of $a+b+c$.

[2020 AMC 12A, problem #13]

$\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}}=\sqrt[a]{N\sqrt[b]{N\cdot N^{\frac{1}{c}}}}=\sqrt[a]{N\sqrt[b]{N^{\frac{c+1}{c}}}}=\sqrt[a]{N\cdot N^{\frac{c+1}{bc}}}=\sqrt[a]{N^{\frac{bc+c+1}{bc}}}=N^{\frac{bc+c+1}{abc}}=N^{\frac{25}{48}}$. *Let* $bc+c+1=25$ *and* $abc=48$. *Then* $c(b+1)=24$. *So* $(c,\ b)=(2,\ 11),\ (3,\ 7),\ (4,\ 5),\ (6,\ 3)$, *or* $(8,\ 2)$. *Only the last one works if* $abc=48$. *So* $a=3$, $b=2$, *and* $c=8$.

Minnesota State High School Mathematics League

2020-21 Meet 5, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 30 minutes, submit only one set of answers. Place your answer to each question on the line provided.

______________ 1. A computer science class consists of both juniors and seniors, with seniors comprising more than 94% of the class. What is the smallest size the class can be?

$[DA'C]=$ ______________ 2. Take a rectangular piece of paper $ABCD$ with $AB=30$, $BC=20$, and M the midpoint of $\overline{AD}$. As in *Figure 2*, make a fold along $\overline{MB}$, reflecting point A to A'. Determine the area of $\triangle DA'C$.

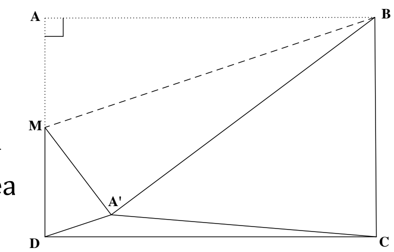

Figure 2

______________ 3. $(a+b+c)^{15}$ is expanded and simplified. **How many** terms are in this simplified expression?

$p+q=$ ______________ 4. ABC is an isosceles right triangle with $BA=BC=10$. Let M be the midpoint of $\overline{AB}$. An arc, centered at M, is drawn external to the triangle, from A to B. Line $\overleftrightarrow{CM}$ intersects this arc at D. The area of $\triangle ABD$ can be written as $p\sqrt{q}$, where q is square-free. Determine the value of $p+q$.

$p+q=$ ______________ 5. In *Figure 5*, R is on $\overline{KL}$ and S is on $\overline{KM}$. Point P is the intersection of $\overline{LS}$ and $\overline{RM}$, with $LP=6$, $PS=2$, $MP=6$, and $PR=3$. $\dfrac{Area\ of\ \triangle LPM}{Area\ of\ \triangle LKM}=\dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

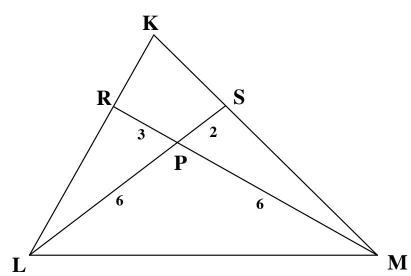

Figure 5

______________ 6. **How many** positive integers N are there such that the least common multiple of N and $4!$ equals 4 times the greatest common divisor of N and $8!$?

Team: ______________________

Minnesota State High School Mathematics League

2020-21 Meet 5, Team Event

SOLUTIONS (page 1)

$\boxed{17}$

1. A computer science class consists of both juniors and seniors, with seniors comprising more than 94% of the class. What is the smallest size the class can be?

$\boxed{30}$

2. Take a rectangular piece of paper $ABCD$ with $AB = 30$, $BC = 20$, and M the midpoint of $\overline{AD}$. As in *Figure 2*, make a fold along $\overline{MB}$, reflecting point A to A'. Determine the area of $\triangle DA'C$.

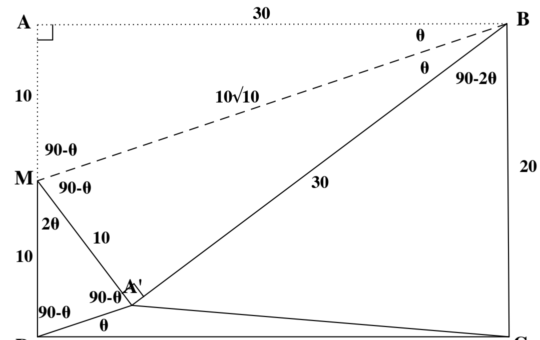

Figure 2.1

$\boxed{136}$

3. $(a+b+c)^{15}$ is expanded and simplified. **How many** terms are in this simplified expression?

$\boxed{15}$

4. ABC is an isosceles right triangle with $BA = BC = 10$. Let M be the midpoint of $\overline{AB}$. An arc, centered at M, is drawn external to the triangle, from A to B. Line $\overleftrightarrow{CM}$ intersects this arc at D. The area of $\triangle ABD$ can be written as $p\sqrt{q}$, where q is square-free. Determine the value of $p+q$.

[2020 AMC 12A, problem #18]

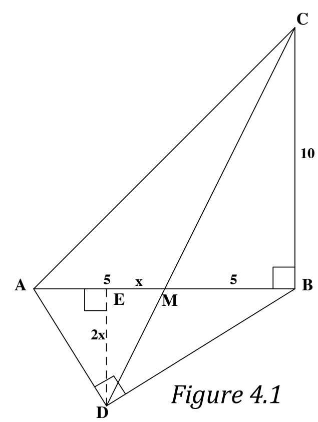

Figure 4.1

$\boxed{17}$

5. In *Figure 5*, R is on $\overline{KL}$ and S is on $\overline{KM}$. Point P is the intersection of $\overline{LS}$ and $\overline{RM}$, with $LP = 6$, $PS = 2$, $MP = 6$, and $PR = 3$. $\dfrac{Area\ of\ \triangle LPM}{Area\ of\ \triangle LKM} = \dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

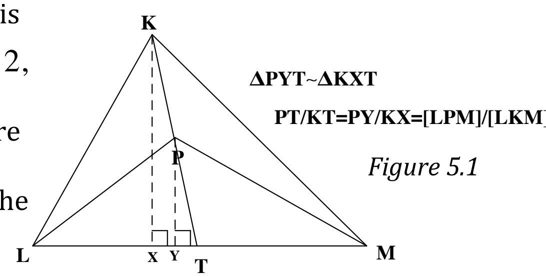

Figure 5.1

$\boxed{16}$

6. **How many** positive integers N are there such that the least common multiple of N and $4!$ equals 4 times the greatest common divisor of N and $8!$?

[2020 AMC 12A, problem #21]

Minnesota State High School Mathematics League

2020-21 Meet 5, Team Event

SOLUTIONS (page 2)

1. *Since the class must be as small as possible, let the number of juniors be 1. If* $x =$ *the number of students in the class, then* $x-1$ *is the number of seniors in the class. Put* $y=(x-1)/x$ *in your calculator and check to see that when* $x=17,\ y=.94118.$

2. *Let* $m\angle ABM=\theta$ *and label the angles as shown in Figure 2.1.* $BM=\sqrt{10^2+30^2}=10\sqrt{10}.$ *So* $\sin\theta=\dfrac{1}{\sqrt{10}}$ *and* $\cos\theta=\dfrac{3}{\sqrt{10}}.$ $DA'=\sqrt{400-400\cos^2\theta}=20\sqrt{1-\cos^2\theta}=20\sin\theta=\dfrac{20}{\sqrt{10}}.$ *So* $[DA'C]=$ $\tfrac12(DA')(DC)\sin\theta=\dfrac12\left(\dfrac{20}{\sqrt{10}}\right)(30)\left(\dfrac{1}{\sqrt{10}}\right)=30.$

3. *Each term in the expansion is* $k\cdot a^pb^qc^r$*, where* $p,\ q,$ *and* r *are integers with* $0\le p,q,r\le 15$ *and* $p+q+r=15.$ *So we are looking for the non-negative integer solutions to the equation* $p+q+r=15.$ *One solution could be represented by* $1111\,|\,11111\,|\,111111=4+5+6.$ *So the total number of solutions is equivalent to the total number of "words" that can be formed using fifteen* $1s$ *and two* $|\,s$*. This is equivalent to* $\dfrac{(15+2)!}{15!\cdot 2!}=136.$

4. *By Thales' Theorem,* ABD *is a right triangle. Draw altitude* $\overline{DE}$ *in* $\triangle ABD.$ $\triangle DME\sim\triangle CMB.$ *Let* $ME=x,$ *then* $DE=2x.$ $\triangle AED\sim\triangle DEB.$ *Therefore,* $\dfrac{5-x}{2x}=\dfrac{2x}{5+x}\Rightarrow 25-x^2=4x^2\Rightarrow x=\sqrt5.$ *So the area of* $\triangle ABD=\tfrac12(10)\left(2\sqrt5\right)=10\sqrt5.$

5. *Draw cevian* $\overline{KT}$ *going through point* $P.$ *Theorem: In any* $\triangle LKM$ *with cevian* $\overline{KT}$*, if* P *is a point on* $\overline{KT}$*, then* $\dfrac{[LPM]}{[LKM]}=\dfrac{PT}{KT}.$ *(Figure 5.1 outlines its proof.) Therefore,* $\dfrac{[LPM]}{[LKM]}=\dfrac{PT}{KT},\ \dfrac{[MPK]}{[MLK]}=\dfrac{PS}{LS},$ *and* $\dfrac{[KPL]}{[KML]}=\dfrac{PR}{MR}.$ *So*
$$\frac{[LPM]}{[LKM]}+\frac{[MPK]}{[MLK]}+\frac{[KPL]}{[KML]}=\frac{PT}{KT}+\frac{PS}{LS}+\frac{PR}{MR}\Rightarrow\frac{[LPM]+[MPK]+[KPL]}{[LKM]}=\frac{PT}{KT}+\frac28+\frac39\Rightarrow 1=\frac{PT}{KT}+\frac{7}{12}\Rightarrow$$
$\dfrac{PT}{KT}=\dfrac{5}{12}.$ *Therefore,* $\dfrac{[LPM]}{[LKM]}=\dfrac{5}{12}.$

6. $4!=2^3\cdot3$ *and* $8!=2^7\cdot3^2\cdot5^1\cdot7^1.$ *Let* $N=2^a\cdot3^b\cdot5^c\cdot7^d$*, then*
$$\mathrm{LCM}(N,4!)=4\cdot\mathrm{GCD}(N,8!)\Rightarrow\mathrm{LCM}\left(2^a\cdot3^b\cdot5^c\cdot7^d,\ 2^3\cdot3\right)=2^2\cdot\mathrm{GCD}\left(2^a\cdot3^b\cdot5^c\cdot7^d,\ 2^7\cdot3^2\cdot5^1\cdot7^1\right).$$
Therefore, looking at 2: $\max(a,3)=2+\min(a,7)\Rightarrow a=1$ or $a=9$*, looking at 3:* $\max(b,1)=\min(b,2)\Rightarrow$ $b=1$ *or* 2*, looking at 5:* $\max(c,0)=\min(c,1)\Rightarrow c=0$ *or* $c=1$*, and looking at 7:* $\max(d,0)=\min(d,1)\Rightarrow d=0$ *or* $d=1.$ *So there are* $2\cdot2\cdot2\cdot2=16$ *possible values for* $N.$