Minnesota State High School Mathematics League

2020-21 Meet 3, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

________________ 1. (x, y) is the intersection of $\frac{3x}{y} - 4 = 11$ and $x - y = 44$. Determine the value of $x + y$.

________________ 2. A large extended family decides to have a catered get-together, with the cost being divided equally among all individuals attending. If everyone that was expected had shown up the cost per person would have been $8.00. As it was, eight people cancelled, and each person's share was now $9.00. How many attended the family get-together?

________________ 3. In one month Kitsu made $4400 selling 60 items of clothing. Shirts were sold for $30, slacks for $80, and blazers for $120. What is the greatest number of blazers Kitsu could have sold?

________________ sec. 4. At the Mall of America, it takes a shopper 24 seconds to walk down to a lower floor on a moving escalator. If the shopper walks down a stationary (non-moving) escalator at the same walking speed, he will be downstairs in 42 seconds. How many seconds will it take to get downstairs, if he just stands on the moving escalator?

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2020-21 Meet 3, Individual Event A

SOLUTIONS

66

1. (x, y) is the intersection of $\frac{3x}{y} - 4 = 11$ and $x - y = 44$. Determine the value of $x + y$.

$3x = 15y \Rightarrow x = 5y$. Therefore, $5y - y = 44 \Rightarrow y = 11$. So $x = 55$ and $x + y = 66$.

64

2. A large extended family decides to have a catered get-together, with the cost being divided equally among all individuals attending. If everyone that was expected had shown up the cost per person would have been $8.00. As it was, eight people cancelled, and each person's share was now $9.00. How many attended the family get-together?

$8(x+8) = 9x \Rightarrow 8x + 64 = 9x \Rightarrow x = 64$.

25

3. In one month Kitsu made $4400 selling 60 items of clothing. Shirts were sold for $30, slacks for $80, and blazers for $120. What is the greatest number of blazers Kitsu could have sold?

Let x = the number of shirts, y = the number of slacks, and z = the number of blazers. Then $x + y + z = 60$ and $30x + 80y + 120z = 4400$. Eliminating x in the system yields $5y + 9z = 260$. Therefore, $z = \frac{260 - 5y}{9}$. Checking for integer values, the maximum for z is when $y = 7$, $z = 25$.

56

4. At the Mall of America, it takes a shopper 24 seconds to walk down to a lower floor on a moving escalator. If the shopper walks down a stationary (non-moving) escalator at the same walking speed, he will be downstairs in 42 seconds. How many seconds will it take to get downstairs, if he just stands on the moving escalator?

Let E = the escalator speed and W = your walking speed. Then $D = (E + W) \cdot 24$, $D = W \cdot 42$, and $D = E \cdot t$. Then $24E + 24W = 42W \Rightarrow 24E = 18W \Rightarrow E = \frac{3W}{4}$. Therefore,

$W \cdot 42 = \frac{3W}{4} \cdot t \Rightarrow t = 42 \cdot \frac{4}{3} = 56$ seconds.

Minnesota State High School Mathematics League

2020-21 Meet 3, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

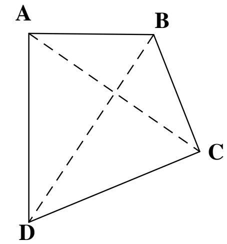


Figure 1

______________ 1. In *Figure 1*, $ABCD$ is a kite with $AC = 4.8$ and $BD = 5$. What is the area of the kite?

$V =$ ______________ 2. In the rectangular prism in *Figure 2*, the areas of the top and bottom faces are each 8, the areas of the front and back faces are each 9, and the area of each side face is 18. What is the volume of the prism?

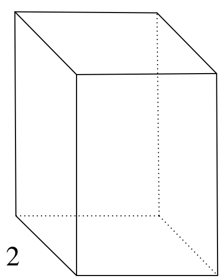


Figure 2

______________ 3. The corners of a 12 inch by 12 inch square are snipped off to form a regular octagon. The side length of the octagon can be written as $a + b\sqrt{2}$. Determine $a + b$.

______________ 4. In *Figure 4*, one base of a trapezoid is 24 and the distance between the midpoints of the diagonals is 4. How long is the other base?

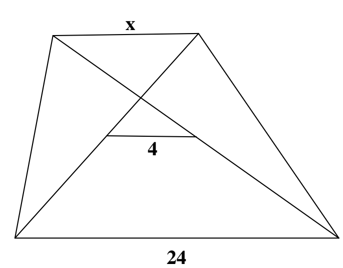


Figure 4

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2020-21 Meet 3, Individual Event B

SOLUTIONS

12

1. In *Figure 1*, $ABCD$ is a kite with $AC = 4.8$ and $BD = 5$. What is the area of the kite?

The area of a kite is half the product of the diagonals, so the area is $\frac{(5)(4.8)}{2} = 12$.

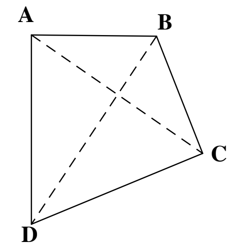


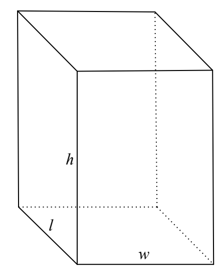
Figure 1

36

2. In the rectangular prism in *Figure 2*, the areas of the top and bottom faces are each 8, the areas of the front and back faces are each 9, and the area of each side face is 18. What is the volume of the prism?

$lw = 8,\ wh = 9,\ and\ lh = 18.\ V = lwh,\ so\ V^2 = l^2w^2h^2 = 8\cdot 9\cdot 18 = 2^3\cdot 3^2\cdot 2\cdot 3^2 = 2^4\cdot 3^4$. *So* $V = 2^2\cdot 3^2 = 36$. *Or notice* $l = 2w$ *and solve the system, yielding* $w = 2,\ l = 4,\ and\ h = \frac{9}{2}$.

Figure 2.1



0

3. The corners of a 12 inch by 12 inch square are snipped off to form a regular octagon. The side length of the octagon can be written as $a + b\sqrt{2}$. Determine $a + b$.

$$As\ in\ Figure\ 3.1,\ \frac{x}{\sqrt{2}} + x + \frac{x}{\sqrt{2}} = 12 \Rightarrow \frac{x\sqrt{2} + 2x + x\sqrt{2}}{2} = 12 \Rightarrow x\sqrt{2} + x = 12 \Rightarrow$$

$$x = \frac{12}{\sqrt{2}+1} = \frac{12\left(\sqrt{2}-1\right)}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)} = 12\left(\sqrt{2}-1\right) = 12\sqrt{2} - 12 = -12 + 12\sqrt{2}.$$

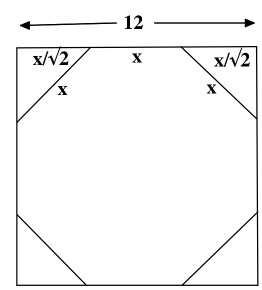


Figure 3.1

16

4. In *Figure 4*, one base of a trapezoid is 24 and the distance between the midpoints of the diagonals is 4. How long is the other base?

Extend the segment connecting the midpoints of the diagonals. This forms the midsegments of the trapezoid $ABCD$ *and triangles* ABC *and* ABD. *Therefore,*

$$\frac{24 + x}{2} = \frac{x}{2} + 4 + \frac{x}{2} \Rightarrow 24 + x = 2x + 8 \Rightarrow x = 16.$$

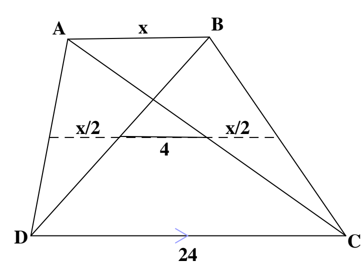


Figure 4.1

Minnesota State High School Mathematics League

2020-21 Meet 3, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

______________ 1. If $\csc^{-1}\left(\frac{2\sqrt{3}}{3}\right)=\theta,$ then $\theta,$ in radians, can be written as $\frac{\pi}{q}.$ Determine the value of q.

______________ 2. The **sum** of all values of θ, in the interval $0\le\theta<2\pi,$ such that $\cos\theta=\cos(2\theta),$ can be written as $p\pi.$ Determine the value of p.

______________ *3.* Determine the value of
$$1+2\left(8-\sin^2(10^\circ)-\sin^2(20^\circ)-\sin^2(30^\circ)-\ldots-\sin^2(170^\circ)\right).$$

______________ 4. Rhombus $ABCD$ has an area of $24\sqrt{3}.$ Using polar coordinates, vertex A is $(0,\ 0)$ and vertex C is $\left(12,\ \frac{\pi}{12}\right).$ If vertex B is in Quadrant I, the polar coordinates $(r,\ \theta)$ of B can be written as $\left(a\sqrt{3},\ \frac{\pi}{b}\right).$ Determine the value of $a+b.$

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2020-21 Meet 3, Individual Event C

SOLUTIONS

$\boxed{3}$ 1. If $\csc^{-1}\left(\frac{2\sqrt{3}}{3}\right)=\theta$, then θ, in radians, can be written as $\frac{\pi}{q}$. Determine the value of q.

Since the range of $\csc^{-1}(x)$ *is* $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\sin\theta=\frac{3}{2\sqrt{3}}=\frac{\sqrt{3}}{2}\Rightarrow\theta=\frac{\pi}{3}$.

$\boxed{2}$ 2. The **<u>sum</u>** of all values of θ, in the interval $0\le\theta<2\pi$, such that $\cos\theta=\cos(2\theta)$, can be written as $p\pi$. Determine the value of p.

$\cos\theta=2\cos^2\theta-1\Rightarrow 2\cos^2\theta-\cos\theta-1=0\Rightarrow(2\cos\theta+1)(\cos\theta-1)=0$. *Therefore,* $\cos\theta=-\frac{1}{2}$ *or* $\cos\theta=1$. *So* $\theta=0$, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$. *The sum is* $0+\frac{2\pi}{3}+\frac{4\pi}{3}=2\pi$.

$\boxed{-1}$ *3.* Determine the value of

$$1+2\left(8-\sin^2\left(10^\circ\right)-\sin^2\left(20^\circ\right)-\sin^2\left(30^\circ\right)-\ldots-\sin^2\left(170^\circ\right)\right).$$

$$1+16-2\sin^2\left(10^\circ\right)-2\sin^2\left(20^\circ\right)-2\sin^2\left(30^\circ\right)-\ldots-2\sin^2\left(170^\circ\right)=$$
$$1-2\sin^2\left(10^\circ\right)+1-2\sin^2\left(20^\circ\right)+1-2\sin^2\left(30^\circ\right)+\ldots+1-2\sin^2\left(170^\circ\right)=$$
$$\cos\left(20^\circ\right)+\cos\left(40^\circ\right)+\cos\left(60^\circ\right)+\ldots+\cos\left(340^\circ\right)=\cos180^\circ=-1.$$

$\boxed{8}$ 4. Rhombus $ABCD$ has an area of $24\sqrt{3}$. Using polar coordinates, vertex A is $(0,\ 0)$ and vertex C is $\left(12,\ \frac{\pi}{12}\right)$. If vertex B is in Quadrant I, the polar coordinates $(r,\ \theta)$ of B can be written as $\left(a\sqrt{3},\ \frac{\pi}{b}\right)$. Determine the value of $a+b$.

See Figure 4.1.

$$[ABCD]=\frac{(AC)(BD)}{2}=\frac{12(BD)}{2}=24\sqrt{3}\Rightarrow BD=4\sqrt{3}\Rightarrow BO=2\sqrt{3}.$$

Therefore, AOB *is a* $30^\circ-60^\circ-90^\circ$ *triangle. So* $AB=4\sqrt{3}$ *and* $m\angle BAO=\frac{\pi}{6}$. $m\angle BAP=\frac{\pi}{12}+\frac{\pi}{6}=\frac{\pi}{4}$. *Therefore,* $B=\left(4\sqrt{3},\ \frac{\pi}{4}\right)$.

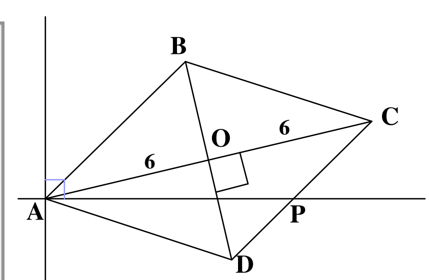


Figure 4.1

Minnesota State High School Mathematics League

2020-21 Meet 3, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

__________ 1. Determine the integer value of $\left(27^{-\frac{2}{3}}\right)\left(9^{\frac{3}{2}}\right)$.

__________ 2. The solution to $5^x + 5^{x+1} = 6\sqrt{5}$ can be written as $\dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

__________ 3. Given that $\log_6 2 = \dfrac{1}{1+\dfrac{1}{1+\dfrac{\log_{10} a}{\log_{10} b}}}$ and that $a = \dfrac{2}{3}$, determine the value of b.

__________ 4. Points A and B lie on the graph of $y = \log x$. Given $A = (2,\ \log 2)$ and $B = (18,\ \log 18)$, let M be the midpoint of $\overline{AB}$. The line through M parallel to the x-axis intersects the y-axis at P. Let O be the origin. The area of $\triangle POM$ can be written as $\log T$. What is the value of the integer T ?

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2020-21 Meet 3, Individual Event D

SOLUTIONS

3

1. Determine the integer value of $\left(27^{-\frac{2}{3}}\right)\left(9^{\frac{3}{2}}\right)$.

$$\left(3^3\right)^{-\frac{2}{3}}\cdot\left(3^2\right)^{\frac{3}{2}}=\left(3^{-2}\right)\left(3^3\right)=3^1.$$

3

2. The solution to $5^x+5^{x+1}=6\sqrt{5}$ can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$$5^x+5^{x+1}=6\sqrt{5}\Rightarrow 5^x\left(1+5\right)=6\cdot 5^{\frac{1}{2}}\Rightarrow 6\cdot 5^x=6\cdot 5^{\frac{1}{2}}\Rightarrow x=\frac{1}{2}.$$

3. Given that $\log_6 2=\cfrac{1}{1+\cfrac{1}{1+\cfrac{\log_{10} a}{\log_{10} b}}}$ and that $a=\frac{2}{3}$, determine the value of b.

3

$\dfrac{\log_{10} a}{\log_{10} b}=\log_b a$. $1+\log_b a=\log_b b+\log_b a=\log_b\left(ab\right)$. $\dfrac{1}{\log_b\left(ab\right)}=\log_{(ab)} b$.

$1+\log_{(ab)} b=\log_{(ab)}\left(ab\right)+\log_{(ab)} b=\log_{(ab)}\left(ab^2\right)$. $\dfrac{1}{\log_{(ab)}\left(ab^2\right)}=\log_{(ab^2)}\left(ab\right)$. *Let*

$ab=2$ *and* $ab^2=6$ *and solving yields* $b=3$ *and* $a=\frac{2}{3}$.

4. Points A and B lie on the graph of $y=\log x$. Given $A=\left(2,\ \log 2\right)$ and $B=\left(18,\ \log 18\right)$, let M be the midpoint of $\overline{AB}$. The line through M parallel to the x-axis intersects the y-axis at P. Let O be the origin. The area of $\triangle POM$ can be written as $\log T$. What is the value of the integer T?

7776

$$M=\left(\frac{2+18}{2},\frac{\log 2+\log 18}{2}\right)=\left(10,\frac{\log 36}{2}\right)=\left(10,\log 6\right).$$ *Therefore,* $P=(0,\log 6)$.

The area of $\triangle POM=\frac{1}{2}(\log 6)(10)=5\log 6=\log\left(6^5\right)=\log 7776.$

Minnesota State High School Mathematics League

2020-21 Meet 3, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 30 minutes, submit only one set of answers. Place your answer to each question on the line provided.

______________ 1. What is the only value of a for which this system of two linear equations will have no solutions?

$$(a+1)x+8y=4a$$
$$ax+(a+3)y=3a-1$$

______________ 2. The set of all x, such that $\log_6\left(x^2-2x-2\right)\le 1$, can be written as $[a,\, b)\cup(c,\, d]$. Determine the value of $a+b+c+d$.

______________ 3. Triangle ABC is isosceles with $AB=AC$. The angle bisector of $\angle B$ intersects $\overline{AC}$ at D. If $\cos\angle BCA=u$, express $\left(\cos(\angle ABD)\right)\cdot\left(\cos(\angle BDA)\right)$ as the polynomial u^2+bu+c. Determine the value of $b+c$.

______________ 4. In isosceles right $\triangle ABC$, with $BA=BC$, point D is on $\overline{AB}$ and E is on $\overline{AC}$, such that $\frac{AD}{DB}=\frac{CE}{AE}=\frac{1}{2}$. The value of $\cos(\angle CED)$ can be written as $\frac{a\sqrt{10}}{10}$. Determine the value of a.

______________ 5. Determine the area of the quadrilateral $ABCD$ bounded by the *x*-axis, the *y*-axis, $x=4$, and the graph of $10^y=25\cdot 2^x$.

______________ cm^2 6. In *Figure 6*, $ABCD$ is a rhombus with $m\angle DAB=120^\circ$ and an area of 100 cm^2. M, N, P, and Q are midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Determine the area of the quadrilateral bounded by $\overleftrightarrow{AP}$, $\overleftrightarrow{BQ}$, $\overleftrightarrow{CM}$, and $\overleftrightarrow{DN}$.

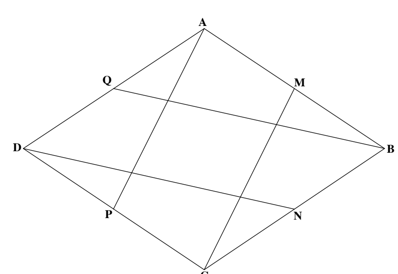


Figure 6

Team: ______________________________

Minnesota State High School Mathematics League

2020-21 Meet 3, Team Event

SOLUTIONS (page 1)

$\boxed{3}$

1. What is the only value of a for which this system of two linear equations will have no solutions?

$$(a+1)x+8y=4a$$
$$ax+(a+3)y=3a-1$$

$\boxed{4}$

2. The set of all x, such that $\log_6\left(x^2-2x-2\right)\le 1$, can be written as $[a,\ b)\cup(c,\ d]$. Determine the value of $a+b+c+d$.

$\boxed{0}$

3. Triangle ABC is isosceles with $AB=AC$. The angle bisector of $\angle B$ intersects $\overline{AC}$ at D. If $\cos\angle BCA=u$, express $\left(\cos(\angle ABD)\right)\cdot\left(\cos(\angle BDA)\right)$ as the polynomial u^2+bu+c. Determine the value of $b+c$.

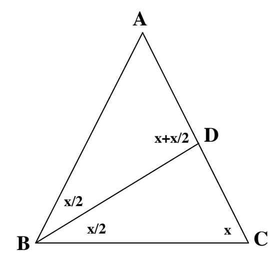


Figure 3.1

$\boxed{-3}$

4. In isosceles right $\triangle ABC$, with $BA=BC$, point D is on $\overline{AB}$ and E is on $\overline{AC}$, such that $\dfrac{AD}{DB}=\dfrac{CE}{AE}=\dfrac{1}{2}$. The value of $\cos(\angle CED)$ can be written as $\dfrac{a\sqrt{10}}{10}$. Determine the value of a.

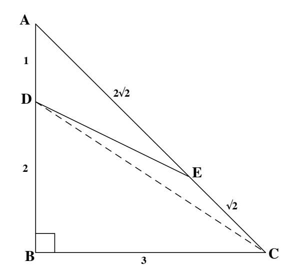


Figure 4.1

$\boxed{8}$

5. Determine the area of the quadrilateral $ABCD$ bounded by the x-axis, the y-axis, $x=4$, and the graph of $10^y=25\cdot 2^x$.

$\boxed{20}$

6. In *Figure 6*, $ABCD$ is a rhombus with $m\angle DAB=120^\circ$ and an area of $100\ cm^2$. M, N, P, and Q are midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Determine the area of the quadrilateral bounded by $\overleftrightarrow{AP}$, $\overleftrightarrow{BQ}$, $\overleftrightarrow{CM}$, and $\overleftrightarrow{DN}$.

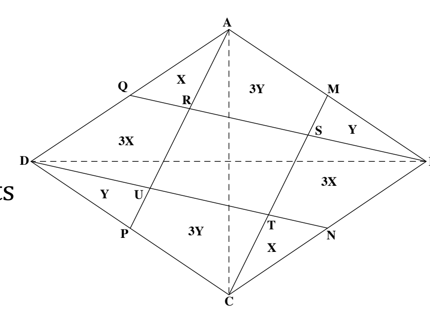


Figure 6.1

Minnesota State High School Mathematics League

2020-21 Meet 3, Team Event

SOLUTIONS (page 2)

1. *Multiply the top equation by* $(-a)$ *and the bottom equation by* $(a+1)$ *Add them together and simplify to get* $y=\dfrac{-(a-1)^2}{(a-1)(a-3)}$. *So* $a \neq 1$ or 3. *But substituting* $a=1$ *into the system yields* $\begin{cases} 2x+8y=4 \\ x+4y=2 \end{cases}$, *which has an infinite number of solutions! Substituting* $a=3$ *into the system yields* $\begin{cases} 4x+8y=12 \\ 3x+6y=8 \end{cases}$, *which yields parallel lines!*

2. *Because the domain of* $y=\log_b x$ *is* $(0, \infty)$ *and* $\log_b b=1$, *the given inequality is equivalent to this system of inequalities:* $\left\{ \begin{array}{l} x^2-2x-2 \le 6 \\ x^2-2x-2>0 \end{array} \right\}$. *The solution of the first inequality is* $[-2, 4]$ *and the solution to the second is* $\left(-\infty, 1-\sqrt{3}\right) \cup \left(1+\sqrt{3}, \infty\right)$. *The intersection of these two sets is* $\left[-2, 1-\sqrt{3}\right) \cup \left(1+\sqrt{3}, 4\right]$.

3. *See Figure 3.1, let* $m\angle C = x$. *Then* $m\angle ABD = \dfrac{x}{2}$ *and* $m\angle BDA = x+\dfrac{x}{2}$. $\cos\left(\dfrac{x}{2}\right)\cdot\cos\left(x+\dfrac{x}{2}\right)=$
$$\left(\sqrt{\frac{1+\cos x}{2}}\right)\left(\cos x\cos\left(\frac{x}{2}\right)-\sin x\sin\left(\frac{x}{2}\right)\right)=\left(\sqrt{\frac{1+\cos x}{2}}\right)\left(\cos x\sqrt{\frac{1+\cos x}{2}}-\sin x\sqrt{\frac{1-\cos x}{2}}\right)=$$
$$\cos x\left(\frac{1+\cos x}{2}\right)-\sin x\sqrt{\frac{1-\cos^2 x}{4}}=\frac{\cos x+\cos^2 x}{2}-\frac{\sin^2 x}{2}=\frac{2\cos^2 x+\cos x-1}{2}=u^2+\frac{1}{2}u-\frac{1}{2}.$$

4. *WLOG, let* $AD=1$, $DB=2$, $BC=3$, $CE=\sqrt{2}$, *and* $EA=2\sqrt{2}$. *Draw* $\overline{DC}$. *In* $\triangle ADE$,
$DE^2=1^2+\left(2\sqrt{2}\right)^2-2(1)\left(2\sqrt{2}\right)\cos 45^\circ=1+8-4\sqrt{2}\left(\dfrac{\sqrt{2}}{2}\right)=5$. *So* $DE=\sqrt{5}$. *In* $\triangle DBC$, $2^2+3^2=DC^2$, *so* $DC=\sqrt{13}$. *In* $\triangle CED$, $\cos(\angle CED)=\dfrac{DE^2+EC^2-DC^2}{2(DE)(EC)}=\dfrac{5+2-13}{2\left(\sqrt{5}\right)\left(\sqrt{2}\right)}=\dfrac{-6}{2\sqrt{10}}=\dfrac{-3\sqrt{10}}{10}$.

5. $\log\left(10^y\right)=\left(\log 25\cdot 2^x\right)\Rightarrow y=\log 25+\log\left(2^x\right)=x\log 2+\log 25$. *This a line with a slope of* $\log 2$ *and a y-intercept of* $\log 25$. *At* $x=4, y=4\log 2+\log 25=\log 400$. *So ABCD is a trapezoid. Let* $A=(0,0)$, $B=(0,\log 25)$, $C=(4,\log 400)$, *and* $D=(4,0)$. *The area of ABCD is* $4\left(\dfrac{\log 25+\log 400}{2}\right)$ $=2\log 10000=8$.

6. *See Figure 6.1.* $\triangle AQR \sim \triangle ADU$. $AQ=\dfrac{1}{2}AD$, $\dfrac{[AQR]}{[ADU]}=\dfrac{1}{4}$. *Let* $[AQR]=x$, *then* $[DQRU]=3x$. *Since* $\triangle ADU \cong \triangle CBS$, *using the same logic,* $[CTN]=x$ *and* $[TSBN]=3x$. $\triangle DPU \sim \triangle DCT$ *and using the same reasoning again, let* $[DPU]=y$ *and* $[PUTC]=3y$ (*and* $[BMS]=y$ *and* $[MSRA]=3y$). $[ADC]=50$, *so* $[ADP]=25=4x+y$. $[ADB]=50$, *so* $[AQB]=25=x+4y$. *Adding the equations, yields* $x+y=10$. *Adding the areas of the nine regions in the rhombus, yields* $8(x+y)+[RSTU]=100 \rightarrow [RSTU]=20$.