Minnesota State High School Mathematics League

2020-21 Meet 1, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

$a+b=$ ______________

1. If $123_4 = \underline{a}\underline{b}_8$, determine the value of $a+b$.

$p+q=$ ______________

2. If $n = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7}}}$, express n as a quotient of two relatively prime integers, $\frac{p}{q}$. Determine the value of $p+q$.

____

3. 3% of 4% of 5% of a **positive** number equals the square of the number, divided by one million. What is the number?

$A+B+C=$ ______________

4. $\underline{A}\underline{B}\underline{A}_C = \underline{B}\underline{A}_{18}$, where A, B, and C are distinct (all different) positive integers. Determine the value of $A+B+C$.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2020-21 Meet 1, Individual Event A

SOLUTIONS

6

1. If $123_4 = \underline{ab}_8$, determine the value of $a+b$.

$123_4 = 1(16) + 2(4) + 3 = 27 = 33_8.$

384

2. If $n = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7}}}$, express n as a quotient of two relatively prime integers, $\frac{p}{q}$.

Determine the value of $p+q$.

$$n = 1 + \frac{2}{3 + \left(\frac{4}{5+\frac{6}{7}}\right)\cdot\frac{7}{7}} = 1 + \frac{2}{3 + \frac{28}{35+6}} = 1 + \left(\frac{2}{3+\frac{28}{41}}\right)\cdot\frac{41}{41} = 1 + \frac{82}{123+28} = 1 + \frac{82}{151} = \frac{233}{151}.$$

60

3. 3% of 4% of 5% of a **positive** number equals the square of the number, divided by one million. What is the number?

$$\frac{3}{100}\cdot\frac{4}{100}\cdot\frac{5}{100}\cdot x = \frac{x^2}{1{,}000{,}000} \Rightarrow \frac{60x}{1000000} = \frac{x^2}{1000000} \Rightarrow 60 = x.$$

10

4. $\underline{ABA}_C = \underline{BA}_{18}$, where A, B, and C are distinct (all different) positive integers.

Determine the value of $A+B+C$.

$A(C^2) + B(C) + A = 18B + A \Rightarrow C(AC+B) = 18B$. *Since* $C > B$, C *must divide 18. Since* A *and* B *are distinct and less than* C, $C \ge 3$. *So* C *must be either 3, 6, or 9. If* $C = 3$, *then* $9A + 3B = 18B \Rightarrow 9A = 15B \Rightarrow 3A = 5B \Rightarrow A = 5$ and $B = 3 \Rightarrow$ *a contradiction since A and B are less than C. If* $C = 6$, *then* $36A + 6B = 18B \Rightarrow 36A = 12B \Rightarrow 3A = B \Rightarrow A = 1$ and $B = 3 \Rightarrow$ *a solution! If* $C = 9$, *then* $81A + 9B = 18B \Rightarrow 81A = 9B \Rightarrow 9A = B \Rightarrow A = 1$ and $B = 9 \Rightarrow$ *a contradiction since B = C. Therefore,* $(1,\ 3,\ 6)$ *is the only solution.*

Check : $131_6 = 1(36) + 3(6) + 1 = 55$ *and* $31_{18} = 3(18) + 1 = 55.$

Minnesota State High School Mathematics League

2020-21 Meet 1, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

$BC =$ ____________

1. In *Figure 1,* in $\triangle ABC$, $AB = 13$, $AC = 20$, and D is a point on $\overline{BC}$ such that $\overline{AD} \perp \overline{BC}$. If $AD = 12$, what is the length of $\overline{BC}$?

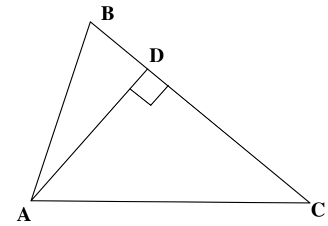
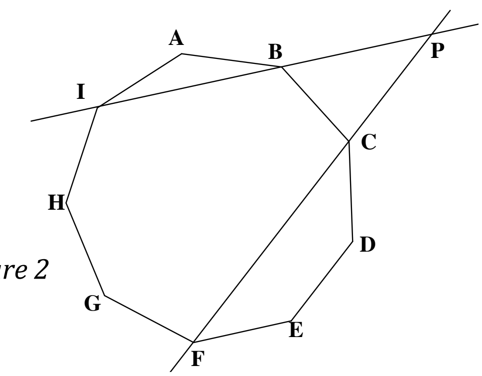


Figure 1

$m\angle P =$ ____________

2. In *Figure 2,* $ABCDEFGHI$ is a regular nonagon (9-sided polygon). Secants $\overleftrightarrow{IB}$ and $\overleftrightarrow{FC}$ intersect at P. What is the measure of acute $\angle P$ in degrees?

Figure 2



$m\angle RST =$ ____________

3. In *Figure 3*, in $\triangle RST$ the bisectors of the exterior angles at R and T intersect at P. If $m\angle RPT = 2m\angle RST$, what is the measure of $\angle RST$ in degrees?

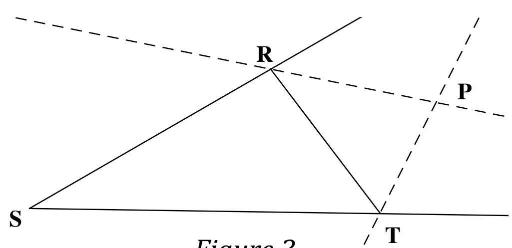
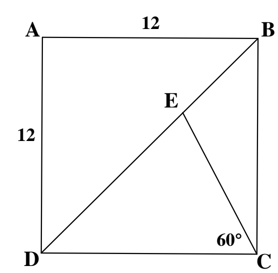


Figure 3

$a+b =$ ____________

4. In *Figure 4,* $ABCD$ is a square with a side length of 12. Point E lies on $\overline{DB}$ such that $m\angle DCE = 60^\circ$. The length of $\overline{DE}$ can be written as $a\sqrt{2} + b\sqrt{6}$. Determine the value of $a+b$.

Figure 4



Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2020-21 Meet 1, Individual Event B

SOLUTIONS

21

1. In *Figure 1,* in $\triangle ABC$, $AB = 13$, $AC = 20$, and D is a point on $\overline{BC}$ such that $\overline{AD} \perp \overline{BC}$. If $AD = 12$, what is the length of $\overline{BC}$?

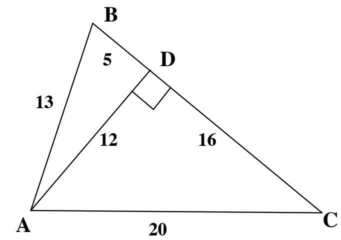


Figure 1.1

See Figure 1.1. Knowing special Pythagorean Triples is helpful. The 12-16-20 triangle is just an enlarged 3-4-5 triangle.

40

2. In *Figure 2,* $ABCDEFGHI$ is a regular nonagon (9-sided polygon). Secants $\overleftrightarrow{IB}$ and $\overleftrightarrow{FC}$ intersect at P. What is the measure of acute $\angle P$ in degrees?

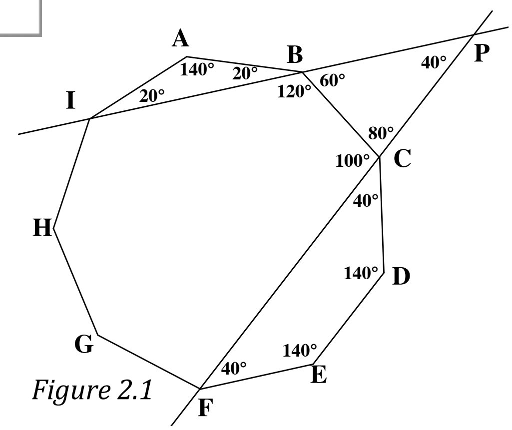


Figure 2.1

The measure of the interior angles of a nonagon is $\dfrac{(9-2)180}{9} = 140^\circ$. *As shown in Figure2.1, with base angles of isosceles triangles and trapezoids being equal, the other angle measures are easily derived.*

36

3. In *Figure 3*, in $\triangle RST$ the bisectors of the exterior angles at R and T intersect at P. If $m\angle RPT = 2m\angle RST$, what is the measure of $\angle RST$ in degrees?

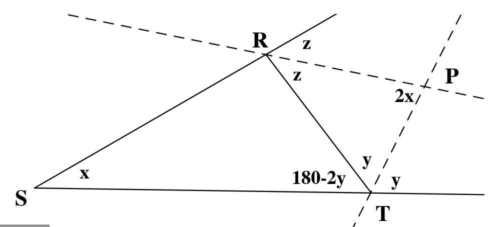


Figure 3.1

Label the angles as in Figure 3.1 In $\triangle RPT$, $z + y + 2x = 180 \Rightarrow z + y = 180 - 2x$. *By the Exterior Angle Theorem,* $2z = x + 180 - 2y \Rightarrow 2(z + y) = x + 180$. *Substituting, gives* $2(180 - 2x) = x + 180 \Rightarrow 5x = 180 \Rightarrow x = 36$.

12

4. In *Figure 4*, $ABCD$ is a square with a side length of 12. Point E lies on $\overline{DB}$ such that $m\angle DCE = 60^\circ$. The length of $\overline{DE}$ can be written as $a\sqrt{2} + b\sqrt{6}$. Determine the value of $a + b$.

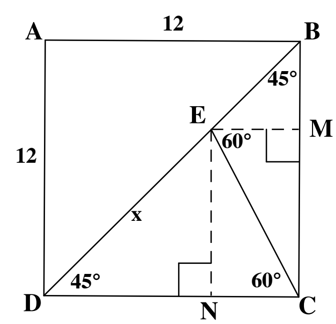


Figure 4.1

As in Figure 4.1, draw $\overline{EM}$ *and* $\overline{EN}$ *perpendicular to sides* $\overline{BC}$ *and* $\overline{DC}$, *respectively. Let* $x = DE$. *Then* $EN = \dfrac{x}{\sqrt{2}} \Rightarrow CN = \dfrac{x}{\sqrt{6}} \Rightarrow CE = \dfrac{2x}{\sqrt{6}}$. $DB = 12\sqrt{2}$, *so* $EB = 12\sqrt{2} - x \Rightarrow EM = \dfrac{12\sqrt{2} - x}{\sqrt{2}} = 12 - \dfrac{x}{\sqrt{2}} \Rightarrow EC = 24 - \dfrac{2x}{\sqrt{2}}$. *Therefore,* $\dfrac{2x}{\sqrt{6}} = 24 - \dfrac{2x}{\sqrt{2}} \Rightarrow \dfrac{x}{\sqrt{6}} + \dfrac{x}{\sqrt{2}} = 12 \Rightarrow x = 18\sqrt{2} - 6\sqrt{6}$.

Minnesota State High School Mathematics League

2020-21 Meet 1, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

$p+q=$ ______________

1. If $\cos\alpha = -\frac{5}{13}$ and $\pi < \alpha < \frac{3\pi}{2}$, the $\tan\alpha$ can be written as a quotient of two relatively prime integers, $\frac{p}{q}$. Determine the value of $p+q$.

$m\angle APB =$ ______________

2. In *Figure 2* (not drawn to scale!), point A is 15 inches from line $\overleftrightarrow{CD}$ and point B is 20 inches from line $\overleftrightarrow{CD}$. Point P is on $\overline{CD}$ such that $m\angle APC = m\angle BPD$. If $AP + PB = 70$ inches, determine the measure of $\angle APB$ in degrees.

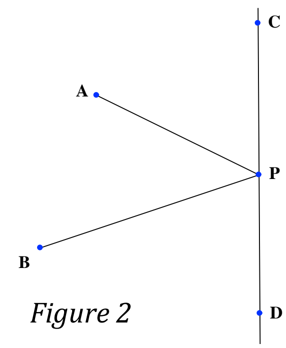


Figure 2

$a+b=$ ______________

3. Determine the smallest positive angle β, in radians, for which the graph of $y = \sin\left(\frac{\pi}{8} - x\right)$ is identical to the graph of $y = \cos(x+\beta)$. If $\beta = \frac{a\pi}{b}$, determine the value of $a+b$.

$p+q=$ ______________

4. Determine the value of :

$$-\sin^2 1^\circ + \sin^2 2^\circ - \sin^2 3^\circ + \ \ldots \ + \sin^2 88^\circ - \sin^2 89^\circ + \sin^2 90^\circ,$$ as a quotient of two relatively prime integers $\frac{p}{q}$. Determine the value of $p+q$.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2020-21 Meet 1, Individual Event C

SOLUTIONS

17

1. If $\cos\alpha = -\frac{5}{13}$ and $\pi < \alpha < \frac{3\pi}{2}$, the $\tan\alpha$ can be written as a quotient of two relatively prime integers, $\frac{p}{q}$. Determine the value of $p+q$.

It's a $5-12-13$ triangle in Quadrant III, so $\tan\alpha = \frac{12}{5}$.

120

2. In *Figure 2* (not drawn to scale!), point A is 15 inches from line $\overleftrightarrow{CD}$ and point B is 20 inches from line $\overleftrightarrow{CD}$. Point P is on $\overline{CD}$ such that $m\angle APC = m\angle BPD$. If $AP + PB = 70$ inches, determine the measure of $\angle APB$ in degrees.

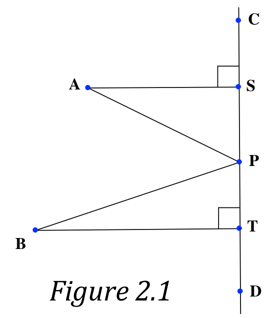


Figure 2.1

As in Figure 2.1, draw $AS \perp CD$ and $BT \perp CD$, then $\triangle APS \sim \triangle BPT$ with a scale factor of 3 to 4. Therefore, $AP = 30''$ and $BP = 40''$. So $\sin(\angle APS) = \frac{15}{30} = \frac{1}{2}$, and $m\angle APS = m\angle BPT = 30^\circ$. So $m\angle APB = 120^\circ$.

11

3. Determine the smallest positive angle β, in radians, for which the graph of $y = \sin\left(\frac{\pi}{8} - x\right)$ is identical to the graph of $y = \cos(x+\beta)$. If $\beta = \frac{a\pi}{b}$, determine the value of $a+b$.

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$. *So* $\sin\left(\frac{\pi}{8} - x\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{8} - x\right)\right) = \cos\left(x + \frac{3\pi}{8}\right)$.

3

4. Determine the value of:

$$-\sin^2 1^\circ + \sin^2 2^\circ - \sin^2 3^\circ + \ \ldots \ + \sin^2 88^\circ - \sin^2 89^\circ + \sin^2 90^\circ,$$ as a quotient of two relatively prime integers $\frac{p}{q}$. Determine the value of $p+q$.

Rearrange the terms:

$-\sin^2 1^\circ - \sin^2 89^\circ + \sin^2 2^\circ + \sin^2 88^\circ - \sin^2 3^\circ - \sin^2 87^\circ + \ \ldots + \sin 44^\circ + \sin 46^\circ \ - \sin^2 45^\circ + \sin^2 90^\circ$.

But $\sin x = \cos(90^\circ - x)$. *So the expression can be rewritten as:*

$-\sin^2 1^\circ - \cos^2 1^\circ + \sin^2 2^\circ + \cos^2 2^\circ - \sin^2 3^\circ - \cos^2 3^\circ + \ \ldots + \sin^2 44^\circ + \cos^2 44^\circ \ - \sin^2 45^\circ + \sin^2 90^\circ =$

$-22 + 22 - \left(\frac{\sqrt{2}}{2}\right)^2 + 1 = -\frac{1}{2} + 1 = \frac{1}{2}$.

Minnesota State High School Mathematics League

2020-21 Meet 1, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

______________ 1. Determine the lesser root of the equation $x^2 - 6x = 7$.

$p+q=$ ______________ 2. The function $f(x) = 2x^2 + px + q$ has zeros 1 and -6. Determine the value of $p+q$.

$p+q=$ ______________ 3. The function $f(x) = 4x^2 - x + 2$ is defined for all $x \geq a$, where a is some real number. If $f(x)$ has an inverse, $f^{-1}(x)$, the least possible value of a can be written as a quotient of two relatively prime integers, $\frac{p}{q}$. Determine the value of $p+q$.

$A+B+C=$ ______________ 4. Jean is thinking of a quadratic equation $Ax^2 + Bx + C = 0$, where A, B and C are digits 1 through 9. Jean tells you the sum of the *squares* of the solutions to this equation is exactly 1.76. Determine the value of $A+B+C$.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2020-21 Meet 1, Individual Event D

SOLUTIONS

-1

1. Determine the lesser root of the equation $x^2-6x=7$.

$x^2-6x-7=(x+1)(x-7)\Rightarrow x=-1$ *or* $x=7$.

-2

2. The function $f(x)=2x^2+px+q$ has zeros 1 and -6. Determine the value of $p+q$.

$1+(-6)=-5\Rightarrow -5=-\frac{p}{2}\Rightarrow p=10$. $(1)(-6)=-6\Rightarrow -6=\frac{q}{2}\Rightarrow q=-12$. *Therefore,* $p+q=-2$.

9

3. The function $f(x)=4x^2-x+2$ is defined for all $x\ge a$, where a is some real number. If $f(x)$ has an inverse, $f^{-1}(x)$, the least possible value of a can be written as a quotient of two relatively prime integers, $\frac{p}{q}$. Determine the value of $p+q$.

The graph of $f(x)$ *is a parabola.* $f^{-1}(x)$ *exists for all* $x\ge a$, *if and only if, the graph of* $f(x)$ *passes "the horizontal line test" on that interval. The axis of symmetry of* $f(x)$ *is* $x=\frac{-b}{2a}=\frac{1}{2(4)}=\frac{1}{8}$. *So* $x\ge\frac{1}{8}$.

15

4. Jean is thinking of a quadratic equation $Ax^2+Bx+C=0$, where A, B and C are digits 1 through 9. Jean tells you the sum of the *squares* of the solutions to this equation is exactly 1.76. Determine the value of $A+B+C$.

Let the solutions be r *and* s. *Then* $r+s=\frac{-B}{A}$ *and* $rs=\frac{C}{A}$. $r^2+s^2=(r+s)^2-2rs=\left(\frac{-B}{A}\right)^2-2\left(\frac{C}{A}\right)=\frac{B^2-2AC}{A^2}=1.76=\frac{44}{25}$. *Therefore,* $A=5$ *and* $B^2-10C=44$. *So* B^2 *is even and greater than 44. So* $B=8$ *and* $C=2$.

Minnesota State High School Mathematics League

2020-21 Meet 1, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 30 minutes, submit only one set of answers. Place your answer to each question on the line provided.

$N =$ __________

1. Let N be the 4-digit number $\underline{a}\underline{b}\underline{c}\underline{d}$, where $a^2 + d^2 = 13$, $b^2 + c^2 = 85$, and $N - 1089 = \underline{d}\underline{c}\underline{b}\underline{a}$, another 4-digit number with the digits of N reversed. Find N.

$k =$ __________

2. In *Figure 2*, $\triangle ABC$ is isosceles with base $BC = 12$. Points B, C and D are collinear. Point E is on $\overline{AC}$ such that $\triangle BED$ is isosceles with base $BD = 16$. If $ED = 8\sqrt{2}$ and $AE = k\sqrt{5}$, determine the value of k.

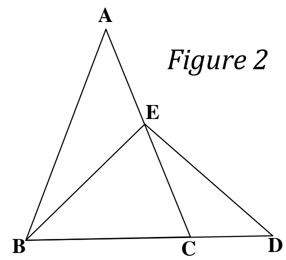


Figure 2

$BC =$ __________

3. In *Figure 3*, in $\triangle ABC$, $AB = AC$. Point D is in the interior of $\triangle ABC$, with perpendiculars $\overline{DE}$, $\overline{DF}$, and $\overline{DG}$ drawn to the sides as shown. $DG = 4\sqrt{3}$ and $DE + DF = 12$. If $m\angle EDF = 60^\circ$, how long is $\overline{BC}$?

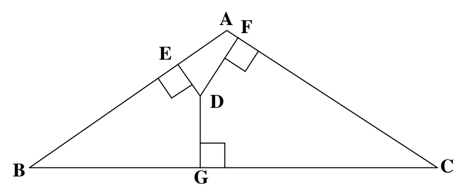


Figure 3

$a + b =$ __________

4. A milk carton, with a 2" × 2" base, contains 1" of milk while sitting on the counter. The carton is then tilted about one edge making an angle of 30° as shown in *Figure 4*, making the milk level $\overline{CD}$. If the exact length of $\overline{AD}$ is written as $\dfrac{a + \sqrt{3}}{b}$, determine the value of $a + b$.

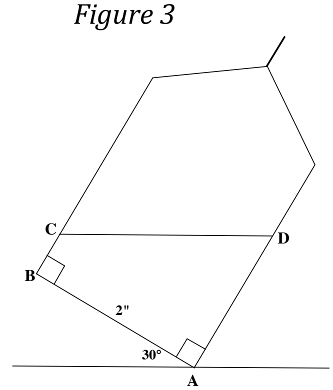


Figure 4

$p + q =$ __________

5. Let $f(x) = 20x^2 - 20x - 1$. Determine exactly the product of all values of x satisfying $\left(f(x)\right)^2 = 4$, expressing the answer as a quotient of two relatively prime integers, $\dfrac{p}{q}$. Determine the value of $p + q$.

____

6. Suppose $\cos\theta = \tan\theta$. Determine the value of $\dfrac{1}{\sin\theta} + \cos^4\theta$.

Team: ____________________

Minnesota State High School Mathematics League

2020-21 Meet 1, Team Event
SOLUTIONS (page 1)

3762

1. Let N be the 4-digit number $\underline{a}\underline{b}\underline{c}\underline{d}$, where $a^2+d^2=13$, $b^2+c^2=85$, and $N-1089=\underline{d}\underline{c}\underline{b}\underline{a}$, another 4-digit number with the digits of N reversed. Find N.

2

2. In *Figure 2*, $\triangle ABC$ is isosceles with base $BC=12$. Points B, C and D are collinear. Point E is on $\overline{AC}$ such that $\triangle BED$ is isosceles with base $BD=16$. If $ED=8\sqrt{2}$ and $AE=k\sqrt{5}$, determine the value of k.

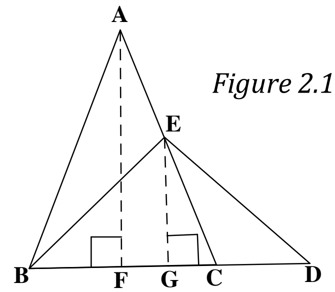


Figure 2.1

48

3. In *Figure 3*, in $\triangle ABC$, $AB=AC$. Point D is in the interior of $\triangle ABC$, with perpendiculars $\overline{DE}$, $\overline{DF}$, and $\overline{DG}$ drawn to the sides as shown. $DG=4\sqrt{3}$ and $DE+DF=12$. If $m\angle EDF=60^\circ$, how long is $\overline{BC}$?

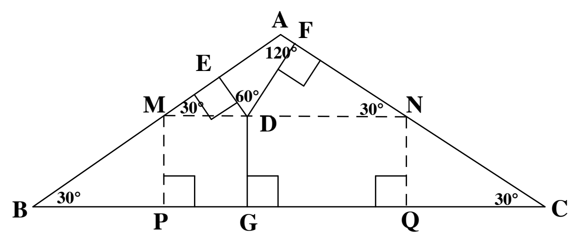


Figure 3.1

6

4. A milk carton, with a 2" $\times$ 2" base, contains 1" of milk while sitting on the counter. The carton is then tilted about one edge making an angle of 30° as shown in *Figure 4*, making the milk level $\overline{CD}$. If the exact length of $\overline{AD}$ is written as $\dfrac{a+\sqrt{3}}{b}$, determine the value of $a+b$.

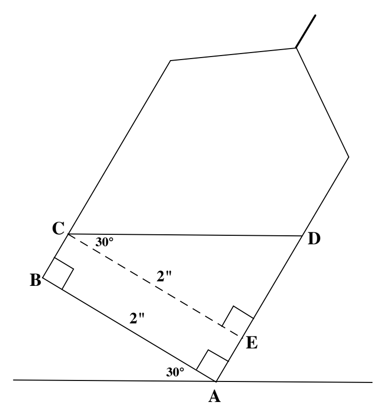


Figure 4.1

397

5. Let $f(x)=20x^2-20x-1$. Determine exactly the product of all values of x satisfying $\left(f(x)\right)^2=4$, expressing the answer as a quotient of two relatively prime integers, $\dfrac{p}{q}$. Determine the value of $p+q$.

2

6. Suppose $\cos\theta=\tan\theta$. Determine the value of $\dfrac{1}{\sin\theta}+\cos^4\theta$.

Minnesota State High School Mathematics League

2020-21 Meet 1, Team Event
SOLUTIONS (page 2)

1. $a^2+d^2=13 \Rightarrow \{a,d\}=\{2,3\}$. $b^2+c^2=85 \Rightarrow \{b,c\}=\{2,9\}$ *or* $\{6,7\}$.

$1000a+100b+10c+d-1089=1000d+100c+10b+a \Rightarrow 999a+90b-90c-999d=1089 \Rightarrow$

$111a+10b-10c-111d=121 \Rightarrow 111(a-d)+10(b-c)=121$. $a-d=1$ *or* -1. *If* $a-d=1$, *then*

$10(b-c)=10 \Rightarrow b-c=1$. *So in this case,* $N=3762$. *If* $a-d=-1$, *then* $10(b-c)=232 \Rightarrow b-c=23.2$, *an impossibilty.*

2. *As in Figure 2.1, draw altitudes* $\overline{AF}$ *and* $\overline{EG}$. $CD=4$, $BF=FC=6$, $BG=GD=8$, *and, therefore,* $GC=4$. *Since* $ED=\sqrt{2}\cdot GD$, $\angle EDG=45^\circ$, *so* $EG=8$. *Using the Pythagorean Theorem in* ΔEGC, $EC=4\sqrt{5}$. $\Delta AFC \sim \Delta EGC$, *and so* $AC=6\sqrt{5}$. *Therefore,* $AE=2\sqrt{5}$.

3. *As shown in Figure 3.1, through point* D, *draw* $\overline{MN}$ *parallel to* $\overline{BC}$ *and drop perpendiculars to* $\overline{BC}$ *from* M *and* N. *Since the angles of a quadrilateral add up to* 360°, *the* $m\angle EAF=120^\circ$, *making* $m\angle B=m\angle C=30^\circ$. *Also* $m\angle AMN=m\angle ANM=30^\circ$. *Let* $ED=x$, *then* $DF=12-x$. $MD=2x=PG$ *and* $DG=MP=NQ=4\sqrt{3}$. *Therefore,* $BP=QC=\left(4\sqrt{3}\right)\cdot\sqrt{3}=12$. *So* $BC=12+2x+(24-2x)+12=48$.

4. *Since, after tilting the carton, the volume of the milk has not changed,* $\dfrac{BC+AD}{2}=1 \Rightarrow BC+AD=2$. *As shown in Figure 4.1, draw* $\overline{CE}$ *parallel to* $\overline{BA}$. *Since* $AB=CE=2$, $ED=\dfrac{2}{\sqrt{3}}$. *Let* $BC=AE=y$. *Then*

$$y+\left(y+\frac{2}{\sqrt{3}}\right)=2 \Rightarrow y=\frac{2\sqrt{3}-2}{2\sqrt{3}}=1-\frac{1}{\sqrt{3}}=\frac{3-\sqrt{3}}{3}.$$ *Therefore,* $AD=\dfrac{3-\sqrt{3}}{3}+\dfrac{2\sqrt{3}}{3}=\dfrac{3+\sqrt{3}}{3}$.

5. *Since* $(f(x))^2=4$, *then* $f(x)=2$ *or* $f(x)=-2$. *These solutions yield two equations:* $20x^2-20x-3=0$ *and* $20x^2-20x+1=0$. *The products of the roots of these equations are* $\dfrac{-3}{20}$ *and* $\dfrac{1}{20}$, *respectively. The product of all four roots then is* $\left(\dfrac{-3}{20}\right)\cdot\left(\dfrac{1}{20}\right)=\dfrac{-3}{400}$.

6. $\cos\theta=\tan\theta \Rightarrow \cos\theta=\dfrac{\sin\theta}{\cos\theta} \Rightarrow \cos^2\theta=\sin\theta$. $\dfrac{1}{\sin\theta}+\cos^4\theta=\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta}+\left(\cos^2\theta\right)^2=$

$\dfrac{\sin^2\theta+\sin\theta}{\sin\theta}+\sin^2\theta=\sin\theta+1+\left(1-\cos^2\theta\right)=\sin\theta+1+1-\sin\theta=2.$