Number and Algebra Review

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Introduction

The unit that we will be recapitulating is Unit 1 Number and Algebra, which includes a portion of chapter 5, specifically 5.3, 5.4, 5.5, 5.6, and all of chapter 12 in the Precalculus Sullivan book. The segments in chapter 5 focus on logarithms and exponentials, where 5.3 discusses logarithmic functions, 5.4 explores exponential functions, 5.5 studies properties of logarithms, and 5.6 concentrates on logarithmic and exponential equations. Despite the fact that chapter 5 on exponential and logarithmic functions is primarily a part of the functions unit, segments 5.3, 5.4, 5.5 and 5.6 are beneficial in understanding log and exponent rules, which are part of the number and algebra unit, therefore they will be included in this investigation. Chapter 12 focuses on sequences and series, induction and binomial theorem, where 12.1 examines sequences as a whole, 12.2 goes into detail about arithmetic sequences, 12.3 explores geometric sequences and series, 12.4 discusses mathematical induction and 12.5 investigates binomial theorem.

In step 1, we will identify and state the key terms, deduce the required formulas, and define their functions in each segment of the unit. The keywords of chapter 5 will only include those that pertain to unit 1, such as **exponent**, **laws of exponents**, **exponential growth**, **growth factor**, **initial value**, **power, base, change of base rule, product rule, quotient rule, power rule and the derivative of a logarithm**. The keywords from Chapter 12 include **arithmetic sequence**, **geometric sequence and series**, **terms**, **general term**, **pattern**, **factorial symbol**, **recursive formula, summation notation, common difference, arithmetic progression, common ratio, geometric progression, principle of mathematical induction, integer, expansion, permutation, combination, pascal's triangle,** and **binomial coefficient**. The key terms will be organized, listed, and defined in correspondence to their segments within each chapter. In addition to the keywords, step 1 will include all of the necessary formulas for unit 1, and will be displayed with the key terms that the formula is related to *see step 1 for all of the formulas included*.

In step 2, we will discuss the key concepts of each segment with worked examples to show demonstration of knowledge in that area. For chapter 5, the key concepts will include **laws of exponents, solving exponential equations, changing exponents to log** and vice versa, **finding values of log expressions, solving logarithms, exercising the properties of logs** *see step two for details on properties*, and **solving logarithmic and exponential equations**. The key concepts in chapter 12.1 will include **writing first several terms of a sequence, determining a sequence from a pattern, using recursively defined sequences, expanding and writing the sum with summation notation, and finding the sum of a sequence**. 12.2 and 12.3 will discuss **determining if a sequence is arithmetic/geometric, finding a term in an arithmetic/geometric sequence, finding the sum of an arithmetic/geometric sequence, and finding the sum of infinity in a geometric sequence**. The key concepts of 12.4 and 12.5 are using the **principle of mathematical induction**, using **Pascal's triangle**, **expanding a binomial**, and **finding a coefficient of a term in a binomial expansion**. Each concept will have worked examples that will be displayed through various technological platforms, such as videos, animations, and digital guided examples.

Step 3 will display a real-world example using key concepts that have been previously discussed, such as solving logarithms and exponential equations. The real-world example investigates how many taxis will be needed in various years, as well as how many people use taxis in various years using exponential equations.

In step 4 we will provide 10 practice problems, 3 easy, 4 medium, and 3 hard and will provide the answers in conclusion to the investigation.

Key Words and Formulas (Step 1)

Keywords	Definitions	Chapter
exponent	a quantity representing the power to which a given number or expression is to be raised, usually expressed as a raised symbol beside the number or symbol beside the number or expression	5
Laws of exponent	A law that states that exponents of numbers are added when the numbers are multiplied, subtracted when the numbers are divided, and multiplied when raised by another exponent. Refer to power, product rule, product rule, and quotient rule $a^x = b \rightarrow x = \log_a b$	5
Exponential growth	Growth whose rate becomes ever more rapid in proportion to the growing total number or size	5
Growth factor	The factor by which a quantity multiplies itself over time	5
Initial value	The point at which a function begins	5
Power	The power (or exponent) of a number says how many times to use the number in multiplication	5
Base	The number or different digits or combinations of digits that a system of counting uses to represent numbers	5

Change of base rule	$\log_a x = \frac{\log_b x}{\log_b a}$	5
Product rule	When multiplying two powers that have the same base, you can add the exponents	5
Quotient rule	When dividing two powers with the same base, you can subtract the exponents	5
Power rule	To raise a power to a power, multiply exponents	5
Arithmetic sequence	A list of numbers with a definite pattern (addition or subtraction) $u_n = u_1 + (n-1)d$	12
Geometric sequence and series	A sequence of numbers where each term after the first term in found by multiplying the previous term $u_n = u_1 r^{n-1}$	12
Sum of n terms of a finite geometric sequence	$S_n = \frac{u_1(r^n - 1)}{r-1}$	12
Infinite Geometric sequence	$S_n = \frac{u_1}{1-r}$	12
General term	a mathematical expression composed of variables and constants that yields the successive terms of a sequence or series when integers are substituted for one of the variables often denoted by k	12
Pattern	a series or sequence that repeats.	12

Factorial	the product of all positive integers less than or equal to a given positive integer and denoted by that integer and an exclamation point	12
Recursive formula	a formula that defines the terms of a sequence using previous terms.	12
Summation notation	a convenient and simple form of shorthand used to give a concise expression for a sum of the values of a variable	12
Common difference	the amount between each number in an arithmetic sequence	12
Common ratio	the amount between each number in a geometric sequence	12
Principle of mathematical induction	a technique for proving a statement - a theorem, or a formula - that is asserted about every natural number.	12
Permutation	Any of the ways we can arrange things, where the order is important	12
Combination	selection of items from a collection, such that (unlike permutations) the order of selection does not matter.	12
Pascal's triangle	a triangle of numbers where each number is the two numbers directly above it added together (except for the edges, which are all "1")	12
Binomial coefficient	the number of ways of picking unordered outcomes from possibilities, also known	12

	as a combination or combinatorial number $\binom{n}{r} = \frac{n!}{r!(n-r)!}$	

Key Concepts (Step 2)

5.4

I. Laws of Exponents

A Key To The Laws Of Exponents

Law	Example
$a^m a^n = a^{m+n}$	$2^3 2^4 = 2^{3+4} = 2^7 = 128$
$(a^m)^n = a^{mn}$	$(2^3)^4 = 2^{3.4} = 2^{12} = 4096$
$(ab)^n = a^n b^n$	$(20)^3 = (2.10)^3 = 2^3.10^3 = 8.1000 = 8000$
$(\frac{a}{b})^n = \frac{a^n}{b^n}$	$(\frac{2}{5})^3 = \frac{2^3}{5^3} = \frac{8}{125}$
$\frac{a^m}{a^n} = a^{m-n}$	$\frac{2^5}{2^3} = 2^{5-3} = 2^2 = 4$
$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$	$\frac{2^3}{2^5} = \frac{1}{2^{5-3}} = \frac{1}{2^2} = \frac{1}{4}$

II.Solving Exponential Equations

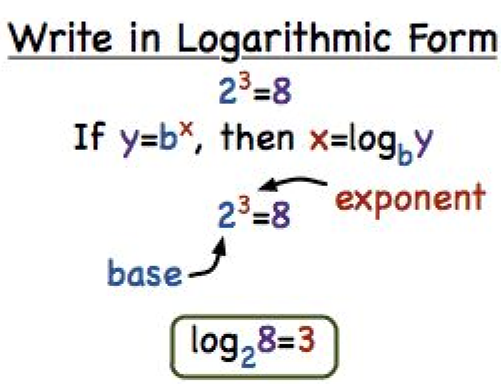
$$5^{3x} = 5^{7x-2}$$

$$3x = 7x - 2 \ \left[since\ both\ sides\ of\ the\ equation\ have\ the\ same\ base,\ set\ the\ exponents\ equal\ to\ each\ other\right]$$

$$2 = 4x \ \left[subtract\ 7x\ from\ both\ sides,\ divide\ both\ sides\ by\ -1\right]$$

$$\frac{1}{2} = x \ \left[divide\ both\ sides\ by\ 4,\ x = \frac{1}{2}\right]$$

III. Changing Exponents to Logarithms



More resources: https://www.youtube.com/watch?v=f0C1KL7GkqY

IV. Finding Values of Logarithmic Expressions

Find the exact value of:

(a) $\log_2 16$ (b) $\log_3 \frac{1}{27}$

(a) To evaluate $\log_2 16$, think "2 raised to what power yields 16." So,

$y = \log_2 16$

$2^y = 16$ Change to exponential form.

$2^y = 2^4$ $16 = 2^4$

$y = 4$ Equate exponents.

Therefore, $\log_2 16 = 4$.

(b) To evaluate $\log_3 \frac{1}{27}$, think "3 raised to what power yields $\frac{1}{27}$." So,

$y = \log_3 \frac{1}{27}$

$3^y = \frac{1}{27}$ Change to exponential form.

$3^y = 3^{-3}$ $\frac{1}{27} = \frac{1}{3^3} = 3^{-3}$

$y = -3$ Equate exponents.

Therefore, $\log_3 \frac{1}{27} = -3$.

More resources: https://www.youtube.com/watch?v=kfJeNbdNlsw

5.5

VI. Properties of Logarithms

Logarithmic Properties	
Product Rule	$\log_a(xy) = \log_a x + \log_a y$
Quotient Rule	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
Power Rule	$\log_a x^p = p \log_a x$
Change of Base Rule	$\log_a x = \frac{\log_b x}{\log_b a}$
Equality Rule	If $\log_a x = \log_a y$ then $x = y$

V. Solving Logarithms

Techniques For Solving Logarithmic Equations

Example: Solve and identify any extraneous solutions.

a) $\log_3(x+1) = 2$

$x + 1 = 3^2$

$x + 1 = 9$

$\therefore x = 8$

b) $\log_5(3x-8) = 2$

$3x - 8 = 5^2$

$3x = 25 + 8$

$3x = 33$

$\therefore x = 11$

c) $\log(x+2) + \log(x-1) = 1$

$\log[(x+2)(x-1)] = 1$

$\log(x^2 + x - 2) = 1$

$x^2 + x - 2 = 10^1$

$x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

$x = -4, 3$

d) $\log x^4 - \log 3 = \log(3x^2)$

More Resources:

https://www.youtube.com/watch?time_continue=1261&v=fnhFneOz6n8&feature=emb_logo

VII. Using Change of Base Formula

Change of Base Formula

For any base $a > 0$ and $a \neq 1$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

or

$$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

$$\begin{aligned}
\log_4 8 &= \frac{\log_2 8}{\log_2 4} \\
&= \frac{\log_2 (2)^3}{\log_2 (2)^2} \\
&= \frac{3 \cdot \log_2 2}{2 \cdot \log_2 2} \\
&= \frac{3 \cdot 1}{2 \cdot 1} \\
\log_4 8 &= \frac{3}{2}
\end{aligned}$$

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$$\begin{aligned}
\log_7 (9) &= \frac{\log_e (9)}{\log_e (7)} \\
&= \frac{\ln (9)}{\ln (7)} \\
&= \frac{2.1972245773...}{1.9459101490...} \\
\log_7 (9) &\approx 1.13
\end{aligned}$$

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More Resources: https://www.youtube.com/watch?v=OkFdDqW9xxM

VIII. Solving Logarithmic and Exponential Equations

Solving Exponential and Logarithmic Equations

$$\log_2 x + \log_2 (x - 3) = 2$$

$$\log_2 (x^2 - 3x) = 2$$

$$x^2 - 3x - 4 = 0$$

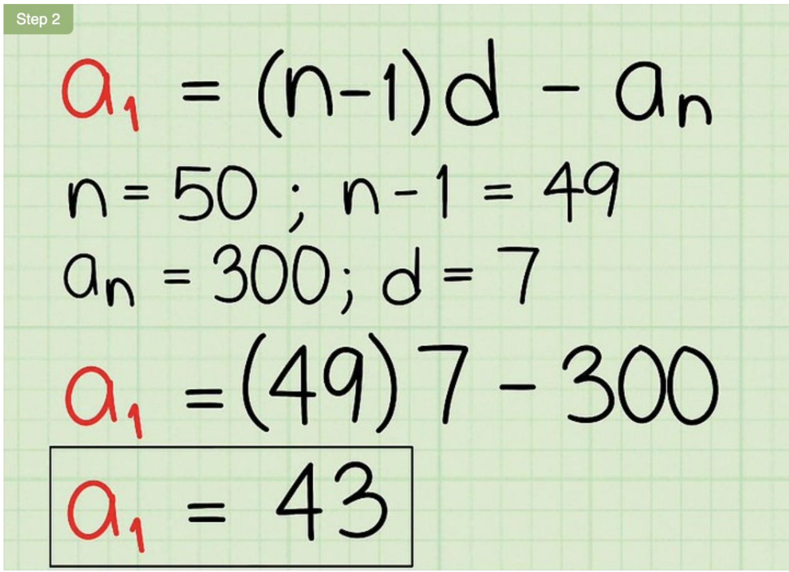
$$(x - 4)(x + 1) = 0$$

$$x = 4, -1$$

More Resources: https://www.youtube.com/watch?v=10I_TVuYLkQ

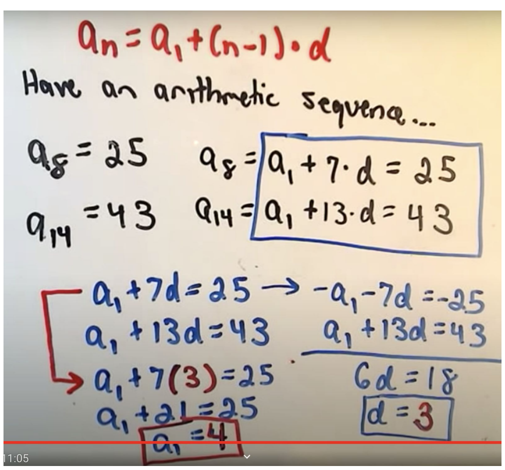
12.1

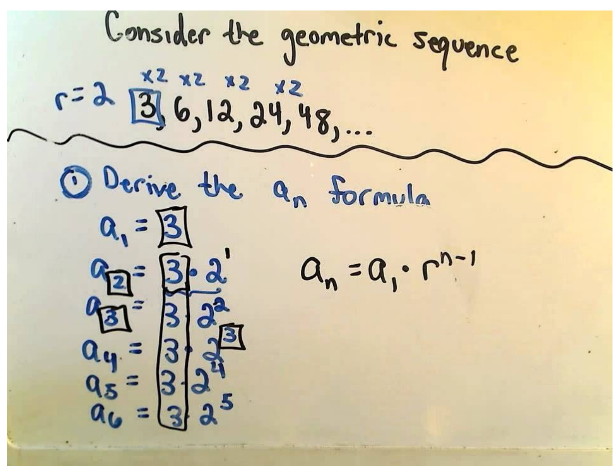
I. Calculating the first several terms in a sequence



More Resources: https://www.youtube.com/watch?v=VGaUKSuaQMw

II. Determining a Sequence from a Pattern

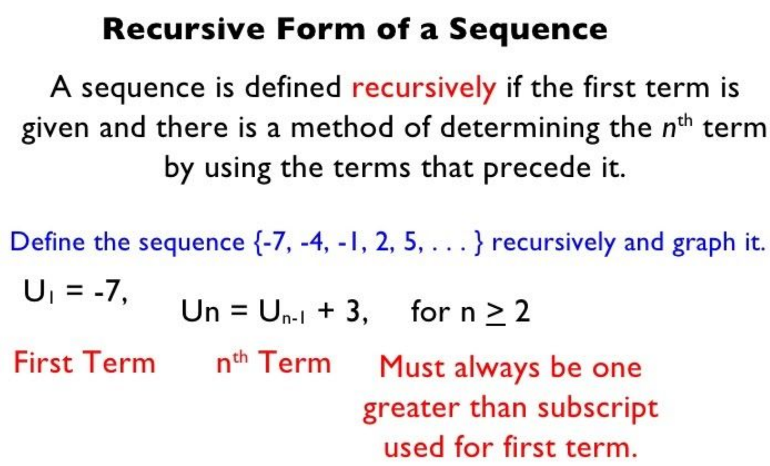




More resources to help determine an arithmetic sequence from a pattern:
https://www.youtube.com/watch?v=lj_X9JVSF8k

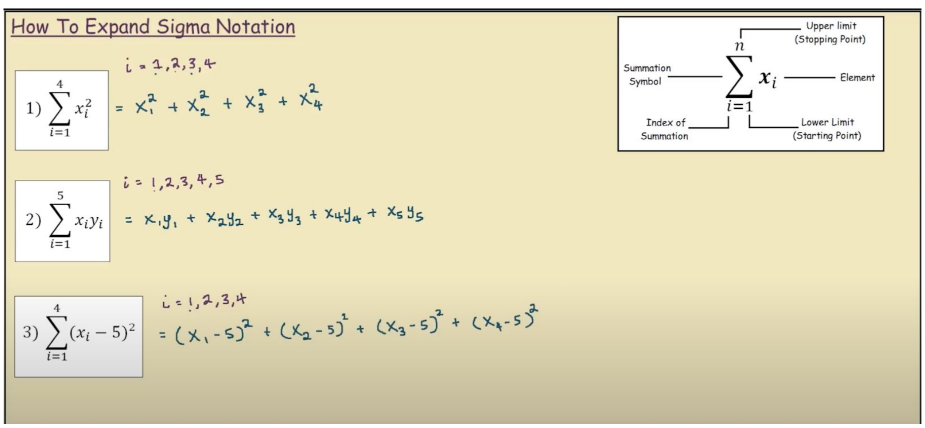
More resources to help determine an geometric sequence from a pattern:
https://www.youtube.com/watch?v=IGFQXInm-co

III. Using Recursively Defined Sequences



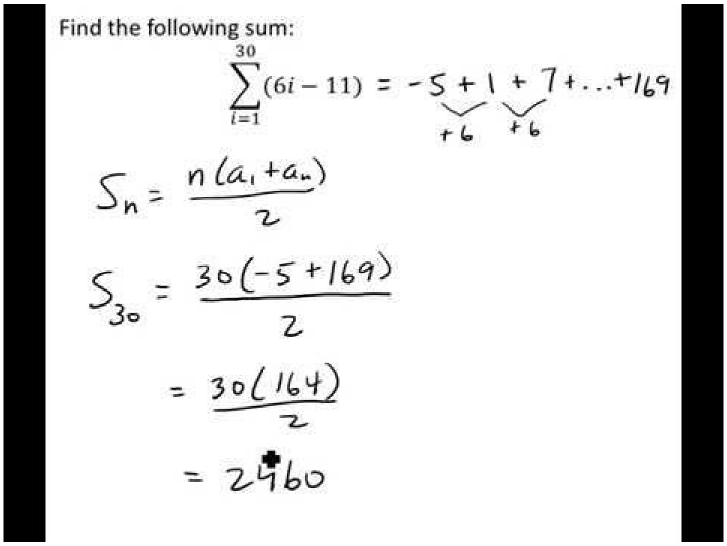
More Resources:
https://www.khanacademy.org/math/old-ap-calculus-bc/bc-series/bc-inf-sequences/v/term-of-recursive-sequence

IV. Expanding and Writing the Sum with Summation Notation



More resources:
https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-3/v/sigma-notation-sum

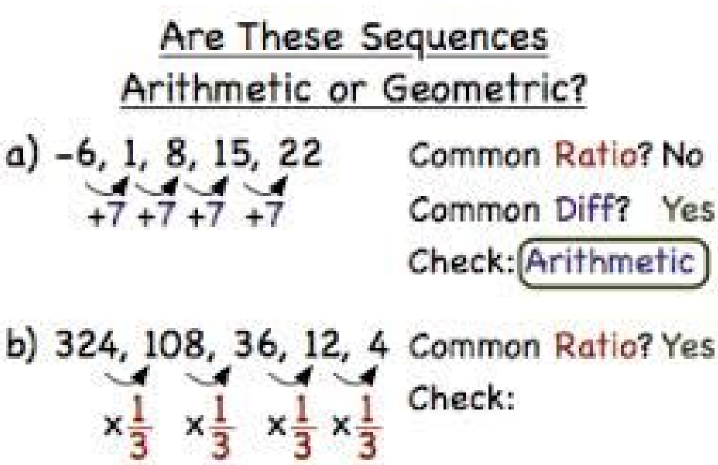
V. Finding the Sum of a Sequence



More resources: https://www.youtube.com/watch?v=gwKocfP60AI

12.2 and 12.3

I. Determining if a sequence is arithmetic or geometric



More resources: https://www.youtube.com/watch?v=JsPMNgg4fTc

II. Finding a term in an arithmetic/geometric sequence

Formulas:

Arithmetic	$a_n = a_1 + (n-1)d$ 1. Find the common difference 2. Substitute the term you are finding for n, the common difference for d, and the first term for a_1 3. Solve for a_n	Find the 20th Term 2, 4, 6, 8, . . . d = 2 a_1 a_2 $a_n = a_1 + (n-1)d$ $a_{20} = 2 + (20-1)\cdot 2$ $= 2 + 19\cdot 2$ $= 2 + 38$ $= \boxed{40}$
Geometric	$a_n = a_1 r^{(n-1)}$ 1. Find the common ratio 2. Substitute the term you are finding for n, the first term for a_1, and the common ratio for r 3. Solve for a_n	nth Term in a Geometric Sequence 2, 8, 32, 128, ... $\boxed{A(n) = a\cdot r^{n-1}}$ n=term number A(n)=nth term a=first term r=common ratio $A(n) = a\cdot r^{n-1}$ r=4 a=2 n=9 $A(9) = 2\cdot 4^{9-1}$ $A(9) = 2\cdot 4^{8}$ $A(9) = 2\cdot 65{,}536$

III. Finding the sum of an arithmetic/geometric sequence

Arithmetic:

Formula

$$\sum_{k=0}^{n-1}(a+kd) = \frac{n}{2}(2a+(n-1)d)$$

$$\begin{aligned}
\frac{n}{2}(2a+(n-1)d) &= \frac{10}{2}(2\cdot 5+(10-1)2)\\
&= \frac{10}{2}(2\cdot 5+(9)2)\\
&= \frac{10}{2}(2\cdot 5+18)\\
&= \frac{10}{2}(10+18)\\
&= \frac{10}{2}(28)\\
&= 5(28)\\
&= 140
\end{aligned}$$

Geometric:

Formula

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Example

Find the sum of the first six terms of the geometric sequence 250, 50, 10, ...

Here $a_1 = 250$, $r = \frac{1}{5}$, and $n = 6$. We substitute these values into the formula for the sum of the first n terms of a geometric sequence and simplify:

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

$$S_n = \frac{250 - 250\left(\frac{1}{5}\right)^6}{1 - \frac{1}{5}}$$

$$= \frac{250 - 250\left(\frac{1}{15{,}625}\right)}{\frac{4}{5}}$$

$$= \frac{5}{4}\left(250 - \frac{250}{15{,}625}\right)$$

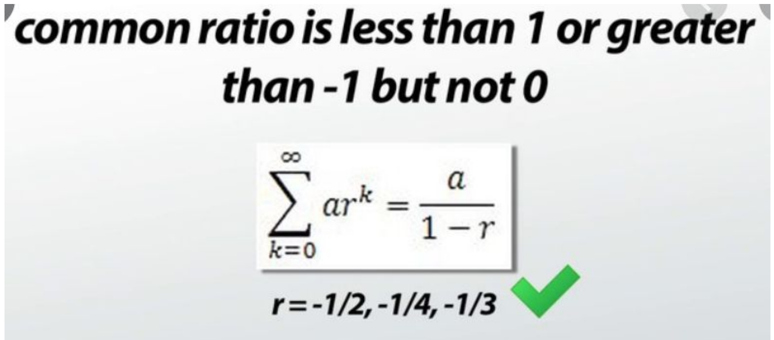
$$= \frac{5}{4}\left(\frac{3{,}906{,}000}{15{,}625}\right)$$

$$= 312.48$$

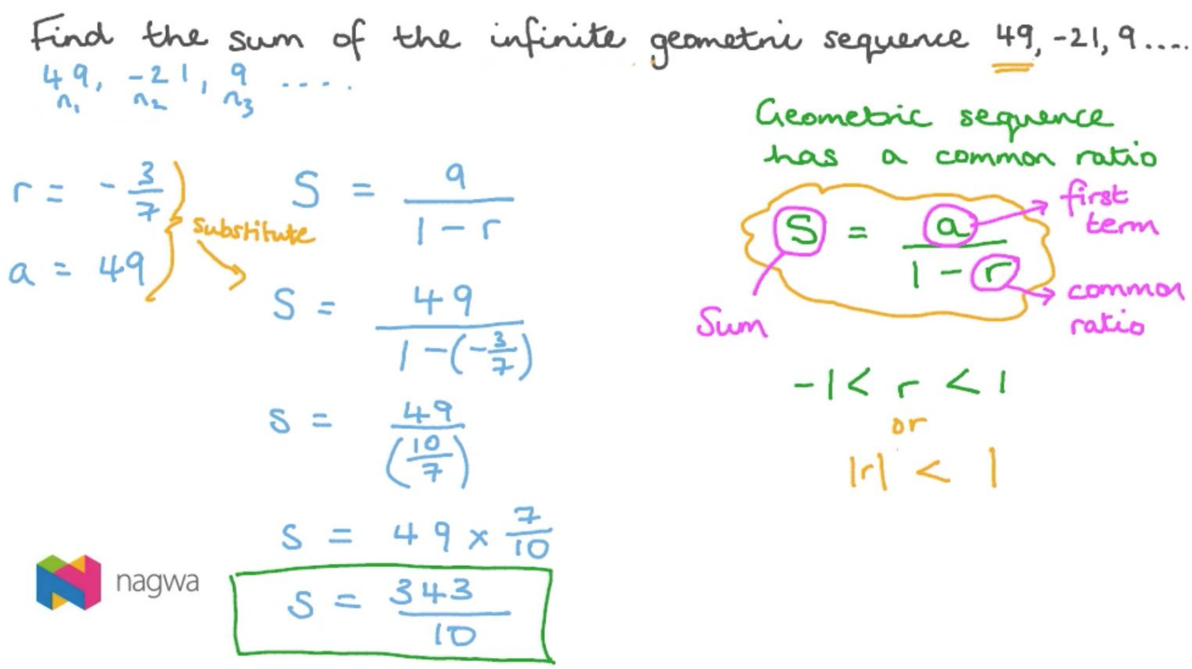
The sum of the first six terms is 312.48.

IV. Finding the sum of infinity in a geometric sequence

Formula



Example



More resources: https://www.youtube.com/watch?v=5NHd7NK1ufk

12.4 and 12.5

I. Principle of mathematical induction

Explanation

3.2 The second principle of mathematical induction

To show that $P(n)$ is true for every positive integer n, show that:

Basis step. $P(1)$ is true.

Inductive step. For every positive integer n, if $P(1), P(2), \ldots, P(n)$ are true, then so is $P(n+1)$.

Generalization: To show that $P(n)$ is true for every integer n with $n \geq k$ (where k can be any integer, including 0 or negative integers), it suffices to show that:

1. *Basis step*: $P(k)$ is true;
2. *Inductive step*: For every integer n with $n \geq k$, if $P(k), P(k+1), \ldots, P(n)$ are true, then so is $P(n+1)$.

Example 1.

Problem: **Prove by the principle of mathematical induction that for all $n \in N$ $\frac{1}{1x2}+\frac{1}{2x3}+\frac{1}{3x4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$**

Solution: Let, $P(n) = \frac{1}{1x2}+\frac{1}{2x3}+\frac{1}{3x4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

$$P(1) = \frac{1}{1x2} = \frac{1}{1+1}$$ (By substituting n=1 on both sides)

$\therefore$ P(1) is true.

Let, P(n) be true for some natural number K.

i.e., $P(K) = \frac{1}{1x2}+\frac{1}{2x3}+\frac{1}{3x4}+\cdots+\frac{1}{K(K+1)}=\frac{K}{K+1}$

Now,

$$P(K+1) = \frac{1}{1x2}+\frac{1}{2x3}+\frac{1}{3x4}+\cdots+\frac{1}{K(K+1)}+\frac{1}{(K+1)(K+2)}$$

$$= \frac{K}{K+1}+\frac{1}{(K+1)(K+2)}$$ [$\because$ P(K) is true]

Example 2.

Problem: **Prove by the principle of mathematical induction that for all $n \in N$ $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$**

Solution:

$$= \frac{K(K+2)+1}{(K+1)(K+2)}$$

$$= \frac{K^2+2K+1}{(K+1)(K+2)}$$

$$= \frac{(K+1)^2}{(K+1)(K+2)}$$

$$= \frac{K+1}{K+2}$$

$\Rightarrow$ P(K+1) is true

Hence by principle of mathematical induction, P(n) is true for all $n \in N$.

II. Pascal's triangle

Diagram 1.

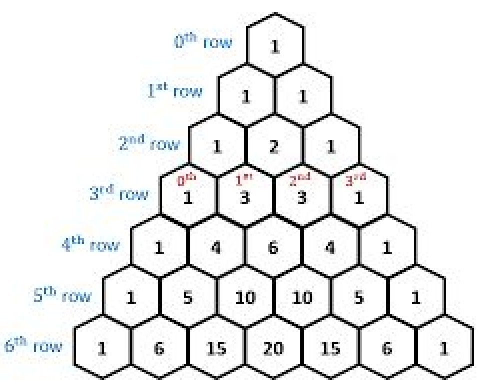
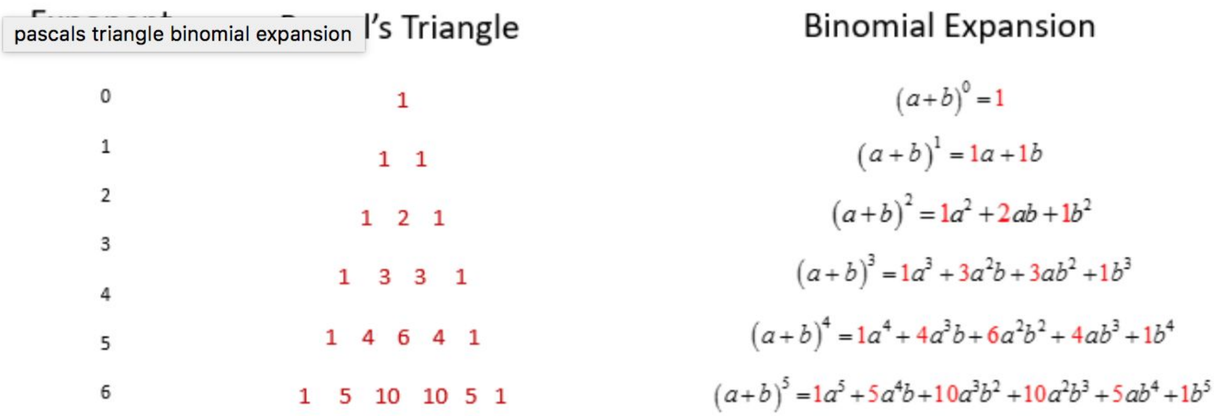


Diagram 2.



Example

Example 3 Expand: $(x^2 - 2y)^5$.

Solution We have $(a + b)^n$,where $a = x^2$, $b = -2y$, and $n = 5$. Then using the binomial theorem, we have

$$
\begin{aligned}
(x^2 - 2y)^5 &= \binom{5}{0}(x^2)^5 + \binom{5}{1}(x^2)^4(-2y) + \binom{5}{2}(x^2)^3(-2y)^2 \\
&\quad + \binom{5}{3}(x^2)^2(-2y)^3 + \binom{5}{4}x^2(-2y)^4 + \binom{5}{5}(-2y)^5 \\
&= \frac{5!}{0!\,5!}x^{10} + \frac{5!}{1!\,4!}x^8(-2y) + \frac{5!}{2!\,3!}x^6(4y^2) + \frac{5!}{3!\,2!}x^4(-8y^3) \\
&\quad + \frac{5!}{4!\,1!}x^2(16y^4) + \frac{5!}{5!\,0!}(-32y^5) \\
&= 1 \cdot x^{10} + 5x^8(-2y) + 10x^6(4y^2) + 10x^4(-8y^3) \\
&\quad + 5x^2(16y^4) + 1 \cdot (-32y^5)
\end{aligned}
$$

Finally $(x^2 - 2y)^5 = x^{10} - 10x^8y + 40x^6y^2 - 80x^4y^3 + 80x^2y^4 - 32y^5$.

III. Expanding a binomial

Formula

Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}a^0 b^n$$

$$\text{where } \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Example

Example:

Expand

a) $(a + b)^5$
b) $(2 + 3x)^3$

Solution:

$$\text{a) } (a+b)^5 = a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + b^5$$
$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$
$$\text{b) } (2+3x)^3 = 2^3 + \binom{3}{1}2^2(3x) + \binom{3}{2}2(3x)^2 + (3x)^3$$
$$= 8 + 36x + 54x^2 + 27x^3$$

IV. Finding a coefficient of a term in a binomial expansion

Example

Illustrative Examples

Find the term independent of x in the expansion of $\left(\frac{1}{2}x^{\frac{1}{3}} + x^{\frac{-1}{5}}\right)^8$.

Solution :

$$T_{r+1} = {}^8C_r\left(\frac{1}{2}x^{\frac{1}{3}}\right)^{8-r} \cdot \left(x^{\frac{-1}{5}}\right)^r = {}^8C_r\left(\frac{1}{2}\right)^{8-r} x^{\frac{8-r}{3}-\frac{r}{5}}$$
$$= {}^8C_r\left(\frac{1}{2}\right)^{8-r} x^{\frac{40-5r-3r}{15}} = {}^8C_r\left(\frac{1}{2}\right)^{8-r} x^{\frac{40-8r}{15}}$$

For the term to be independent of x $\quad \frac{40-8r}{15} = 0 \Rightarrow r = 5$

Hence sixth term is independent of x and is given by

$$T_6 = {}^8C_5\left(\frac{1}{2}\right)^3 = \frac{8!}{5!\,3!}\cdot\frac{1}{8} = 7 \qquad T_6 = 7$$

More resources: https://www.youtube.com/watch?v=gdmKkRYdiyA

Real-life Application (Step 3)

An real life example where concepts in this unit is applied is when looking at modelling. The problem below is an example of a real world math problem.

A city is concerned about pollution, and decides to look at the number of people using taxis. At the end of the year 2000, there were 280 taxis in the city. After “n” years the number of taxis, “T”, in the city is given by:

$$T = 280 \times 1.12^n$$

(a) (i) Find the number of taxis in the city at the end of 2005.

$$T = 280 \times 1.12^5$$

[The difference between the years 2000 and 2005 is 5, therefore substitute 5 for number of years, n]

T= 493. 46 → 493 taxis

[Using a calculator, calculate the number of taxis after 5 years, round answer to a whole number because there cannot be a part of a taxi]

(ii) Find the year in which the number of taxis is double the number of taxis there were at the end of 2000.

280 x 2 = 560 taxis

[At the end of the year 2000, there were 280 taxis in the city. Multiply 280 x 2 to find double the number of taxis at the end of the year 2000.]

$$560 = 280 \times 1.12^n$$

[Substitute 560 taxis for the number of taxis, T, in the given equation]

$$560 = 280 \times 1.12^n$$

$$2 = 1.12^n$$

$$n = \log_{1.12}(2)$$

$$n = 6.116 \ years$$

The year that the number of taxis was double that of the year 2000 was 2007.

[Divide both sides by 280, then take the logarithm of both sides of the equation to isolate n, use a calculator to calculate $n = \log_{1.12}(2)$. The number of years is a little over 6 years, add 6.116 years to the year 2000, since the question asked in which year was the number of taxis double, the answer is 2007 because n was more than 6 years, therefore the actual year was in 2007.]

(b) At the end of 2000, there were 25,600 people in the city who used taxis. After "n" years the number of people, "P", in the city who used taxis is given by:

$$P = \frac{2560000}{10 + 90e^{-0.1n}}$$

(i) Find the value of P at the end of 2005, giving your answer to the nearest whole number.

$$P = \frac{2560000}{10 + 90e^{-0.1(5)}}$$

$$P = 39635.993 \rightarrow 39636 \ people$$

[The difference between the years 2000 and 2005 is 5 years, therefore substitute 5 for number of years, n. Using a calculator, calculate the number of people using taxis at the end of 2005. Round answer to the nearest whole number because there cannot be a part of a person.]

(ii) After seven complete years, will the value of P be double its value at the end of 2000? Justify your answer.

$$P = \frac{2560000}{10 + 90e^{-0.1n}}$$

$$P = \frac{2560000}{10 + 90e^{-0.1(7)}}$$

$$P = 46806.997 \rightarrow 46807 \ people$$

$$25600 \times 2 = 51200$$

$$46807 \neq 51200$$

[Substitute 7 for number of years, n and solve for P using a calculator. Multiply the number of people that used taxis in the year 2000 by 2 (25600 x 2 = 51200). The number of people that used taxis after 7 years is not double the number of people that used taxis at the end of 2000 because $46807 \neq 51200$].

(c) Let "R" be the ratio of the number of people using taxis in the city to the number of taxis. The city will reduce the number of taxis if $R < 70$.

(i) Find the value of "R" at the end of 2000.

$$R = \frac{25600}{280} = \frac{640}{7}$$

$$R = 91.4$$

[Set the ratio of the number of people using taxis (25600) to the number of taxis (280) as a fraction, display the value of R as the decimal number of the fraction]

(ii) After how many complete years will the city first reduce the number of taxis?

$$P = \frac{2560000}{10 + 90e^{-0.1n}}$$

$$T = 280 \times 1.12^n$$

$$\frac{P}{T} < 70$$

$$\frac{\left(\frac{2560000}{10 + 90e^{-0.1n}}\right)}{280 \times 1.12^n} < 70$$

$$\frac{2560000}{\left(10 + 90e^{-0.1n}\right) 280 \times 1.12^n} < 70$$

$$n > 9.31 \ years$$

$$After\ 10\ years,\ the\ city\ will\ reduce\ the\ number\ of\ taxis.$$

[Set the equations for P and T in a ratio that is less than 70. Multiply the equation for P by $\frac{1}{T}$, take the logarithm of both sides to find n. Since the answer is more than 9, round up, therefore after 10 years the city will reduce the number of taxis].

Practice Problems (Step 4)

The 10 practice problems range from easy to difficult. Answers are provided on page 7.

EASY:

1. *Find the term in* x^3 *in the expansion of* $\left(\frac{2}{3}x-3\right)^8$.

2. *Determine where the sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, find the common difference and the sum of the first n terms. If the sequence is geometric, find the common ration and the sum of the first n terms.*

$(a)\ 6, 12, 36, 144\ldots \quad (b) -2, -10, -18, -26, \ldots \quad (c)\ 25, 10, 4, \frac{8}{5}, \ldots$

3. *The coefficent of* x^6 *in the expression of* $(x+3)^{10}$

MEDIUM:

4. *Find the inverse of* $f(x) = \frac{1}{x-1}$.

5. *Expand* $(3m + 2)^5$ *using the Binomial Theorem.*

6. *Solve the equation* $2e^x = 5$.

7. $f(x) = \frac{3x}{x-2}$ *and* $g(x) = 2x+1$

Find:

$(a)\ (f \circ g)(2)$

$(b)\ (f \circ g)(x)$

$(c)\ (g \circ f)(x)$

HARD:

8. *The first four terms of a sequence are 18, 54, 162, 486.*

(a) *Use all four terms to show that this is a geometric sequence.*

(b) (i) *Find an expression for the* n^{th} *term of theis geometric sequence.*

(ii) *If the* n^{th} *term of the sequence is 1,062,822 , find the value of* n.

9. *Let* $p = \log_{10} x$, $q = \log_{10} y$ *and* $r = \log_{10} z$.

Write the expression $\log_{10}\left(\frac{x}{y^2\sqrt{z}}\right)$ *in terms of* p, q, *and* r.

10. *A company offers its employees a choice of two salay schemes A and B over a period of 10 years.*

Scheme A offers a starting salary of \$11,000 in the first year and then an annual increase of \$400 per year.

(a) (i) *Write down the salary paid in the second year and in the third year.*

(ii) *Calculate the* ***total*** *salary paid over ten years*

Scheme B offers a starting salary of \$10,000 in the first year and then an annual increase of 7% of the previous year's salary.

(b) (i) *Write down the salary paid in the second and third year*

(ii) *Calculate the salary paid in the tenth year*

(c) *Noah works for n complete years under scheme A. Bill works for n complete years under under scheme B. Find the minimum number of years so that the total earned by Bill exceeds the total earned by Noah.*

ANSWERS:

1. $-4032x^3$
2. (a) Neither (b) Arithmetic; d = -8; $S_n = n(2 - 4n)$ (c) Geometric; $r = \frac{2}{5}$; $S_n = \frac{125}{3}\left[1 - \left(\frac{2}{5}\right)^n\right]$
3. 17, 010
4. $f^{-1}(x) = \frac{x+1}{x}$
5. $243m^5 + 810m^4 + 1080m^3 + 720m^2 + 240m + 32$
6. $\left\{\ln\left(\frac{5}{2}\right)\right\}$
7. (a) 5 (b) $\frac{6x+3}{2x-1}$ (c) $\frac{7x-2}{x-2}$
8. (a) Show work. (b) i. $U_n = 18(3^{n-1})$ ii. 11
9. p - 2q - ½ r
10. (a) i. 2nd year: \$11,400; 3rd year: \$11, 800
    ii. \$12,800

    (b) i. 2nd year: \$10,700; 3rd year: \$11,449
    ii. \$18,384

    (c) 7 years

Unit 2 Review: Function

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IB MYP Pre-Calculus Extended

Table of Contents

Introduction

The goal of this project is to provide an overview and review of Unit 2, which focuses on functions. This includes aspects such as graphing, transforming, and finding the asymptotes of different types of functions. Some of the function types used for graphing are quadratic, logarithmic, exponential, absolute value, cubic, and rational. The transformations in this unit are based off of the parent functions, in which a step by step method is used to transform the parent function into the intended graph. Various types of asymptotes include vertical asymptotes, horizontal asymptotes, and oblique asymptotes. Generally, the graphs that have asymptotes are rational functions, reciprocal functions, logarithmic functions, and square root functions. Composite and inverse functions will also be included. Composite functions are solved by applying one function to another and solving while inverse functions find the converse of a function. Functions is a broad topic with many aspects that are listed above. In-depth coverage of the content of Unit 2 is detailed below.

Glossary

<u>Key Terms</u>

Absolute or global extrema	← The highest or lowest value that a function can take
Amplitude	← Half the distance between the minimum and maximum values of the range of a periodic function
Asymptote	← A straight line that a curve approaches but never meets
At a constant rate	← With no change in speed. The gradient of a graph representing constant rate is a straight diagonal line
Average rate of change	← The slope, or gradient, of a secant line
Axis of revolution	← An axis in a plane, about which the are bound by a curve and the axis is revolved to form a solid of revolution
Axis of Symmetry	← A line dividing a shape such that the two parts on either side of the line are identical
Bearings	← The position of an object, with reference to the angle it makes with a set point or pole
Cartesian Equation	← The equation of a line of curve expressed in terms of x and y
Completing the Square	← Rearranging a function into $(ax+b)^2+c$ where c is constant
Composite Function	← Resultant function when two or more functions are combined
Constant Function	← Function taking only one fixed value
Continuous	← A function able to have all values within a given range
Correlation	← A measure of linear association between two variables. It takes values between 1 and -1, and is independent of any linear change of scale of variables.
Critical Number	← A point where f(x) = 0 or is undefined
Cubic Function	← Function in which highest power of variable is 3

Decreasing Function	← If f(x) has a negative gradient, it is a decreasing function
Dilation	← A stretch or compression of a graph. Result of multiplying x or y by constant a
Discontinuity	← A point where a function is undefined, and the graph of the function is broken at that point
Discontinuous	← A function that is not continuous
Discrete	← Able to have only certain values
Discriminant	← In a quadratic equation $ax^2 + bx + c$, the discriminant is $b^2 - 4ac$
Domain	← A set of values which define the elements of the independent variable (the x-values)
Elementary Function	← A function that is algebraic, transcendental, or a sum, difference, product, quotient or composition of algebraic and transcendental functions
Equal Roots	← When the solutions of a quadratic function are the same
Even Function	← A function for which f(x) = f(-x) for all values of x
Exponential	← A function of the form a^x where a is a positive constant
Function	← A mapping in which each x-value maps onto one and only one y-value
General Form	← The general form of a straight line is ax+by+c=0 where a, b, and c are constants
Gradient	← The slope of a straight line which includes its direction of sign. Positive gradient is + and negative gradient is -
Gradient Function	← A function expressing the gradient of a function
Gradient Intercept	← y=mx+b is the gradient-intercept form of the straight line

	equation, where m is the gradient of the line and b is the point at which the line intercepts the y-axis
Horizontal	← A line that has zero gradient
Horizontal Asymptote	← A horizontal line that the graph of a function approaches but never meets as x tends to infinity
Horizontal Translation	← A movement of a graph in the direction (positive or negative) of the horizontal axis
Increasing Function	← If f(x) has a positive gradient, it is a increasing function
Intersect (of Lines)	← The point where two or more lines meet
Inverse	← The inverse function $f(x)$ is $f^{-1}(x)$. It reverses the action of that function
Linear	← A linear relationship can be represented by a straight line
Linear Combination	← A summation of functions which are multiplied by constants
Linear Function	← A function given in terms of a linear expression
Local Maximum	← The name given to a maximum point of a curve which has more than one of these points
Local Minimum	← The name given to a minimum point of a curve which has more than one of these points
Logarithm	← The power to which the base must be raised to give that number
Logarithmic Function	← If $b = a^x$, then $log\ _a b = x$
Many-To-One Function	← a function which maps all elements in its domain onto one value in its range
Maximum Point	← The point at which the gradient of the curve changes from positive to negative.
Minimum point	← The point at which the gradient of a curve changes from negative to positive

Natural Logarithm ← A logarithm in base e, written as either $log_e x$ or lnx

Non-Linear ← A relationship that can not be represented graphically be a straight line

Odd Function ← A function where f(-x) = -f(x) for all values of x

One-To-One Function ← A function which maps each element in the domain onto a unique value in the range

Optimization problems ← Practical problems involving finding maximum or minimum values

Parabola ← A smooth $\bigcup$ or $\bigcap$ shaped curve. The corresponding function is quadratic

Parameter ← A the variable t in a parametric equation $x = f(t)$, $y = g(t)$ of a curve b. The variable t associated with the points on a line used to determine the vector equation of the line

Period of a Function ← The interval in which repeats itself

Periodic Function ← A function which repeats itself after a given interval

Properties of Limits as $x \rightarrow \pm\infty$ ← Properties to find the limits of a function as $x \rightarrow \infty$ algebraically

Properties of Polynomial Addition ← Properties showing how to combine polynomials under addition and when multiplying with real constants

Properties of Polynomial Multiplication ← Properties showing how to combine polynomial under multiplication

Quadratic Equation ← An equation in which the highest power of the variable is 2. A quadratic in the variable x has the general form $ax^2 + bx + c = 0$ where a, b, and c are constant and $a \neq 0$

Quadratic Formula ← The formula used to solve a quadratic equation if the equation does not factorise easily. For the equation $ax^2 + bx + c = 0$ the formula is $x = -b \pm \sqrt{b^2} - \frac{4ac}{2a}$

Quadratic Function ← A function given in terms of a quadratic expression

Quartic Function ← A function in which the highest power of the variable is 4

Quintic Function ← A function in which the highest power of the variable is 5

Range ← A set of values which define the elements of the dependent variable (the y-axis)

Rate of Change (of f with respect to x) ← The rate at which a variable changes over time. The instantaneous rate of change applies to an instant of time. The average rate of change applies over a period of time.

Rational Function ← A function of the form $f(x) = \frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomials

Reciprocal Function ← A function of the form $y = \frac{1}{(x-a)}$

Relative Maximum Point ← When the function changes from increasing to decreasing

Relative Minimum Point ← When the function changes from decreasing to increasing

Restrict the Domain ← A restriction given to the domain so that the relation is a function

Roots of an Equation ← The roots, or solutions, of an equation $f(x) = 0$ are the values of x which satisfy the equation. In other words when a root is substituted for x in f(x) the answer is zero

Same Degree ← Two functions have the same degree if the highest power of the variable in each function is equal

Self-Inverse Function ← A function such that f(f(x)) = x

Simultaneous Equations ← Several equations in several variables which you can solve to give a common solution

Substitution Method ← Replace a variable in a formula by a particular value. Often used to solve simultaneous equations

Transcendental Function	← Functions that can not be expressed as sums, differences, products, quotients, and radicals involving x^n
Undefined	← Without a solution
Vertex	← A point where two lines intersect
Vertical Asymptote	← Occurs when the value of y tends to infinity as x tends to 0
Vertical Stretch of Scale Factor	← A value which shows the extent of a vertical stretch or compression
Vertical Translation	← A movement parallel to the vertical axis
x-intercept	← The coordinates of the point where a line intersects the x-axis. The y coordinate of this point is always zero
y-intercept	← The coordinates of the point where a line intersects the y-axis. The x coordinate of this point is always zero
Zero Polynomial	← The function $\theta(x) = 0$. The graph is the x-axis itself

<u>Key Formulas</u>

$f(x) = ax^2 + bx + c$ ← quadratic equation

$x = \frac{-b}{2a}$ ← equation for axis of symmetry

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ← quadratic formula

$\Delta = b^2 - 4ac$ ← discriminant formula

$a^x = e^{x \ln a}$ ← relation of exponential and logarithmic functions

$\log_a a^x = x = a^{\log_a x}$ ← relation of exponential and logarithmic functions

$m = \frac{y_2 - y_1}{x_2 - x_1}$ ← gradient formula

<u>Parent Function Equations</u>

$f(x) = x^2$ ← quadratic function

$f(x) = x$ ← linear function

$f(x) = |x|$ ← absolute value function

$f(x) = \frac{1}{x}$ ← reciprocal function

$f(x) = x^3$ ← cubic function

$f(x) = \sqrt{x}$ ← square root function

$f(x) = \log(x)$ ← logarithmic function

$f(x) = \ln(x)$ ← natural logarithmic function

Examples of Concepts

Main Concepts Include:
- Graphing Functions
- Graph Transformations
- Finding Asymptotes of Functions
- Composite Functions
- Inverse Functions
- Solving for the discriminant of a quadratic

Below, there are one example problem and one reference video for each concept listed above.

1) Graphing Functions (Quadratic) and Transforming
   Graph the function: $f(x) = 5x^2 - 20x + 15$
   Method 1: Change the function to vertex form by completing the square

   $f(x) = 5x^2 - 20x + 15$
   $f(x) = 5(x^2 - 4x) + 15$
   $f(x) = 5(x-2)^2 + 15$
   $f(x) = 5(x-2)^2 - 5$

   With the function in vertex form, you can easily conduct the transformation of the parent graph into this function. Translate the parent graph 2 units to the right, and 5 units down. The numbers 2 and -5 in the vertex-form function indicate the vertex. Then, do a vertical dilation by a factor of 5.

   Method 2: Solve for x-intercepts, y-intercept, and axis of symmetry

   First, factor the function to solve for the x-intercepts:
   $f(x) = 5x^2 - 20x + 15$
   $f(x) = (5x-5)(x-3)$
   $5x - 5 = 0 \qquad x - 3 = 0$
   $x = 1 \qquad x = 3$
   X-intercepts at (1,0) and (3,0)

   Second, solve for the axis of symmetry:
   AoS = $\frac{-b}{2a}$
   AoS = $\frac{-(-20)}{2(5)}$
   AoS = $\frac{20}{10}$
   AoS = 2
   Axis of Symmetry at x = 2

Third, use the x value of the AoS to find its corresponding y-value. This will be the coordinate of the vertex:

$f(x) \;=\; 5x^2 - 20x + 15$

$y \;=\; 5x^2 - 20x + 15$

$y \;=\; 5(2)^2 - 20(2) + 15$

$y \;=\; -5$

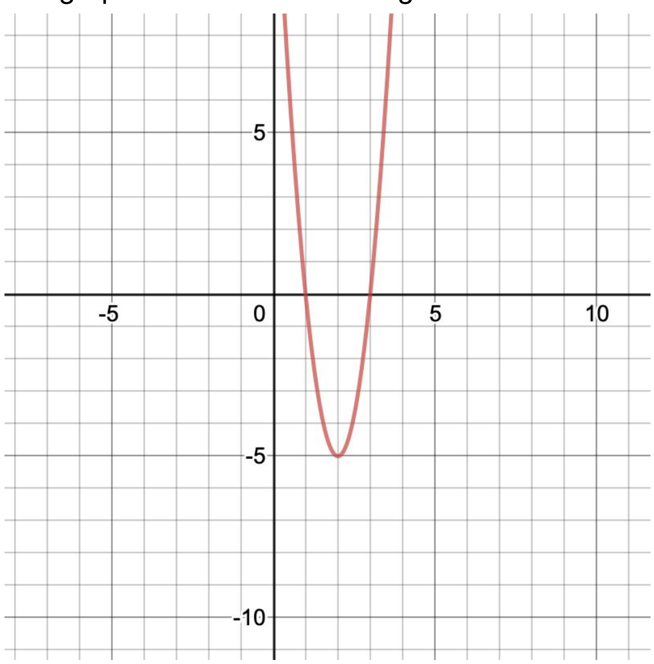
Coordinate for vertex = (2, -5)

Fourth, write down the y-intercept. Since the y-intercept is when x = 0, the y-coordinate of the intercept will be equal to the constant in the function. In this equation, 15 is the constant.

Y-intercept = (0,15)

Lastly, plot the vertex, y-intercept, and x-coordinates on a graph and connect with a line in the shape of a quadratic function.

The graph for the function using either method should look like this:



For further explanation on graphing functions, reference this video link:
https://www.youtube.com/watch?v=Cn1aFaxRyeU

2) Finding Asymptotes of functions

Find the vertical and horizontal asymptotes of the function f(x)

$f(x) = \frac{x^2+3x+1}{4x^2-9}$

First, start with the vertical asymptotes. Vertical asymptotes are located at values of x when the function is undefined. When the denominator equals 0, the function is undefined. Therefore, solve for the value of x that makes the denominator equal to 0.

$4x^2 - 9 = 0$

$4x^2 = 9$

$x^2 = \frac{9}{4}$

$x = \pm\frac{3}{2}$

Next, solve for the horizontal asymptote. Since the greatest power in the numerator is not larger than the greatest power in the denominator, there will not be an oblique asymptote. The highest powers are the same in the numerator and denominator, so there is a horizontal asymptote. To solve for it, divide the coefficient of the highest power in the number by that of the denominator.

$y = \frac{x^2}{4x^2}$

$y = \frac{1}{4}$

For further explanation on how to solve for vertical and horizontal asymptotes, reference this video link: https://www.youtube.com/watch?v=HeqfhnKncjc

3) Composite Functions

$f(x) = 2x - 1$

$g(x) = 3x$

Simplify $(f \circ g)(x)$

Composite functions can be written in two different formats. The first one is shown above, and the second one is f(g(x)). To solve a composite function, the function that is listed second in the parentheses, in this case g(x) is substituted into f(x). If a number is in the place where the x currently is, then that number would be substituted into g(x), then simplified, and then the simplified g(x) would be substituted into f(x)

$(f \circ g)(x) = 2(3x) - 1$

$(f \circ g)(x) = 6x - 1$

For more on composite functions, reference this video link:
https://www.khanacademy.org/math/precalculus/x9e81a4f98389efdf:composite/x9e81a4f98389efdf:composing/v/evaluating-composite-functions

4) Inverse Functions
Solve for the inverse function of f(x)
$f(x) = 3x + 2$

To solve for the inverse of a function, switch the location of the x and y variables in the function, and then rearrange it so that y is given in terms of x. It is important to note that when graphed. A function and its inverse will be exact reflections about the line y = x. Also, the notation to show that a function is the inverse, is to write it as $f^{-1}(x)$, while the original function is f(x).

$f(x) = 3x + 2$
$y = 3x + 2$
$x = 3y + 2$
$x - 2 = 3y$
$y = \frac{x-2}{3}$
$f^{-1}(x) = \frac{x-2}{3}$

For more on inverse functions, reference this video link:
https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions/x2f8bb11595b61c86:inverse-functions-intro/v/introduction-to-function-inverses

5) Solve for the discriminant of f(x) and then state how many real solutions it has
$f(x) = 5x^2 - 20x + 15$

The discriminant of a function is solved for, using the formula $d = b^2 - 4ac$. This formula is derived from the quadratic equation, is it the portion underneath the radical. It is helpful in determining the number of solutions, or x-intercepts, for the function. If the value of d is greater than 0, then there are 2 real solutions. If the value of d is equal to 0, then there is only one real solution, so the graph of f(x) has its vertex on the x-axis. If the value of d is less than 0, then there are no real solutions, only imaginary solutions. In that case, there would be no x-intercepts.

a = 5
b = -20
c = 15

$d = b^2 - 4ac$

$d = -20^2 - 4(5)(15)$

$d = 100$

Because d is greater than 1, there are 2 real solutions for the function f(x).

For more on discriminants, reference these two videos:
https://www.khanacademy.org/math/in-in-grade-10-ncert/x573d8ce20721c073:in-in-chapter-4-quadratic-equation/x573d8ce20721c073:in-in-10-quadratic-discriminant-and-number-of-solutions/v/discriminant-of-quadratic-equations

https://www.khanacademy.org/math/in-in-grade-10-ncert/x573d8ce20721c073:in-in-chapter-4-quadratic-equation/x573d8ce20721c073:in-in-10-quadratic-discriminant-and-number-of-solutions/v/discriminant-for-types-of-solutions-for-a-quadratic

Real Life Applications

Example: Alligator Modeling

Length in inches	Weight in Pounds
58	28
61	44
63	33
68	39
69	36
72	38
72	61
74	54
74	51
76	42
78	57
82	80
85	84
86	83
86	80
86	90
88	70
89	84
90	106
90	102
94	110
94	130
114	197
128	366
147	640

Wildlife populations can be observed with aerial photographs. Through these pictures, certain characteristics of animal populations can be monitored. It is possible to accurately determine the length of an alligator from these photographs, but determining weight is more difficult. Using the data provided, develop a model from which the weight of an alligator can be predicted from its length. Using 4 functions (linear, quadratic, exponential, and power), determine the regressions of these models for the data provided and determine the line of best fit for the data. The line of best fit will be used to predict and interpret further alligator data.

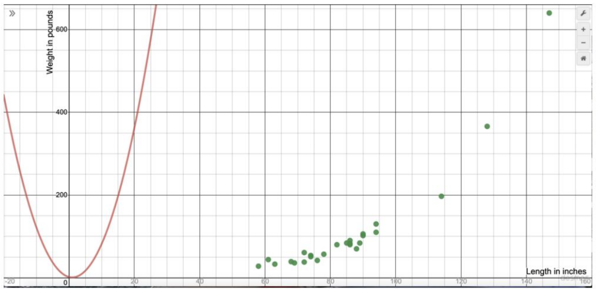
For each model:

- ❏ Find the y-intercept and interpret it. Does it make sense in context?
- ❏ Using your model, find the weights of alligators that are 20 inches and 100 inches long. Are these values reasonable?
- ❏ Determine the weight of the largest alligator found in Florida's Everglade National Park, which was 17 feet 5 inches long, based on your model.
- ❏ State strengths and limitations of the model.

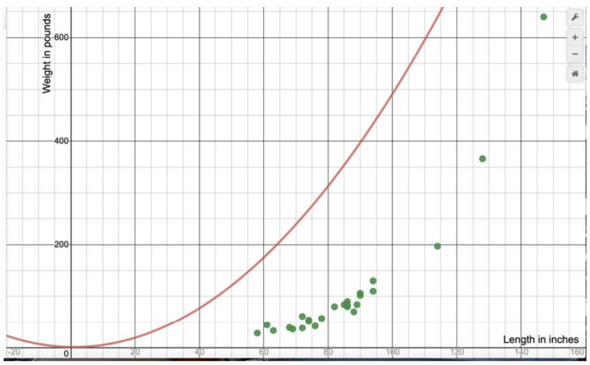
Write a conclusion to analyze your findings and state which function produces the most accurate model of the data which can assist in determining the future weights of alligators based on length.

At least one model must be analytically developed.

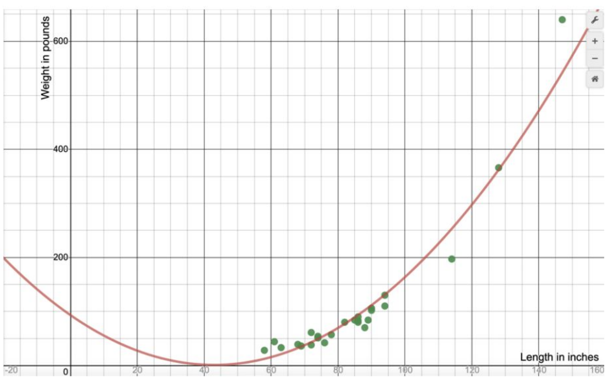
The following model is an example of how to analytically develop a model.



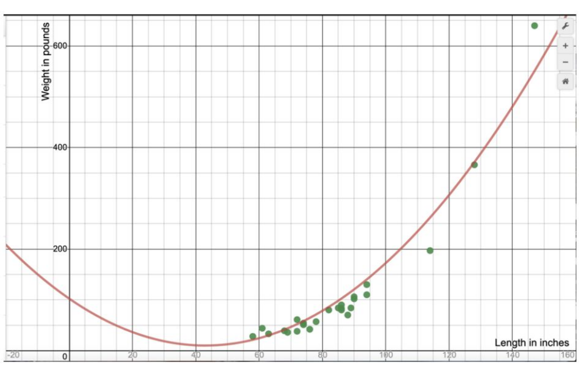
The process began with the parent graph of a quadratic function where a, b, and c are all equal to 1.



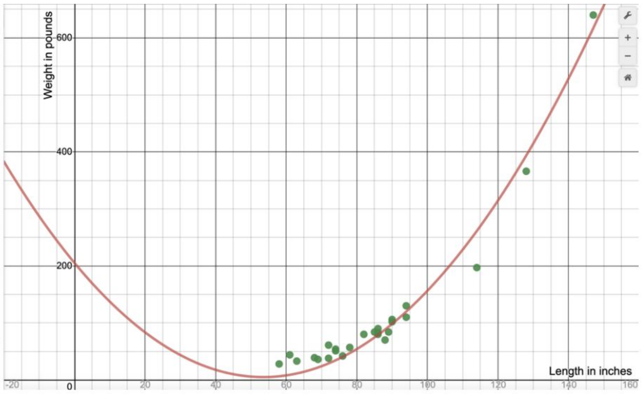
Here, a was manipulated from 1 to 0.05 to compress the model and roughly simulate the curve of the data.



b was manipulated so the model was translated 43 units left to align with the data.

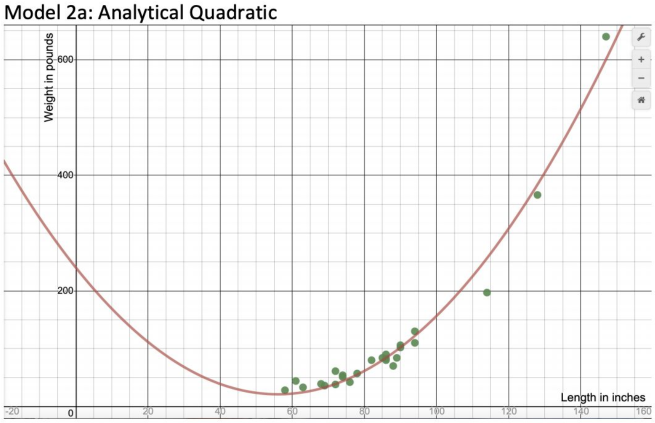


c was adjusted upwards 10 units, shifting the model 10 units up. Now because the model is basically fitted to the data, the value of r^2 now matters. Only small adjustments will be needed to create the most fitting model, so the value of r^2 will guide how the values of a, b, and c are adjusted. Here, $r^2 = 0.9533$.



All of the values of the variables were adjusted based on the value of r^2. If the value of r^2 rose, that adjustment was made to that variable. Here, $r^2 = 0.9725$.

Model 2a: Analytical Quadratic



This is the final analytical quadratic model. Its correlation to the data with r^2 is 0.9808.

Analytically Developed Model

Model	Equation	r^2 value
Quadratic	$y = 0.070(x - 56.000)^2 + 21.000$	0.9808

Other models can be developed using technology, but all must have their strengths and weaknesses assessed and you must identify the best model for the data and situation in the conclusion.

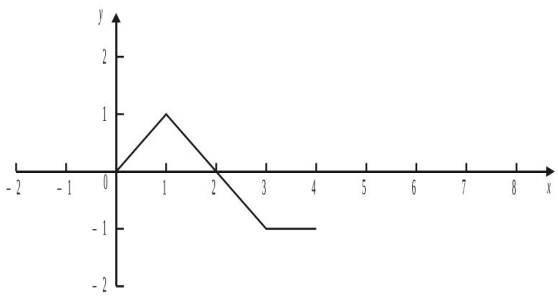
Practice Problems

Easy:

1. Let $g(x) = x^4 - 2x^3 + x^2 - 2$
   a. Solve g(x) = 0.

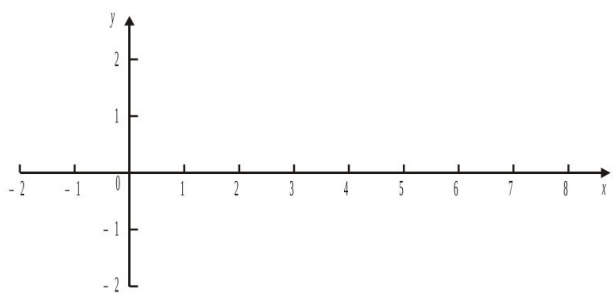
2. Let $f(x) = 2^x$ and $g(x) = \frac{x}{x-2}$ $(x \neq 2)$
   Find
   a. $(g \circ f)(3)$
   b. $(g)^{-1}(5)$

3. The graph of $y = f(x)$ is shown in the diagram.

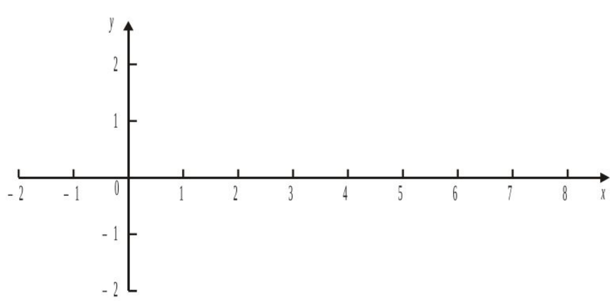


   a. On each of the following diagrams draw the required graph,

(i) $y = 2f(x)$;



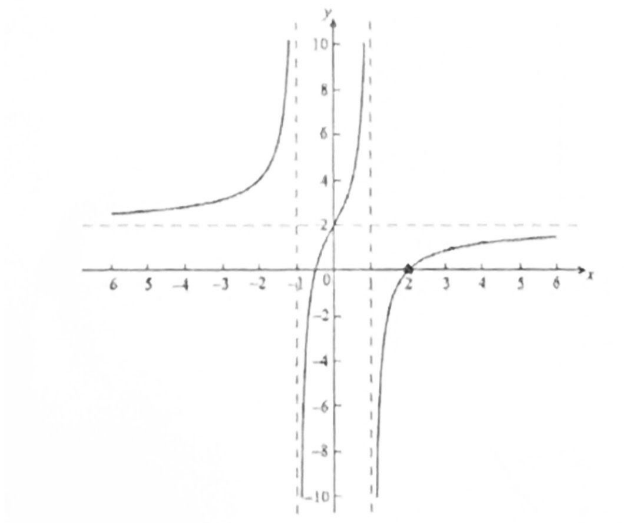
(ii) $y = f(x-3)$.



Medium:

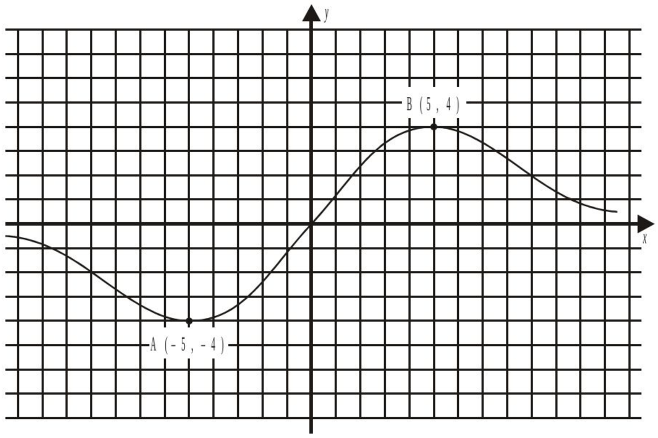
1. The function *f* is given by $f(x) = x^2 - 6x + 13$ for $x \geq 3$.
    a. Write down $f(x)$ in the form $(x-a)^2 + b$.
    b. Find the inverse function f^{-1}.
    c. State the domain of f^{-1}.

2. Let $f(x) = p - \frac{3x}{x^2 - q^2}$, where $p, q \in R^+$.
   Part of the graph of f, including the asymptotes is shown below.



    a. The equations of the asymptotes are $x = 1, x = -1, y = 2$. Write down the value of:
        i. p;
        ii. q;

3. The diagram below shows the graph of $y = f(x)$, with the x-axis as an asymptote



   a. On the same axes, draw the graph of $y = f(x+2) - 3$, indicating the coordinates of the images of the points A and B.
   b. Write down the equation of the asymptote to the graph of $y = f(x+2) - 3$.

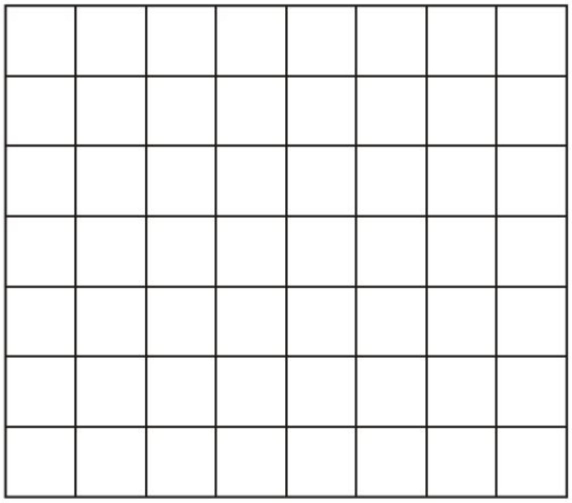
4. Let $f(x) = ln(x+2),\ x > -2$ and $g(x) = e^{(x-4)},\ x > 0$
   a. Write down the x-intercept of the graph of f.
   b. i. Write down $f(-1.999)$.
      ii. Find the range of f.
   c. Find the coordinates of the point of intersection of the graphs of f and g.

Hard:

1.

The function f is defined by $f(x) = \dfrac{3}{\sqrt{9 - x^2}}$, for $-3 < x < 3$.

(a) On the grid below, sketch the graph of f.



(b) Write down the equation of each vertical asymptote.

(c) Write down the range of the function f.

2. Consider the function $f(x) = \frac{16}{x-10} + 8,\ x \neq 10$.
   a. Write down the equation of
      i. The vertical asymptote,
      ii. The horizontal asymptote.
   b. Find the
      i. y-intercept,
      ii. x-intercept.
   c. Sketch the graph of f, clearly showing the above information.
   d. Let $g(x) = \frac{16}{x},\ x \neq 0$.
      The graph of g is transformed into the graph of f using two transformations.
      The first is a translation with vector $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$. Give a full geometric description of the second transformation.

3.

10. Let $f(x)=2x^2-12x+5$

(a) Express $f(x)$ **in the form** $f(x)=2(x-h)^2+k$ **.** **(3)**

(b) Write down the vertex of the graph of $f(x)$ **.**
(2)

(c) Write down the equation of the axis of symmetry of the graph of $f(x)$ **.** **(1)**

(d) Find the y**-intercept of the graph of** $f(x)$. **(2)**

(e) The **x**-intercepts of $f(x)$ can be written as $\frac{p\pm\sqrt{q}}{r}$, where $\boldsymbol{p}$ **,** $\boldsymbol{q}$ **,** $\boldsymbol{r} \in$ ¢
Find the exact values of p, of q, and of r. **(7)**

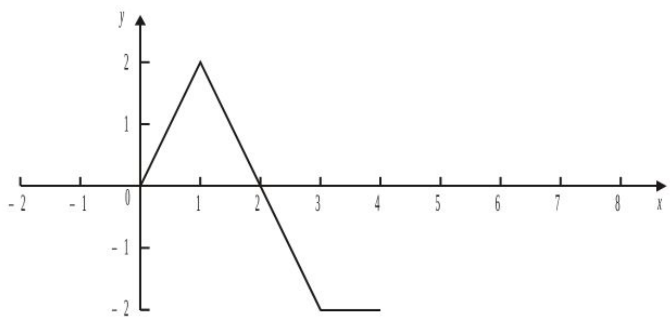
(Total 15 marks)

Answer Key

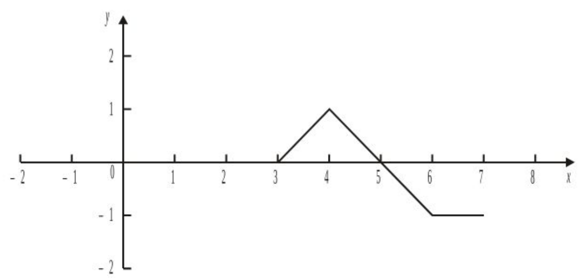
Easy

1) $x = \{0.5 \pm 1.079i,\ -0.79,\ 1.79\}$
2)
   a) $(g \circ f)(3) = \frac{4}{3}$
   b) $(g)^{-1}(5) = 2.5$
3)
   a)

(i)



(ii)

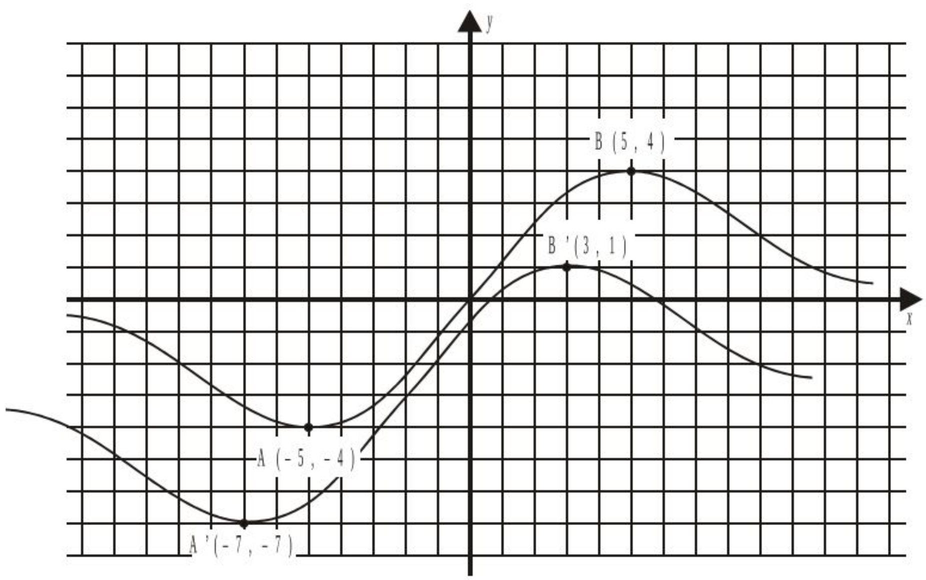


Medium

1)
   a) $f(x) = (x-3)^2 + 7$
   b) $f^{-1}(x) = 3 + \sqrt{x-7}$
   c) domain of x $= \{x|\ x \geq 7,\ x \in R\}$

2) a)
   i) p = 2
   ii) q = 1

3)
   a)

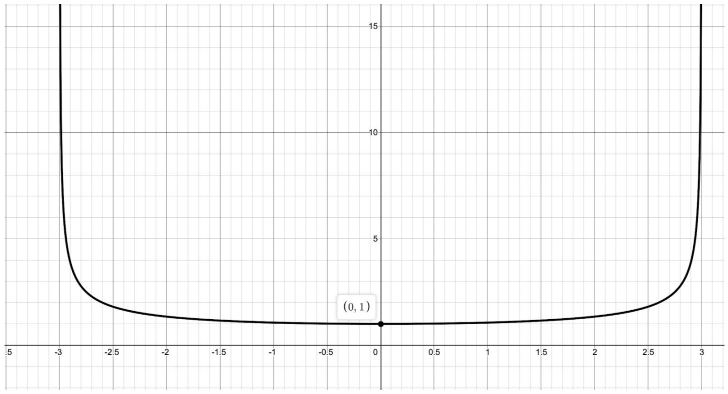


   b) Asymptote of new graph: $y = -3$

4)
   a) (-1,0)
   b)
      i) $f(-1.999) = -6.908$
      ii) Range: $\{y|\ -\infty \leq y \leq \infty,\ y \in R\}$
   c) Intersection of f and g: (4.638, 1.893)

Hard

1)

a) 

b) $x = 3$

$x = -3$

c) Range = $\{y \mid y \geq 1,\ y \in R\}$

2)

a)

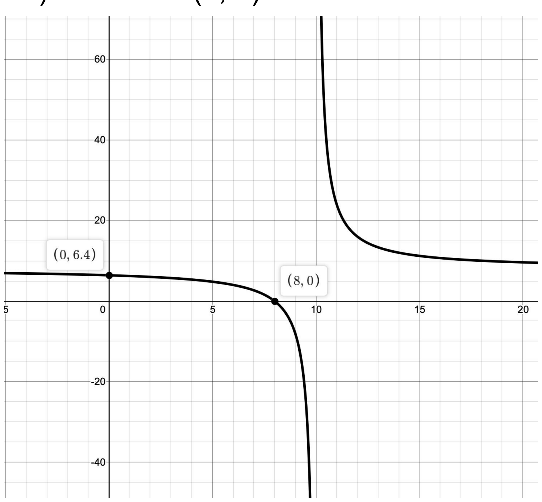
i) x = 10

ii) y = 8

b)

i) y-int at (0, 6.4)

ii) x-int at (8, 0)

c) 

d) A translation of the graph in the positive y-direction of 8 units

3) e

a) $f(x) = 2(x-3)^2 - 13$

b) Vertex of $f(x) = (3,-13)$

c) Axis ox Symmetry: $x = 3$

d) y-intercept : (0,5)

e) $p = 6$

$q = 26$

$r = 2$

Trigonometry Review

Suhani and Alexa

Introduction

Trigonometry, which is Unit 3, focuses on the relationships between sides and angles of triangles. The primary goals of this unit are:

- Understanding how to measure angles
  - Degrees, radians, unit circle
- Using trigonometry to solve unknown information about right and non-right triangles
- Applying trig functions and identities to equations and expressions
- Understanding how to create and interpret graphs of trig functions

Key Terms

➢ Trigonometric Function
  - A function that relates one of the non-right angles in a right triangle to a ratio of two sides of the triangle
  - There are six basic trigonometric functions:

$$Sine = \frac{opposite}{hypotenuse}$$

$$Cosine = \frac{adjacent}{hypotenuse}$$

$$Tangent = \frac{opposite}{adjacent}$$

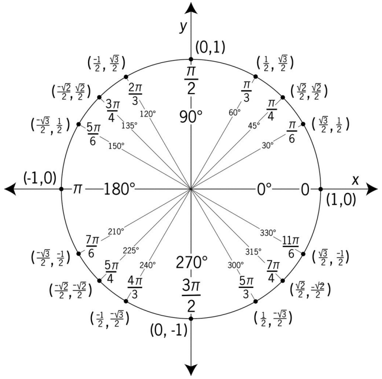
$$Cosecant = \frac{1}{sine}$$

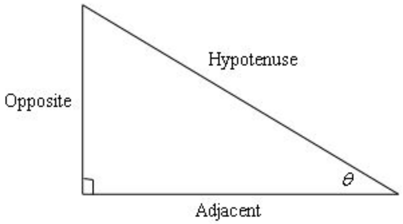
$$Secant = \frac{1}{cosine}$$

$$Cotangent = \frac{1}{tangent}$$

Video Resource by Khan Academy demonstrating how to apply these basic trigonometric functions - https://youtu.be/F21S9Wpi0y8

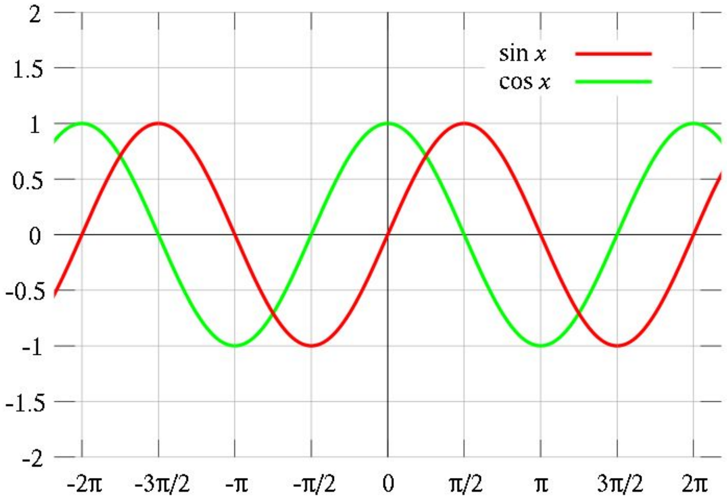
- ## Unit Circle
  - A circle with a radius of 1 centered in the coordinate plane
- ## Coterminal angles
  - Angles that share the same initial side and terminal side
- ## Reference Angles
  - The smallest angle made from another angle's terminal side with the x-axis
- ## Standard Position
  - An angle whose vertex is at the origin and the initial side coincides with the positive x-axis
- ## Radians
  - A form of measuring angles where a circle is equal to 2π and just over 6 radians (6.28)
- ## Angle of Elevation
  - An angle formed with the horizontal beaming upwards
- ## Angle of Depression
  - An angle formed with the horizontal leaning downwards
- ## Bearings
  - An angle measured clockwise from the north





Video Resource regarding trigonometry on the unit circle and how angles correlate by Khan Academy - https://youtu.be/ZffZvSH285c

- **Period**
  - The distance a trig function travels to complete one full cycle
- **Amplitude**
  - The distance from a function's rest height to its maximum or minimum
- **Phase Shift**
  - How far a function is translated horizontally from its original position
- **Vertical Shift**
  - How far a function is translated vertically from its original position



https://youtu.be/yHo0CcDVHsk -

Resource from Khan Academy on how to find *a*, *b*, *c*, and *d* in the equation - $y = a\sin b(x-c) + d$ from a graph

Key Formulas

$Length\ of\ an\ arc \rightarrow l = \theta r$

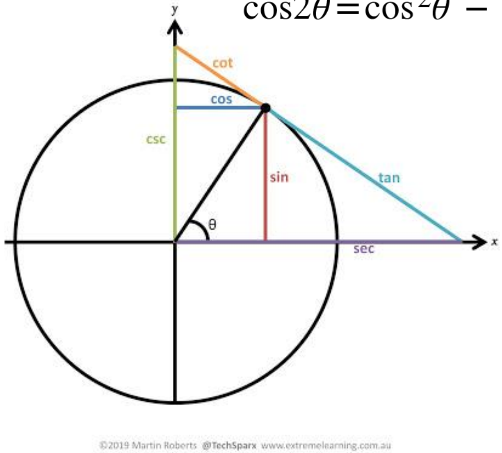
$Area\ of\ a\ sector \rightarrow A = \frac{1}{2}\theta r^2$

$Trigonometric\ Identity \rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta}$

$Pythagorean\ Identity \rightarrow \cos^2\theta + \sin^2\theta = 1$

$Double\ Angles \rightarrow \sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$



$\boldsymbol{Cosine\ Rule} \rightarrow c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

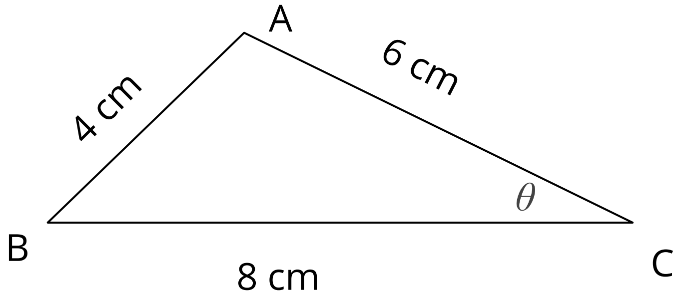
$\boldsymbol{Sine\ Rule} \rightarrow \frac{a}{\sin A} = \frac{\mathbf{b}}{\mathbf{sinB}} = \frac{\mathbf{c}}{\mathbf{sinC}}$

$\boldsymbol{Area\ of\ a\ Triangle} \rightarrow \frac{1}{2}ab\sin C$

Easy Practice Problems

1. *Convert* $57°$ *to radians*.

2. *Find* $\sin\theta$ *in the triangle ABC. Answer as a fraction.*



3. *Find* $\cos\theta$ *if* $\sin\theta = 3/5$ *and* θ *is obtuse.*

Medium Practice Problems

4. *If A is an obtuse triangle and* $\sin A = \dfrac{5}{13}$, *calculate the exact value of* $\sin 2A$.

5. *In triangle ABC*, $\widehat{B} = 43°$, $AC = 6.8\ cm$, *and* $AB = 4.3\ cm$. *Find the size of* $\widehat{A}$, *giving your answer to the nearest degree*.

6. *Find the radius of a sector of a circle that has an area of* $5\pi\ in^2$ *and a central angle of* $36°$.

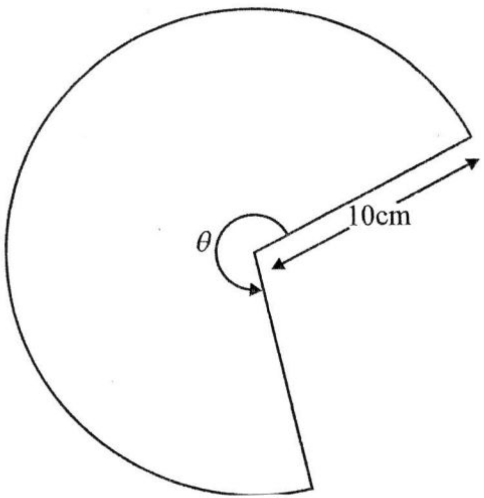
7. a) *Sketch the graph of* $y = \sin(x)$. *Then sketch the graph of* $y = 4\sin 0.5\pi\,(x) - 2$.
   b) *Describe the transformation.*

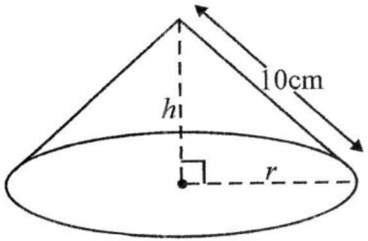
Hard Practice Problems

8. *The first diagram shows a circular sector of radius* 10 *cm and an angle* θ *radians.*

The second diagram shows it formed into a cone of slant height 10 *cm. The vertical height,* h*,*

of the cone is equal to the radius, r*, of its base. Find the size of angle* θ *in radians.*





9. *Prove*

$$\sin 3x = 3\sin x - 4\sin^3 x$$

10. *Find* tan(105º) *and show work*. Hint*: what does* tan(θ) *equal in relation to other trig functions?*

Answer Key

1. $0.995 \; radians$

2. $\sin\theta = ⅔$

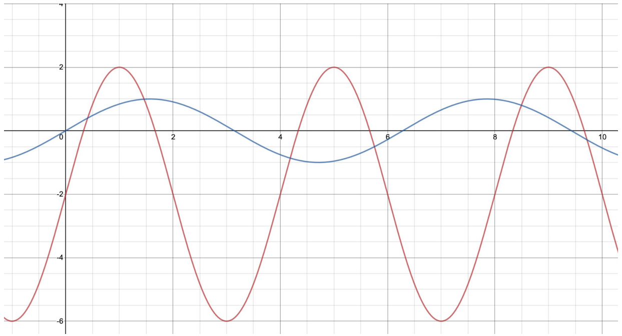
3. $\cos\theta = -⅘$ *or* -0.8

4. $\sin 2\theta = -\dfrac{120}{169}$

5. $\widehat{A} = 112°$

6. $r = 5$ inches

7. a) Graph of -
$y = \sin(x)$
$y = 4\sin 0.5\pi(x) - 2$



b) amplitude increases by 3 units; period decreases to 4 units long; vertical translation negative 2 units

8. $\theta = 4.44 \; radians$

9. $\sin(2x + x) = 3\sin x - 4\sin^3 x$

$3\sin x - 4\sin^3 x = 3\sin x - 4\sin^3 x$

10. $\tan(105°) = 2 + \sqrt{3}$

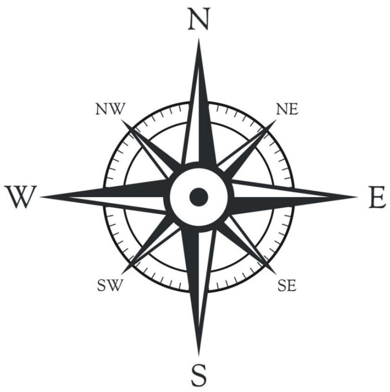
Real Life Application

As mentioned before, bearings are real life applications of trigonometry and its relationships. Bearings are angles measured clockwise from the North because they are most commonly used when utilizing a compass. This is different than angles measured with the unit circle, as 0º rather begins where 90º on a unit circle would be and the measurements are taken going clockwise.

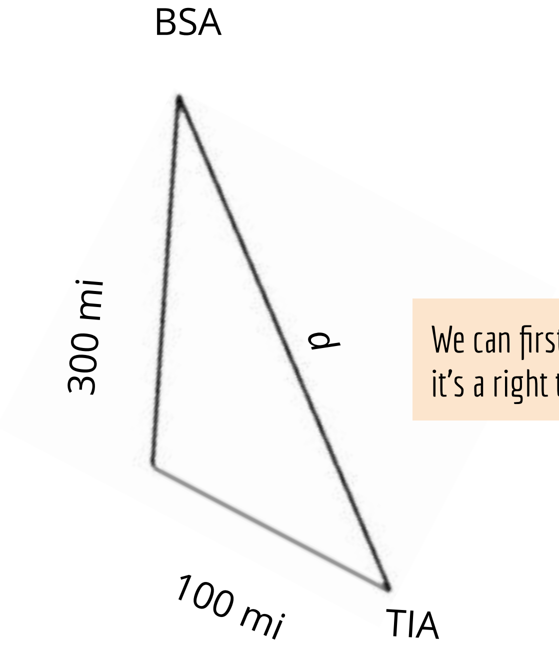
Example 1. An airplane leaves Tampa International Airport and initially travels 100 miles northwest. The plane then travels 300 miles north to reach its destination of Birmingham-Shuttlesworth Airport. Find the distance (to three decimal places) and bearing (to the nearest degree) that the airplane needs to travel at in order to reach TIA again.

TIA BSA

Solving the Application



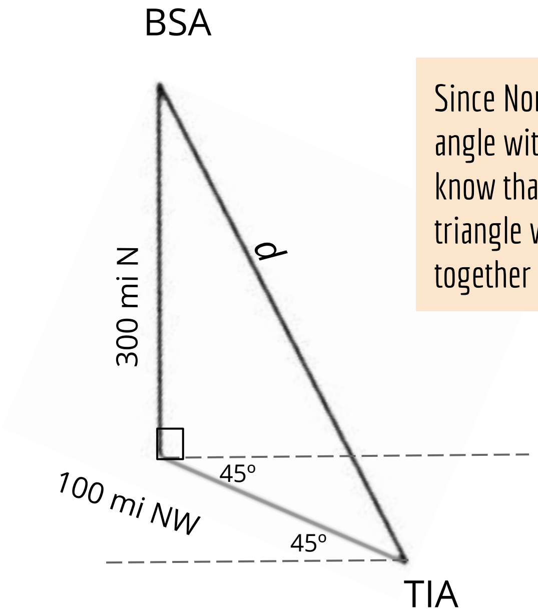
1. First, draw a diagram to properly understand what the situation looks like.



We can first solve for *d*, but we will need an angle measurement since we do not know if it's a right triangle.

Solving the Application

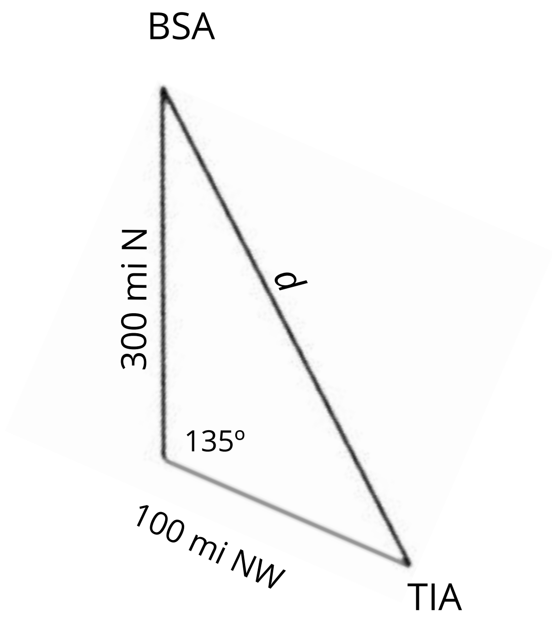
2. We were given directions of Northwest and North. Since we know the angles these directions form, we can solve for one of the angles in the triangle.



Since Northwest is in the middle of directions North and West, we know it creates a 45º angle with a horizontal line. Additionally, since North is directed straight upwards, we know that it creates a 90º angle from a horizontal line. To find the interior angle in the triangle where the plane switches directions, all we have to do is add these angles together to get 135º.

Solving the Application

3. With one of the angle measures, we can now use Cosine Rule. We have two sides and the angle opposite the unknown side, and with Cosine rule we can solve for this last side.

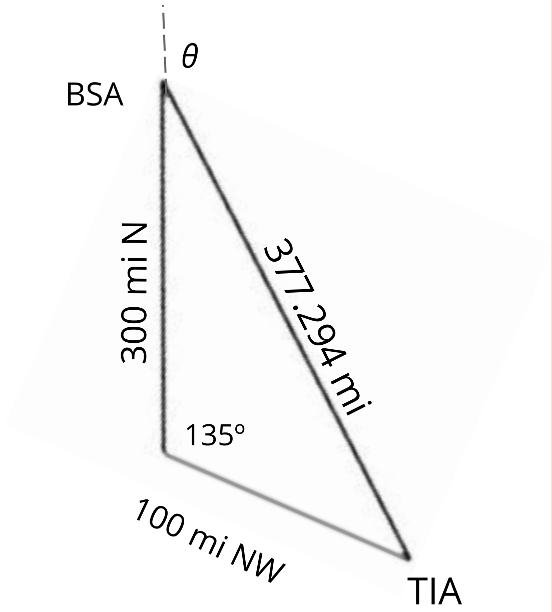


Cosine Rule:
$c^2 = a^2 + b^2 - 2ab\cos C$

$d^2 = 100^2 + 300^2 - 2(100)(300)\cos(135°)$
$d^2 = 100000 - 60000\cos(135°)$
$d^2 = 142426.407$
$d = 377.294$ miles

Solving the Application

4. Now that we know the magnitude of the flight back to TIA, we can decipher the bearing needed for the pilots to return there. We need to find the angle from the North that will take us from BSA ro TIA. To find this angle theta, we can find the supplementary angle to the interior angle at BSA. To find this interior angle, we can use sine rule.



Sine Rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin(135^\circ)}{377.294} = \frac{\sin(BSA)}{100}$$

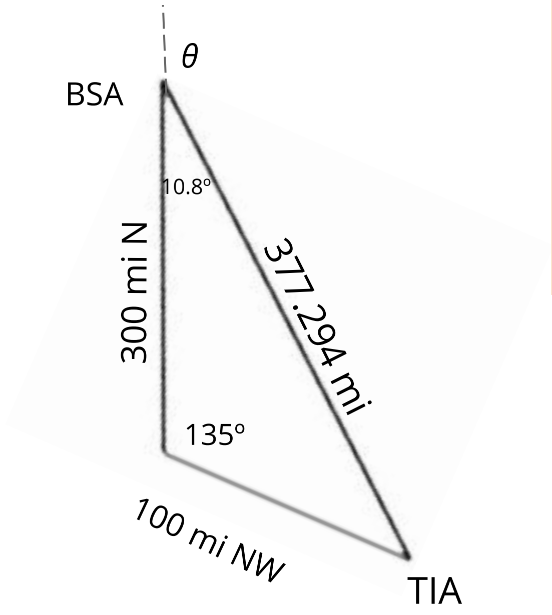
$$\sin(BSA) = \frac{100\sin(135^\circ)}{377.294}$$

$$\sin(BSA) = 0.1874$$

$$BSA = 10.8^\circ$$

Solving the Application

5. To find the bearing, we have to find the interior angle's supplementary angle. This way, the reality of the scenario with the usage of bearings is accomplished by transforming the trigonometric concepts to real life.



supplementary angles add to become 180º

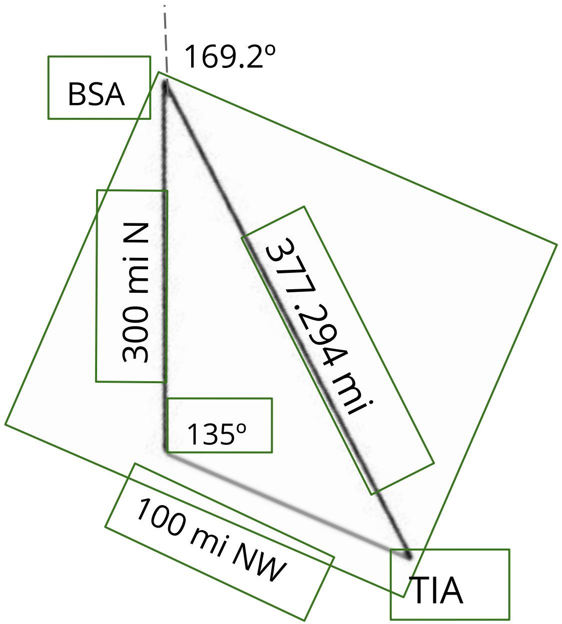
$10.8^\circ + \theta = 180^\circ$

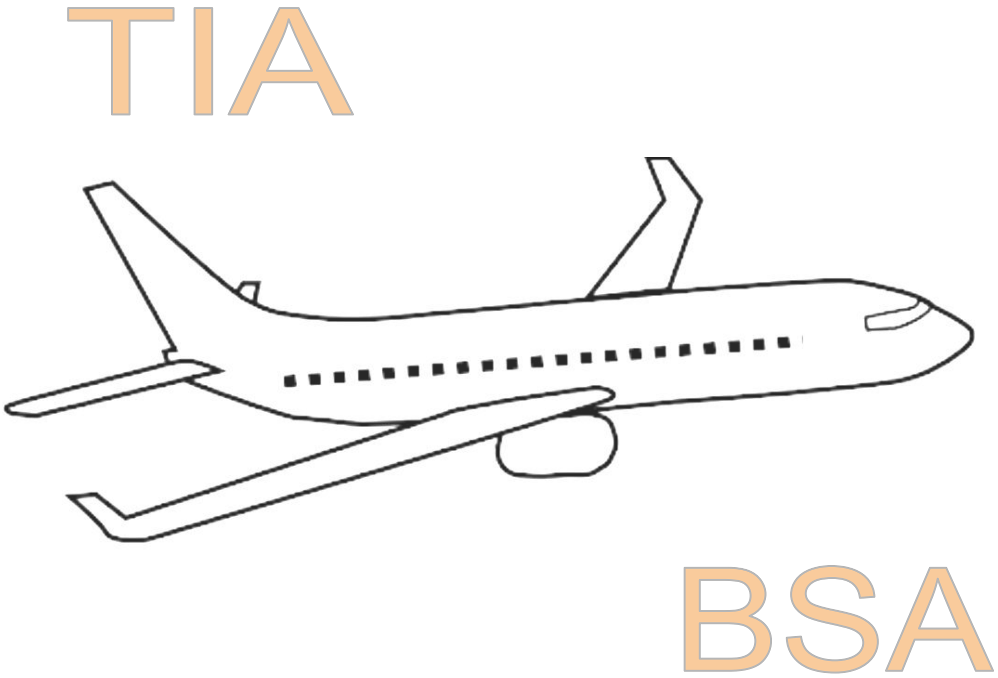
$\theta = 180^\circ - 10.8^\circ$

$\theta = 169.2^\circ$

Solving the Application

Therefore, the answer to this application is that the plane must travel 377.294 miles at a bearing of 169º in order to reach TIA again. This answer takes into account the precision that the problem asks for and also includes units.





Works Cited

Khan, Sal, director. *Basic Trigonometry. Khan Academy*, Youtube, 2007, https://www.youtube.com/watch?v=F21S9Wpi0y8&feature=youtu.be.

Khan, Sal, director. *Determining the Equation of a Trig Function. Khan Academy*, Youtube, 2013, https://www.youtube.com/watch?v=yHo0CcDVHsk&feature=youtu.be.

Khan, Sal, director. *Unit Circle Definition of Trig Functions. Khan Academy*, Youtube, 2007, www.youtube.com/watch?v=ZffZvSH285c.

Step 0

Overview

Our topic that we will be discussing and reviewing is **Probability**. Probability is the likeliness or how likely something will occur. Probability is very useful as it occurs in real life in different situations such as poker, games, and even weather reports. In our review, we will cover key concepts and terms, formulas and equations, practice problems, different real world problems and how probability is specifically used in them. This review should be able to explain the unit of probability as a whole and even teach how to do certain probability problems.

In **step one**, we will first be discussing different key terms. Some of the key terms include factoricals, combinations, permutations, tree diagrams, mutually exclusive, independent, dependent, mutually exhaustive, conditional probability, intersection, and union. We will identify and define these terms, and give examples in order for the reader to fully understand the concepts of them. Secondly, we will be identifying and explaining some of the formulas in the unit of probability. Some of these formulas include, the conditional probability formula, and the mutually exclusive formula.

In **step two**, we will identify and define the key concepts of the probability unit. For each of the key concepts, we will provide different worked out examples for the readers to get a greater understanding of the topics. Additionally, we will also include different technological learning strategies, such as videos, keynotes and songs, to help the reader really grasp the knowledge that we will be providing in this step.

In **step three** of this process, we will provide an example of the concepts mentioned in steps one and two within the contents of the real world. We will also overview what the process of completing this real world problem would look like.

Finally, in **step four** we will give ten problems pertaining to the key concepts of the unit of Probability. These problems will be easy at first, then grow harder as they progress. There will be three problems that are easy, then four medium level problems, and finally three difficult problems. There will also be worked solutions to all of the problems given.

Step 1

Key Words and Formulas

Key words:

Factoricals - Multiplies the number by every positive integer under it, represented by using an exclamation point.
EX: 4! = (4)(3)(2)(1) = 24

Combinations - A group of items (these can be numbers, letters, cards, etc.) where it does not matter the position or the order of the items.
EX: Four students are chosen for a group project, the students can be rearranged but it is still the same group of kids.

Permutations - A group of items (these can be numbers, letters, cards etc.) where the order and position of items does matter.
EX: A password. If a password is 1234, you cannot rearrange it and make it 4321.

Tree diagrams - A diagram that is used to help calculate and determine the likelihood or the probability of a certain event. Usually used with items that can have only two outcomes.
EX: the probability of flipping heads three times in a row (here you would use a tree diagram)

Mutually exclusive - Two groups of events that have no intersection and it is impossible for both of the groups can be true.
EX: Picking a red card or picking a black card. There is no intersection between these two events because there are no cards that are black and red.

Independent - Events that do not have direct correlation. The events do not affect the outcome of the other event.
EX: Flipping a coin and picking a card out of a deck.

Dependent - Events that are directly correlated and the outcome of one event affects the outcome of the other event.
EX: Drawing two cards from the deck, without replacement.

Mutually exhaustive - Two or more events that cannot coincide.
EX: It's raining outside and it's not raining outside

Conditional probability - The probability that one event will happen given that another event happened
EX: The probability that a card is a jack given its a face card

Intersection - a set of elements found in two other sets or the overlapping elements of two sets
EX: the intersection between even numbers and numbers 1-10 is 2,4,6,8,10

Union - a set made by combining elements of two other sets
EX: the union of the set (1,2,3,4,5) and the set (6,7,8,9,10) is the set (1,2,3,4,5,6,7,8,9,10)

Formulas:

Independent Events - $P(A \cap B) = P(A) \times P(B)$

Explanation: This is because independent events do not affect the outcome of the other event. So in order to get the probability of A and B, you would just need to multiple the probability of A, by the probability of B.

Conditional Probability - $P(A \cap B) = P(A) \times P(B| A)$

Explanation: Since event B is dependent on event A the probability of B happening in general is the probability of A multiplied by the probability of B

Combined Events - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Explanation: Here, you are adding the probability of A to the probability of B, and then you are subtracting the intersection of A and B. You need to subtract this intersection because when you are adding A to B, their intersection is in both of them (that's what makes it an intersection). So, you need to subtract one of the intersections so you do not have two of them.

Mutually Exclusive Events - $P(A \cup B) = P(A) + P(B)$

Explanation: This is because mutually exclusive events do not have any intersection. So, you do not need to subtract the intersection, because there is none.

Permutations - P(n, r) = n!/(n - r)!

Explanation: r is the number of "slots" or the total number of elements in the permutation and n is the number of unique elements

Combinations - C(n, r) = n!/((n-r)!r!)

Explanation: r is the number of "slots" or the total number of elements in the combination and n is the number of unique elements the r! In the denominator factors out combinations with the same elements but in a different order.

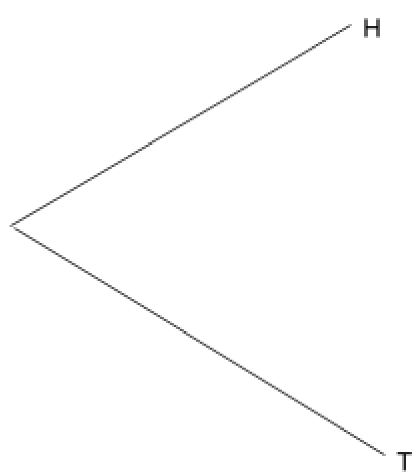
Step 2

Key Concepts

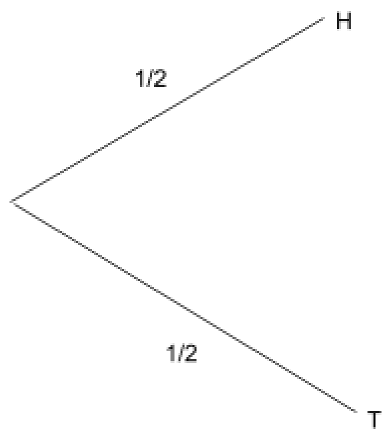
Tree Diagram - Used to help determine the probability of a specific event happening.

Example Problem: What is the probability of throwing tails two times in a row?

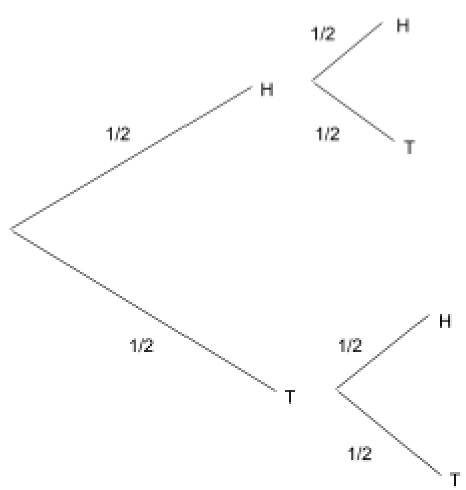
First, we know that flipping a coin only has two outcomes, heads and tails. We can begin to draw the diagram like this:



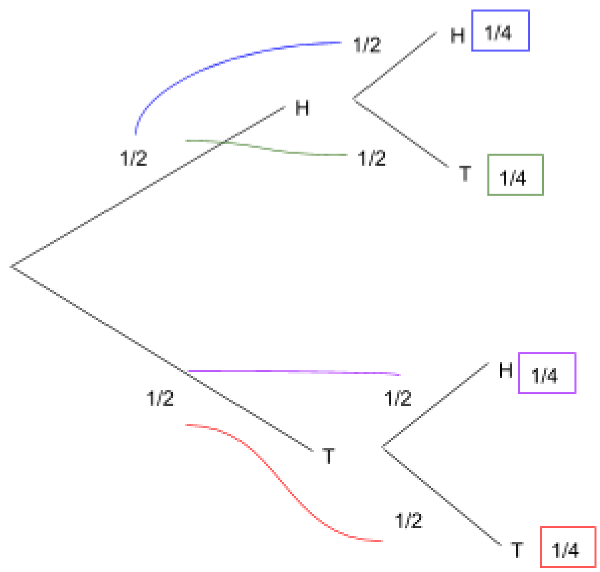
Now, we know that there is a fifty percent chance of throwing tails, and there is a fifty percent chance of throwing heads. We can show this on the diagram like this:



This diagram represents only the first throw. We now have to include the second throw for both of the outcomes of the first throw:



Finally, now that the both of the throws are seen on the diagram, we need to calculate the probability of each outcome. We do this by multiplying the each every number on its path, like this:



Finally, in order to get the answer to the question, you have to follow the path that it states. In this question, it asks the probability of getting two tails in a row. If you follow the path of the two T's on this diagram, you will reach the probability of 1/4, with the red box around it.

<u>Conditional Probability:</u> The probability that one event will happen given that another event happened

Example Problem: You roll a 6 sided die. What is the probability that the number that you roll is a 6, given that an even number is rolled.

P(A/Given B) = P(A)/P(B)

First we know that in order to solve this, we need to substitute our example into this equation above.

P(6 is rolled/number is even) = P(A)/P(B)

Now, we need to figure out the probabilities of both of the events. The probability of rolling a 6, is 1/6. The probability of the number being even is 3/6, or 1/2. Now we need to plug this in to the equation as well.

P(6 is rolled/number is even) =(⅙)/(½).

Now, simplify the equation, and you get your answer.

P(6 is rolled/number is even) = 1/3.

Combinations: A group of items (these can be numbers, letters, cards, etc.) where it does not matter the position or the order of the items.

Example Problem: How many committees of 3 people can be made with a pool of 7 possible candidates?

Since the question presents a scenario that doesn't require elements to be in specific positions within each committee, we know it's a combination. After assessing its combination we can plug in 7 for n since there's 7 unique candidates and plug 3 for r since there are three spots on the committee. This would give us an equation of

$$\frac{7!}{(7-3)!\cdot 3!}$$

Then use a calculator to solve the equation to get 35 different different committees

Permutations: A group of items (these can be numbers, letters, cards etc.) where the order and position of items does matter.

Example Problem: a horse race has 12 different horses racing. How many different ways can 1st, 2nd, and 3rd occur?

Since order matters in which horses finished st, 2nd, and 3rd, we can use the permutation equation. Since there are 12 different horses n is 13 and r is 3 since we only care about the first 3 horses. That equation would look like so

$$\frac{12!}{(12-3)!}$$

And if you plug this into a calculator you get 1320 different permutations.

Step 4

Ten Practice Problems

Easy:

1. For events *A* and *B*, $p(A) = 0.6$, $p(B) = 0.8$, $p(A \cup B) = 1$. Find $p(A \cap B)$.

   $p(A \cap B) = 0.4$

2. In a survey, 100 teachers were asked if they preferred their morning classes or their evening classes better. Of the 46 male teachers in the survey, 33 said that they prefer their evening classes, while 29 female teachers made this choice. Fill Out the rest of this table

	Male teachers	Female teachers	total
Morning classes	13	25	38
Evening classes	33	29	62
Total	46	54	100

3. Using the table above, find the probability that a teacher chosen at random prefers their morning classes. In addition, given that the teacher is male, the teacher prefers their morning classes.

a. p(morning classes) = 0.38
b. p(morning classes/male) = 13/46

Medium:

4. The following Venn diagram shows a sample space *U* and events *A* and *B.* $n(U) = 36$, $n(A) = 11$, $n(B) = 6$, *and* $n(A \cup B)' = 21$.
    a. Find $n(A \cap B)$
    b. Find $P(A \cap B)$
        a. $n(A \cap B) = 2$
        b. $P(A \cap B) = 1/18$

5. A painter has 12 tins of paint. Seven tins are red and five tins are yellow. Two tins are chosen at random. Calculate the probability that both tins are the same color.

p(both tins are same color) = 31/66

6. The class contains 13 girls and 11 boys. The teacher randomly selects 4 students. Find the probability that at least 2 of the selected students are girls.

p(at least 2 girls) = 76.7%

7. Let A and B be events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$, $P(A \cup B) = \frac{7}{8}$. Calculate $P(A \cap B)$.

$P(A \cap B) = 3/8$

Hard:

8. What is the probability that in a group of 10 people at least 2 people share a birthday? (Assume there is 365 days in a year)

p(at least 2 share a birthday) = 0.117 (3sf)

9. In a certain lottery there are ten numbers (1-10). Of these numbers five are drawn in order. If you guess the numbers drawn in order then you will win. What are the odds of winning?

p(winning) = 0.000033069

10. There is a combination lock that displays 50 numbers. To open it you turn to a number, then turn clockwise to a second number, then turn counterclockwise to a third number. What are the odds of guessing the combination?

p(guessing combination) = 0.000008

Unit 5: Statistics

Statistics is a branch of math in which analysis and models are commonly used. These models and analytical devices are commonly used to explain, predict, and describe experimental data and real life scenarios. The use of experimental data or data collected from real situations suggests that statistics focuses on expected values or occurrences in realistic and possible conditions. There are two types of statistics, descriptive and inferential. Descriptive statistics includes measuring frequency, variation, position, and central tendency while inferential focuses on the analysis of regressions, variance, co-variance, and correlation. Both of them are very useful in jobs such as weather forecasting, marketing, medical predictions, and many more.

Vocab and Formulas

Mean- The average of a set of numbers and their values.

FORMULA: *sum of numbers ÷ amount of numbers*

Median- If the numbers given were organized in order from smallest to largest, the median is the number that appears in the middle of the set or the average of the two numbers in the center.

Mode- The number in a set of data that shows up most often.

Population- The total number of things within a large group that share a common characteristic or trait. The population is used for data collection and analysis.

Sample- A small portion of a population that is used to estimate or predict results of a large or unmeasurable population.

Discrete data- Data that can be categorized or that is an integer.

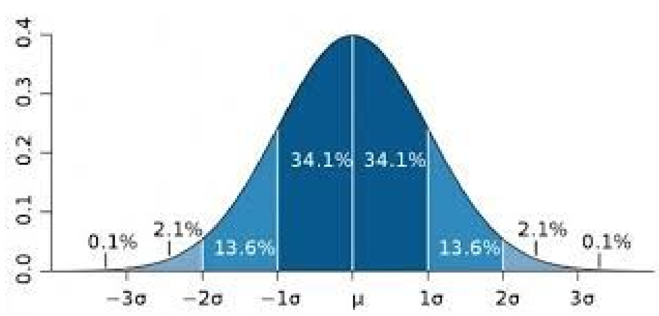
EXAMPLE: 75 colors/ 3 roses.

Continuous data- Data that is not necessarily an integer and is most likely measured.

EXAMPLE: 3.5 miles per hour/ 61.83 meters.

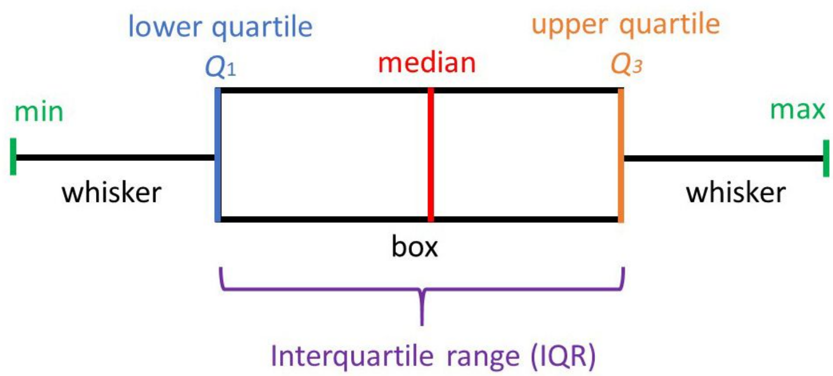
Probability Distribution- A function that describes the possibility of scenario or possible value within a scenario. This function is in the shape of a bell curve meaning the extremes have a lower probability while the mean has the highest.

EXAMPLE: (https://trumpexcel.com/bell-curve/)



Box and Whisker- A form of graphing that organizes values of data into four sections known as quartiles.

EXAMPLE: https://www.simplypsychology.org/boxplots.html



Quartile- A sort of barrier that separates a set of data into 4ths that represent a specific 25% of the data. There is the lower quartile/ Q_1 , middle quartile/ Q_2 ,and the upper quartile/ $Q3$.

Interquartile Range- A range or percentage of values in a set of data that represent the middle 50% of the data.

FORMULA: $IQR = Q_3 - Q_1$

Variance- This describes how the values are distributed compared to the mean or average of the set.

FORMULA: $\sigma^2 = (\sum(x - \mu)^2 \div N$

Standard Deviation- This value shows how close the data is to the mean or average of the set.

FORMULA: $\sigma = \sqrt{}(\sum(x - \mu)^2 \div N$

Regression- An estimated model of a set of data points that shows the predicted relationship between the independent and dependent variable.

Correlation- The relationship between two random data points or variables.

Correlation Coefficient- The numerical value that describes the relationship between two random data points or variables.

FORMULA: $r = \frac{n(\Sigma xy)-(\Sigma x)(\Sigma y)}{\sqrt{[n(\Sigma x^2)-(\Sigma x)^2][n(y^2)-(\Sigma y)^2]}}$

Bias- This is the difference between an estimated value and the actual value.

Outliers- a number in a data set that is significantly different from the rest of the data. It can either be much less or much more than the set.

Binomial Distribution- The probability of an experiment being successful or unsuccessful or the probability of a scenario occurring

FORMULA: $P(k) = {}_nC_k p^k(1 - p)^{n-k}$

Mean=np- The mean of the binomial distribution.

FORMULA: $\mu = n \cdot p$

Variance=npq- The variance of the binomial distribution.

FORMULA: $\sigma^2 = n \cdot p \cdot q$

Standard Deviation=sqrt(npq)- The standard deviation of the binomial distribution.

FORMULA: $\sigma = \sqrt{n \cdot p \cdot q}$

Z score- Also known standard score, this value represents the number of standard deviations that a raw score is above or below the mean.

FORMULA: $Z = \frac{x-q}{\sigma}$

Concepts

Types of data:

Discrete data- Data that can be categorized or that is an integer.

EXAMPLE: 75 colors/ 3 roses.

Continuous data- Data that is not necessarily an integer and is most likely measured.

EXAMPLE: 3.5 miles per hour/ 61.83 meters.

Question- Is the amount of time it takes to ride a roller coaster an example discrete or continuous data?

Answer: Continuous because the answer is measured and would not be an integer.

Question- Is the amount of people that can ride the roller coaster at one time an example of discrete or continuous data?

Answer: Discrete because you cannot have a fraction of a person, so the answer would be an integer.

Video:https://www.youtube.com/watch?v=6IdJ1aPFDCs

Population vs. Sample:

Population- The total number of things within a large group that share a common characteristic or trait. The population is used for data collection and analysis.

Sample- A small portion of a population that is used to estimate or predict results of a large or unmeasurable population.

Question- 100 people are chosen at random from a mall to represent the total amount of customers that shop there during the week. Which is the sample and which is the population?

Sample- The 100 people because it is the smaller portion of a larger group.

Population- The customers that shop there during the week because it is the much larger group.

Video:https://www.youtube.com/watch?v=eIZD1BFfw8E

Mean, Median, and Mode:

Mean- The average of a set of numbers and their values.

FORMULA: $sum\ of\ numbers \div amount\ of\ numbers$

Median- If the numbers given were organized in order from smallest to largest, the median is the number that appears in the middle of the set or the average of the two numbers in the center.

Mode- The number in a set of data that shows up most often.

Question- What is the mean, median, and mode of the data below?

3,5,8,6,3,4,9,5,2,5

Mean- $sum\ of\ numbers \div amount\ of\ numbers = mean$

$(3+5+8+6+3+4+9+5+2+5) \div 10 = mean$

$50 \div 10 = mean$

$5 = mean$

Median- 3,5,8,6,3,4,9,5,2,5

2,3,3,4,5,5,5,6,8,9

2,3,3,4,*5,5*,5,6,8,9

$(5+5) \div 2 = median$

$(10) \div 2 = median$

$5 = median$

Mode- 2,3,3,4,5,5,5,6,8,9

2- one

3- two

4- one

5- three

6- one

8-one

9- one

$5 = mode$

Video: https://www.youtube.com/watch?v=aLpjoo0A1ag

Quartiles and Ranges:

Quartile- A sort of barrier that separates a set of data into 4ths that represent a specific 25% of the data. There is the lower quartile/ Q_1 , middle quartile/ median/ Q_2 ,and the upper quartile/ Q_3 .

Interquartile Range- A range or percentage of values in a set of data that represent the middle 50% of the data.

FORMULA: $IQR = Q_3 - Q_1$

Question- What is the middle, lower, and upper quartile of the data set below? What is the interquartile range of the set?

6, 6, 8, 9, 9, 12, 14, 14, 14, 15

Middle- 6, 6, 8, 9, *9, 12*, 14, 14, 14, 15

$(9+12) \div 2 = median/middle\ quartile$

$21 \div 2 = median/middle\ quartile$

$10.5 = median/middle\ quartile$

Lower- 6, 6, *8,* 9, 9, | 12, 14, 14, 14, 15

$8 = lower\ quartile$

Upper- 6, 6, 8, 9, 9, | 12, 14, *14,* 14, 15

$14 = upper\ quartile$

Interquartile Range- $IQR = Q_3 - Q_1$

$IQR = 14 - 8$

$IQR = 6$

Video:https://www.youtube.com/watch?v=K3wsOqIqA6k

Variance and Standard Deviation:

Variance- This describes how the values are distributed compared to the mean or average of the set.

FORMULA: $\sigma^2 = \sum(x - \mu)^2 \div N$

Standard Deviation- This value shows how close the data is to the mean or average of the set.

FORMULA: $\sigma = \sqrt{}(\sum(x - \mu)^2 \div N)$

Question- What is the variance and standard deviation of the data set below?
12,13,15,17

Variance- $\sigma^2 = \sum(x - \mu)^2 \div N$

$(12 + 13 + 15 + 17) \div 4 = mean$

$57 \div 4 = mean$

$14.25 = mean$

$(12 - 14.25)^2 \div 4 = 1.27$

$(13 - 14.25)^2 \div 4 = 0.39$

$(15 - 14.25)^2 \div 4 = 0.14$

$(17 - 14.25)^2 \div 4 = 1.89$

$\sigma^2 = 1.27 + 0.39 + 0.14 + 1.89$

$\sigma^2 = 3.69$

Standard deviation- $\sigma = \sqrt{}(\sum(x - \mu)^2 \div N)$

$\sigma = \sqrt{}(variance)$

$\sigma = \sqrt{}(3.69)$

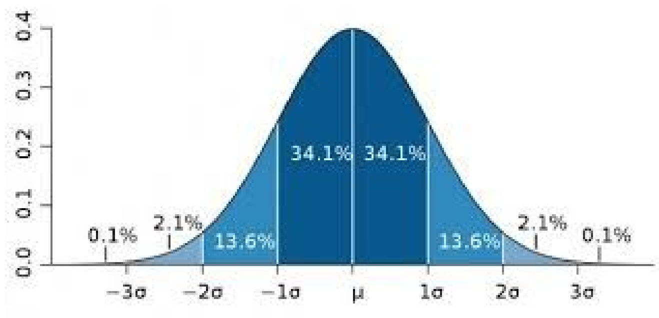
$\sigma = 1.92$

Video:https://www.youtube.com/watch?v=qqOyy_NjflU

Graphs and Regressions:

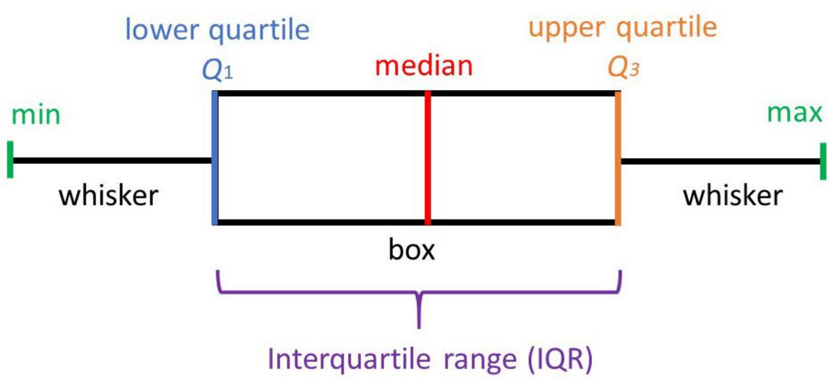
Probability Distribution- A function that describes the possibility of scenario or possible value within a scenario. This function is in the shape of a bell curve meaning the extremes have a lower probability while the mean has the highest.

EXAMPLE: (https://trumpexcel.com/bell-curve/)



Box and Whisker- A form of graphing that organizes values of data into four sections known as quartiles.

EXAMPLE: https://www.simplypsychology.org/boxplots.html



Common graphs/devices used for displaying data in statistics: Probability Distribution, Box and Whisker, Bar Graph, and Scatter Plot.

Question: Label the Box and Whisker diagram using the data set below.

2,2,5,6,8,8,8,11,13,16

Median- 2,2,5,6,8,*8*,8,8,11,13,16

median=8

Upper quartile- 2,2,5,6,8,8,8,8,11,13,16

8,8,*11*,13,16

upper quartile= 11

Lower quartile-2,2,5,6,8,8,8,8,11,13,16

2,2,*5*,6,8

lower quartile= 5

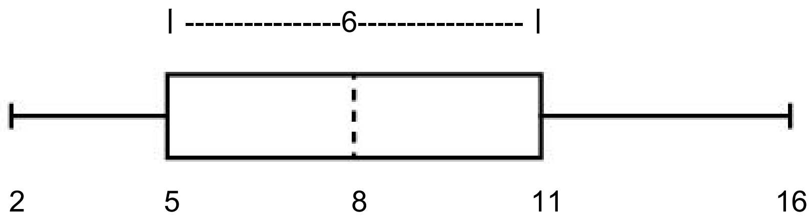
minimum- 2

Maximum-16

Interquartile range- $11 - 5 = IQR$

interquartile range= 6

Diagram-



Video:https://www.youtube.com/watch?v=fJZv9YeQ-qQ

Correlation:

Correlation- The relationship between two random data points or variables.

Correlation Coefficient- The numerical value that describes the relationship between two random data points or variables.

FORMULA: $r = \frac{n(\Sigma xy)-(\Sigma x)(\Sigma y)}{\sqrt{[n(\Sigma x^2)-(\Sigma x)^2][n(y^2)-(\Sigma y)^2]}}$

Question: What is the correlation coefficient of the data set below?

20	26
51	28
62	45
85	60
99	42

Correlation coefficient- $r = \frac{n(\Sigma xy)-(\Sigma x)(\Sigma y)}{\sqrt{[n(\Sigma x^2)-(\Sigma x)^2][n(y^2)-(\Sigma y)^2]}}$

x	y	xy	x^2	y^2
20	26	520	400	676
51	28	1428	2601	784
62	45	2790	3844	2025
85	60	5100	7225	3600
99	42	4158	9801	1764

$\Sigma x = 317 \quad \Sigma y = 201 \quad \Sigma xy = 13996 \quad \Sigma x^2 = 23871 \quad \Sigma y^2 = 8849$

$$r = \frac{5(13996)-(317)(201)}{\sqrt{[5(23871)-(23871)^2][5(8849^2)-(8849)^2]}}$$

$r = 0.735$

Video:https://www.youtube.com/watch?v=atLZNGsTN6k

Binomial Distribution:

Binomial Distribution- The probability of an experiment being successful or unsuccessful or the probability of a scenario occurring

FORMULA: $P(k) = {}_nC_k p^k (1-p)^{n-k}$

Mean=np- The mean of the binomial distribution.

FORMULA: $\mu = n \cdot p$

Variance=npq- The variance of the binomial distribution.

FORMULA: $\sigma^2 = n \cdot p \cdot q$

Standard Deviation=sqrt(npq)- The standard deviation of the binomial distribution.

FORMULA: $\sigma = \sqrt{n \cdot p \cdot q}$

Z score- Also known standard score, this value represents the number of standard deviations that a raw score is above or below the mean.

FORMULA: $Z = \frac{x-q}{\sigma}$

Question: if 47% of students in 2nd grade like chocolate ice cream and 3 students are chosen at random, what is the mean, variance, and standard deviation of students that like chocolate ice cream?

Mean of the binomial distribution- $\mu = n \cdot p$

$\mu = 3 \cdot 0.47$

$\mu = 1.41$

Variance of the binomial distribution- $\sigma^2 = n \cdot p \cdot q$

$\sigma^2 = 3 \cdot 0.47 \cdot (1 - 0.47)$

$\sigma^2 = 0.7473$

Standard deviation of the binomial distribution- $\sigma = \sqrt{n \cdot p \cdot q}$

$\sigma = \sqrt{(3 \cdot 0.47 \cdot (1 - 0.47))}$

$\sigma = 0.8645$

Question: 83% of ice cream in a store contains dairy. If 3 cartons of ice cream are chosen what is the probability that 2 of them contain dairy?

Probability: $P(k) = {}_nC_k p^k (1-p)^{n-k}$

$P(2) = {}_3C_2 0.83^2 (1 - 0.83)^{3-2}$

$P(2) = 3 \cdot 0.83^2 (0.17)^1$

$P(2) = 3 \cdot 0.83^2 (0.17)^1$

$P(2) = 0.351$

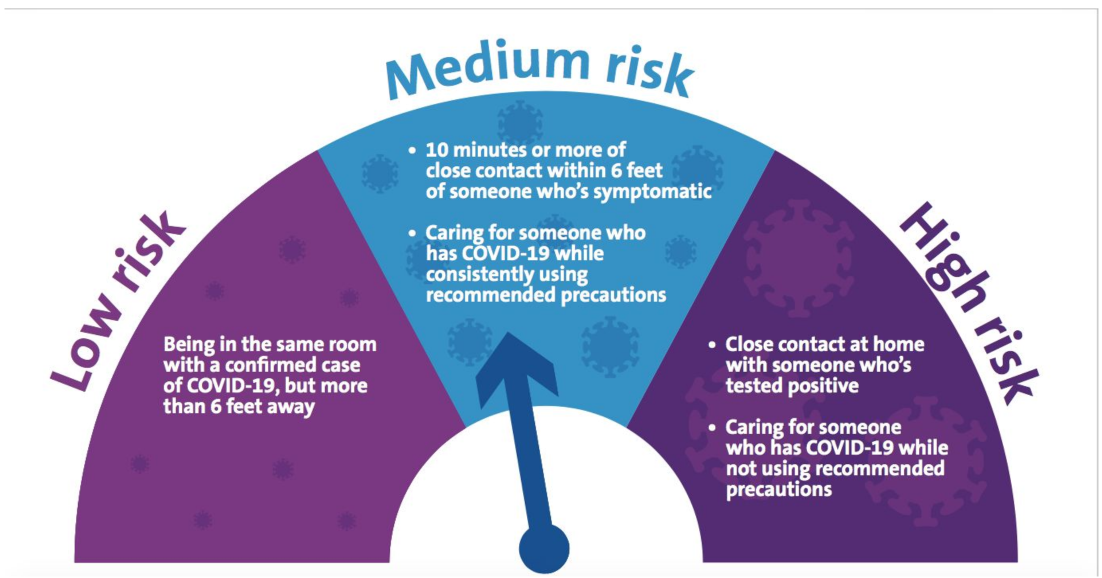
Video:https://www.youtube.com/watch?v=R1IDgN3ill8

Real Life Application

In light of the recent pandemic, a lot of data is being recorded to help people keep track and make guidelines for the pandemic. Statistics is very useful in such situations to monitor risk and make predictions. The questions below are extremely relevant as all three are situations that could be happening around the world right now during the current Covid Crisis. The CDC has posted their risk assessment along with the relevant probabilities for getting sick. Not everyone is at a high threat of getting Coronavirus, there are measures and guidelines that people can follow to reduce their risk of getting infected. Statistics can be used here to calculate the probabilities of outcomes of situations in the real world.

Question 1: According to the CDC, there are three levels of risk for people in the US. Low Risk which means a person has a 25% or less chance of getting infected. Medium Risk which equates to around a 50% chance of being infected. High Risk which is a 75% or higher chance of getting infected. Use graphic 3-1 to answer A,B,and C.

Graphic 3-1



Source: CDC risk assessment

A: County hospitals have 400 healthcare workers in close contact with patients who are symptomatic. They work 48 hour shifts with most of the hours within 6 feet of Corona positive patients. All of them maintain proper safety protocols and guidelines while taking care of the patients. What is the probability that exactly 211 healthcare workers contract the virus.

B: 200 people pass through a grocery store maintaining 6 feet social distancing and not staying in close contact with anyone for more than 5 minutes. Assume there was always at least one person with Covid in the store at all times. What is the probability that less than 10 shoppers contract Covid-19.

C: An estimated 32 families of 4 have a member of the family who has tested positive for Corona but is still at home because the local hospital is full. Those families have to take care of the infected family member, and since they live in Corona infested neighbourhoods, they cant find many masks, alcohol wipes, etc in the local stores. What is the probability that more than 85 family members get infected.

In conclusion, statistics can be very useful in making predictions from previous data. These types of statistics can help in real life: calculating the probability that half a community of 2000 using binomial and normal distribution people will get Covid can help hospitals to be prepared for the number of sick people who will need beds and doctors, allowing more efficient use of resources. Calculating the average number of people getting infected in a week with social distancing or a complete lockdown by comparing the most likely outcomes for both can help the government decide when to increase or decrease restrictions. This type of statistics is closely related to probability, but it allows for more specific calculations for more specific situations making it more versatile.

Sources:

"Public Health Guidance for Potential COVID-19 Exposure Associated with International Travel or Cruise Travel." *Centers for Disease Control and Prevention*, Centers for Disease Control and Prevention, 3 May 2020, www.cdc.gov/coronavirus/2019-ncov/php/risk-assessment.html.

Practice Problems:

Easy:

1: The maths teacher randomly selects the test scores of 12 HL students, all of whom had been in HL math for 2 years.
a: Is this sample of HL maths students biased, or unbiased?
b: Select the outlier from the test scores in table 1-1

Scores	91	93	94	90	86	89	100	72	99	98	87	95

2: A school is collecting data on the amount of time taken to finish a maths summative based on the number of questions on the test so that they can provide adequate time to students.

a. X is the number of questions on a specific math test. Is X discrete or continuous?
b. Y is the amount of time taken by a student to complete a maths test. Is Y discrete or continuous.

3: Table 1-2 notes the average amount of time a student spends in online class every day during the covid lockdown.
Table 1-2

Average hours on Zoom Class per Day	**Number of Students**
1.3	6
6.5	19
8.0	5
3.2	12
4.0	26
5.2	22

a. Write the probability distribution in the table below. Write each probability as a decimal rounded to the nearest thousandth.

X	P(X)
1.3	
6.5	

8.0	
3.2	
4.0	
5.2	

b. Calculate the expected value of X from the values obtained from the previous problem.

Medium:

4: A Ring Toss manufacturer claims that 10% people who toss the ring can get 3 consecutive tosses to win a prize. If this pattern holds, and 3 players play the game in the opening 5 minutes of the fair, what is the probability that exactly 2 of the players can get prizes.

5: An apple farmer has 9 orchards, the data in Table 2-1 contains the number of trees in each orchard.

Table 2-1

Number of Trees	31	22	45	36	62	45	33	56	21

a. Find the mean, median, and mode of the number of trees
b. Find the quartiles and calculate the interquartile range.
c. Make a box and whisker plot using the answers from b.

6: The average resting heart rate of olympic runners is 40 bpm with a standard deviation of 8 bpm. The bottom 30% are required to have extra screening before a race to ensure the athletes are healthy. At what heart rate is extra screening required rounded to the nearest whole number.

7: A researcher conducted a survey on the hours of sleep in a week based on the number of classes giving homework.

Table 2-2

Hours Slept in a Week	Number of classes giving HW
51.7	3
49.3	4
38.9	6

18.6	7

a: calculate the mean, variance, and standard deviation for the hours slept in a week rounded to the nearest thousandth(use GDC for variance and standard deviation)
b: Using the information from part a, find the z score of someone who slept:
i. 27.5 hours
ii. 62.0 hours
iii. 45.3 hours

Hard:

8:There is a baseball game between CDS and Berkeley. Using past statistics, the team analyst concluded that there is a 38% chance that a homerun will be hit by CDS batters against a pitch by Berkeley's best pitcher. If CDS's batters face 2 pitches from Berkeley's best pitcher, what is the mean, variance and standard deviation of the number of Homeruns hit rounded to the nearest thousandth. (Hint: Binomial Distribution)

9: One of the staff members has tested positive for Covid 19 on ABC Cruise ships. There are 950 passengers on the ship out of which 72% are predicted to test positive for Covid according to calculations made by the CDC. Unfortunately, there are only 100 test kits, if 100 passengers are tested, what is the probability that less than 80 passengers test positive for Covid?

10: A study is being conducted under controlled conditions to determine the impact of sodas on weight gain. A group of volunteers who all have the same age, gender, height, weight, and similar metabolic rates are monitored on their soda intake and weight gain per week.
Table 3-1

250ml Cans of Soda per week(x)	**Weight Gain in kg per week(y)**
3.5	-0.2
7	0.5
10.5	0.8
28	1.5

a. Find the mean cans of soda and mean weight gain separately
b. Find the standard deviation of both x and y separately **no GDC**
c. The correlation coefficient for the data is 0.928,use the answers from a and b combined with the correlation coefficient to find the equation of linear regression y=mx+b.

Practice Problem Answers:

Easy:

1a: Unbiased because all the students take the same level of math for the same time. Because they were picked randomly, they represent the HL math student population fairly.
b:

Scores	91	93	94	90	86	89	100	**72**	99	98	87	95

2a: Discrete, because the number of problems is round and concrete.
b: Continuous, because time is complex and measured in intervals.

3a:

X	P(X)
1.3	6/90 = 0.067
6.5	19/90 = 0.211
8.0	5/90 = 0.056
3.2	12/90 = 0.133
4.0	26/90 = 0.289
5.2	22/90 = 0.244

b. Expected value of X = $(P(x_1)*X_1)+(P(x_2)*X_2)+...(P(x_n)*X_n)$
= (0.067*1.3)+(0.211*6.5)+(0.056*8.0)+(0.133*3.2)+(0.289*4.0)+(0.244*5.2)
= (0.0871)+(1.3715)+(0.448)+(0.4256)+(1.156)+(1.2688)
Expected Value of X= 4.757

Medium:

4:Need Binomial Distribution formula $(nCx)p^x(1-p)^{(n-x)}$
p=0.1 probability
n=3 trials
x=2 successes
Binomial Distribution= $(3C2)0.1^2(1-0.1)^{(3-2)}$
= $(3)0.01\ (0.9)^{(1)}$
= 0.027
Binomial Distribution=0.027

5: Rearrange data from lowest to highest

21	22	31	33	**36**	45	45	56	62

a: Mean = $sum\ of\ numbers \div amount\ of\ numbers$
=(21+22+31+33+36+45+45+56+62)/9
=351/9
Mean = 39
Median = 36 in the 5th position
Mode=45 appears twice

b: Q1 Position = ¼(n+1)
= ¼(9+1)
= ¼(10)
Q1 Position = 2.5

Q3 Position =¾(n+1)
=¾(9+1)
=¾(10)
Q3 Position =7.5

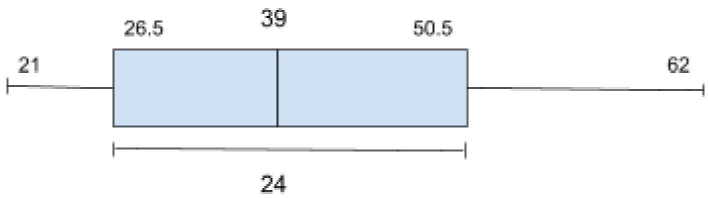
Quartile 1=(22+31)/2
=53/2
Quartile 1=26.5

Quartile 3=(45+56)/2
=101/2
Quartile 3=50.5

IQR=Q3-Q1

=50.5-26.5
IQR=24

c: Box and Whisker Graph



6:formula $x = \mu + Z\sigma$
μ = 40
$\sigma = 8$
$P(Z <- 0.524) = 0.3$

$x = \mu + Z\sigma$
$x = 40 + (-0.524)8$
$x = 35.808$
Max bpm under 30% = 35 bpm
If the athletes resting heartbeat falls under 36 bpm, then further screening is required.

7a: Use GDC to calculate Variance and Standard Deviation
Variance = 180.798
Standard Deviation = 13.446
Mean = (51.7*(3/20))+(49.3(4/20))+(38.9(6/20))+(18.6(7/20))
Mean=35.795

b. z score formula: $Z = \frac{x-q}{\sigma}$
i. x=27.5
$Z = \frac{27.5-35.795}{13.446}$
$Z = \frac{-8.295}{13.446}$
$Z =- 0.617$
ii. x=62.0
$Z = \frac{62.0-35.795}{13.446}$
$Z = \frac{26.205}{13.446}$
$Z = 1.949$
iii. x=45.3
$Z = \frac{45.3-35.795}{13.446}$
$Z = \frac{9.505}{13.446}$

$Z = 0.707$

Hard:

8:Mean

$\mu = n \cdot p$

$= 2 \cdot 0.38$

$\mu = 0.760$

Variance

$\sigma^2 = n \cdot p \cdot q$

$= 2 \cdot 0.38 \cdot 0.62$

$\sigma^2 = 0.471$

Standard Deviation

$\sigma = \sqrt{n \cdot p \cdot q}$

$= \sqrt{2 \cdot 0.38 \cdot 0.62}$

$= \sqrt{0.4712}$

$= 0.68644$

$\sigma = 0.686$

9:Calculate mean and standard deviation using $\mu = n * p$ and $\sigma = \sqrt{n \cdot p \cdot q}$

$\mu = n * p$

$\mu = 100 * 0.72$

$\mu = 72$

$\sigma = \sqrt{n \cdot p \cdot q}.$

$\sigma = \sqrt{100 \cdot 0.72 \cdot 0.28}$

$\sigma = \sqrt{20.16}$

$\sigma = 4.490$

Calculate the Z score using $Z = \frac{x-q}{\sigma}$

$Z = \frac{x-q}{\sigma}$

$Z = \frac{80-72}{4.490}$

$Z = \frac{8}{4.490}$

$Z = 1.782$

$P(Z < 1.782) = 0.963$

The probability that less than 80 people would test positive is 0.963

10:

a: Find Means of x and y

Mean of x = (3.5+7+10.5+28)/4

= (49)/4

Mean of x = 12.25

Mean of y= (-0.2+0.5+0.8+1.5)/4
= (2.6)/4
Mean of y= 0.65

b: Find standard deviations of x and y no gdc

$\sigma = \sqrt{}(\sum(x-\mu)^2 \div N$

$\sigma = \sqrt{}(((3.5-12.25)^2 + (7-12.25)^2 + (10.5-12.25)^2 + (28-12.25)^2)/4)$
$\sigma = \sqrt{}((355.25) \div 4)$
$\sigma = \sqrt{}(88.8125)$
$\sigma_x = 9.424$

$\sigma = \sqrt{}(\sum(y-\mu)^2 \div N$

$\sigma = \sqrt{}(((-0.2-0.65)^2 + (0.5-0.65)^2 + (0.8-0.65)^2 + (1.5-0.65)^2)/4)$
$\sigma = \sqrt{}((1.49) \div 4)$
$\sigma = \sqrt{}(0.3725)$
$\sigma_y = 0.610$

c: Calculate the equation of linear regression
$y = mx + b$

$m = r * (\sigma_y / \sigma_x)$
$m = 0.928 * (0.610/9.424)$
$m = 0.060$

$b = \mu_y - m\mu_x$
$b = 0.610 - (0.060)12.25$
$b = 0.610 - (0.060)12.25$
$b =- 6.863$

$y = 0.060x - 6.863$

Differential Calculus

IB MYP 10 Pre-Calculus Extended Unit 6

Aidan Ragan and Gabe Slater

What is Calculus?

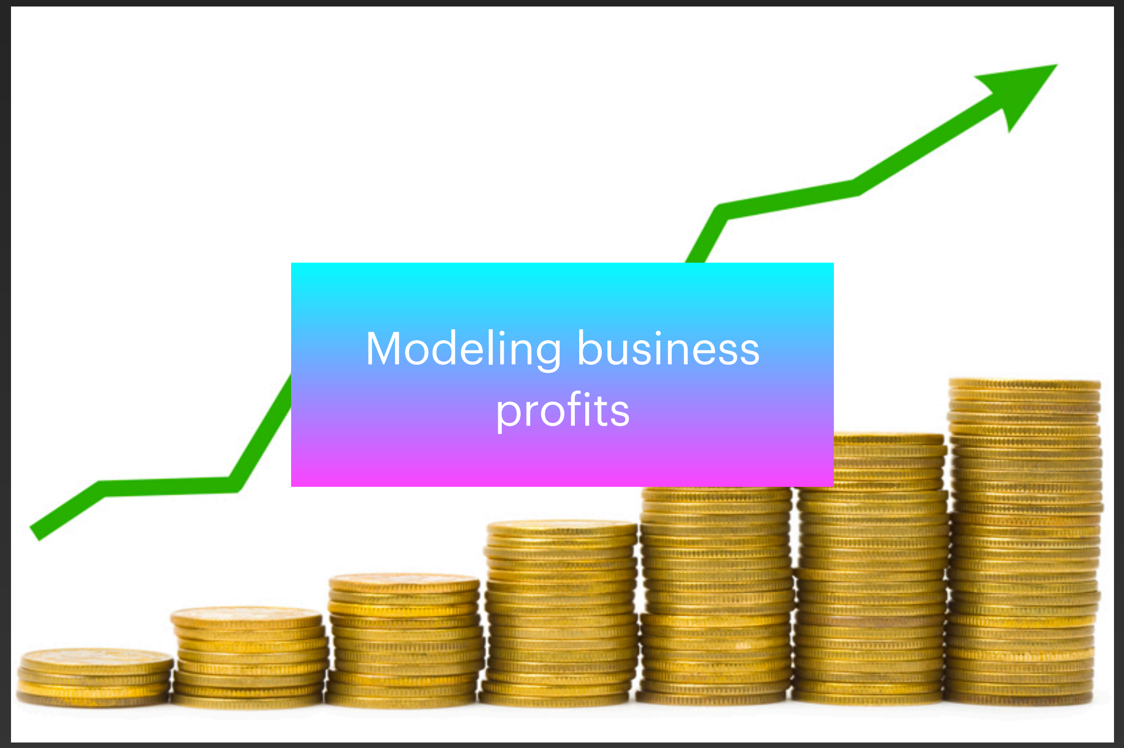
https://www.carrollwooddayschool.org/about/pop

Watch Mr. Horsington Teach Calculus in Only ONE MINUTE!

Introduction to Differential Calculus

What is Differential Calculus?

- Differential Calculus: The mathematical study of differentiation, i.e., how quickly one variable is changing with respect to another (rate of change)
- Differential calculus can be applied to any situation involving change, such as…



AND MANY MORE………..

Key Concepts

- Tangents
- Chords and Secants
- Normal Line
- Derivatives—The Basics
- Extrema
- Optimization

- Derivative Rules
  - Power Rule
  - Constant Rule
  - Constant Multiple Rule
  - Sum/Difference Rule
  - Product Rule
  - Quotient Rule
  - Chain Rule

Tangents

- **Definition:** A line that shares a common point with a curve, but does not pass through the curve.
- **Calculus Application:** A line drawn tangent to a curve of a function by nature has the same slope of the function at a specific point and therefore a tangent line represents the instantaneous rate of change (i.e., gradient/slope) of the function at a point.

- **Graphing a Tangent Using a TI-84 Plus GDC**
  1. Enter function into the "y =" function
  2. Hit "Graph"
  3. Hit "2nd" and then "PRGM"
  4. Select "Tangent(" (#5)
  5. Type in x-coordinate of desired point and hit "enter"


A(a, f(a))
tangent
x

Tangents: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide)
- https://www.youtube.com/watch?v=O_cwTAfjgAQ (Watch First 5 Minutes)
- https://www.desmos.com/calculator/tkyexmkhyg (interactive desmos graph)

Auditory

- https://www.youtube.com/watch?v=O_cwTAfjgAQ (Watch First 5 Minutes)

Kinesthetic

- Worked Example (next slide)
- https://www.desmos.com/calculator/tkyexmkhyg (interactive desmos graph)

Worked Example

Find the instantaneous velocity of a baseball 5 seconds after being launched from a high powered gun where the velocity of the baseball is represented by the function $d(t) = -t^2 + 76t - 3$ (where d(t) is distance in meters and t is time in seconds). Use a GDC.

1. Enter "$-x^2 + 76x - 3$ into the "y =" function
2. Hit "Graph"
3. Hit "2nd" and then "PRGM"
4. Select "Tangent(" (#5)
5. Type in 5 and hit "enter"
6. The calculator will display the equation of the tangent: $y = 66x + 22$
7. The equation for a line in slope-point form is $y = mx + b$, where m is the slope
8. Hence, the instantaneous velocity after 5 sec is 66 m/s

Chords and Secants

- **Chord Definition:** A line segment joining two points on a graph.
- **Secant Definition:** A line that passes through two points on a graph
- **Calculus Application:** The gradient of a chord/ secant represents the average rate of change between two points. Additionally, the gradient of a tangent line at a point serves as a limit to a secant at that same point.
- Secants are found by completing a linear systems of equations with the desired two points.

- **Calculating a Secant Using a TI-84 Plus GDC**
  1. Hit “APPS”
  2. Hit “PlySmlt2” (#9)
  3. Hit “Simult Eqn Solver” (#2)
  4. Select 2 equations w/ 2 unknowns
  5. Hit “Graph”
  6. In the first row, insert the x-coordinate of the first point in the first column, 1 in the second column, and the y-coordinate of the first point in the last column; repeat with the second point in row 2.
  7. Hit “Graph”
  8. X1 = the slope and X2 = y-intercept of the secant

10
5
0
-1
1
2
3
4
x

Secants: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide)
- https://www.youtube.com/watch?v=0z_MDIWMBwU
- https://www.desmos.com/calculator/tljjbh7vms (interactive Desmos graph)

Auditory

- https://www.youtube.com/watch?v=0z_MDIWMBwU

Kinesthetic

- Worked Example (next slide)
- https://www.desmos.com/calculator/tljjbh7vms (interactive Desmos graph)

Worked Example

Manually find the average rate of change between the points (0, 1) and (1.62, 5.053) on the function $f(x) = e^x$.

1. Insert given points into slope-point formula
2. Solve system for *m using* substitution, re-writing the first equation in terms of *b*

1. $1 = 0m + b \mid 5.053 = 1.62m + b$
2. $1 = b \mid 5.053 = 1.62m + b$

$$5.053 = 1.62m + 1$$

$$4.053 = 1.62m$$

$$2.50 = m$$

Derivatives—The Basics

- **Definition:** The equation representing the slope of a tangent line to a function at a specific point
- **Notation:** The equation for the first derivative of f(x) is written as f′(x)
- There are many levels of derivatives, with the first derivative being the derivative of f(x), the second derivative being the derivative of the first derivative, and so on.
- Derivatives can also be solved using derivative rules (discussed later)

- **Equation:**

$$f'(x) = \frac{lim}{h -> 0} \frac{f(x+h) - f(x)}{h}$$

  - This equation expresses the derivative as the limit of the secant at a point, i.e., the slope of the tangent.
  - **Solving:** Substitute f(x) into the equation and simplify. Then substitute in 0 for *h* to roughly express the tangent as the limit of the secant approaches 0. The derivative is this simplified excluding any terms with x^0

Derivatives: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide)
- https://www.youtube.com/watch?v=Khz7fZOLV_Y
- https://www.desmos.com/calculator/bwgbjze1zr (interactive Desmos graph)

Auditory

- https://www.youtube.com/watch?v=Khz7fZOLV_Y

Kinesthetic

- Worked Example (next slide)
- https://www.desmos.com/calculator/bwgbjze1zr (interactive Desmos graph)

Worked Example

Find the first derivative of the function $f(x) = x^2 + 9$

1. Substitute f(x) into the equation for f′(x)
2. Simplify as much as possible

1. $f'(x) = \frac{lim}{h->0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \frac{lim}{h->0} \frac{(x+h)^2 + 9 - x^2 + 9}{h}$$

2. $f'(x) = \frac{lim}{h->0} \frac{x^2 + 2xh + h^2 + 9 - x^2 + 9}{h}$

$$f'(x) = \frac{lim}{h->0} \frac{2xh + h^2 + 18}{h}$$

$$f'(x) = \frac{lim}{h->0} \frac{h(2x+h) + 18}{h}$$

$$f'(x) = \frac{lim}{h->0} 2x + h + 18$$

Worked Example Cont.

Find the first derivative of the function $f(x) = x^2 + 9$

3. Substitute 0 for *h*
4. Remove x⁰ term

3.

$$f'(x) = \frac{lim}{h -> 0} 2x + h + 18$$

$$f'(x) = \frac{lim}{h -> 0} 2x + 0 + 18$$

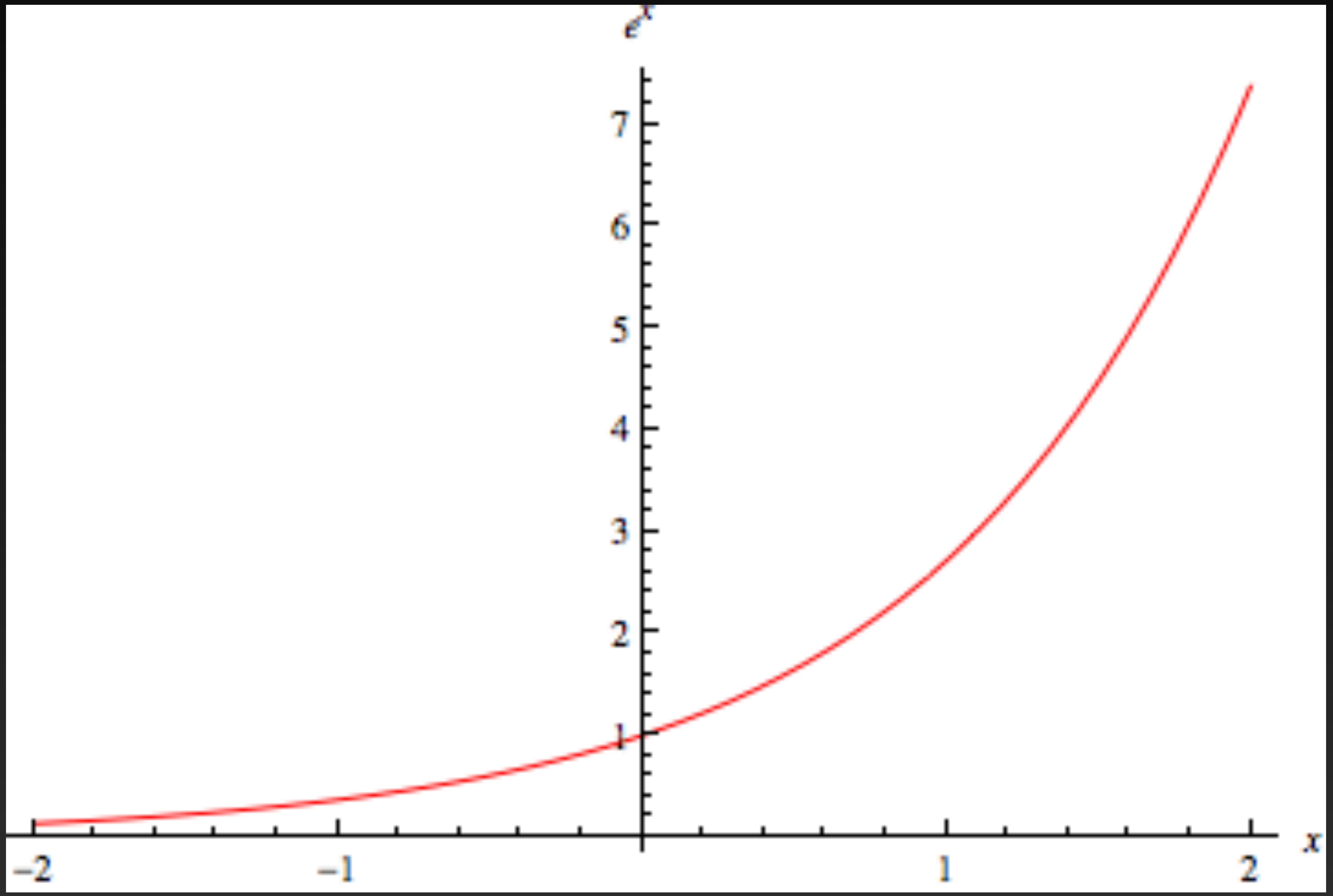
$$f'(x) = 2x + 18$$

4. $f'(x) = 2x$

Power Rule

- **Definition:** $\frac{d}{dx}(x^n) = nx^{n-1}$

- **Calculus Application:** The power rule is used to differentiate functions whenever a real number is present. Polynomials are differentiated using this rule.



Power Rule: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide)
- https://www.youtube.com/watch?v=9Yz-RCdS2Tg
- https://www.desmos.com/calculator/qtzamwcdrk (interactive Desmos graph)

Auditory

- https://www.youtube.com/watch?v=9Yz-RCdS2Tg

Kinesthetic

- Worked Example (next slide)
- https://www.desmos.com/calculator/qtzamwcdrk (interactive Desmos graph)

Worked Example

Find the derivative of the function $f(x) = 2x^2 + x$ using the power rule

1. Use the power rule to differentiate the polynomial.
2. The coefficient of the variable is multiplied by the exponential and the exponential is subtracted by 1.
3. Simplify the new equation.

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $2x^2 + x = 2(2)x + 1(1)$
3. $\frac{d}{dx}(2x^2 + x) = 4x + 1$

Constant Rule/Constant Multiple Rule

Constant Rule

- **Definition:** $\frac{d}{dx}(k) = 0$

- **Calculus Application:** The Constant Rule is used when a constant is present. It states that when a function is a number multiplied by another function, the number can be ignored as a constant.

Constant Multiple Rule

- **Definition:** $\frac{d}{dx}[(k)f(x)] = (k)f'(x)$

- **Calculus Application:** The Constant Multiple Rule states that the derivative of a constant multiplied by a function is equal to the constant multiplied by the derivative of that function.

Constant/Constant Multiple Rule: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide)
- https://www.youtube.com/watch?v=cyn6CDnAlcA
- https://www.youtube.com/watch?v=ut-AFj60OSO

Auditory

- https://www.youtube.com/watch?v=ut-AFj60OSO
- https://www.youtube.com/watch?v=cyn6CDnAlcA

Kinesthetic

- Worked Example (next slide)
- https://www.emathhelp.net/notes/calculus-1/differentiation-rules/constant-multiple-rule/

Worked Example

Find the derivative of the function $f(x) = 3x^5$ using the constant rule

1. Take out the 3 as a constant
2. Differentiate the rest of the function leaving the constant.
3. Simplify the new equation.

1. $\frac{d}{dx}(3)x^5$
2. $\frac{d}{dx}(3)x^5 = (3)5x^4$
3. $\frac{d}{dx}(3x^5) = 15x^4$

Sum/Difference Rule

Sum Rule

- **Definition:**

$$\frac{d}{dx}[f(x) + g(x) = f'(x) + g'(x)$$

- **Calculus Application:** The sum rule states that the derivative of a function plus another function is equal to the derivative of the first function plus the derivative of the second function.

Difference Rule

- **Definition:**

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

- **Calculus Application:** The difference rule states that the derivative of a function subtracted by another function is equal to the derivative of the first function subtracted by the derivative of the second function.

Sum/Difference Rule: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide)
- https://www.youtube.com/watch?v=fIOMrpVLtBs
- https://www.youtube.com/watch?v=d1vGGbm2PQg

Auditory

- https://www.youtube.com/watch?v=d1vGGbm2PQg

Kinesthetic

- Worked Example (next slide)
- https://www.ck12.org/c/calculus/sum-and-difference-differentiation-rules/lesson/Differentiation-Rules:-Sums-and-Differences-CALC/

Worked Example

Find the derivative of $f(x) + g(x)$ using the sum rule when $f(x) = 3x^4$ and $g(x) = 2x^4 + 2x$

1. Use the sum rule
2. Use the power rule to find the derivative of the first function.
3. Simplify the new equation.
4. Find the derivative of the second function using the power rule.
5. Simplify the new equation.
6. Use the sum rule equation and substitute the derivatives of both functions into the equation.
7. Simplify the new equation

1. $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
2. $\frac{d}{dx}f(x) = 3(4)x^3$
3. $f'(x) = 12x^3$
4. $\frac{d}{dx}g(x) = 2(4)x^3 + 2(1)$
5. $g'(x) = 8x^3 + 2$
6. $\frac{d}{dx}[f(x) + g(x)] = 12x^3 + 8x^3 + 2$
7. $\frac{d}{dx}[f(x) + g(x)] = 2(10x^3 + 1)$

Product/Quotient Rule

Product Rule

- **Definition:**
  $$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

- **Calculus Application:** The Product rule states that the derivative of a function multiplied by another function is equal to the first function multiplied by the derivative of the second function plus the second function multiplied by the derivative of the first function.

Quotient Rule

- **Definition:**
  $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

- **Calculus Application:** The Quotient rule states that the derivative of one function divided by another function is equal to the derivative of the first function multiplied by the second function subtracted by the first function multiplied by the derivative of the second function, all of this is over the second function squared.

Product/Quotient Rule: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide)
- https://www.youtube.com/watch?v=8jVDEcQ0wXk
- https://www.youtube.com/watch?v=17X5g9QArTc

Auditory

- https://www.youtube.com/watch?v=17X5g9QArTc
- https://www.youtube.com/watch?v=8jVDEcQ0wXk

Kinesthetic

- Worked Example (next slide)
- https://tutorial.math.lamar.edu/Problems/CalcI/ProductQuotientRule.aspx

Worked Example

Find the derivative of $f(x)g(x)$ using the product rule when $f(x) = 3x^4$ and $g(x) = 2x^4 + 2x$

1. Use the product rule
2. Use the power rule to find the derivative of the first function.
3. Simplify the new equation.
4. Find the derivative of the second function using the power rule.
5. Simplify the new equation.
6. Use the product rule equation and substitute the derivatives of both functions into the equation.
7. Simplify the new equation

1. $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
2. $\frac{d}{dx}f(x) = 3(4)x^3$
3. $f'(x) = 12x^3$
4. $\frac{d}{dx}g(x) = 2(4)x^3 + 2(1)$
5. $g'(x) = 8x^3 + 2$
6. $\frac{d}{dx}[f(x)g(x)] = (3x^4)(8x^3 + 2) + (2x^4 + 2x)(12x^3)$
7. $\frac{d}{dx}[f(x)g(x)] = 48x^7 + 30x^4$

Chain Rule

Chain Rule

- **Definition:** $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

- **Calculus Application:** The Chain rule shows how to find the derivative of of composite functions where one function is inside of another. It states that the derivative of a composite function is equal to the second function inside the derivative of the first function multiplied by the derivative of the second function.

Product/Quotient Rule: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide)
- https://www.youtube.com/watch?v=HaHsqDjWMLU

Auditory

- https://www.youtube.com/watch?v=HaHsqDjWMLU

Kinesthetic

- Worked Example (next slide)
- https://tutorial.math.lamar.edu/Problems/CalcI/ChainRule.aspx

Worked Example

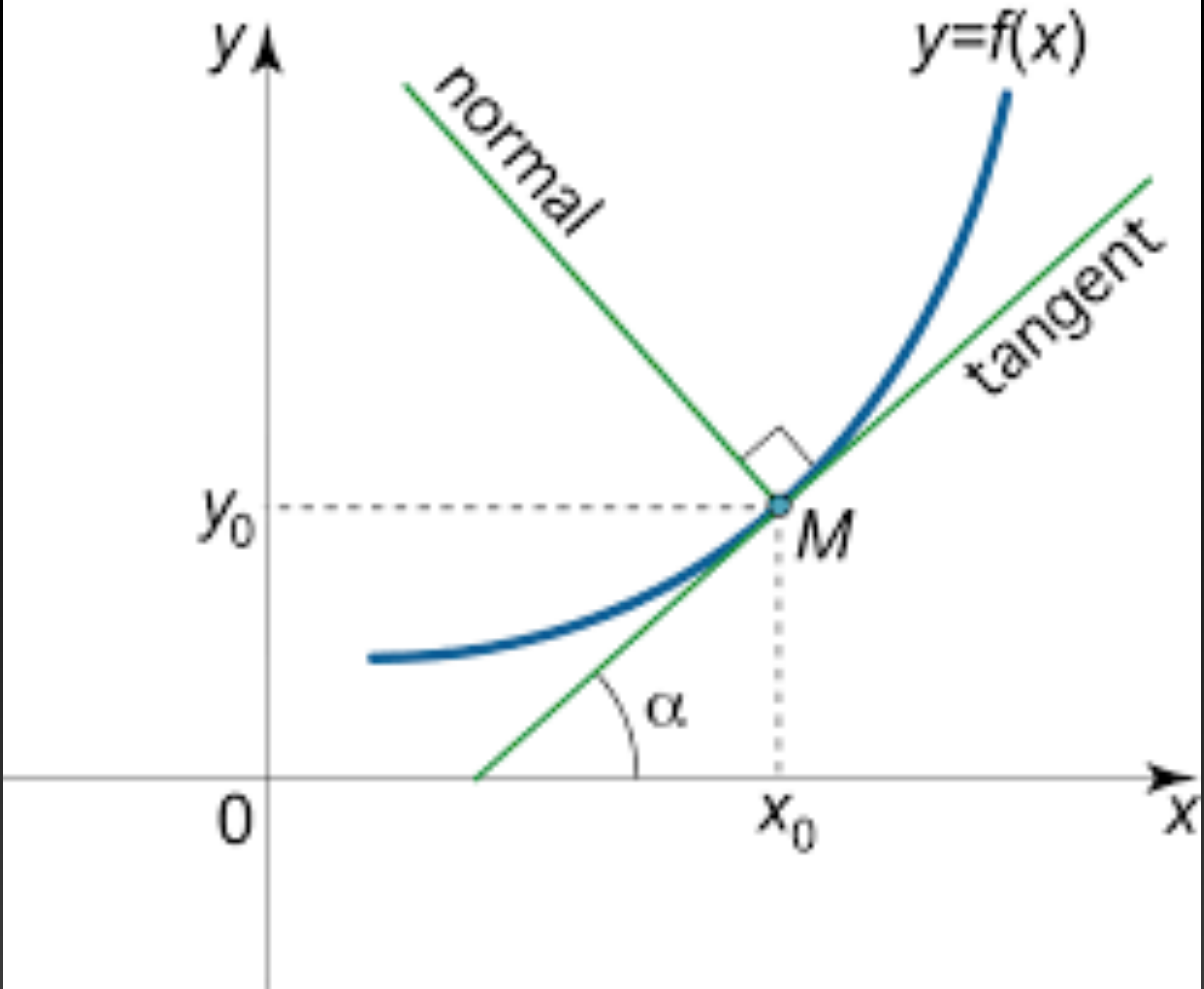
Find the derivative of $f(g(x))$ using the chain rule when $f(x) = 3x^4$ and $g(x) = 2x^4 + 2x$

1. Use the chain rule
2. Use the power rule to find the derivative of the first function.
3. Simplify the new equation.
4. Find the derivative of the second function using the power rule.
5. Simplify the new equation.
6. Use the chain rule equation and substitute the derivatives of both functions into the equation.
7. Simplify the new equation

1. $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$
2. $\frac{d}{dx}f(x) = 3(4)x^3$
3. $f'(x) = 12x^3$
4. $\frac{d}{dx}g(x) = 2(4)x^3 + 2(1)$
5. $g'(x) = 8x^3 + 2$
6. $\frac{d}{dx}[f(g(x))] = 12(2x^4 + 2x)^3(8x^3 + 2)$
7. $\frac{d}{dx}[f(g(x))] = 768x^{15} + 2496x^{12} + 2880x^9 + 1344x^6 + 192x^3$

Normal Line

- **Definition:** A normal line is a line that is perpendicular to a tangent.
- **Solving Given Tangent:**
    1. Substitute in the tangent's point of contact and the opposite-reciprocal of the tangent's slope into the point-slope equation (if the tangent's slope is not given, calculate using the derivative and the point of contact)
    2. Solve for the y-intercept
    3. Rewrite general equation for the normal line given the slope and y-intercept
- **Solving for All Points of Normalcy Whose Line Passes Through a Given Point:**
    1. Take a general point on a given function (x, y) and substitute the equation of the function for y
    2. Find the first derivative of the function: the opposite-reciprocal of the slope of the derivative is the slope of the normal line(s)
    3. Substitute the given point, the general point (step 1), and the slope (step 2) into the slope formula: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
    4. Solve for all values of *x*, these are the x-coordinates of the points of contact of the normal lines
    5. Plug the x-coordinates (step 4) into the original function to find the y-coordinates of the points of contact between the normal line and function



y
y=f(x)
normal
tangent
y_0
M
α
0
x_0
x

Normal Line: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide)
- https://www.youtube.com/watch?v=lEF2mmR3CWU
- https://www.desmos.com/calculator/37qkihsspj (interactive Desmos graph)

Auditory

- https://www.youtube.com/watch?v=lEF2mmR3CWU

Kinesthetic

- Worked Example (next slide)
- https://www.desmos.com/calculator/37qkihsspj (interactive Desmos graph)

Worked Example

Find the Slope of the Normal Line to the Point (5, 97) when $f(x) = 3x^2 + 4x + 2$.

1. Calculate the derivative of f(x) using the power rule and constant rule
2. Substitute x=5 into the equation of the derivative (this is the slope of the tangent at x=5)
3. Calculate the opposite-reciprocal of the slope of the tangent (this is the slope of the normal line at x=5)
4. Substitute in the tangent's point of contact and the normal line's slope into the point-slope equation and solve for the y-intercept
5. **Rewrite general equation for the normal line given the slope and y-intercept**

1. $f(x) = 3x^2 + 4x + 2$

   $f'(x) = 2(3)x^{2-1} + 1(4)x^{1-1} + 0$

   $f'(x) = 6x + 4$

2. $f'(5) = 6(5) + 4$

   $f'(5) = 34$

3. $34 -> \frac{-1}{34} = m$

4. $8 = 8(\frac{-1}{34}) + b$

   $b = 8.24$

5. $y = \frac{-1}{34}x + 8.24$

Extrema

- **Definition:** The extrema of a function can be defined as either the absolute or relative maximum and minimum points.
- **Calculus Application:** Maximums and minimums are stationary points, i.e., they have a slope of 0. Hence, they can be calculated using derivatives.
- **Solving Using Calculus (can also be solved using TI-84 Plus w/o calculus)**
  1. Find the first derivative of the function
  2. Set the derivative equal to 0 and solve for x; these are the x-coordinates of the maxs/mins
  3. Substitute solutions into original function to determine y-coordinates of each extrema
  4. Determine whether extrema are maxs or mins by substituting in 1 more than the solution and one less for x in the original function. If both y-coordinates are more than that of the extrema, it is a minimum, but if both are less than it is a maximum

Extrema: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide)
- https://www.khanacademy.org/math/ap-calculus-ab/ab-diff-analytical-applications-new/ab-5-4/v/relative-minima-maxima (Khan Academy Lesson Including Videos and Practice Problems)
- https://www.desmos.com/calculator/y8zzzuuih3 (interactive desmos graph)

Auditory

- https://www.khanacademy.org/math/ap-calculus-ab/ab-diff-analytical-applications-new/ab-5-4/v/relative-minima-maxima (Khan Academy Lesson Including Videos and Practice Problems)

Kinesthetic

- Worked Example (next slide)
- https://www.desmos.com/calculator/y8zzzuuih3 (interactive desmos graph)
- https://www.khanacademy.org/math/ap-calculus-ab/ab-diff-analytical-applications-new/ab-5-4/v/relative-minima-maxima (Khan Academy Lesson Including Videos and Practice Problems)

Worked Example

Find the absolute extrema of the function $f(x) = x^3 + x^2 - 3x + 2$

1. Find the first derivative of the function using the power and constant rules
2. Set the derivative equal to 0 and solve for x using a GDC; these are the x-coordinates of the maxs/mins
3. Substitute solutions into original function to determine y-coordinates of each extrema
4. Determine whether extrema are maxs or mins by substituting in 1 more than the solution and one less for x in the original function. If both y-coordinates are more than that of the extrema, it is a minimum, but if both are less than it is a maximum.

1. $f(x) = x^3 + x^2 - 3x + 2$

$f'(x) = 3x^{3-1} + 2x^{2-1} - 1(3)x^{1-1} + 0$

$f'(x) = 3x^2 + 2x - 3$

2. $0 = 3x^2 + 2x - 3$

$0 = \{-1.39, 0.721\}$

3. $f(-1.39) = -1.39^3 + -1.39^2 - 3(-1.39) + 2$

$f(-1.39) = 5.42$

$(-1.39, 5.42)$

$f(0.721) = 0.721^3 + 0.721^2 - 3(0.721) + 2$

$f(0.721) = 0.732$

$(0.721, 0.732)$

Worked Example Cont.

Find the absolute extrema of the function $f(x) = x^3 + x^2 - 3x + 2$

4. Determine whether extrema are maxs or mins by substituting in 1 more than the solution and one less for x in the original function. If both y-coordinates are more than that of the extrema, it is a minimum, but if both are less than it is a maximum.

4.

$f(-1.39 - 1) = -2.39^3 + -2.39^2 - 3(-2.39) + 2$	$f(-1.39 + 1) = -0.39^3 + -0.39^2 - 3(-0.39) + 2$
$f(-2.39) = 1.23$	$f(-0.39) = 3.26$
$(-2.39, 1.23)$	$(-0.39, 3.26)$

$f(0.721 - 1) = -0.279^3 + -0.279^2 - 3(-0.279) + 2$	$f(0.721 + 1) = 1.721^3 + 1.721^2 - 3(1.721) + 2$
$f(-0.279) = 2.89$	$f(1.721) = 4.90$
$(-0.279, 2.89)$	$(1.721, 4.90)$

Because 5.42 > 1.23 and 5.42 >3.26, the point (-1.39, 5.42) is a local maximum

Because 0.732 < 2,89 and 5.42 <34.90, the point (-1.39, 5.42) is a local minimum


Abs. & Rel. Max.
Rel. Max.
a
b
c
d
e
x
Rel. Min.

Optimization

- **Definition:** Optimization in Calculus is maximizing or minimizing a specific factor within a complex system to achieve the "best" results possible. For example, optimization of a companies profits could be determined by finding the minimum amount of expenses with the maximum amount of profit.
- Optimization is the real world application of finding extrema using calculus
- This topic will be demonstrated in the real world application problem

Optimization: Additional Resources for Different Learning Styles

Visual

- Worked Example (next slide—Real Life App.)
- https://www.youtube.com/watch?v=Ef22yTJDUZI
- https://teacher.desmos.com/activitybuilder/custom/55c384de3c35758b2b2ea102 (desmos lesson)

Auditory

- https://www.youtube.com/watch?v=Ef22yTJDUZI

Kinesthetic

- Worked Example (next slide—Real Life App.)
- https://teacher.desmos.com/activitybuilder/custom/55c384de3c35758b2b2ea102 (desmos lesson)

Real Life Application—Optimization

A Florida man lives on a lake and just bought a dog named Sparky. Because he lives on a lake in Florida, the man wants to build a fence in his back yard to prevent Sparky from going near the water where there are alligators. In addition to protecting his dog, however, the man wants to maintain as much of his backyard as possible. The man bought 125 meters of fencing and wants to arrange it into a rectangle. What dimensions of the fencing enclosure will maximize the area of his backyard?

See worked solution on following slide

Real Life Application—Optimization (Solution)

1. Express the area (A) of the yard in terms of the length (L) and width (W)
2. Express the perimeter (P) of the yard in terms of the length (L) and width (W). You already know that the perimeter must be 125 meters.
3. Solve for the width in the perimeter equation
4. Substitute the width found in step 3 into the area equation
5. This can now be written as the function $A(L) = L(62.5 - L)$. Find the derivative of A(L) using the power rule.

1. $$A = LW$$

2. $$P = 2L + 2W$$
$$125 = 2L + 2W$$

3. $$125 = 2L + 2W$$
$$125 - 2L = 2W$$
$$62.5 - L = W$$

4. $$A = LW$$
$$A = L(62.5 - L)$$

5. $$A(L) = L(62.5 - L)$$
$$A(L) = 62.5L - L^2$$
$$A'(L) = 62.5 - 2L$$

Real Life Application—Optimization (Solution Cont.)

6. The maximum area will be a maximum point in the function A(L) and at extreme points the slope is 0. Therefore, solve for L when A′(L) = 0.

7. Substitute the length into perimeter equation and solve for width.

6.
$$A'(L) = 62.5 - 2L$$
$$0 = 62.5 - 2L$$
$$-62.5 = -2L$$
$$31.25 = L$$

7.
$$125 = 2(31.25) + 2W$$
$$125 = 62.5 + 2W$$
$$62.5 = 2W$$
$$31.25 = W$$

The dimensions of the yard that will maximize the area using 125 meters of fencing are a length and width of 31.25 meters.

Practice Problems

3 Easy, 4 Medium, 3 Hard

Easy Practice Problem 1

Complete the following table for the function $f(x) = x^2 + 2x - 1$ using a graphing calculator:

x	Instantaneous Rate of Change at *x*
-4	
-2	
0	
2	
4	

CORRECT ANSWERS ON NEXT SLIDE

Remember: On a TI-84 Plus calculator, use the "draw" program to graph a tangent line.

Easy Practice Problem 1

Complete the following table for the function $f(x) = x^2 + 2x - 1$ using a graphing calculator:

x	**Instantaneous Rate of Change at *x***
-4	-6
-2	-2
0	2
2	6
4	10

Remember: On a TI-84 Plus calculator, use the "draw" program to graph a tangent line.

Easy Practice Problem 2

Find the first derivative of these polynomial functions using the prime differentiation formula.

(show all work)

$y =$	**First Derivative**
$4x^3$	
$2x^2$	
$-5x$	
$7x^0$	
$4x^3 + 2x^2 - 5x + 7$	

CORRECT ANSWERS ON NEXT SLIDE

Hint: The formula is $\frac{dy}{dx} = nax^{n-1}$

Easy Practice Problem 2

Find the first derivative of these polynomial functions using the prime differentiation formula.

(show all work)

$y =$	**First Derivative**
$4x^3$	$12x^2$
$2x^2$	$4x$
$-5x$	-5
$7x^0$	0
$4x^3 + 2x^2 - 5x + 7$	$12x^2 + 4x - 5$

Hint: The formula is $\frac{dy}{dx} = nax^{n-1}$

Easy Practice Problem 3

Find the first derivative of the equation $y = 2x^2 - x + \frac{1}{x} - \frac{3}{x^2} + 3$

CORRECT ANSWERS
ON NEXT SLIDE

Easy Practice Problem 3

Find the first derivative of the equation $y = 2x^2 - x + \frac{1}{x} - \frac{3}{x^2} + 3$

ANSWER: $\boxed{y = 4x - 1 - \frac{1}{x^2} - \frac{6}{x^3}}$

Medium Practice Problem 1

Find the derivative and the gradient of the tangent at the point where x=15 of the following function: $6x^4 + 3x^2 - 8x + 12$

Derivative	Gradient

CORRECT ANSWERS ON NEXT SLIDE

Medium Practice Problem 1

Find the derivative and the gradient of the tangent at the point where x=15 of the following function: $6x^4 + 3x^2 - 8x + 12$

Derivative	Gradient
$24x^3 + 6x - 8$	81082

Medium Practice Problem 2

The function f(x) is used to model the profit of a toy company where x is the number of toys produced in millions. Calculate the maximum profit and output for the toy company using the function of the derivative based on f(x):

$$f(x) = -5x^2 + 75x - 150.$$

Maximum Profit	Maximum Toy Output

CORRECT ANSWERS ON NEXT SLIDE

Medium Practice Problem 2

The function f(x) is used to model the profit of a toy company where x is the number of toys produced in millions. Calculate the maximum profit and output for the toy company using the function of the derivative based on f(x):

$$f(x) = -5x^2 + 75x - 150.$$

Maximum Profit	Maximum Toy Output
\$131.25 million	7.5 million

Medium Practice Problem 3

Using correct notation, find the derivative of the function $f(x) = x^2 + 1$ at the point $(2,5)$

CORRECT ANSWERS ON NEXT SLIDE

Hint: Use the limit formula

$$\lim_{h->0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Medium Practice Problem 3

Using correct notation, find the derivative of the function $f(x) = x^2 + 1$ at the point $(2,5)$

ANSWER: $\boxed{f'(2) = 4}$

Hint: Use the limit formula

$$\lim_{h->0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Medium Practice Problem 4

Find the derivatives of these trigonometric functions:

$f(x) =$	**Derivative**
$sinx$	
$cosx$	
$tanx$	
$cscx$	
$secx$	
$cotx$	
$x^3 + sinx + 4cosx$	

Use this link if you need help!

https://www.youtube.com/watch?v=_niP0JaOgHY

CORRECT ANSWERS ON NEXT SLIDE

Medium Practice Problem 4

Find the derivatives of these trigonometric functions:

$f(x) =$	**Derivative**
$sinx$	$cosx$
$cosx$	$-sinx$
$tanx$	sec^2x
$cscx$	$-cscx(tanx)$
$secx$	$secx(tanx)$
$cotx$	$-csc^2x$
$x^3 + sinx + 4cosx$	$3x^2 + cosx - 4sinx$

Use this link if you need help!

https://www.youtube.com/watch?v=_niP0JaOgHY

Hard Practice Problem 1

A can of Campbell's chicken noodle soup contains 662.45 mL of soup. Campbell's wants to ensure that they can maximize their chicken noodle soup profits by ensuring that the surface area of the can as small as possible to reduce manufacturing costs. Calculate the radius and height of the optimal soup can.

Radius	Height

CORRECT ANSWERS ON NEXT SLIDE

NOTE: 1 mL = 1 cm^3, Volume $= \pi r^2 h$, and Surface Area $= 2\pi rh + 2\pi r^2$

Hard Practice Problem 1

A can of Campbell's chicken noodle soup contains 662.45 mL of soup. Campbell's wants to ensure that they can maximize their chicken noodle soup profits by ensuring that the surface area of the can as small as possible to reduce manufacturing costs. Calculate the radius and height of the optimal soup can.

Radius	Height
4.72 cm	9.46 cm

NOTE: 1 mL = 1 cm^3, Volume $= \pi r^2 h$, and Surface Area $= 2\pi rh + 2\pi r^2$

Hard Practice Problem 2

Find the value of $f''(37)$ when $f(x) = 15x^4 - 36x^2 + 467 + \dfrac{8}{x^3} - \dfrac{48}{4x^4}$

CORRECT ANSWERS ON NEXT SLIDE

Hard Practice Problem 2

Find the value of $f''(37)$ when $f(x) = 15x^4 - 36x^2 + 467 + \dfrac{8}{x^3} - \dfrac{48}{4x^4}$

$$f''(37) = 246348$$

Hard Practice Problem 3

Prove that $\frac{d}{dx}[secx] = secx(tanx)$

CORRECT ANSWERS ON NEXT SLIDE

Hint: Use the chain rule, trigonometric identities, and the power rule

Hard Practice Problem 3

Prove that $\dfrac{d}{dx}[secx] = secx(tanx)$

Full solution:

$$secx = \frac{1}{cosx} = (cosx)^{-1}$$

$$= -1(cosx)^{-2}(-sinx)$$

$$= \frac{sinx}{(cosx)^2} = \frac{1}{cosx} \cdot \frac{sinx}{cosx}$$

$$= secx(tanx)$$

Hint: Use the chain rule, trigonometric identities, and the power rule