LESSON 21.2

Solving Equations by Factoring $ax^2 + bx + c$

Common Core Math Standards

The student is expected to:

CA CC **F-LE.6**

Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity. Also A-REI.4b, A-SSE.2, A-SSE.3a

Mathematical Practices

CA CC **MP.5 Using Tools**

Language Objective

Explain to a partner how to factor a trinomial in the form $ax^2 + bx + c$.

ENGAGE

Essential Question: How can you use factoring to solve quadratic equations in standard form for which $a \neq 1$?

Factor the quadratic equation. Set each linear factor equal to 0. Solve each linear equation. The solutions are the solutions of the original quadratic equation.

PREVIEW: LESSON PERFORMANCE TASK

View the Engage section online. Have students demonstrate how a car accelerating from a stopped position could first be passed by a bus traveling at a constant speed, then overtake and pass the bus. Then preview the Lesson Performance Task.

Name ____________ Class ________ Date ______

21.2 Solving Equations by Factoring $ax^2 + bx + c$

Essential Question: How can you use factoring to solve quadratic equations in standard form for which $a \neq 1$?

Resource Locker

Explore Factoring $ax^2 + bx + c$ When $c > 0$

When you factor a quadratic expression in standard form $(ax^2 + bx + c)$, you are looking for two binomials, and possibly a constant numerical factor whose product is the original quadratic expression.

Recall that the product of two binomials is found by applying the Distributive Property, abbreviated sometimes as FOIL:

$$(2x + 5)(3x + 2) = \underbrace{6x^2}_{F} + \underbrace{4x}_{O} + \underbrace{15x}_{I} + \underbrace{10}_{L} = 6x^2 + 19x + 10$$

F The product of the coefficients of the first terms is a.

O, I The sum of the coefficients of the outer and inner products is b.

L The product of the last terms is c.

Because the a and c coefficients result from a single product of terms from the binomials, the coefficients in the binomial factors will be a combination of the factors of a and c. The trick is to find the combination of factors that results in the correct value of b.

Follow the steps to factor the quadratic $4x^2 + 26x + 42$.

(A) First, factor out the largest common factor of 4, 26, and 42 if it is anything other than 1.

$4x^2 + 26x + 42 = \boxed{2}(2x^2 + 13x + 21)$

(B) Next, list the factor pairs of 2:

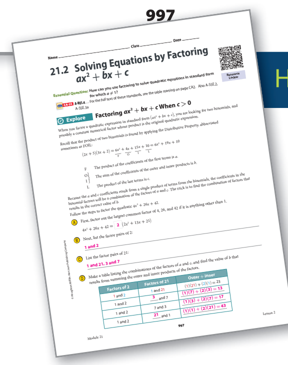
1 and 2

(C) List the factor pairs of 21:

1 and 21, 3 and 7

(D) Make a table listing the combinations of the factors of a and c, and find the value of b that results from summing the outer and inner products of the factors.

Factors of 2	Factors of 21	Outer + inner
1 and 2	1 and 21	$(1)(21) + (2)(1) = 23$
1 and 2	**3** and 7	$(1)(7) + (2)(3) = 13$
1 and 2	7 and 3	$(1)(3) + (2)(7) = 17$
1 and 2	**21** and 1	$(1)(1) + (2)(21) = 43$



HARDCOVER PAGES 781–790

Turn to these pages to find this lesson in the hardcover student edition.

Ⓔ Copy the pair of factors that resulted in an outer + inner sum of 13 into the binomial factors. Be careful to keep the inner and outer factors from the table as inner and outer coefficients in the binomials.

$2x^2 + 13x + 21 = (\boxed{1}x + \boxed{3})(\boxed{2}x + \boxed{7})$

Ⓕ Replace the common factor of the original coefficients to complete the factorization of the original quadratic.

$4x^2 + 26x + 42 = \boxed{2}(x + 3)(2x + 7)$

Reflect

1. **Critical Thinking** Explain why you should use negative factors of c when factoring a quadratic with $c > 0$ and $b < 0$.
   A negative value of b tells you at least one of the factors of c must be negative, but since $c > 0$, in fact both must be negative.

2. **What If?** If none of the factor pairs for a and c result in the correct value for b, what do you know about the quadratic?
   The quadratic cannot be factored into the product of two binomials.

3. **Discussion** Why did you have to check each factor pair twice for the factors of c (3 and 7 versus 7 and 3) but only once for the factors of a (1 and 2, but not 2 and 1)? Hint: Compare the outer and inner sums of rows two and three in the table, and also check the outer and inner sums by switching the order of both pairs from row 2 (check 2 and 1 for a with 7 and 3 for c).
   It would be redundant to switch the order of both sets of factor pairs. Switching one set represents a different set of possible binomial factors, while switching both pairs corresponds to exchanging the order of the two binomial factors, which is not a different answer. The second and third rows of the table (switching the factor order of c only) have different outer plus inner sums (17 versus 13). Switching both pairs results in a sum of $(2)(3) + (1)(7) = 13$, which is the correct value. The corresponding factored equation would be written as $4x^2 + 26x + 42 = 2(2x + 7)(x + 3)$, which is the same set of binomial factors as the original answer, just written in a different order.

Explain 1 Factoring $ax^2 + bx + c$ When $c < 0$

Factoring $x^2 + bx + c$ when $c < 0$ requires one negative and one positive factor of c. The same applies for expressions of the form $ax^2 + bx + c$ as long as $a > 0$. When checking factor pairs, remember to consider factors of c in both orders, **and** consider factor pairs with the negative sign on either member of the pair of c factors.

When you find a combination of factors whose outer and inner product sum is equal to b, you have found the solution. Make sure you fill in the factor table systematically so that you do not skip any combinations.

If $a < 0$, factor out -1 from all three coefficients, or use a negative common factor, so that the factors of a can be left as positive numbers.

EXPLORE

Factoring $ax^2 + bx + c$ When $c > 0$

INTEGRATE TECHNOLOGY

Students have the option of completing the factoring activity either in the book or online.

QUESTIONING STRATEGIES

? If there are three factor pairs for a and one factor pair for c, how many possible arrangements of factor pairs are there? Explain. **6; the factor pair for c can be paired with each factor pair for a in either order.**

? After finding the combination of factors of a and c that give the correct value of b, how do you know which factor to put in each position in the binomial factors? **The factors of a are the coefficients of x, and the factors of c are the constant terms. The factors you used to find the inner product should be the two inner values, and the factors you used to find the outer product should be the two outer values.**

EXPLAIN 1

Factoring $ax^2 + bx + c$ When $c < 0$

AVOID COMMON ERRORS

Make sure students understand that the order of the coefficients in the binomials matters. Encourage students to check for errors by making sure that the product of the x-coefficients is a, the product of the constant terms is c, and the sum of the products of the inner and outer terms is b.

PROFESSIONAL DEVELOPMENT

Math Background

Quadratic equations were probably first examined in Babylonia about 4000 years ago. Around 800 BCE in India, geometric methods were used to solve quadratic equations. These methods were similar to the method of completing the square, which was developed between 300 and 200 BCE. Both Euclid and Pythagoras grappled with finding a formula or procedure for solving quadratic equations. It was not until around 600 CE that the first explicit solution to the standard form of the equation was given, although it was not completely general. The practical notation using the symbols and methods currently in use was developed during the 15th century.

QUESTIONING STRATEGIES

? How many possible arrangements of factor pairs are there when $a = 3$ and $c = -2$? What are they? **4;**

Factors of a	Factors of c
1 and 3	1 and -2
1 and 3	2 and -1
1 and 3	-1 and 2
1 and 3	-2 and 1

? What pairs of binomial factors result from these arrangements of factor pairs? **$(x + 1)(3x - 2)$, $(x + 2)(3x - 1)$, $(x - 1)(3x + 2)$, and $(x - 2)(3x + 1)$**

Example 1 **Factor the quadratic by checking factor pairs.**

(A) $6x^2 - 21x - 45$

Find the largest common factor of 6, 21, and 45, and factor it out, keeping the coefficient of x^2 positive.

$6x^2 - 21x - 45 = 3(2x^2 - 7x - 15)$

Factors of a	Factors of c	Outer Product + Inner Product
1 and 2	1 and -15	$(1)(-15) + (2)(1) = -13$
1 and 2	3 and -5	$(1)(-5) + (2)(3) = 1$
1 and 2	5 and -3	$(1)(-3) + (2)(5) = 7$
1 and 2	15 and -1	$(1)(-1) + (2)(15) = 29$
1 and 2	-1 and 15	$(1)(15) + (2)(-1) = 13$
1 and 2	-3 and 5	$(1)(5) + (2)(-3) = -1$
1 and 2	-5 and 3	$(1)(3) + (2)(-5) = -7$
1 and 2	-15 and 1	$(1)(1) + (2)(-15) = -29$

Use the combination of factor pairs that results in a value of -7 for b.

$2x^2 - 7x - 15 = (x - 5)(2x + 3)$

Replace the common factor of the original coefficients to factor the original quadratic.

$6x^2 - 21x - 45 = 3(x - 5)(2x + 3)$

(B) $20x^2 - 40x - 25$

Factor out common factors of the terms.

$20x^2 - 40x - 25 = \boxed{5}(4x^2 - 8x - 5)$

Factors of a	Factors of c	Outer Product + Inner Product
1 and 4	1 and -5	$(1)(-5) + (4)(1) = \boxed{-1}$
1 and 4	5 and -1	$\boxed{(1)(-1)} + \boxed{(4)(5)} = \boxed{19}$
1 and 4	-1 and 5	$\boxed{(1)(5)} + \boxed{(4)(-1)} = \boxed{1}$
1 and 4	-5 and 1	$\boxed{(1)(1)} + \boxed{(4)(-5)} = \boxed{-19}$
2 and 2	1 and -5	$\boxed{(2)(-5)} + \boxed{(2)(1)} = \boxed{-8}$
2 and 2	-1 and 5	$\boxed{(2)(-1)} + \boxed{(2)(5)} = \boxed{8}$

Use the combination of factor pairs that results in a value of $\boxed{-8}$ for b.

$4x^2 - 8x - 5 = (\boxed{2}x + \boxed{1})(\boxed{2}x + \boxed{-5})$

Replace the common factor of the original coefficients to factor the original quadratic.

$20x^2 - 40x - 25 = \boxed{5}(2x + 1)(2x - 5)$

COLLABORATIVE LEARNING

Peer-to-Peer Activity

Have student pairs discuss how to use the signs of b and c to decide whether binomial factors should contain + or − signs and then complete this chart:

b	c	Example	Sign in Factors	Factors
+	+	$2x^2 + 9x + 4$	both +	$(2x + 1)(x + 4)$
−	+	$3n^2 - 11n + 6$	both −	$(3n - 2)(n - 3)$
−	−	$4t^2 - t - 10$	product with greater absolute value has −	$(2t - 5)(t + 2)$
+	−	$6q^2 + q - 7$	product with greater absolute value has +	$(6q + 7)(q - 1)$

Reflect

4. **What If?** Suppose a is a negative number. What would be the first step in factoring $ax^2 + bx + c$?
Factor out a negative common factor from all of the coefficients, even if the common factor is -1. This will result in a new quadratic to factor with a positive value of a to which the previous methods can be applied.

Your Turn

5. Factor. $-5x^2 + 8x + 4$
$-1(5x^2 - 8x - 4) = -1(x - 2)(5x + 2)$

Explain 2 Solving Equations of the Form $ax^2 + bx + c = 0$ by Factoring

For a quadratic equation in standard form, $ax^2 + bx + c = 0$, factoring the quadratic expression into binomials lets you use the Zero Product Property to solve the equation, as you have done previously. If the equation is not in standard form, convert it to standard form by moving all terms to one side of the equation and combining like terms.

Example 2 **Change the quadratic equation to standard form if necessary and then solve by factoring.**

(A) $2x^2 + 7x - 2 = 4x^2 + 4$

Convert the equation to standard form:

Subtract $4x^2$ and 4 from both sides. $-2x^2 + 7x - 6 = 0$

Multiply both sides by -1. $2x^2 - 7x + 6 = 0$

Consider factor pairs for 2 and 6. Use negative factors of 6 to get a negative value for b.

Use the combination pair that results in a sum of -7 and write the equation in factored form. Then solve it using the Zero Product Property.

$$(x - 2)(2x - 3) = 0$$

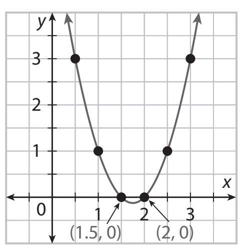
$$x - 2 = 0 \quad \text{or} \quad 2x - 3 = 0$$

$$x = 2 \qquad 2x = 3$$

$$x = \frac{3}{2} = 1.5$$

The solutions are 2 and $\frac{3}{2}$, or 1.5.

The solution can be checked by graphing the related function, $f(x) = 2x^2 - 7x + 6$, and finding the x-intercepts.



EXPLAIN 2

Solving Equations of the Form $ax^2 + bx + c = 0$ by Factoring

INTEGRATE MATHEMATICAL PRACTICES

Focus on Communication

MP.3 Ask students to describe the general form of solutions to a factored quadratic equation of the form $(ax + b)(cx + d) = 0$. Students should find that the solutions are $x = -\frac{b}{a}$ and $x = -\frac{d}{c}$. Therefore, when the coefficients of x in the factored form are not 1, the solutions are likely to be proper or improper fractions.

DIFFERENTIATE INSTRUCTION

Multiple Representations

Show students this alternate factoring method. Any trinomial can be written as a polynomial with four terms and then factored by grouping. To factor $6x^2 + 19x + 15$, first find ac: $6 \cdot 15 = 90$. Then find factors of ac that sum to b: 9 and 10 are factors of 90 whose sum is 19. Rewrite the trinomial using those factors: $6x^2 + 10x + 9x + 15$. Finally, factor by grouping:
$(6x^2 + 10x) + (9x + 15) = 2x(3x + 5) + 3(3x + 5) = (3x + 5)(2x + 3)$

QUESTIONING STRATEGIES

What is the first step for solving $7x^2 + 20x = x + 6$? **Convert the equation to standard form by subtracting $x + 6$ from both sides of the equation, which results in $7x^2 + 19x - 6 = 0$.**

Why is this step necessary for solving the equation by factoring? **In order to use the Zero Product Property to solve the equation, the expression that you factor must be equal to zero.**

(B) $3(x^2 - 1) = -3x^2 + 2x + 5$

Write the equation in standard form and factor so you can apply the Zero Product Property.

$$\boxed{3}x^2 - \boxed{3} = -3x^2 + 2x + 5$$

$$\boxed{6}x^2 - 2x - \boxed{8} = 0$$

$$\boxed{3}x^2 - x - 4 = 0$$

Use the combination pair that results in a sum of $\underline{-1}$.

$$(x + 1)\left(\boxed{3}x + \boxed{-4}\right) = 0$$

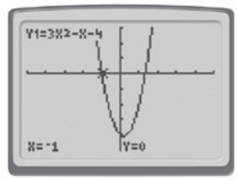
$x + 1 = \boxed{0}$ or $3x - 4 = 0$

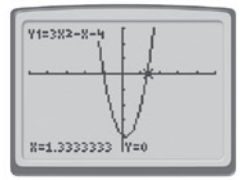
$x = \boxed{-1}$ $\qquad \boxed{3}x = 4$

$x = \boxed{\frac{4}{3}}$

The solutions are -1 and $\frac{4}{3}$.

Use a graphing calculator to check the solutions.





Reflect

6. In the two examples, a common factor was divided out at the beginning of the solution, and it was not used again. Why didn't you include the common term again when solving x for the original quadratic equation?
Solutions are arrived at by applying the Zero Product Property to the factors of the quadratic expression. Although the common factor is a factor of the quadratic expression, it cannot equal 0.

Your Turn

7. $12x^2 + 48x + 45 = 0$

$3(4x^2 + 16x + 15) = 0$

$(2x + 3)(2x + 5) = 0$

$2x + 3 = 0$ **or** $2x + 5 = 0$

$x = -\frac{3}{2}$ **and** $x = -\frac{5}{2}$

LANGUAGE SUPPORT EL

Connect Vocabulary

English learners may need numerous exposures and structured practice with new vocabulary to truly acquire it for their own use. Before describing the new terminology for this lesson, review more basic terms such as *sum*, *product*, and *factor*. Students will need to rely on their understanding of these words to grasp phrases such as *outer products*, *inner products*, *sum of inner and outer products*, and *common factors*. It may be helpful for students to use diagrams and examples to record the meaning of new terminology.

Explain 3 Solving Equation Models of the Form $ax^2 + bx + c = 0$ by Factoring

A projectile is an object moving through the air without any forces other than gravity acting on it. The height of a projectile at a time in seconds can be found by using the formula $h = -16t^2 + vt + s$, where v is in the initial upwards velocity in feet per second (and can be a negative number if the projectile is launched downwards) and s is starting height in feet. The a term of -16 accounts for the effect of gravity accelerating the projectile downwards and is the only appropriate value when measuring distance with feet and time in seconds.

To use the model to make predictions about the behavior of a projectile, you need to read the description of the situation carefully and identify the initial velocity, the initial height, and the height at time t.

Example 3 **Read the real-world situation and substitute in values for the projectile motion formula. Then solve the resulting quadratic equation by factoring to answer the question.**

(A) When a baseball player hits a baseball into the air, the height of the ball at t seconds after the ball is hit can be modeled with the projectile motion formula. If the ball is hit at 3 feet off the ground with an upward velocity of 47 feet per second, how long will it take for the ball to hit the ground, assuming it is not caught?

Use the equation $h = -16t^2 + vt + s$. Find the parameters v and s from the description of the problem.

$v = 47$ $\quad s = 3$ $\quad h = 0$

Substitute parameter values. $\quad -16t^2 + 47t + 3 = 0$

Divide both sides by -1. $\quad 16t^2 - 47t - 3 = 0$

Use the combination pair that results in a sum of -47.

$$(t - 3)(16t + 1) = 0$$

$$t - 3 = 0 \quad \text{or} \quad 16t + 1 = 0$$

$$t = 3 \qquad 16t = -1$$

$$t = -\frac{1}{16}$$

The solutions are 3 and $-\frac{1}{16}$.

The negative time answer can be rejected because it is not a reasonable value for time in this situation. The correct answer is 3 seconds.

EXPLAIN 3

Solving Equation Models of the Form $ax^2 + bx + c = 0$ by Factoring

QUESTIONING STRATEGIES

? To model the motion of a projectile when distance is measured in meters instead of feet, you use the formula $h = -4.9t^2 + vt + s$ instead of $h = -16t^2 + vt + s$. By what factor does the coefficient of t^2 change? Why does that factor make sense? **The coefficient is reduced by a factor of 3.3, because there are 3.3 feet in a meter.**

INTEGRATE MATHEMATICAL PRACTICES

Focus on Modeling

MP.4 When using an equation of the form $h = -16t^2 + vt + s$ to determine how many seconds it takes for an object to reach the ground, remind students that v represents the initial *upward* velocity. If a problem states that an object is thrown downward, students must use a negative value for v.

INTEGRATE MATHEMATICAL PRACTICES

Focus on Technology

MP.5 Have students use a graphing calculator to graph an equation that represents the height of an object that is thrown upward, then falls. Discuss the shape of the graph, and ask students at what time the projectile reaches its maximum height. Students should recognize that the time of the object's maximum height is the x-value for the vertex of the parabola. Because the graph is symmetric, they can also find this value by determining the x-value that is half way between the x-intercepts, or zeros, of the function.

ELABORATE

QUESTIONING STRATEGIES

How is factoring a trinomial in the form $ax^2 + bx + c$ similar to factoring a trinomial in the form $x^2 + bx + c$? How is it different? **For both forms, you find factors of the coefficient of x^2 and of c such that the sum of products of factors is equal to b. When the trinomial is in the form $x^2 + bx + c$, the coefficient of x^2 is 1, so the products of the factors are simply the factors of c.**

(B) A child standing on a river bank ten feet above the river throws a rock toward the river at a speed of 12 feet per second. How long does it take before the rock splashes into the river?

Find the parameters v and s from the description of the problem.

$\boxed{v} = -12 \qquad s = \boxed{10} \qquad h = \boxed{0}$

Substitute parameter values. $\boxed{-16}t^2 + \boxed{-12}t + \boxed{10} = 0$

Divide both sides by $\boxed{-2}$. $8t^2 + \boxed{6}t + \boxed{-5} = 0$

Use the combination pair that results in a sum of 6.

$$(\boxed{2}t - 1)(\boxed{4}t + 5) = 0$$

$$2t - 1 = 0 \quad \text{or} \quad 4t + 5 = 0$$

$$2t = \boxed{1} \qquad 4t = \boxed{-5}$$

$$t = \boxed{\tfrac{1}{2}} \qquad t = \boxed{-\tfrac{5}{4}}$$

The solutions are $\boxed{\frac{1}{2}}$ and $\boxed{-\frac{5}{4}}$.

The only correct solution to the time it takes the rock to hit the water is $\boxed{\frac{1}{2}}$ second.

Your Turn

8. How long does it take a rock to hit the ground if thrown off the edge of a 72-foot tall building roof with an upward velocity of 24 feet per second?

$v = 24$; $s = 72$; $h = 0$

$$-16t^2 + 24t + 72 = 0$$

$$-8(2t^2 - 3t - 9) = 0$$

$$(t - 3)(2t + 3) = 0$$

$$t - 3 = 0 \quad \text{or} \quad 2t + 3 = 0$$

$$t = 3 \qquad t = -\frac{3}{2}$$

The rock lands 3 seconds after it is thrown.

Elaborate

9. **Discussion** What happens if you do not remove the common factor from the coefficients before trying to factor the quadratic equation?

If you do not remove the common factor from the coefficients, you can still find binomial factors, but you may need to check many more factor pairs before you find the correct answer. Additionally, when you do find a set of factor pairs that work, one of the binomial factors will not be fully factored, such as $(2x + 4)$, which should be written as $2(x + 2)$ if you are asked to find the factors of a quadratic expression.

10. Explain how you can know there are never more than two solutions to a quadratic equation, based on what you know about the graph of a quadratic function.

The graph of a quadratic equation is shaped like a parabola and can only cross the x-axis twice. The solutions of a quadratic equation in standard form are the x-intercepts of the corresponding quadratic function, and it can have up to two.

11. **Essential Question Check-In** Describe the steps it takes to solve a quadratic equation by factoring.

Change the quadratic equation to standard form if it is not there already, and then find the factors of the quadratic expression by checking factor pairs of a and c. When the quadratic is factored, set each binomial equal to zero to find a possible solution.

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

Factor the following quadratic expressions.

1. $6x^2 + 5x + 1$

$(3x + 1)(2x + 1)$

2. $9x^2 + 33x + 30$

$3(3x^2 + 11x + 10) = 3(x + 2)(3x + 5)$

3. $4x^2 - 8x + 3$

$(2x - 1)(2x - 3)$

4. $24x^2 - 44x + 12$

$4(6x^2 - 11x + 3) = 4(2x - 3)(3x - 1)$

5. $3x^2 - 2x - 5$

$(x + 1)(3x - 5)$

6. $-10x^2 + 3x + 4$

$-1(10x^2 - 3x - 4) = -1(2x + 1)(5x - 4)$

7. $12x^2 + 22x - 14$

$2(6x^2 + 11x - 7) = 2(2x - 1)(3x + 7)$

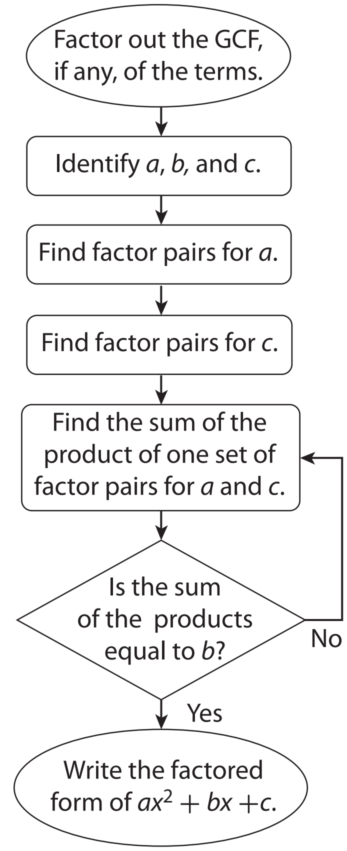
8. $-15x^2 + 21x + 18$

$-3(5x^2 - 7x - 6) = -3(x - 2)(5x + 3)$

Exercise	Depth of Knowledge (D.O.K.)	CA CC Mathematical Practices
1–16	2 Skills/Concepts	MP.4 Modeling
17–20	2 Skills/Concepts	MP.4 Modeling
21–22	2 Skills/Concepts	MP.4 Modeling
23	1 Recall of Information	MP.2 Reasoning
24	2 Skills/Concepts H.O.T.	MP.4 Modeling
25–26	3 Strategic Thinking H.O.T.	MP.4 Modeling

SUMMARIZE THE LESSON

Create a flowchart like the one below for factoring trinomials in the form $ax^2 + bx + c$.



EVALUATE

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ASSIGNMENT GUIDE

Concepts and Skills	Practice
Explore Factoring $ax^2 + bx + c$ When $c > 0$	Exercises 1–4, 22
Example 1 Factoring $ax^2 + bx + c$ When $c < 0$	Exercises 5–8, 21, 23, 25–26
Example 2 Solving Equations of the Form $ax^2 + bx + c = 0$ by Factoring	Exercises 9–16
Example 3 Solving Equation Models of the Form $ax^2 + bx + c = 0$ by Factoring	Exercises 17–20, 24