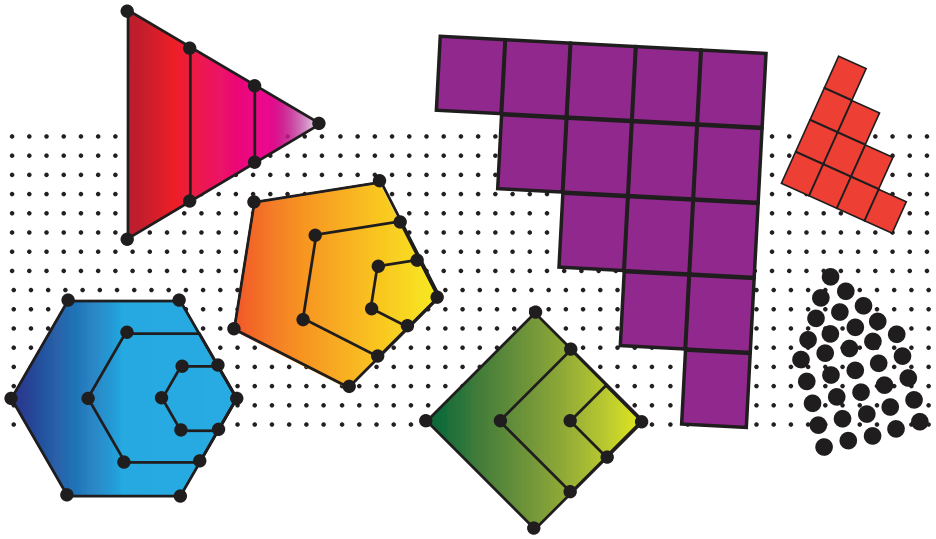
Student Explorations in Mathematics

What Shapes Do You See?



What Are Figurate Numbers?

While the rest of humanity was seemingly occupied with the more practical uses of mathematics, the Greeks were among the first to develop a mathematical world that was not necessarily tied to real-world applications, including what were to become figurate numbers.

A. Consider the three arrangements in figure 1.

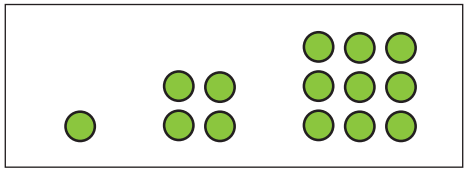


Figure 1

1. Predict the next two arrangements in this growing pattern.

2. How many dots would be in arrangements 4 and 5?

3. Identify the shapes you notice in the arrangements.

4. Write a description of how these arrangements are constructed. Share your answers with a partner.

We acknowledge Kathy Clark, Florida State University, for her submission of the idea for this issue.

5. Invent a name for these figurate numbers.

6. Consider the shapes you found in question A3. Sketch another way to group these dots into geometric shapes.

B. Consider the three arrangements in **figure 2.**

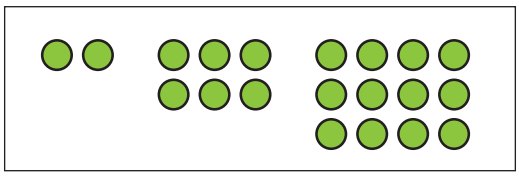


Figure 2

1. Predict the next two arrangements in this growing pattern.

2. How many dots would be in arrangements 4 and 5?

3. Identify the shapes you notice in the arrangements.

4. Explain what you observe about how these arrangements are constructed. Share your answers with a partner.

5. Invent a name for these figurate numbers.

6. Consider the shapes you found in question B3. Sketch another way to group these dots into geometric shapes.

C. The shapes in **figure 3** are arrangements 4 and 5 in the sequence of a growing pattern.

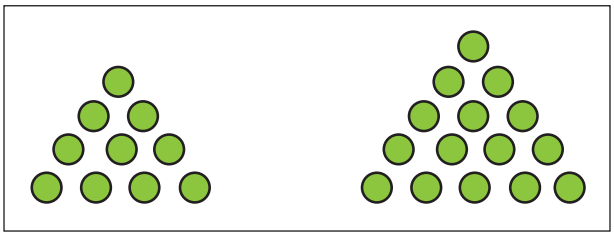


Figure 3

1. What are the first three arrangements in this growing pattern? (Remember that **figure 3** shows arrangements 4 and 5.)

2. How many dots would be in arrangements 1, 2, and 3?

3. Identify the shapes you notice in the arrangements.

4. Explain what you observe about how these arrangements are constructed. Share your answers with a partner.

5. Invent a name for these figurate numbers.

6. Consider the shapes you found in question C3. Sketch another way to group these dots into geometric shapes.

The patterns in **figures 1–3** reveal what mathematicians call *figurate* numbers. Figurate numbers get their name from the fact that they model geometric shapes or figures.

Now, make up your own arrangement of some figurate numbers and share with another student in your classroom.

D. Consider the figurate number sequence in figure 4.

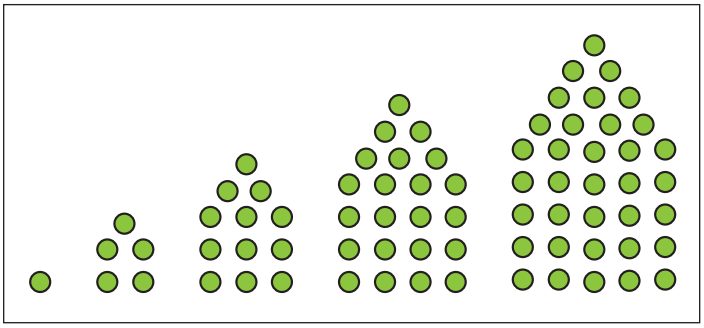


Figure 4

1. What are the next two arrangements in this growing pattern?

2. How many dots would be in arrangements 6 and 7?

3. Sketch the shapes you notice in the arrangements.

4. Describe how these arrangements are constructed.

5. You may see two or more shapes within the overall shapes. Sketch the shapes inside of other shapes and share your drawings with a partner.

6. Use your observations to predict how many dots would be in arrangement 10 of this sequence.

7. Tell a partner how you determined your answer to question D6.

8. Invent a name for these figurate numbers.

9. Consider the shapes you found in question D3. Sketch another way to group these dots into geometric shapes.

E. The figurate number sequence in **figure 5** contains arrangements 4 and 5.

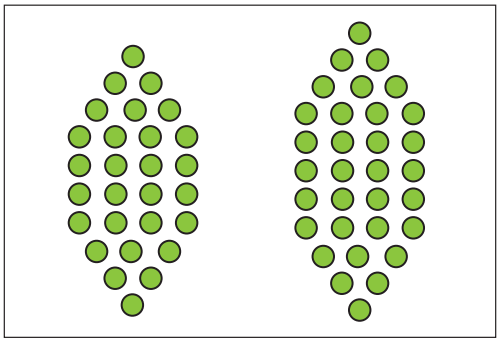


Figure 5

1. Sketch the first three arrangements in this growing pattern.

2. Sketch what you think arrangements 6 and 7 would look like if this were a growing pattern.

3. Describe how these arrangements are constructed.

4. You may see two or more shapes within the overall shapes. Sketch the shapes inside of other shapes that you notice in this sequence.

5. Use your observations to predict how many dots would be in arrangement 20 of this sequence.

6. Tell a partner how you determined your answer to question E5.

7. Invent a name for these figurate numbers.

8. Consider the shapes you found in question E4. Sketch another way to group these dots into geometric shapes.

F. Consider all the sequences in **figure 6**, which is actually **figures 1–5** combined.

1. How are the sequences the same?

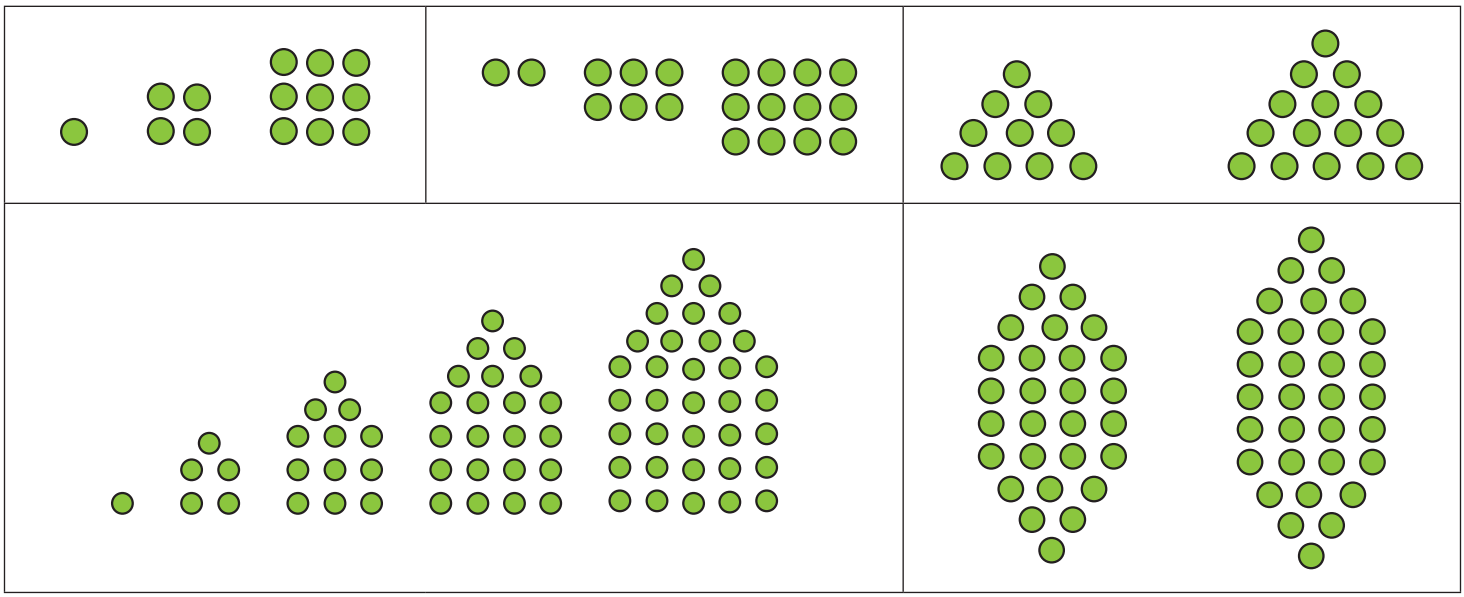


Figure 6 (Combined Figures 1–5)

2. How are the sequences different?

3. Explain what you observe about how these arrangements are constructed.

4. Which sequences are embedded in other sequences? Share your answers with a partner.

G. Suppose that all you knew was the number of dots in a sequence of arrangements in a growing pattern.

1. If the number of dots in the first four arrangements were 1, 5, 13, and 25, respectively, sketch how you think the dots might be arranged and what shapes might be possible.

2. Sketch how the dots can be arranged. What shapes or figures emerge? Sketch what the fifth arrangement might look like. Share your sketches with a partner.

H. Consider the sequence in **figure 7**.

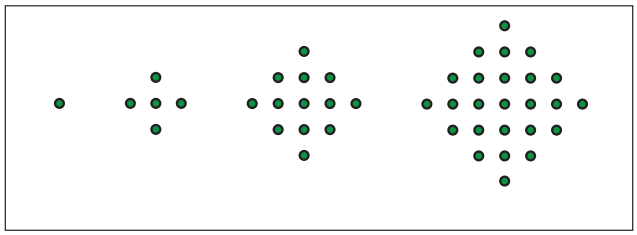


Figure 7

1. How is **figure 7** the same as or different from your sketches in part G?

Mathematicians named the sequence in **figure 7** *centered square* numbers.

I. Using sketch paper or an electronic device, create your own arrangements to illustrate a figurate number sequence. Figure 8 shows an example of centered square numbers created using Doodle Buddy on an iPad®.

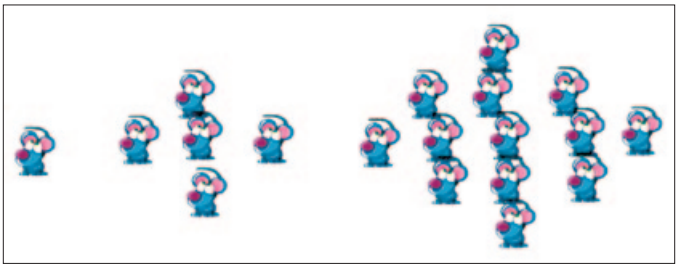


Figure 8

1. Explain the mathematics that describes your sequence.

2. How do your arrangements change from one arrangement to the next?

3. Describe how you can use what you know about your sequence to predict the 100th arrangement; the 568th arrangement; the 1000th arrangement; and any arrangement.

J. On the basis of what you have explored and learned in this activity, complete table 1 (see p. 7).

1. Share your numerical and verbal representations of figurate number tables with other students, in pairs or in groups of three.

2. Explain your understanding of patterns created by another student in your group.

3. How are these patterns the same? How are they different?

4. Some types of figurate numbers can be even or odd. Can any type of figurate number never be even? Why, or why not?

5. Concerning the values and verbal explanations in **table 1**, what are you sure about? What are you unsure of? What do you wonder about the values and verbal explanations in **table 1**?

Table 1. Numerical and Verbal Representations of Figurate Numbers

Figurate Number Type	Square Numbers	Pentagonal Numbers	Centered Square Numbers	Student-Created Pattern
Number of dots in each arrangement	1, 4, 9, 16, 25	1, 5, 12, 22, 35	1, 5, 13, 25, 41	
Verbal: Briefly describe how each figurate number is determined. (Note: There may be more than one way to express this, which we will address later in the exploration.)				

K. Create your own figurate number notation.

1. Think about how you described the number sequence in **table 1**. Invent a way to use notation to represent the sequences of square numbers, pentagonal numbers, and centered square numbers.

2. Using words, describe how your notation works.

3. Illustrate samples of how your notation works. Consult with others in your group and find other notations that are similar to or different from your notation.

L. Explore common figurate number notation agreed on by mathematicians.

T_4 means the *fourth triangular number*: ***T*** stands for *triangular*; the subscript **4** represents the *fourth* such number. (As a side note, the subscript is sometimes called the *index*.)

The subscripted notation for representing figurate numbers is useful. It is easier to write T_4 than *the fourth triangular number*.

There is another interesting way to represent figurate numbers, which is called the *symbolic* (or *algebraic*) representation. We can represent the *n*th triangular (or any figurate number) by determining the algebraic formula for it. For example, we can examine how to calculate the *n*th triangular number T_n, where n is a natural number, by using analysis: T_8 = the sum of the first eight natural numbers, or $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$.

Complete **table 2** to reflect the total value of each triangular number.

1. Explain how the numbers in **table 2** are determined.

Table 2. Triangular Numbers

Triangular no. index *n*	1	2	3	4	5	…	21	22	23	24
Triangular no. T_n	1	3								

2. Examine **table 2** for a pattern in determining the value of each triangular number. Determine the value of the fiftieth triangular number using your pattern.

3. Write an equation that would predict the number of dots for any triangular number.

M. Explore a representation of the Handshake problem.

If everyone in the room shook hands with everyone else, how many handshakes would there be?

Suppose we represent triangular number 4, or T_4, with ladybugs rearranged to form a staircase-like figure (see **fig. 9**).

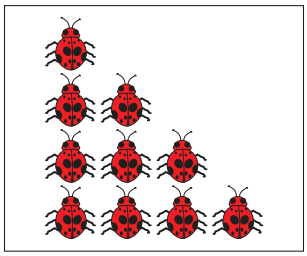


Figure 9

Now suppose you rotate the bugs (see **fig. 10**).

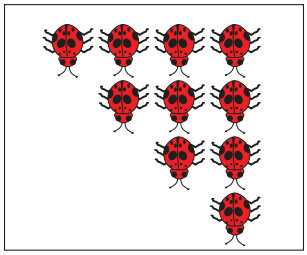


Figure 10

Now imagine that the two groups of bugs are joined to form a rectangular arrangement of all the bugs (see **fig. 11**). Analyze the dimensions of the "bug rectangle" and examine the characteristics of the rectangle to see if a formula for the total number of bugs emerges.

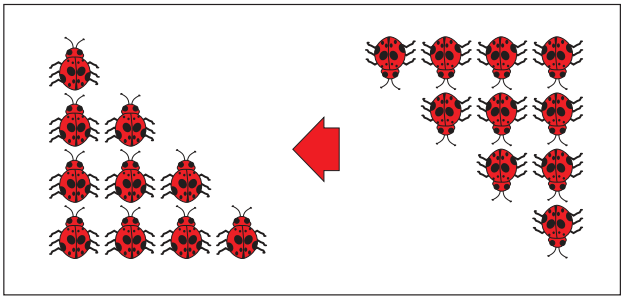


Figure 11

N. Consider these points.

1. How is the Handshake problem similar to or different from what you have seen about figurate numbers?

2. Predict how many handshakes there would be in a room with 50 people, 100 people, 1000 people, and any number of people.

O. Let's make a few conjectures.

Let's examine some conjectures about relationships among the different figurate numbers. A *conjecture* is a mathematical statement that is believed to be true but that has not been proven to be true—yet.

Conjecture 1: *One more than eight times a triangular number is a square number.*

1. Express this conjecture using subscript notation.

2. Will this conjecture always, sometimes, or never be true? Why, or why not?

Now, with a partner or your group, use the various figurate numbers (triangular, square, pentagonal, hexagonal, inspirational, oblong)—and any of the representations that you desire—to formulate conjectures about relationships among the different figurate numbers. Identify and prove (a proof is considered a convincing argument) at least three of your own conjectures.

Did you know that ….

- the Pythagoreans were well invested in the theory of figurative numbers? The credo of their brotherhood implied *All is number*. Later scholars, however, also studied such numbers. For example, in 1665 Blaise Pascal (1601–1665) wrote his *Treatise on Figurative Numbers*, which contains the famous result that every positive integer can be written as the sum of three or fewer triangular numbers.

  For more on Pythagoras, see http://www.learner.org/courses/mathilluminated/interactives.

  For more on adding something to a figurate number to transform it to the next larger figure, see "Gnomon" at http://en.wikipedia.org/wiki/Figurate_number.

- "the set of numbers that form Pascal's triangle were well-known before Pascal? However, Pascal developed many applications of it and was the first one to organize all the information together in his treatise, *Traité du Triangle Arithmétique* (1653). The numbers originally arose from Hindu studies of combinatorics and binomial numbers and the Greeks' study of figurate numbers." See http://www.facebook.com/pages/Pascals-triangle/102793529757111?sk=info.

- the triangular numbers are found in the third diagonal of Pascal's triangle?

- a centered cube number is a centered figurate number that represents a cube? Centered cube numbers have applications in shell models found in atomic structures.

- although numerous possibly flawed stories exist about Pythagoras and his brotherhood, most reliable sources agree that the Pythagorean Brotherhood was composed of individuals who were both religious and philosophical (Katz 1998, p. 48)? Mathematically speaking, the Pythagoreans' doctrine centered on positive integers because the Pythagoreans believed such numbers represented the "organizing principle of the universe" (Katz 1998, p. 48). Also, since the Pythagoreans interpreted all physical phenomena using numbers, "it is only natural that they studied the properties of positive integers, what we would call the elements of the theory of numbers" (Katz 1998, p. 49). Note: Biographies of famous (and not so famous) mathematicians can be found on the MacTutor History of Mathematics archive.

References

Katz, Victor J. 1998. *A History of Mathematics: An Introduction*. Reading, MA: Addison-Wesley Educational Publishers.

O'Connor, John J., and Edmund F. Robertson. August 2011. The MacTutor History of Mathematics Archive. University of St. Andrews, Scotland, School of Mathematics and Statistics. http://www-history.mcs.st-andrews.ac.uk/index.html.

Posamentier, Alfred S. 2003. *Math Wonders to Inspire Teachers and Students*. Alexandria, VA: Association for Supervision and Curriculum Development.

Student Explorations in Mathematics is published electronically by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 20191-1502. The five issues per year appear in September, November, January, March, and May.

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS