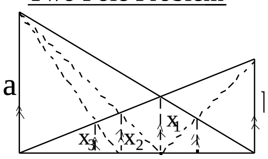
"The Noah Sheets" [Begun by Noah Rosenberg, IMSA Class of 1995, Edited and enhanced by George Milauskas]

Triangles

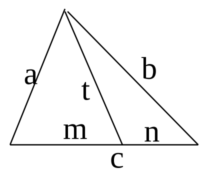
Two Pole Problem



$\frac{1}{x_1} = \frac{1}{a} + \frac{1}{b}$ so

$x_1 = \frac{a \cdot b}{a + b}$ $\quad x_k = \frac{a \cdot b}{a + kb}$

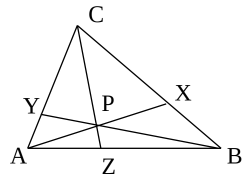
Stewart's Theorem



$a^2 n + b^2 m = t^2 c + m \cdot n \cdot c$

(Proven by using Law of Cosines twice)

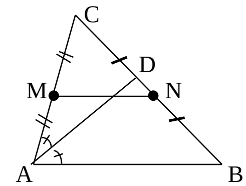
Ceva's Theorem



$\frac{AZ}{ZB} \bullet \frac{BX}{XC} \bullet \frac{CY}{YA} = 1, \ \frac{PX}{AX} + \frac{PY}{BY} + \frac{PZ}{CZ} = 1$

$\Leftrightarrow$ AX, BY, & CZ are concurrent

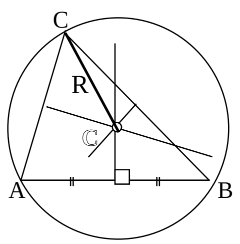
Basic Theorems



AC:AB = CD:DB (bis thm)

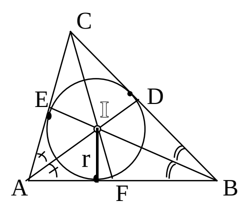
MN || & $\frac{1}{2}$ AB [Midline Thm]

Circumcenter [− Bisectors]



$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Incenter [- Bisectors]

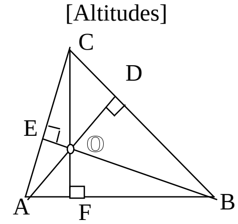


Area (ΔABC) $= \frac{1}{2}$ r P

$=$ r·s (s=semi-perimeter)

Orthocenter

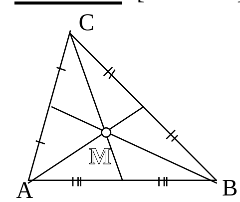
[Altitudes]



Watch for similar triangles.

eg: ΔADB ΔCFB

Centroid [Medians]

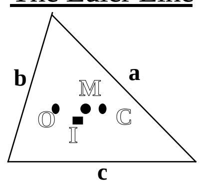


All six areas are equal.

𝕄 splits each median in ratio 2:1.

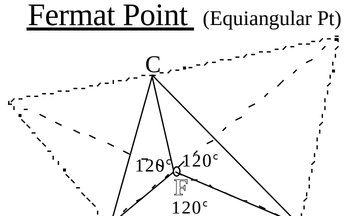
Coordinates of 𝕄 = avg of vertices.

The Euler Line



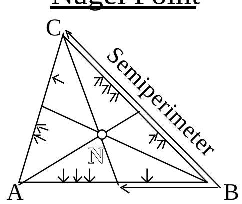
𝕆, 𝕄, & ℂ are collinear, such that 𝕆𝕄 : 𝕄ℂ = 2:1 and $9 \cdot (\mathbb{OC})^2 = a^2 + b^2 + c^2$

Fermat Point (Equiangular Pt)



The sum A𝔽 + B𝔽 + C𝔽 is a minimum. (Found by, putting equilateral Δ's on sides)

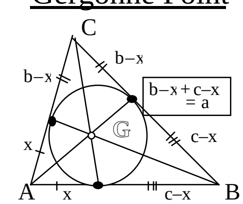
Nagel Point



[Joins Semi-Perimeter Pts to Vertices]

Notice resulting segments

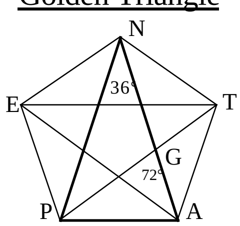
Gergonne Point



[Tangency Pts to Vertices]

Notice segments & "walkaround" labeling.

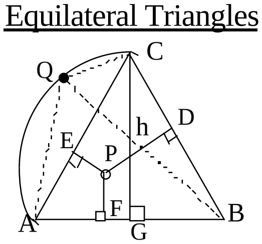
Golden Triangle



In a regular pentagon,

PN:PA = PG:GA = $\frac{1+\sqrt{5}}{2}$

Equilateral Triangles

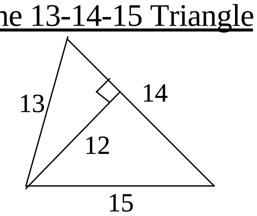


Sum of dist from any **P** to sides = **h**.

Any **Q** on Circum-⊙: QB = QC + QA

ΔCGB, has sides in a ratio, $1 : \sqrt{3} : 2$

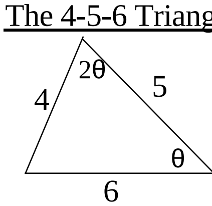
The 13-14-15 Triangle



[An altitude and three sides are consecutive integers.]

Area = 84, r = 4, R = $\frac{65}{8}$

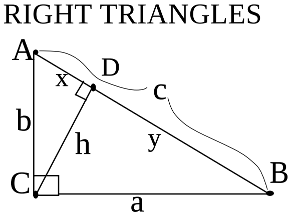
The 4-5-6 Triangle



One angle is twice the other.

Area = $6\sqrt{6}$

RIGHT TRIANGLES



$(a + b)^2 = c^2 + 4(\text{Area})$

$\frac{y}{a} = \frac{a}{c}$ $\quad a^2 = y \cdot c$

$\frac{x}{b} = \frac{b}{c}$ $\quad b^2 = x \cdot c$

$\frac{x}{h} = \frac{h}{y}$ $\quad h^2 = x \cdot y$

$a \cdot b = c \cdot h$

$a^2 + b^2 = c^2$ Pythagorean Thm

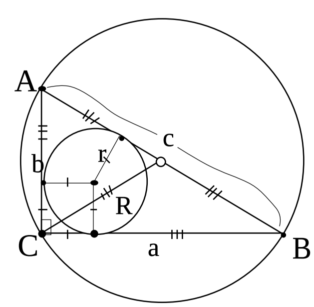
$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2}$

$2 \cdot r = a + b - c$

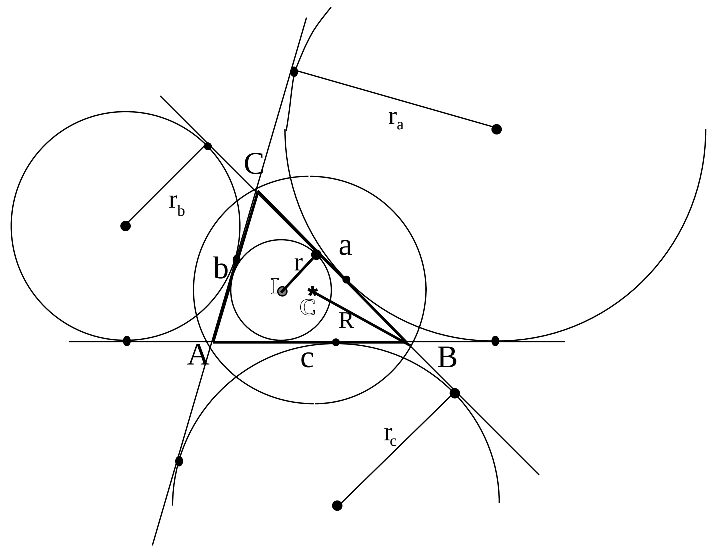
$2 \cdot R = c \quad \frac{a+b}{2} = R + r$

median = $\frac{1}{2}$ hypotenuse

$(m^2 - n^2, 2mn, m^2 + n^2)$ is Pythagorean triple for m, n positive integers



A Triangle And Its Circles: PROPERTIES



Δ ABC has sides: **c**, **b**, and **a**,
and angles A, B, and C.

The radii of the:
- Inscribed circle, **r**.
- The three escribed circles: $\mathbf{r_a}$, $\mathbf{r_b}$, & $\mathbf{r_c}$
- and the circumscribed circle, **R**.

The area of the triangle is **K**.

The semiperimeter is **S**.

Drawn to sides **a**, **b**, and **c**, respectively:
- Let $\mathbf{m_a}$, $\mathbf{m_b}$, & $\mathbf{m_c}$ be the medians.
- Let $\mathbf{t_a}$, $\mathbf{t_b}$, & $\mathbf{t_c}$ be the angle bisectors.
- Let $\mathbf{h_a}$, $\mathbf{h_b}$, & $\mathbf{h_c}$ be the altitudes.

The following relationships are true for triangles as labeled above:

$K = \sqrt{S\,(S-a)\,(S-b)\,(S-c)}$ Heron's Formula

$$K = \frac{1}{2}\cdot a\cdot b\cdot \sin C = \frac{a^2 \sin B \cdot \sin C}{2 \sin A} = r\cdot S = \frac{a\cdot b\cdot c}{4\cdot R}$$

$R = \dfrac{a\cdot b\cdot c}{4\cdot K}$ $\quad 2\cdot R\cdot r = \dfrac{a\cdot b\cdot c}{a+b+c}$ $\quad 2\cdot r \leq R$ in all Δ's

$$t_c = \frac{2\cdot a\cdot b\cdot \cos\frac{C}{2}}{a+b} = \frac{2\sqrt{a\cdot b\cdot S\cdot(S-c)}}{a+b}$$

$\dfrac{1}{r} = \dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c}$ $\qquad r^2 = \dfrac{(s-a)(s-b)(s-c)}{s}$

$c^2 = 2\,a^2 + 2\,b^2 - 4\,m_c^{\,2}$ and its permutations

Law of Cosines:
$c^2 = a^2 + b^2 - 2\cdot a\cdot b \cos C$ (& permutations)

Law of Sines:
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Law of Tangents: $\dfrac{a-b}{a+b} = \dfrac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$

Other

Point-Line(plane) Distance

between (x_0,y_0) and line $ax + by + c = 0$:

dist is: $\dfrac{|\,a\cdot x_0 + b\cdot y_0 + c\,|}{\sqrt{a^2 + b^2}}$

between (x_0,y_0,z_0) and line $ax + by + c\cdot z + d = 0$:

is: $\dfrac{|\,a\cdot x_0 + b\cdot y_0 + c\cdot z_0 + d\,|}{\sqrt{a^2 + b^2 + c^2}}$

also dist $= \dfrac{a\cdot b\cdot c}{a\cdot b+b\cdot c+a\cdot c}$

Logarithms

$\log_b N = p \;\Leftrightarrow\; b^p = N$

$\log_b N = \dfrac{\log_a N}{\log_a b}$ "change base"

$\log \dfrac{m\cdot n}{q} = \log m + \log n - \log q$

$\log N^p = p \log N$

$\log_b a = \dfrac{1}{\log_a b}$

Series

Series: $S = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

•Arithmetic: Constant Difference $d = a_{n+1} - a_n$

$a_n = a_1 + (n-1)\cdot d$ & $S_n = \frac{n}{2}(a_1 + a_n)$

•Geometric: Constant ratio $r = \dfrac{a_{n+1}}{a_n}$

$a_n = a_1\cdot r^{n-1}$, $S_n = \dfrac{a - a\,r^{n-1}}{1-r}$ & $S_\infty = \dfrac{a_1}{1-r}$

$(-1 < r < 1)$

If you find any errors, or have any worthy additions to "The Noah Sheets" please contact
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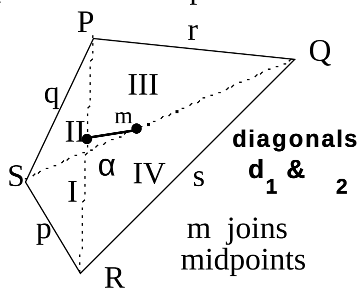
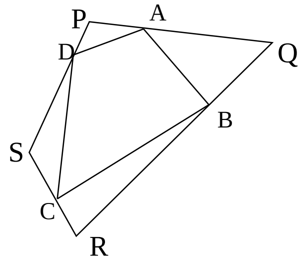
Quadrilateral Properties: **K** = Area, **r** = inradius, **R** = circumradius, **P** = perimeter, **S** = semiperimeter



Areas: $A_I \cdot A_{III} = A_{II} \cdot A_{IV}$

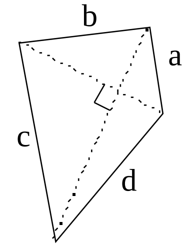
$\mathbf{K}_{PQRS} = \frac{1}{2} d_1 d_2 \sin \alpha$

$p^2 + q^2 + r^2 + s^2 = d_1^{\,2} + d_2^{\,2} + (2m)^2$



A,B,C,D are midpoints, ABCD is a parallelogram.

If $\frac{PA}{AQ} = \frac{QB}{BR} = \frac{RC}{CS} = \frac{SD}{DP} = n$ then the ratio of areas, $\mathbf{K}(ABCD) : \mathbf{K}(PQRS) = \frac{n^2+1}{(n+1)^2}$



If Diagonals are perpendicular,

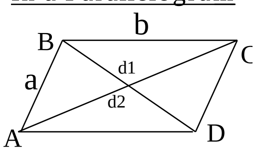
$\mathbf{K} = \frac{1}{2}(\text{diag}_1)(\text{diag}_2)$

$a^2 + c^2 = b^2 + d^2$

⇔ (if and only if)

the diagonals are .

In a Parallelogram



$2(a^2 + b^2) = d_1^{\,2} + d_2^{\,2}$

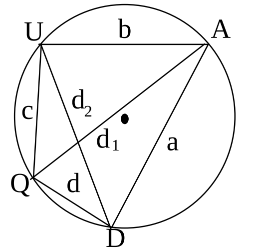
$BD = a^2 + b^2 - 2ab \cos A$

$\mathbf{K} = a \cdot b \cdot \sin A$.

$\frac{d_1}{d_2} = \frac{ab + cd}{ad + bc}$

(2nd Ptolemy's Theorem)

Cyclic Quadrilaterals

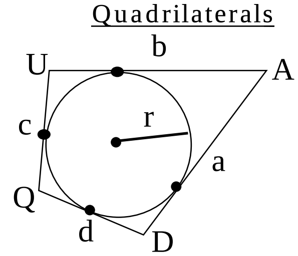


A + Q = U + D = 180°

$d_1 \cdot d_2 = a \cdot c + b \cdot d$ (Ptolemy)

$\mathbf{K} = \sqrt{(S-a)(S-b)(S-c)(S-d)}$

Circumscribed Quadrilaterals

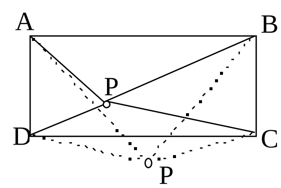


a+c = b+d ⇔ QUAD has in⊙

$r = \frac{\mathbf{K}}{a+c}$ and $\mathbf{K} = \frac{1}{2}\ \mathbf{r} \cdot \mathbf{P}$

If QUAD is both inscribed. and circumscr., then $K = \sqrt{a \cdot b \cdot c \cdot d}$

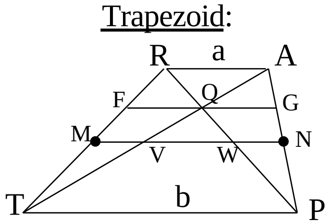
The Rectangle



For any point, P, & rectangle ABCD

$(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$

Trapezoid:



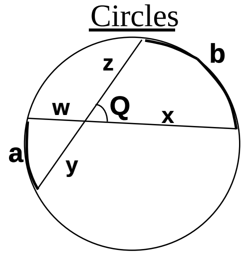
If MN = median, $MN = \frac{a+b}{2}$

$VW = \frac{b-a}{2}$ $FG = \frac{2 \cdot a \cdot b}{a+b}$

If $\overline{MN}$ is ***any*** parallel,

$MN = \frac{RM \cdot b + MT \cdot a}{RT}$

Circles

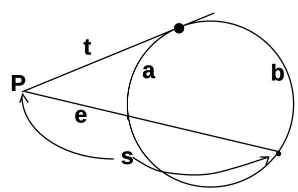


Angle-Arc Property

$Q = \frac{b + a}{2}$

Power Theorem

$w \cdot x = y \cdot z$

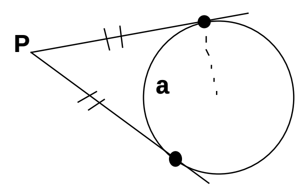


Angle-Arc Property

$P = \frac{b - a}{2}$

Power Theorem

$t^2 = s \cdot e$



A tangent to a circle is perpendicular to a radius.

Two tangents to a circle from an outside point are equal.

tan-tan angle = supp of inner arc. P + a = 180°

Volume and Surface Areas of Solids:

Prismatic solids: (prism, box, cylinder)

Lateral Area = (base perimeter)(height)

Total Area = lateral area + 2 bases

Volume = (Area of Base)(height)

Pointed Solids: (pyramid, cone)

[Linearly related cross sections]

Lateral Area (add lat faces), $LA_{(cone)} = \pi \cdot r \cdot l$

l = lateral edge (slant height)

Total Area = lateral area + one base

Volume = $\frac{1}{3}$(Area of Base)(height)

Spheres:

Total Surface Area = $4 \cdot \pi \cdot r^2$

Volume = $\frac{4}{3}\pi r^3$

Ellipsoid: Volume = $\frac{4}{3} a \cdot b \cdot c \cdot \pi$

Prismoidal Volume Formula: $V = \frac{h}{6}(B_1 + 4M + B_2)$

[For solids with quadratically related cross sections, height h, upper bases B_1, B_2 and mid section M]

Trigonometry:

$\sin A = \dfrac{\text{opp leg}}{\text{hypotenuse}}$

$\cos A = \dfrac{\text{adj leg}}{\text{hypotenuse}}$

$\tan A = \dfrac{\text{opp leg}}{\text{adj leg}} = \dfrac{\sin A}{\cos A}$

$\csc A = \dfrac{\text{hypotenuse}}{\text{opp leg}} = \dfrac{1}{\sin A}$

$\sec A = \dfrac{\text{hypotenuse}}{\text{adj leg}} = \dfrac{1}{\cos A}$

$\cot A = \dfrac{\text{adj leg}}{\text{opp leg}} = \dfrac{1}{\tan A}$

Values to Memorize:

$\sin 30° = \dfrac{1}{2} = \cos 60°$

$\cos 30° = \dfrac{\sqrt{3}}{2} = \sin 60°$

$\tan 30° = \dfrac{\sqrt{3}}{3} = \cot 60°$

$\sin 45° = \cos 45° = \dfrac{\sqrt{2}}{2}$

$\tan 45° = 1$

$\sin 15° = \dfrac{\sqrt{6} - \sqrt{2}}{4} = \cos 75°$

$\cos 15° = \dfrac{\sqrt{6} + \sqrt{2}}{4} = \sin 75°$

$\tan 15° = 2-\sqrt{3}$, $\tan 75° = 2+\sqrt{3}$

Golden rectangle & regular pentagon.

$\sin 18° = \cos 72° = \dfrac{\sqrt{5} - 1}{4}$

$\cos 36° = \sin 54° = \dfrac{\sqrt{5} + 1}{4}$

Pythagorean Identities

$\sin^2 A + \cos^2 A = 1$

$1 + \tan^2 A = \sec^2 A$

$1 + \cot^2 A = \csc^2 A$

Odd-Even Functions:

$\sin(-A) = -\sin(A)$

$\cos(-A) = \cos(A)$

$\tan(-A) = -\tan(A)$

Complements A & B

$\sin^2 A + \sin^2 B = 1$

$\sin A = \cos B$, etc

Sum & Difference Identities

$\sin(A \pm B) = \sin A\cdot\cos B \pm \cos A\cdot\sin B$

$\cos(A \pm B) = \cos A\cdot\cos B \mp \sin A\cdot\sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Double Angle Identities

$\sin 2A = 2\sin A \cdot \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

or $= 1 - 2\sin^2 A = 2\cos^2 A - 1$

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Sum to Product

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\sin\dfrac{A-B}{2}\cos\dfrac{A+B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\tan A \pm \tan B = \dfrac{\sin(A \pm B)}{\cos A\cdot\cos B}$

Triple Angle Identities

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$

$\tan 3A = \dfrac{\tan A\cdot(\tan^2 A - 3)}{3\tan^2 A - 1}$

Half Angle Formulas

$\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$

$\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$

$\tan\dfrac{A}{2} = \dfrac{\sin A}{1+\cos A} = \dfrac{1-\cos A}{\sin A}$

Product to Sum

$\sin A\cdot\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

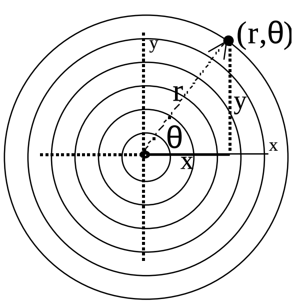
$\cos A\cdot\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$\sin A\cdot\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$\tan A\cdot\tan B = \dfrac{\cos(A-B) - \cos(A+B)}{\cos(A-B) + \cos(A+B)}$

Polar Coordinates

Points are represented in terms of (r,θ) rather than (x,y)



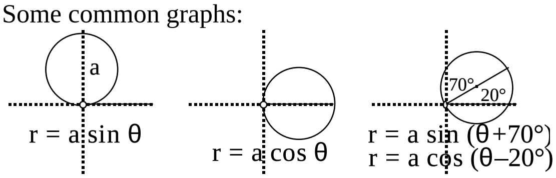
$x^2+y^2 = r^2$

$\tan\theta = \dfrac{y}{x}$

$x = r\cos\theta$

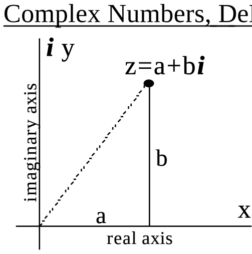
$y = r\sin\theta$

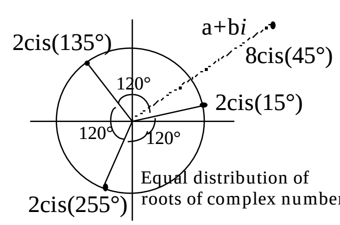
Some common graphs:



$r = a\sin\theta$ $\quad$ $r = a\cos\theta$ $\quad$ $r = a\sin(\theta+70°)$, $r = a\cos(\theta-20°)$

Complex Numbers, DeMoivre's Thm, Euler's Thm & CIS





$Z = a + b\,i = r\,\text{cis}\,\theta$ (polar form of complex number)

The magnitude, $r = |a + b\,i| = \sqrt{a^2 + b^2}$

$e^{i\theta} = \cos\theta + i\sin\theta = \text{cis}\,\theta$ (Euler)

$\text{cis}(A + B) = \text{cis}\,A\cdot\text{cis}\,B$ $\qquad$ $\text{cis}(A - B) = \dfrac{\text{cis}\,A}{\text{cis}\,B}$

DeMoivre's Theorems: (see illustration above)

$(a+b\,i)^n = (r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\cdot\theta)$ for n = pos int

$\sqrt[n]{r\cdot\text{cis}\,\theta} = \sqrt[n]{r}\cdot\text{cis}\dfrac{2\pi k+\theta}{n}$ for $k = 0,1,2,3\ldots,n-1$