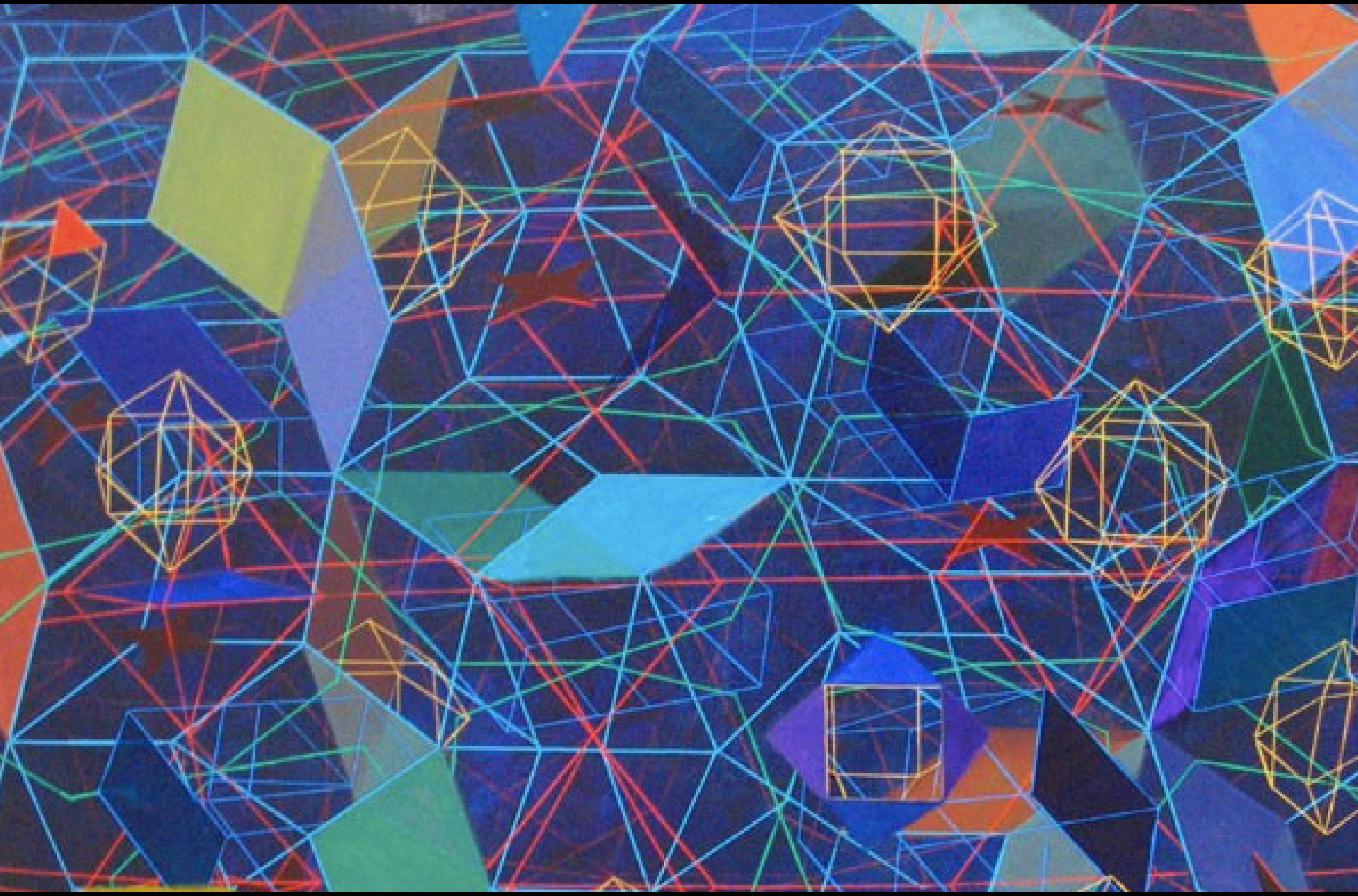
PRIME



VOLUME 2 , ISSUE 2

Issue 2, Volume 2

Letter from the editors

Hello Horace Mann!

Thanks for picking up a copy of Prime! We're so excited to share some of the interesting mathematical ideas our writers have been thinking about. From pure math topics, like topology, non Euclidean geometry and number theory, to in-depth perspectives of math you might encounter on a competition or in your classes, such as the golden ratio, inequalities and the brachistochrone, to applied math topics like the Fourier Transform, quantum computing, or even mathematical art, we have a ton of super cool topics and articles for this issue. We hope you'll enjoy the articles, challenge problems and more that we've put together!

We'd like to thank several people who have helped us to create this issue: first, Dr. Delanty, Dr. Kelly, Dr. Levenstein and all of the administrative faculty and staff without whom this publication would have been impossible, and second, Mr. Worrall, our wonderful advisor. Finally, we'd like to thank every student who took the time to write or edit an article for this issue. You guys rock!

Sincerely,

Dora Woodruff and Mandy Liu

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The Brachistochrone Problem

By Zachary Brooks

Today the math community is predicated on collaboration. However, in the 17th century that was hardly the case. A common practice at the time was for mathematicians to pose problems that only they knew the answer to, and intentionally leave out their solution in order to perplex the rest of the community while flexing their own abilities. One problem that came out of this was the Brachistochrone (Greek for "shortest-time") Problem, which was prompted by Johann Bernoulli in 1696. Unfortunately for him, several of his contemporaries, such as Newton and Jacob Bernoulli (Johann's brother and rival), were able to crack the problem before Johann could claim victory and announce his own solution. Nonetheless, the question has become one of the most fascinating in the history of mathematics.

The question was simple: given two points A and B of different elevations, what is the path that allows a rolling object to get from A to B in the shortest amount of time possible? "What a stupid question, it's a straight line!" one might think. However, that turns out to be one of the worst possible answers. Because you want your rolling object to pick up the most speed possible, our path would clearly benefit from having curvature. The question is, what type of curve?

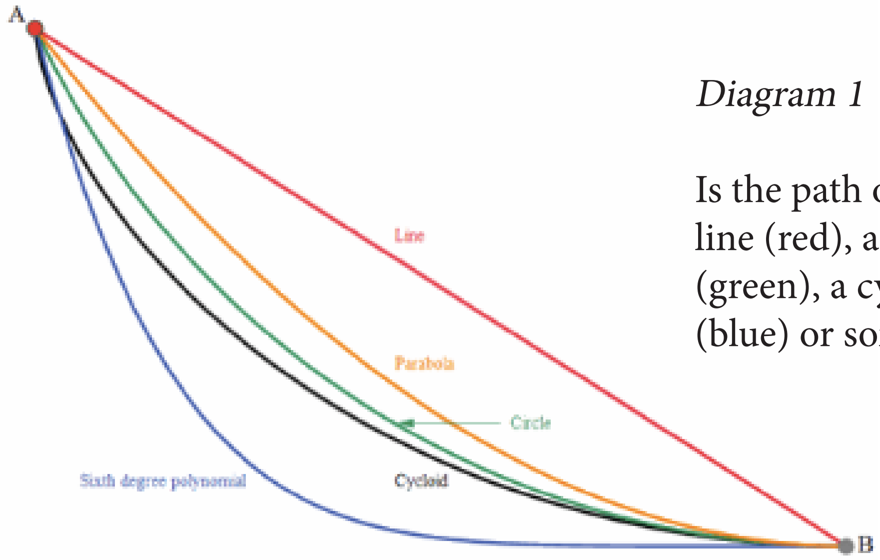
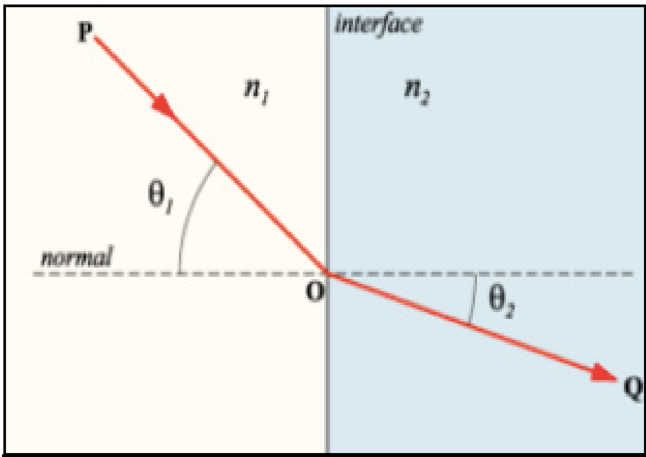


Diagram 1

Is the path of shortest duration a straight line (red), a parabola (yellow), a circle (green), a cycloid (black), a polynomial (blue) or something else?

In order to get ot the bottom of this, let's take a look at a different yet related problem. Say you're at the beach, and you see something you really want floating in the water (say, for the sake of example, it is the opportunity to replace the alma-mater with Digital Love by EMatt). What would be the path you can take to get from where you are to the floating object (whatever it may be) as fast as possible? Straight line? Wrong again! You can travel much faster on land than water, so you want to maximize how much time you spend on land versus water. So in order to get to the floating object as quickly as possible, at what angle from the beach should you approach the water at? The answer is dependant on something called Snell's Law, which states that the angle you should approach the shore at is equal to the expression $\sin(\theta_1)/n_1=\sin(\theta_2)/n_2$, where n_1 and n_2 represent the speed at which you can travel through that medium and your represents the angle your path should make with the orthogonal, or "normal," line to the barrier before and after you transfer mediums. *(See diagram 2 on the next page.)*

How does this relate to the brachistochrone? Well, as an object falls, it moves faster the further down it travels. This is analogous to moment where our person moves from the beach to the shore, however instead of happening once, this process continues infinitely, at every moment in time. This would produce a curve where, if a vertical line were drawn across the resulting curve, the sine of the angle to the right of the curve divided by the velocity of the rolling object would be equal to the angle the curve (or more precisely, the tangent to the curve) makes with the vertical to the left divided by the new velocity. This fact would have to hold true not just for one particular vertical line, but for any drawn through the curve. *(See diagram 3 on the next page.)*



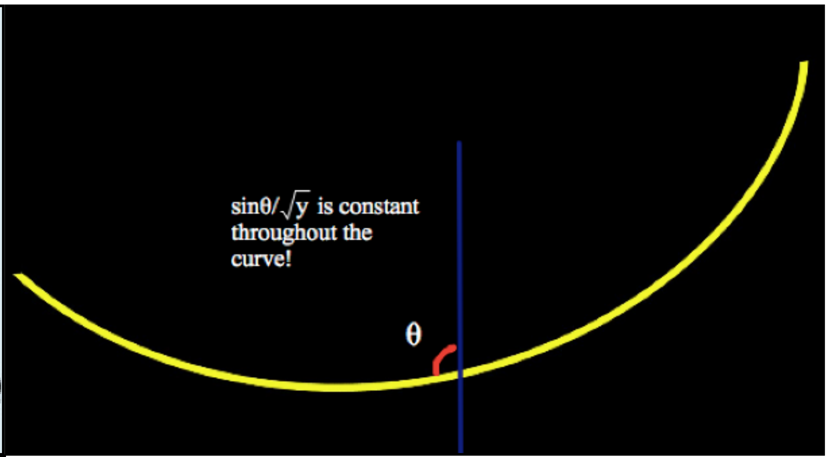


Diagram 2: Snell's Law in action! An object at point P takes the shortest possible path to point Q. Its path bends so that it spends more time in the medium it can travel fastest in.

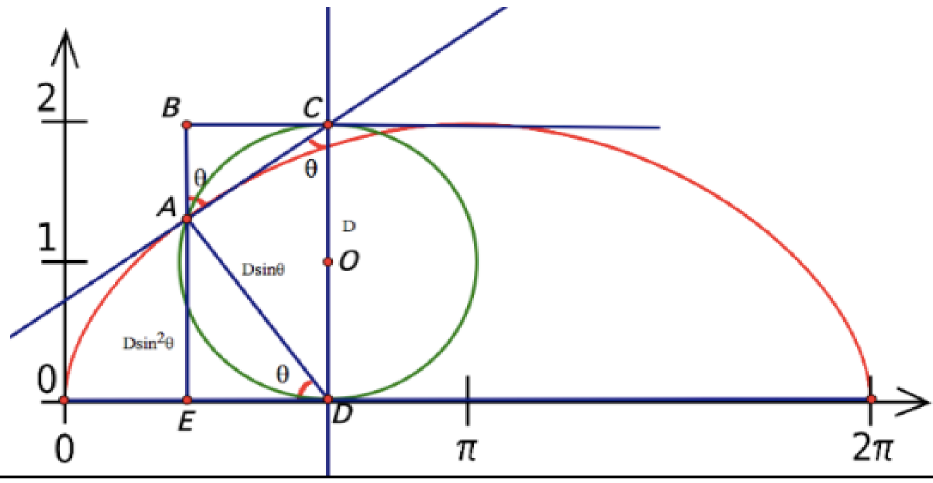
Diagram 3: The path of the object changes instantaneously every moment, with the change of slope dictated by Snell's Law. Using Snell's Law, we get that the sine of angle theta over the square root of the velocity is constant for every point on the curve. So, for what curve is this property true???

If we think of a falling object, the velocity of the object will always be proportional to the square-root of the distance fallen, more specifically $\sqrt{(2yg)}$ (I am not going to go over why this is true, but if you want to see for yourself think about finding the velocity knowing that the $\Delta PE=\Delta KE$). Essentially, what this means is that $\sin\theta/\sqrt{y}$ (where θ represents the angle the curve makes with the vertical at any given instant) is constant throughout the curve. Lo and behold, this is the description of a CYCLOID! This is the curve generated when you trace a point as a circle rolls along a line. To be fair, this does seem somewhat arbitrary; there does not appear to be any obvious explanation as to why the cycloid has this particular property. There is, however, a nice geometric proof of why $\sin\theta/\sqrt{y}$ is constant for cycloids.

In the diagram below, angle θ at BAC represents the “incident” angle that the vertical axis has with the curve at any given moment. Angles ABC and DEA are right, and DC is constructed so that is passes from the “rolling” point D on the circle to the other side of the circle creating a diameter (because line D is tangent to circle O, DE is perpendicular to the diameter). Knowing this allows us to do some angle chasing and find three other angles equal to θ. If we let the diameter constructed be some value D, than AD is equal to $D\sin\theta$ and AE is equal to $D\sin^2(\theta)$. Since AE is also our Δy, we can do some algebra to find:

$$y = D * \sin^2(\theta)$$
$$\sqrt{y} = \sqrt{(D)} * \sin(\theta)$$
$$1/\sqrt{D} = \sin(\theta)/\sqrt{y}$$

And because the diameter is constant throughout the circle, we know that $\sin(\theta)/\sqrt{y}$ is always constant, which meets the requirement for our Snell’s law curve! This is a beautiful explanation to one of the all-time famous math problems. The shortest time it takes from an object to role from two points on earth is a cycloid, which although does not seem obvious, has its roots in some very clean geometry. Cycloids also have a greater significance beyond their practical application; they demonstrate that even the most allusive and esoteric results in math have their foundation in eloquent, logical thinking.

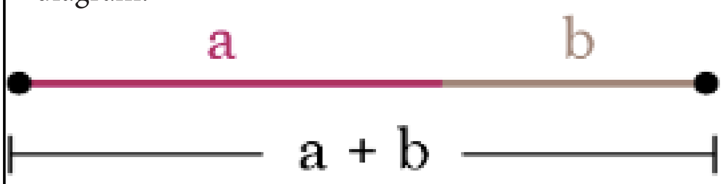


2018: A Golden Year?

By Alexis Fry

Although 2018 wasn't a great year for our country, it was a great year for math geeks! 2018 was the only year in the 21st century when the shorthand notation for the date, month/day/year, matched the first four digits of the golden ratio - January 6th, 2018 or 1/6/18. Most of you probably know that the value of the golden ratio, symbolized by the Greek letter phi (φ), is 1.618033988749895... or $(1 + \sqrt{5})/2$. But, what exactly is the golden ratio, and how does one derive its value? Nothing other than simple algebra can be used to reveal φ's true identity and the beautiful phenomena within.

We can find the golden ratio when we divide a line into two segments such that the ratio of the length of the long segment to the length of the short segment is equal to the ratio of the length of the whole segment to the length of the long segment. In other words, given a line of length a + b, where 'a' is the long segment, we need to find the value for a and b such that the ratio of a to b is equal to the ratio of a + b to a. This idea is illustrated in the following diagram:



When a and b satisfy the equation $a/b = (a + b)/a$, we have found the golden ratio for the line of length a + b.

As noted above, the golden ratio is symbolized by the Greek letter φ, so $\varphi = a/b = (a+b)/a$. And now, let's do math!

If: $\varphi = a/b = (a + b)/a$, then rewrite $(a + b)/a$ as $(a/a + b/a)$ which can be recast as $1 + 1/\varphi$ (since $1/\varphi = b/a$). Now, $\varphi = 1 + 1/\varphi$.

There are a few interesting properties given $\varphi = 1 + 1/\varphi$.
1. Let's take $\varphi = 1 + 1/\varphi$ and subtract 1 from both sides, so when we subtract 1 from both sides of this we get the multiplicative inverse of φ, neat!

$\varphi = 1 + 1/(1+(1/(1+(1/(1+\varphi)\ldots$ and we can continue to substitute φ for its value $1 + 1/\varphi$ forever! This is a recursive definition of a variable where it is defined in terms of itself.

But the question still remains: what is the value of φ? Let's use the quadratic equation to figure this one out.

Let's take phi $= 1 + 1/\varphi$ and multiply both sides by phi so that $\varphi^2 = \varphi + 1$. (A brief tangent: when we take the square root of $\varphi^2 = \varphi + 1$, we get $\varphi = \sqrt{(1 + \varphi)}$, and we can now uncover another recursive definition by substituting for φ!

$\varphi = \text{sqrt}(1 + \text{sqrt}(1+\text{sqrt}(1+\text{sqrt}(1+\ldots$

Anyways, let's go back to solving for φ. Return to the equation $\varphi^2 = \varphi + 1$ and rearrange to get $\varphi^2-\varphi-1 = 0$. Now, we can use the quadratic formula to solve for phi, where a = 1, b = -1 and c = -1. Additionally, we can negate the result of a negative solution, since we are determining the distance represented by segments a and b as noted above (and all distances are nonnegative.)

Using the quadratic formula, phi evaluates to $(1 + \sqrt{5})/2$ which is equal to 1.618033988... and that number looks quite familiar! Another interesting circumstance of φ is when we revisit $\varphi - 1 = 1/\varphi$, we can solve for $1/\varphi$ in our heads! $1/\varphi = 0.618033988$ (the inverse of φ is really the decimal remaining once we subtract 1 from it!)

The golden ration surrounds us, occuring in nature, art, architecture, the solar system and even DNA.

For example, we know that $\varphi = a/b = (a+b)/a$, so let's create a rectangle where the ratio of the width to the height is the golden ratio. Then, separate out a 'b by b' square so that the remaining distance of the side is a-b (see Figure 1 on the next page for a visual of this process.)

Wouldn't it be interesting if the b by a-b rectangle is also a golden ratio rectangle? Let's find the ratio of b to a-b.

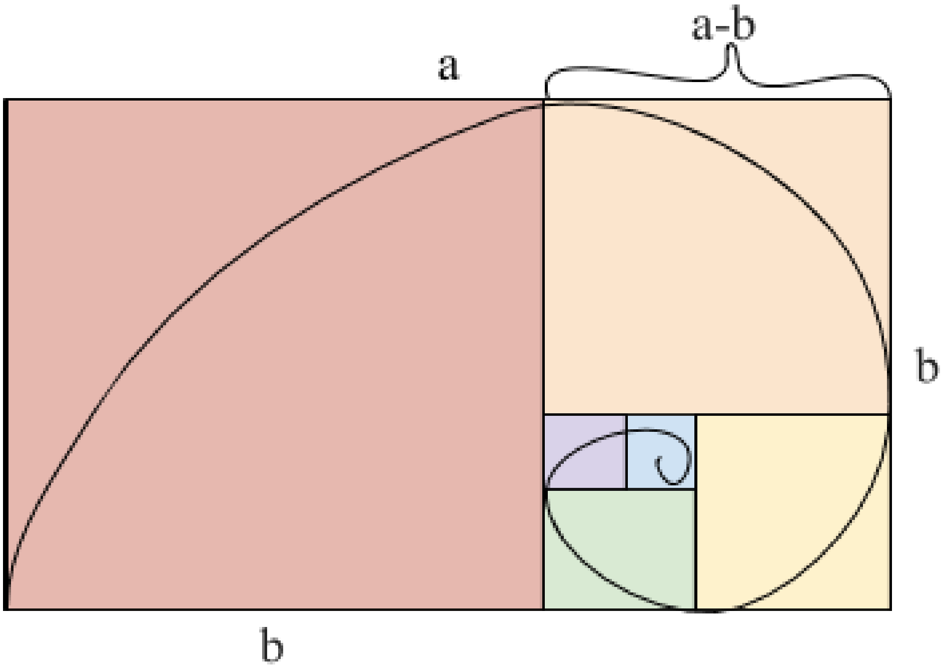
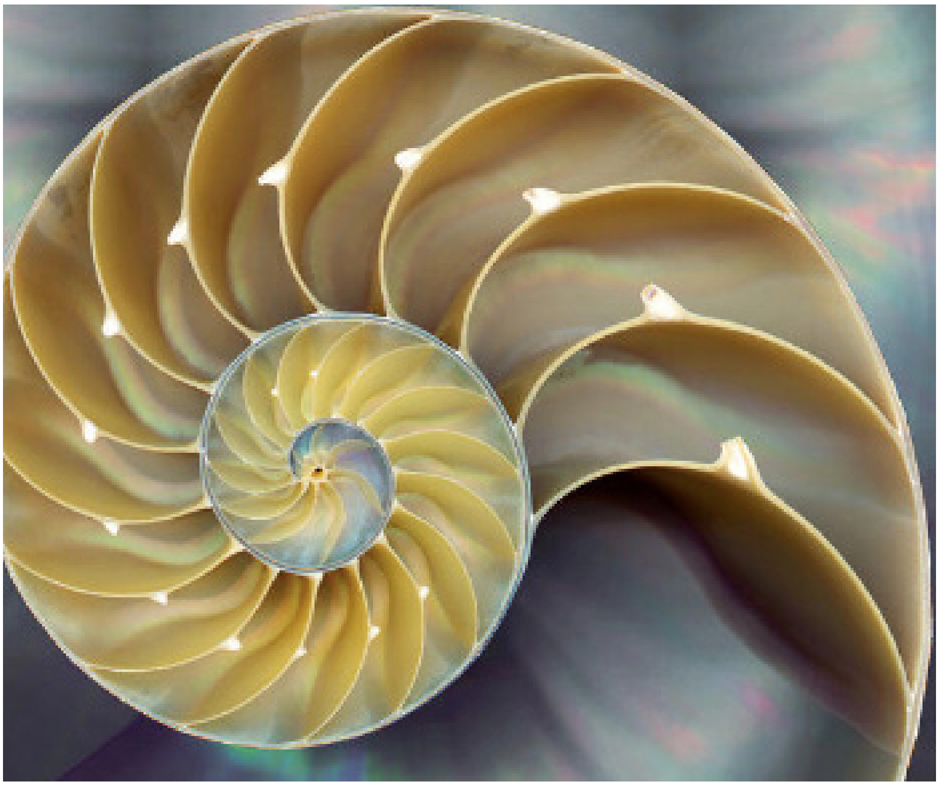


Figure 1: an interesting property of golden ratio rectangles, and the Golden Spiral

Finding the ratio of b to a-b:

b/(a-b) =	
1/((a-b)/b) =	(by rearranging)
1/((a/b)-1) =	(by expanding the fraction)
1/(φ-1) =	(since a/b = φ)
1/(1/φ) =	(since φ - 1 = 1/φ)
φ.	(by rearranging)

Right: the nautilus shell is an example of where we can find golden spirals in nature!



Non-Euclidean Geometry

By Danielle Paulson

Non-Euclidean geometry begins with the proposal of Euclidean geometry by Euclid, a geometer from Alexandria during the 3rd century BC. He declared five basic postulates:

1. A straight line segment can be drawn joining any two given points.

2. Any straight line segment can be extended indefinitely in a straight line.

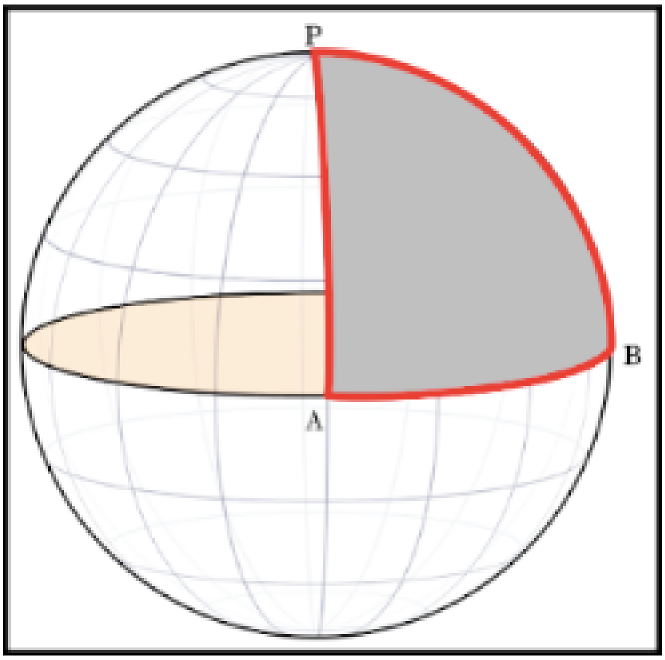
3. Given any straight line segment, a circle can be drawn having the segment as the radius and one endpoint as the center.

4. All right angles are congruent.

5. Given any straight line and a point not on it, there exists only one straight line which passes through that point and is parallel to the first line.

Euclid's fifth postulate became known as the parallel postulate, and although attempted by many mathematicians, cannot be proven as a theorem. In 1823, Janos Bolyai and Nikolai Lobachevsky realized that "non-Euclidean" geometries could be created in which the parallel postulate does not hold. If the phrase "exists only one straight line" of the fifth postulate is replaced by "exists no line" or "exists at least two lines," the postulate describes two other types of geometries known as spherical and hyperbolic geometries, respectively. In this case, we're going to focus on spherical geometry.

Spherical geometry refers to the study of figures on the surface of a sphere. The equivalent of a line in spherical geometry is not defined in the sense of a "straight line" as in Euclidean geometry, but rather in the sense of the shortest distance between two points, which is along the arc of a great circle. A great circle is a circle on the surface of a sphere that lies in a plane passing through the sphere's center. Thus, a great circle is the largest possible circle that can be drawn around a sphere. Because all lines are great circles, any two lines will always meet at two points, suggesting that parallel lines don't exist in spherical geometry. This explains why Euclid's fifth postulate doesn't hold true in spherical geometry.



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A spherical triangle is a triangle that is mapped onto a sphere and bound by three great circles. The angles of a planar triangle sum to 180°; however, the angles of a spherical triangle have a sum that is greater than 180° and less than 540°. Consider figure 1 with a spherical triangle on the surface of the Earth.

Figure 1

To trace the sides of the triangle, you would start at the north pole at P, travel south to the equator at A, turn 90°, travel a quarter of the way around the equator to B, turn 90°, and travel back to the north pole at P. The sum of the three angles in this spherical triangle is 270°!

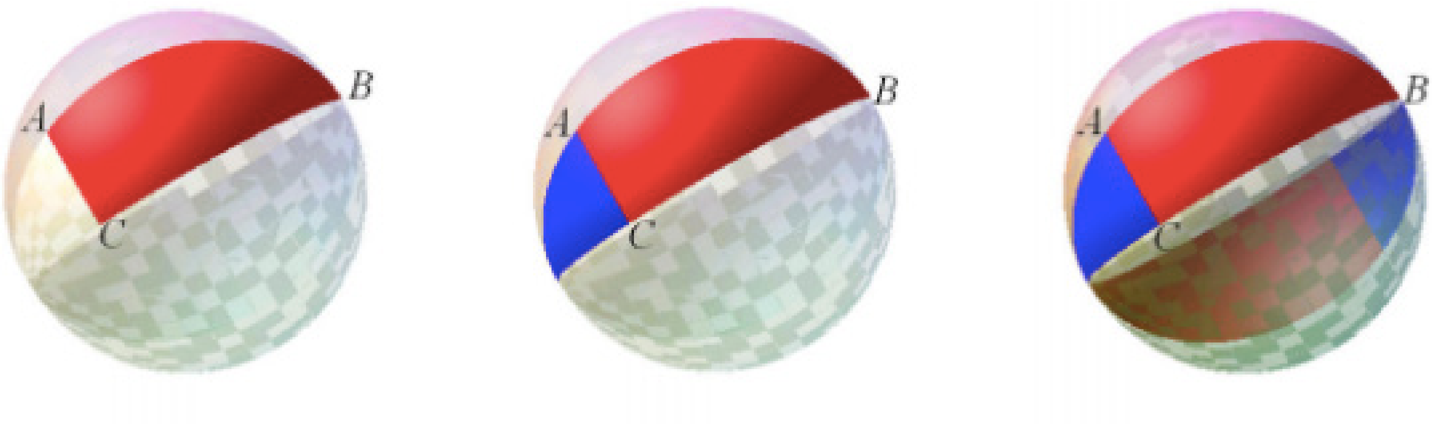
The minimum measure of angle APB is ever so slightly greater than 0°.

In this case, the sum of the three angles of spherical triangle APB would be slightly greater than 180°. The maximum measure of angle APB is ever so slightly less than 360°. In this case, the sum of the three angles would be slightly less than 540°. Thus, the sum of the three angles of a spherical triangle must be greater than 180° and less than 540°. Spherical excess refers to the difference between the sum of the angles A, B, and P and 180°. In other words, it determines how much the sum of the angles deviates from 180°. Since the sum of the angles must be greater than 180° and less than 540°, the spherical excess must be greater than 0° and less than 360°. The equation for spherical excess represented by E is shown below.

$$E = A + B + P - 180$$

Girard's theorem describes the area of a spherical triangle drawn on the surface of a sphere of radius r and having angles A, B, and C. First, it is important to establish how a spherical triangle relates to 'lunes.' Two great circles intersect at two points on the surface of a sphere. These points are antipodal, and they form a crescent shape on the surface of the sphere called a lune. A lune stretches exactly halfway around the sphere, from one pole to another (not necessarily the North and South poles). A third great circle will cut this lune into two spherical triangles. Notice that in the figure at (a), spherical triangle ABC is formed by the great circles QAB, through A and B, QCB, through C and B, and QAC, through A and C. The figure at (b) shows the great circle QAC cutting the line formed by the great circles QAB and QCB into two spherical triangles depicted in red and blue. The figure at (c) shows a second pair of these spherical triangles on the 'back' of the sphere. These spherical triangles are congruent to the first pair and located antipodally.

(See the diagram below)



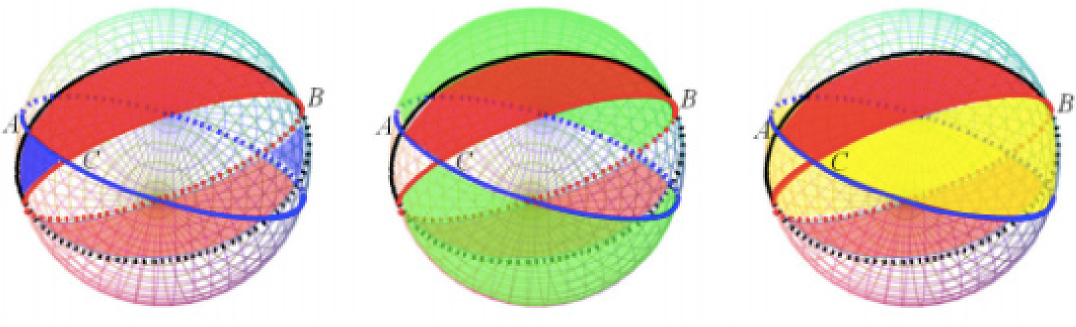
(a) (b) (c)

Now that we have all of the basic ideas for Girard's theorem, consider a sphere of radius r with a surface area of $4\pi r^2$. If two great circles meet at a lunar angle of θ where 0<θ<2, the proportion of the surface of the sphere occupied by the lune is θ/(2π). Thus, we have that the

Area of a lune with a lunar angle of $\theta = (\theta/(2\pi))*(4\pi r^2) = 2r^2\theta$

Three intersecting great circles give us three antipodal pairs of lunes all overlapping at triangle ABC. In figure (a), the original pair of lunes is shown with a lunar angle at B. Figure (b) shows the second pair of lunes with a lunar angle at C. Finally, figure (c) shows the third pair of lunes with a lunar angle at A. Let LA, LB and LC represent the areas of the lunes formed by angles A, B, and C respectively. Recall that each of these three lunes has its own antipodal duplicate. Let Trepresent the area of triangle ABC, which also has its own antipodal duplicate.

(Article continued on page 21)

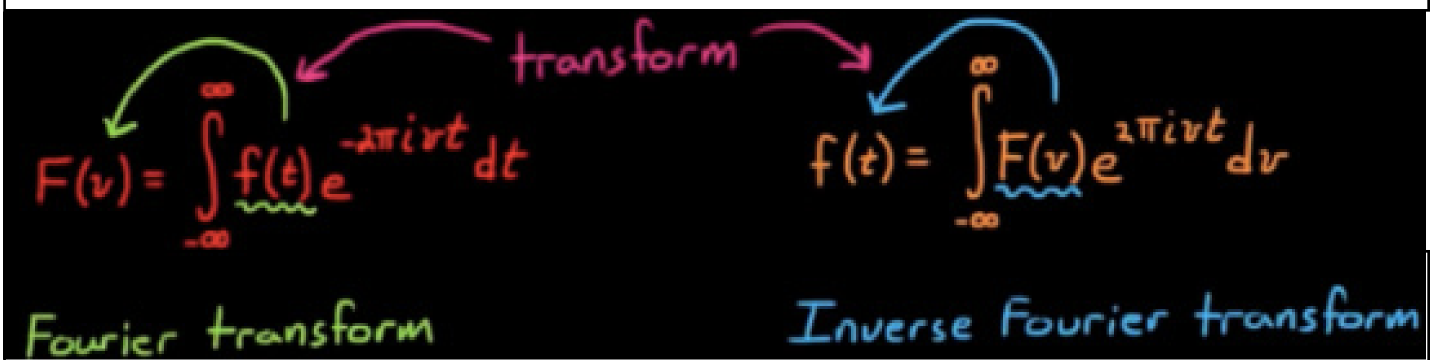


(a) (b) (c)

Intro to the Fourier Transform

By Mandy Liu

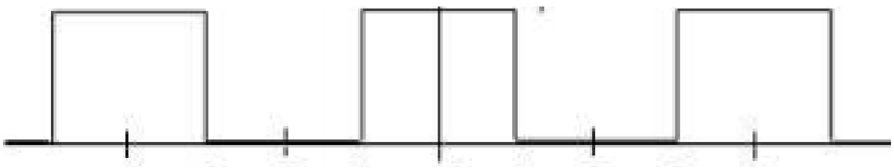
Almost everything can be described by means of a waveform–a variable that fluctuates with time. And all waveforms are actually just the sum of sinusoids, or sine waves, of various frequencies.
The Fourier Transform, named after the French mathematician and physicist Joseph Fourier, breaks down a waveform into sinusoids. In order to understand that a little bit better, here's a metaphor:
"What does the Fourier Transform do? Given a smoothie, it finds the recipe."
"How?"
"Run the smoothie through filters to extract each ingredient."
"Why?"
"Recipes are easier to analyze, compare, and modify than the smoothie itself."
"How do we get the smoothie back?"
"Blend the ingredients."
The fundamental idea of the Fourier Transform is that adding up different sinusoids gives you functions that look very different - for example, we know what sin(x) and sin(2x) look like, but what if you added them together? (Try adding the functions on Desmos.) And, given only a graph like that, is there a way to decomose it and figure out what the original pure sinusoids were?



The first equation is the Fourier Transform. The yellow arrow shows that the first equation turns a function of time (t) into a function of frequency (v). And, the blue arrow in the second equation shows the vice versa.

The Fourier transform changes the information in the time domain f(t) into information in the frequency domain f(v). The data in the two domains look different, but they both represent the same information. The Fourier transform also expresses your output or function in the frequency domain using cosines and sines. Sinusoids are described by their amplitude, frequency and phase. These three basic parameters are everything we need to know in order to describe the signal. We can plot the amplitude and phase at every frequency and that would be the frequency domain representation of the signal. Sinusoids are great because they are the only waveform that doesn't change shape, which is a property that makes sinusoids easier to work with.

Here is an example of a problem that can be solved with these ideas. Let's take a look at this rectangular pulse, X(t):



(Assume that this function continues infinitely in both directions)

First, come up with a piecewise linear function that represents X(t). Put your answer into the equations given above for the Fourier Transform; if you know a bit of calculus, try solving the integral to see what you get! Your answer will represent the composition of sinusoids that, when combined, give back the original function.

Brouwer's Fixed Point Theorem

By Dora Woodruff

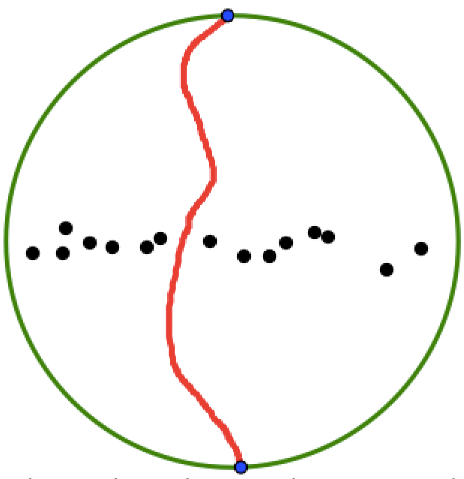
Brouwer's Fixed Point Theorem is one of the most beautiful theorems in topology. Not only is it an interesting theorem in and of itself, but it can also be used in surprising ways to prove seemingly unrelated statements. The theorem itself essentially says that if you have a continuous function that takes points from a disk (the inside of a circle, including the circle's perimeter) to other points in the disk, that function always has to have a fixed point, that is, a point whose position does not change under the function. An example of such a function is one that takes every point in the disk and rotates it about the disk's center by a fixed angle; in this case, the fixed point is the center. (This theorem also applies to n-dimensional spheres as well as circles, but in this article, we will mostly just focus on circles.) There are many ways to interpret this theorem physically - for example, it means that if you shake a glass of water, then no matter how you shake it, there will always be at least one water molecule that ends up in exactly the same place as it started, because a water bottle is topologically the same as a sphere, and shaking it sends points in the water bottle to other points in the water bottle continuously, so there must be a fixed point.
So, how do we prove this theorem? There are several proofs, but this article will cover two: one which uses an analytic approach and one which uses a rather surprising, combinatorial approach.

Proof 1:

Imagine that the disk is in the Cartesian plane, that is to say, every point on the disk has an x and y coordinate. We want to prove that there is a point in this disk whose x and y coordinates both stay the same when a continuous function f sends each point to another point of the disk. First, we will define another function, which we'll call F, on the circle. This function will send points to two dimensional coordinates. If the original coordinates of point p are (x, y) and the coordinates under f become (f(x), f(y)), then F(p) = (x-f(x), y-f(y)). Since f is a continuous function, Fmust be as well. We want to prove that there is a point whose output of F is (0,0).

Next, consider the point in the circle that, before f is applied, has the highest y coordinate (we will call this point a.) This point's y coordinate cannot increase, so the x coordinate of F(a) must be positive. Similarly, if point bis the point that starts out with the lowest y coordinate, the x coordinate of F(b) must be negative. If you travel from ato balong any continuous path, since Fvaries continuously, by the intermediate value theorem, there must be a point c along that path such that the x coordinate of F(c) is 0.

In fact, there must be a continuous path of such points (whose x coordinate of F is 0) that stretches from one side of the disk to the other (see the diagram below.) Because, if there was not such a path, then we could draw a curve (like the curve drawn in the diagram) that goes from point ato point bbut does not contain any of those points, and we proved above that every such curve must contain one. (Very technically, for this step to work, we need to use the Jordan Curve Theorem here. This theorem says that a closed 'Jordan' curve, which is a non-intersecting continuous closed loop in the plane, divides the plane into an exterior and an interior, such that any path from an exterior point to an interior point intersects the Jordan curve somewhere. Although this theorem seems really intuitive and obvious, it is actually very difficult to prove formally and requires a lot of heavy machinery, so we won't prove it here.)



In the diagram above, the red curve shows a possible path from point a to point b that does not have a point whose y coordinate does not change (represented by black points.) This diagram shows that the black points must form a continuous path, by the Jordan Curve Theorem

Now, consider the endpoints of this path. We'll call them cand d. WLOG, the y coordinate of F(c)is greater than or equal to the y coordinate of F(d).So, again by the intermediate value theorem and the reasoning above, there is a path of points 'separating' cand d whose y coordinates F are 0. These two paths, of points whose x coordinates are 0 and whose y coordinates are 0, must intersect somewhere since they are both continuous. This point of intersection is the fixed point we are looking for; since both its x and y coordinates of Fare 0, there is no difference between its starting location and its final location under f. (This proof as it's written only works for two-dimensional circles, but it can be extended to higher dimensions. This is left as an exercise for the reader!)

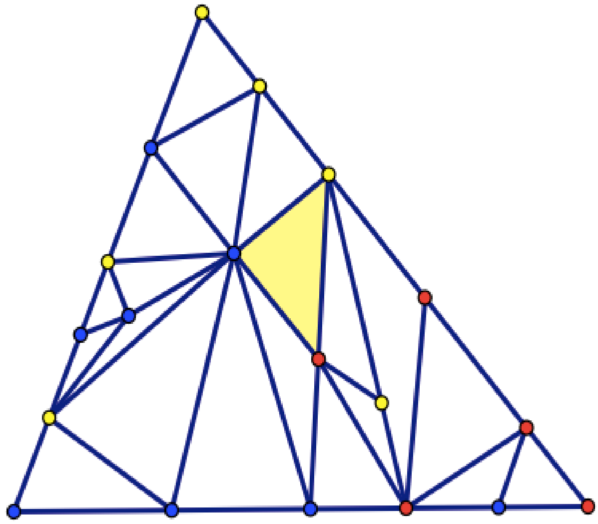
Something interesting about this proof is that it is similar to a different, seemingly unrelated math problem, called the Hex Problem. Hex is a game, played on a board made of tessellating hexagons (usually arranged in an 11 by 11 rhombus), with two players: one who colors hexagons red, and one who marks hexagons blue. The red (or blue) player wins when there is a connected path of red (or blue) hexagons from one side of the rhombus to an opposite side. The Hex Problem is to prove that if every hexagon is filled in, the game cannot possibly end in a tie. It turns out that the Hex Theorem is logically equivalent to Brouwer's Fixed Point Theorem (although we won't prove this.)

There is an interesting, informal way to see why the Hex Problem should be true: once the game has been played, cut out all the blue hexagons. Either the left edge of the rhombus and the right edge of the rhombus fall apart from each other or they do not. If they do not, there must be a path of red hexagons connecting them - in which case, red has won. If they do, then there must be a path of blue hexagons from the top to the bottom of the rhombus that separated the two edges, in which case blue wins. Either way, it is never a tie.

Proof 2

The second proof uses a surprising lemma. First, say you have triangulated a triangle. Now, we're going to use three colors to color the points in a specific way: the three vertices of the triangle are all colored differently, and then any points on the segment between one vertex and another vertex are colored either with the color of the first vertex or the second vertex.

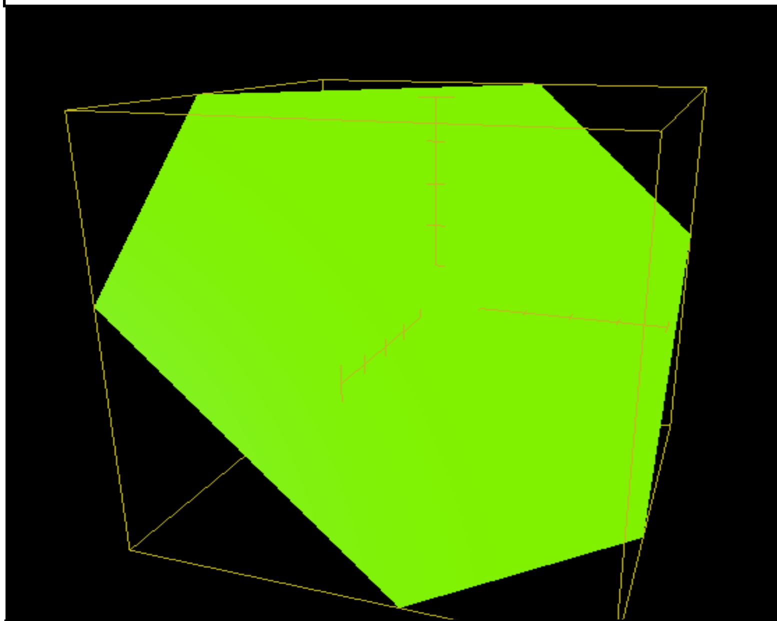
The points on the inside of the triangulation can be any of these three colors. The lemma then says that there must be a triangle in the triangulation whose vertices are also three different colors (see the diagram below.) Proving this lemma is not too hard; if you want to try proving it yourself, try proving that there must be an odd number of such triangles in the triangulation, rather than proving that there are not 0, and use the fact that on each edge of the triangle, there is an odd number of segments whose vertices are colored differently (can you see why?)

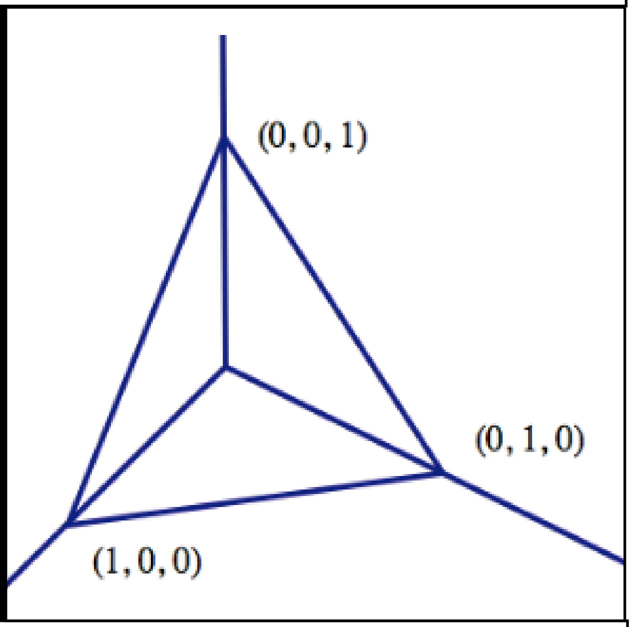


Above: one possible coloring of a triangulation that matches all of our requirements!

This lemma might seem completely unrelated to Brouwer's Fixed Point Theorem, but it is actually quite simple to complete the proof from here. First of all, it doesn't matter whether we prove the theorem for a triangle or for a circle since triangles and circles are topologically the same. Now, we're going to use 3D geometry. If we consider the plane $x+y+z=1$, if x, y and z are all positive, this equation gives an equilateral triangle (see the diagram below.) We will consider a continuous function from the triangle to itself, and also triangulate the triangle in some random way. Now, we want to 'color' all the points in the triangulation in a certain way, using the function. Let's say we have a point in the triangle whose coordinates are (a, b, c) and whose coordinates under the function are (f(a), f(b), f(c)). Since $a+b+c=1=f(a)+f(b)+f(c)$(since both the point and its image are in the plane $x+y+z=1$), which means that either $a<f(a)$, $b<f(b)$ or $c<f(c)$(unless they are all equal, in which case, there is definitely a fixed point.)

So, if the first time a coordinate is less than or equal to its image is the first coordinate, we'll 'color' that point with a 1 (if the first time is with the second or third coordinate, the point is colored with a 2 or 3.) For example, if the point (0.1, 0.3, 0.6)is sent to (0.05, 0.2, 0.75), 0.1>0.05 and 0.3>0.2,so the first time that a coordinate is less than its image is the third coordinate, 0.6. Thus, this point would be colored with a 3.





Above: if you look at only the positive x, positive y, positive z 'quadrant' of 3D space, the equation x + y + z = 1 looks like an equilateral triangle, intersecting the axes at (0, 0, 1), (1, 0, 0) and (0, 1, 0). The image to the right shows what the plane x + y + z = 1 looks like in only that quadrant, the image on the left shows the plane as a whole (where the three yellow axes in the middle of the cube are the x, y and z axes.)

This coloring satisfies the requirements for the lemma - that the three vertices of the triangle are colored with 1, 2 and 3, and that points on the segment between one of the triangle's vertices and another vertex are colored with one of those two vertices' colors (can you see why this is true?) So, there must be a triangle in the triangulation whose vertices are all colored differently. Now comes the odd step: you take this triangle, triangulate it again, and get another smaller triangle whose vertices are colored differently (there must be one - can you see why?) Continue this process infinitely many times. The area of the triangle continues to get smaller, converging to one point (again, to prove this statement more formally, you have to use a couple of techniques outside the scope of this article, but it intuitively it makes sense). The point at which the three differently colored vertices of these triangles converge at therefore must exist. I claim that this point is, in fact, our fixed point: according to the way we colored the points in the triangle, the x, y and z coordinates are all less than or equal to the coordinates of the point this point is sent to.

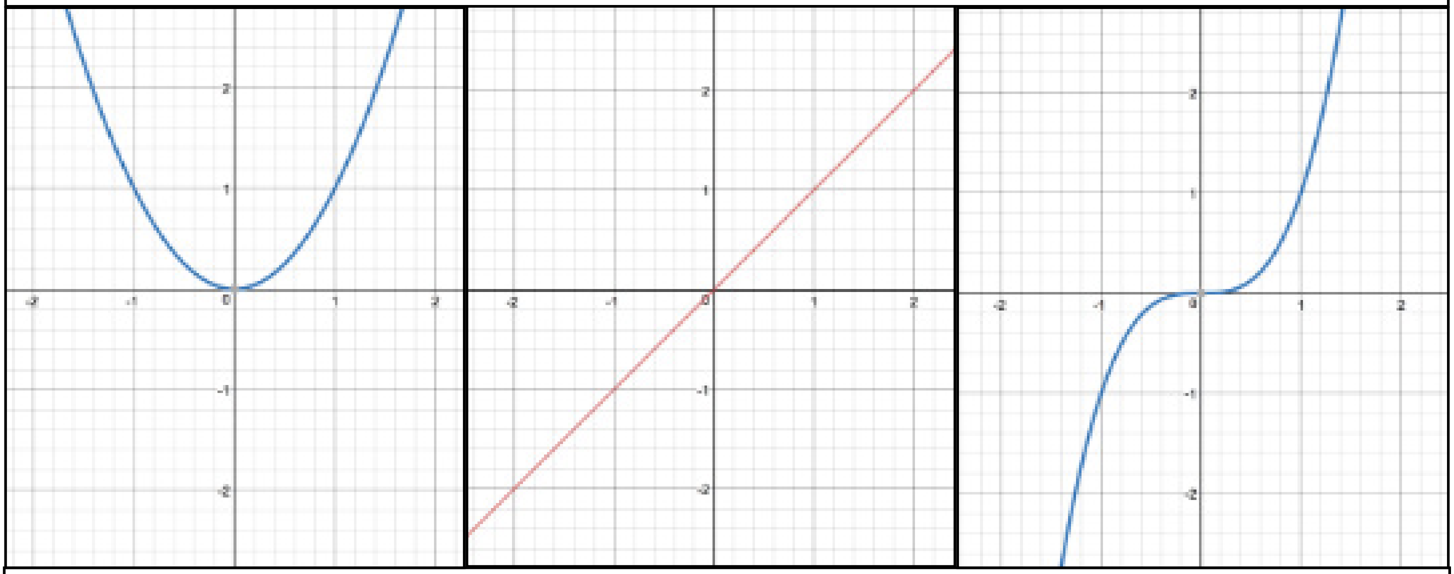
None of the first coordinates can be less than the second coordinates, , because if that were true, then the sum of the first coordinates would not equal the sum of the second coordinates, and both of the sums should be 1 since the plane the triangle is in is x + y + z = 1. Therefore, the coordinates of the point and the point's image are the exact same, which is the definition of a fixed point.

Brouwer's Fixed Point Theorem is a very important, well-known theorem in topology, and it can be used to prove many other results (such as Borsuk Ulam and the Ham Sandwich Lemma - articles explaining these two theorems, which are just as beautiful as Brouwer's Fixed Point Theorem, will be in the next issue, to keep an eye out for them!) Brouwer's Fixed Point Theorem is even used in economics and game theory, to prove theorems about the Nash Equilibrium. It also has many other proofs - to read through some of them, look up the citations listed in the citations section on page 22. All in all, it is a fascinating theorem, with many beautiful proofs and surprising, useful applications.

Cool Graphs!

By Henry Bloom

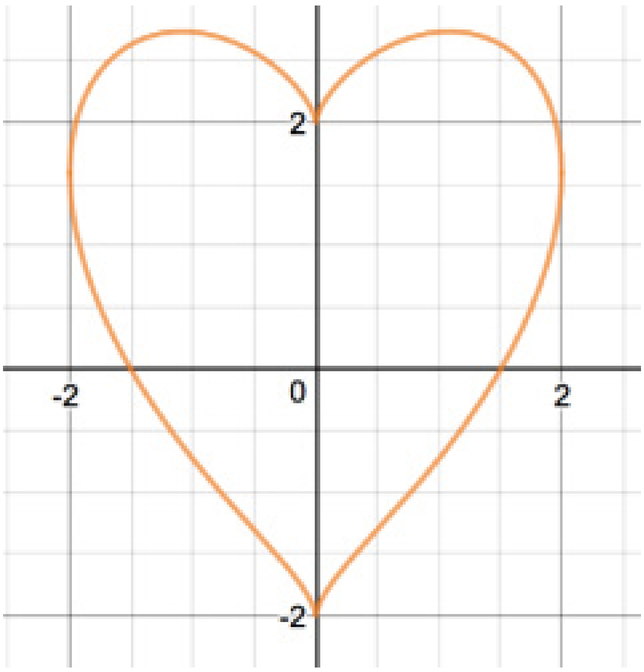
To be blunt, I think that graphing is really cool. There are just so many possibilities and many of them create interesting shapes, which can be intriguing to people who love geometry, algebra, or both. Some graphs are quite simple, such as y=x, y=x^2, and y=x^3, shown below.



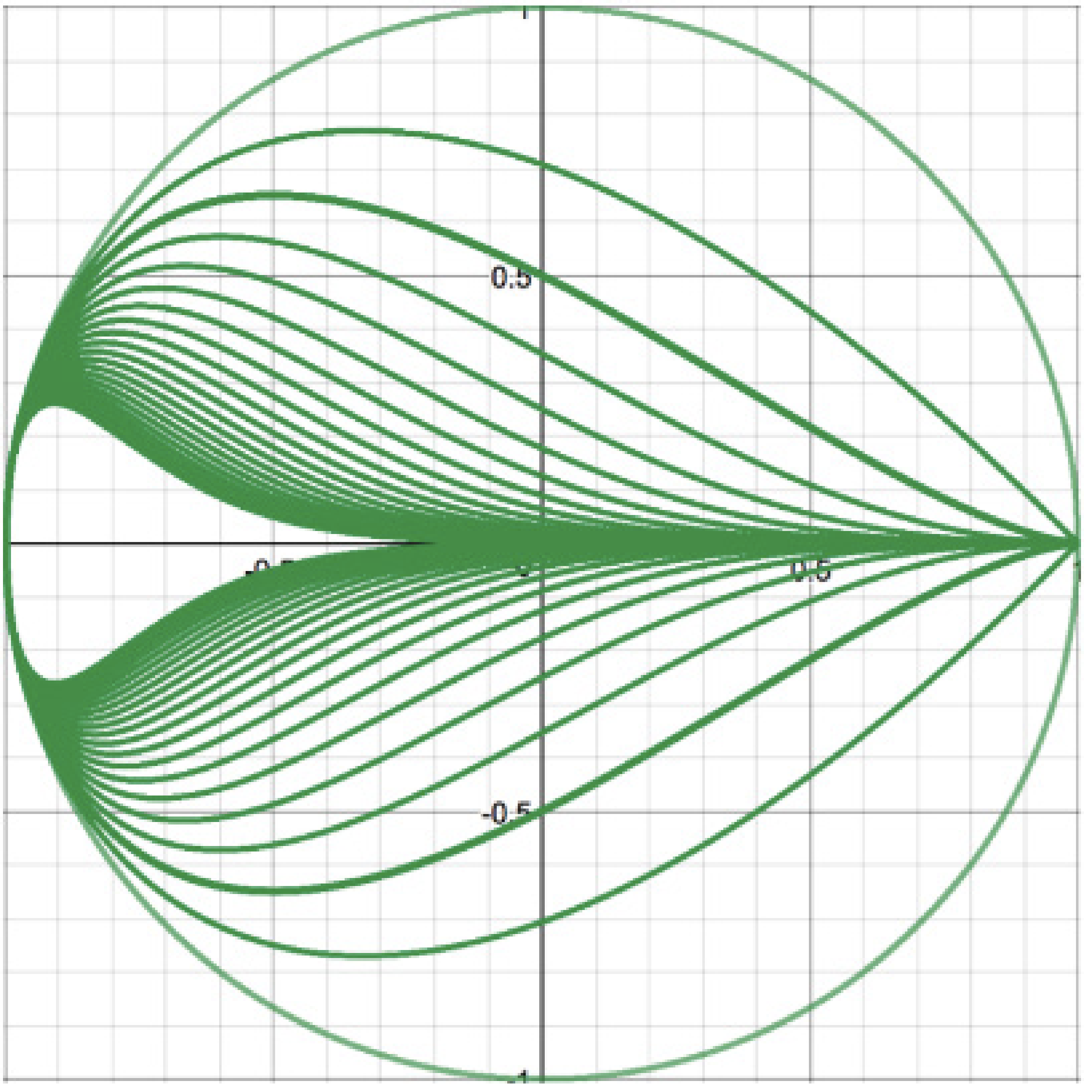
Some graphs, however, are much more interesting. Using trigonometry is a strong tool in creating these graphs as it brings many different functions that can alter our graphs' forms. Particularly, sine and cosine are used in many of these graphs, and the graphs which utilize them are very cool. Desmos, a website which is based on graphing, has many neat graphs on their website, (found on https://www.desmos.com/calculator/h68atkr5ic) and I will showcase a few of them.

As hearts are quite similar in shape to circles, the formula for a heart that is used in Desmos is similar to the formula for a circle, $(x-h)^2+(y-k)^2=r^2$. The formula they use is $x^2+(y-3\sqrt{(x)}2)^2 = a$. The y-intercepts of the heart will be equivalent to $\pm\sqrt{a}$. For example, when a = 4, the graph looks like this:

Left: the graph of a heart on Desmos

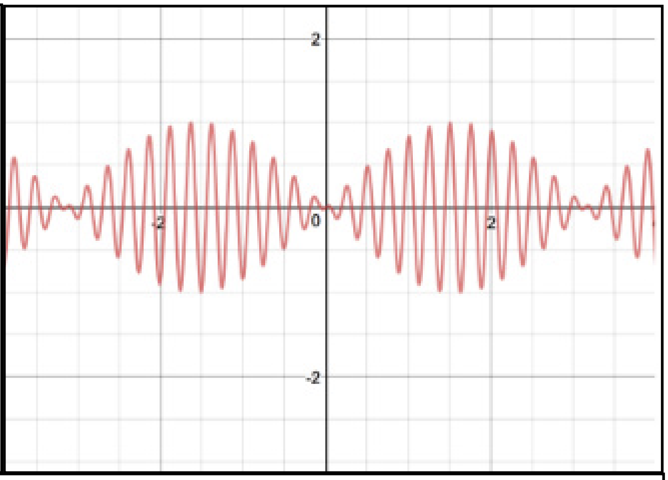


Another graph Desmos showcases, called “The Leaf,” uses sine and cosine to graph multiple shapes which come together into a shape that looks like a leaf:



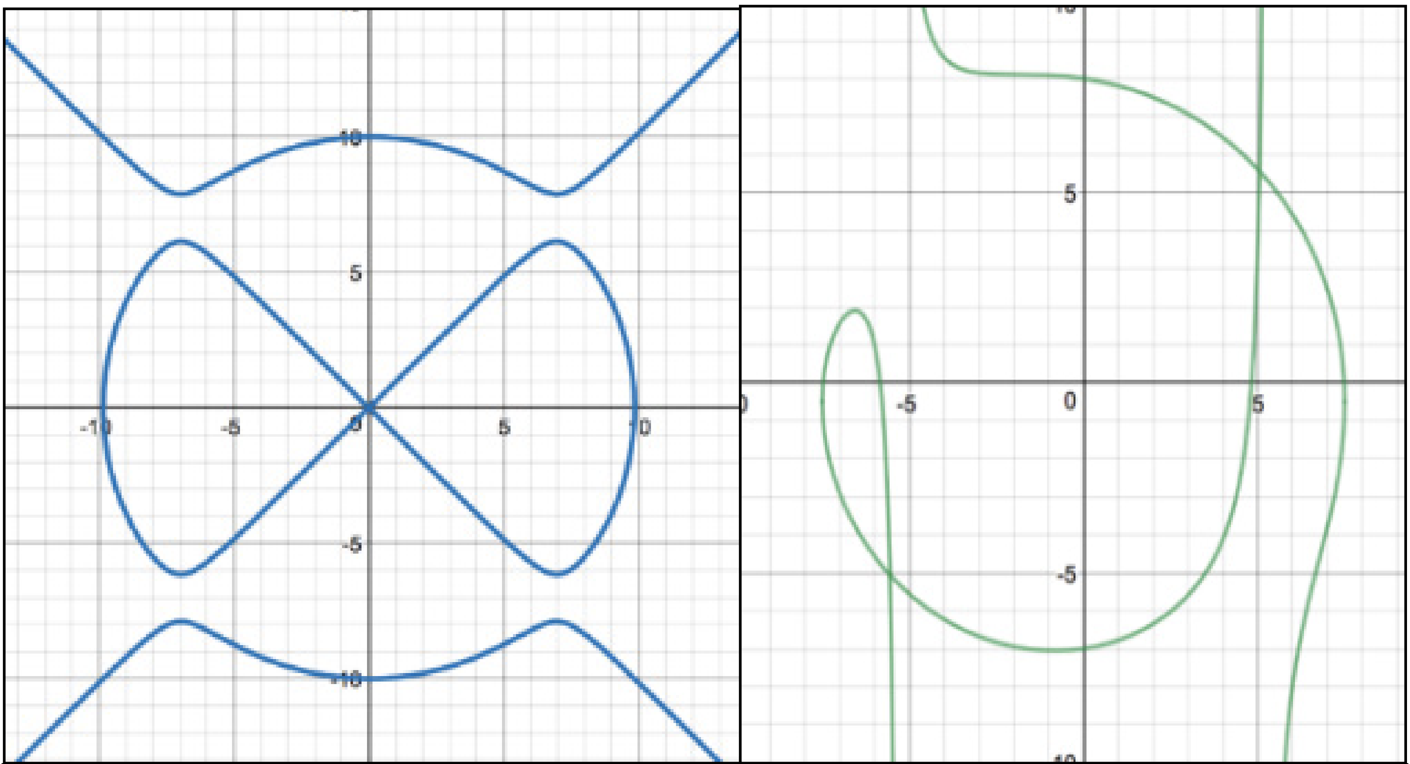
The only difference between each shape is that they are multiplied by different powers of sin(t/2), which changes the way that they curve.

The next graph Desmos shows off is one they call “Sinusoidal-Function-Seption,” which shows a bunch of oval-like shapes formed by lines which consistently go up and down at differing heights. The equation takes the form sin(ax)cos(bx)=y, where a controls the frequency of the ovals and b controls the frequency of the lines which create the ovals. When a=1 and b=25, the graph looks like this:



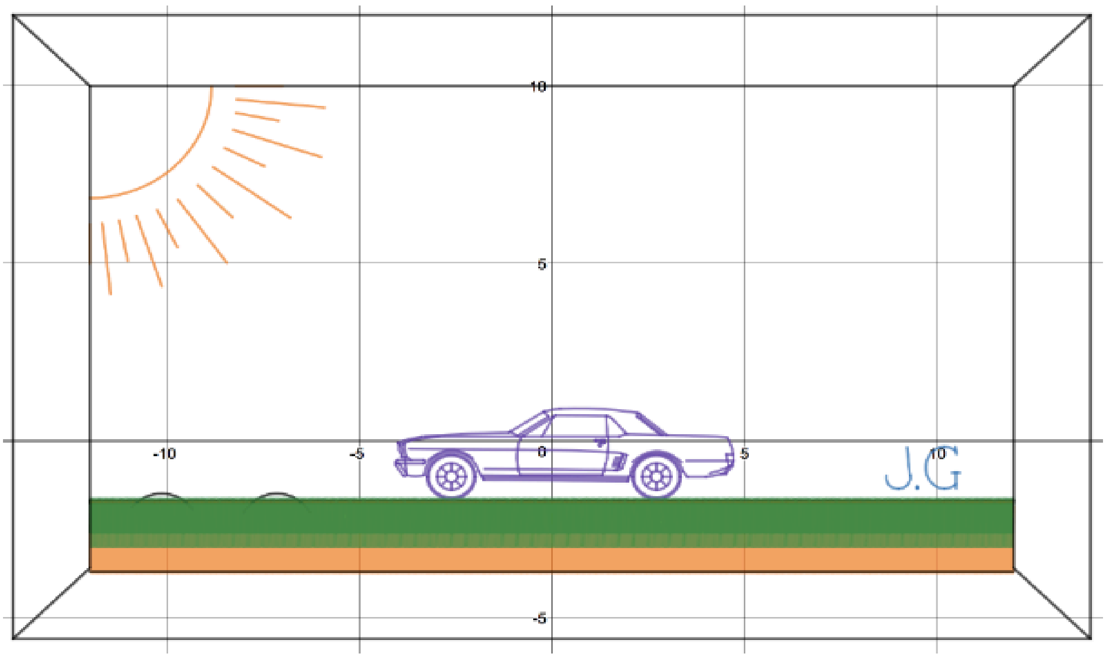
There are a few other graphs Desmos shows here, but I am going to move on to a few cool curves which we can look at. While most people know what a circle, oval, and parabola are, there are many other curves, some of which can be quite interesting to graph.

First, a curve called “The Devil’s Curve,” (which isn’t a fluid curve), expressed by the equation $y^4-x^4+ay^2+bx^2 = 0$, creates an unusual graph. Simply watching the graph change when the values for a and b change is very interesting. When a=-100 and b=97, the graph looks like this: *(graph on the next page)*

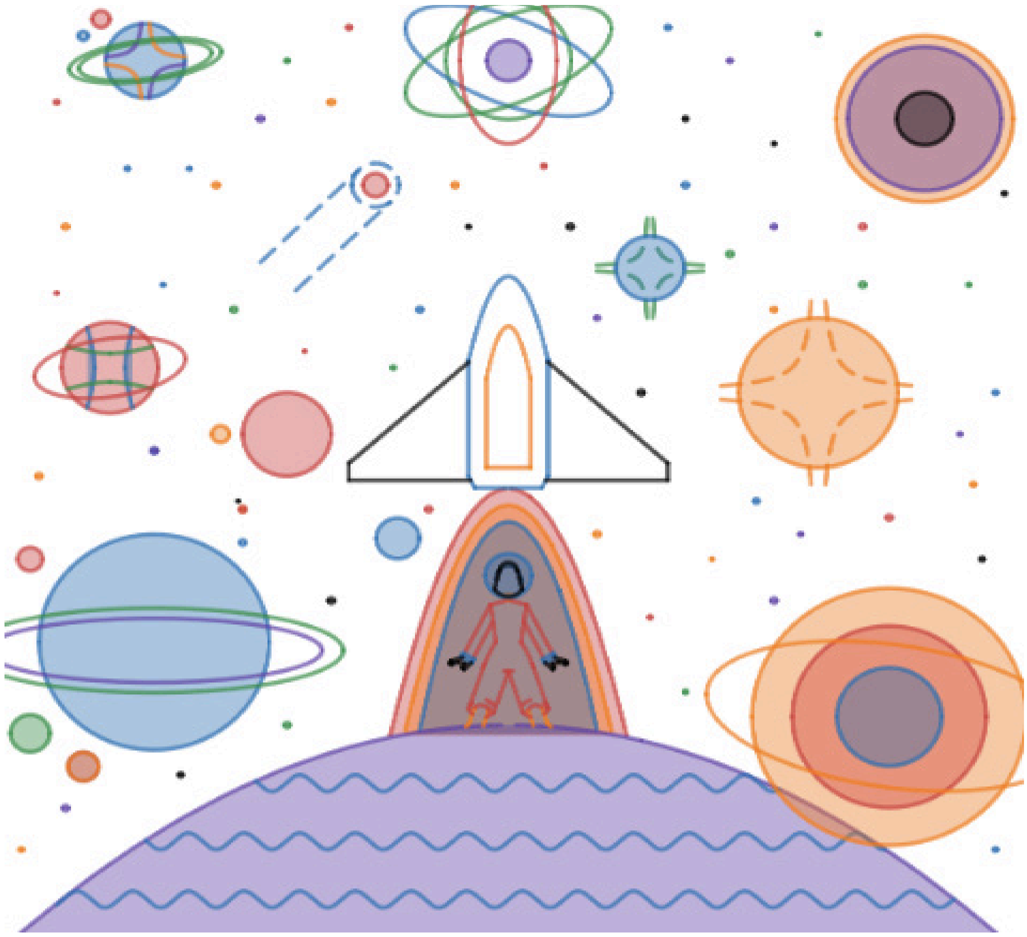


Left: The Devil's Curve. Right: one of Dürer's Shell Curves

Another curve, called Dürer's Shell Curves, expressed $(x^2+xy+ax-b^2)^2=(b^2-x^2)(x-y+a)^2$, forms a very cool graph that changes a lot when the values for a and b change. When a=0.5 and b=7.5, the graph looks like the image above. There are many more curves, many of which are included on a list found here, on http://www-groups.dcs.st-and.ac.uk/~history/Curves/Curves.html; however, I will not be looking at any more curves and will not return to Desmos to look at a few more cool graphs. Desmos has a list of interesting graphs which showcase mathematical ideas, (https://www.desmos.com/math), which I recommend looking at, however I will showcase the Desmos list of creative art people have graphed (https://www.desmos.com/art). This art is incredible, as people have managed to create recognizable forms that are constructed by graphs that the artists needed to fit together perfectly. It is incredibly to look at some of these and realize that the artists needed to find the exact numbers to input as coefficients and for other values in their graphs, and the ability they had to figure out what graphs they needed to use to create their forms. I will include a few of these and mention how many separate equations, variables, or other lines in Desmos were necessary to create the images:



Mustang, by Justin Gauthier (155 lines of equations)



Graphing Poster, by Coffee Tiramisu (215 lines of equations)

Snow Globe, by Lisa Wermers (1285 lines of equations)

To conclude, while there are many people who think graphing is just a simple, boring tool in science and math, it is an incredibly complex topic with endless possibilities for artistic purposes and just as an interesting topic.

Quantum Computing

By Ashley Dai

From graphing calculators to smartphones to computers calculating rocket trajectories at NASA, all classical computing systems use pieces of data called 'bits'. These bits exist as either 0s or 1s- up or down, on or off, true or false- stored in transistors using electrical currents; when current flows through the transistor, the transistor stores a '1' bit. Classical computers take a string of bits and break up complicated algorithms into simple logical operations- AND, OR, NOT- to arrive at a final answer. These operations, called logic gates, are combinations of smaller gates known as transistors. A transistor can either block or pass information by closing or opening a circuit to allow for electron flow. In the past 50 years, the size of transistors has decreased exponentially, now approaching the size of atoms. This advancement has allowed us to build smaller computer chips with more transistors. But when transistors become too small, they become effectively useless, as electrons can simply flow through the transistor through a process called quantum tunnelling.

Quantum computing uses the principles of quantum mechanics to circumvent this issue. A branch of physics, quantum mechanics studies the nature of atoms and subatomic particles, which exhibit properties such as wave-particle duality that cannot be explained by classical Newtonian physics. Quantum computers use the concept of superposition to store data in qubits ("CUE-bits"). Unlike traditional bits, qubits behave like spinning coins. They can be represented as two-dimensional complex vectors, denoted by $|\Phi> = a\,|0> + b\,|1>$ with $|0>$ and $|1>$ as base vectors and a and b as complex scalars such that $|a|^2+|b|^2=1$. While being 'spun' in isolation in the computer's memory, qubits can exist in an infinite number of spin directions at the same time and its value cannot be determined. A system of 6 classical bits can be in only one of 2^6 possible combinations. In a system of 6 qubits, these 2^6 possible states exist at the same time. The state space of a quantum computer, which represents all of its possible states, is denoted by the tensor product ( ) of the individual qubits. The number of possible states increases exponentially; a system of 20 qubits can be in 2^20, or 1,048,576 states simultaneously. When the computer measures the value of the qubit, however, the qubit collapses into either 'heads' or 'tails'- 0 or 1- and the system collapses into one of its possible states.

In multi-qubit systems, the quantum theory of entanglement dictates that entangled qubits will behave in perfect correlation even if separated by long distances. In measuring the properties of one entangled qubit, the properties of the other entangled qubits can be deduced as well. Qubits can be manipulated using quantum gates, which can both change the superposition of and entangle qubits. After passing through a series of quantum gates, qubits can be measured and the value of the system determined.

Since the steps of a quantum computer cannot be checked, can its result be verified by a classical computer? With problems such as the factoring of a large number, the result of a quantum computation can be checked simply by passing it through a classical computer. Solutions to other problems, however, are more difficult to verify. More importantly, how can we know if a quantum computer has done anything quantum at all? For years, quantum computing scientists had struggled with this challenge, known as quantum verification. Several relaxations of the problem showed that verification was possible with two quantum computers; one model allowed a classical computer to work with a smaller quantum computer to verify a general quantum computer, while another gave the verifying classical computer access to two quantum computers solving the same problem.

No one was able to solve the problem using the original model until 2017, when a graduate student at UC Berkeley named Urmila Mahadev developed a verification protocol based on post-quantum cryptography requiring just one quantum computer. In her model, the classical computer uses two-to-one trapdoor claw-free functions to control the computations of the quantum computer. The trapdoor allows the system, given y, to output x0 and x1 such that $f(x_0) = f(x_1) = y$, while claw-free refers to the quality of the function that, without a cryptographic key, makes it difficult to find a claw (x_0, x_1) such that $f(x_0) = f(x_1)$. The classical computer in Mahadev's model first builds a cryptographic key for the function, then asks the quantum computer to create a superposition of all possible inputs for the function. The quantum computer then must apply the function to the superposition, creating two entangled superpositions: one of the inputs and one of the outputs. When the quantum computer measures the output state, the system will collapse into one possible output and a superposition, creating two entangled superpositions: one of the inputs and one of the outputs. When the quantum computer measures the output state, the system will collapse into one possible output and a superposition of its two corresponding inputs. Since the classical computer has the key to find the claw, it can find the two components of the superposition. For the quantum computer, though, this task proves much more difficult. Measuring the superposition of inputs will collapse the values, making it impossible to find the claw; this leaves the quantum computer with no way to find the original inputs but creating a secret state, then entangling it with the input superposition to determine what it should do next. In doing so, the quantum computer is assured have gone through quantum

For now, quantum computers certainly cannot replace classical computers, and the myth of "quantum supremacy" is just a myth. Quantum researchers have found that many quantum algorithms, especially those not involving entanglement or entangling only a certain number of qubits, can be simulated using mathematical techniques on a classical computer. In fact, when a team at Google challenged classical techniques using a form of computation called instantaneous quantum polynomial (IQP) circuits, they found that to account for error correction in the circuit and to match the computing power of a classical system, they would need more gates and hundreds more qubits. Designing quantum algorithms can also be challenging, as quantum computers cannot measure the superpositions of qubits, only 0s and 1s.

In classical computing, the runtime of an algorithm increases exponentially with the number of bits inputted. For all their computational power, problems such as finding prime factors of a 2048 bit number or large-scale optimization (think "travelling salesman" - a popular math problem in which a travelling salesman's route through multiple towns must be optimized such that his path is as short as possible) would take a classical computer billions of years to solve. Quantum algorithms can decrease computing time by exponential factors because of their ability to calculate multiple outcomes at once. Grover's algorithm, which searches unorganized lists and databases, speeds up the process by a quadratic factor. For a classical computer, the complexity of searching through a database is N, the number of elements in the database. Using Grover's algorithm, the complexity (and therefore computing time) decreases to $\sqrt{N}$. The algorithm does so using amplitude amplification, a process in which the amplitude of the qubit associated with a specific item is first flipped, then amplified. In searching through the database, the item will be returned with much higher probability.

This issue can be resolved only by combining series of quantum gates such that the final position of the qubits will closely relate to a measurable value. To expand the capacities of quantum chips, scientists are researching ways to not only entangle more qubits, but manage the way qubits interact with each other and with quantum gates

Because of their ability to simulate quantum systems, quantum computing scientists hope that future systems may be able to assist in medical and chemical research by mimicking the process of chemical bonding and the folding of large proteins. The efficiency of quantum computers could also allow for developments in machine learning, which requires computers to sort through datasets. A commercial quantum computer used for financial modeling is already on the market, and three quantum computers are available for public use through the cloud. A world where classical and quantum computers work side by side is already here.

Three Proofs That There Are Infinitely Many Prime Numbers

By Rohan Buluswar

The existence of infinitely many primes is often one of the first proofs introduced in a number theory course, a theorem with one of the most ancient rigorous proofs that we know of, and a great example of how to use proof by contradiction. Here are three proofs of the theorem:

Euclid's Proof:

Euclid was a Greek mathematician, born ca. 300 BCE in Alexandria, Egypt. While he was most famous for his developments in the field of geometry, he also devised this clever little proof that there are infinite prime numbers. This is the first known proof of infinite primes.

Proof:

Suppose there is a finite number of primes, and let the set of those primes be P.

$P = \{p_1, p_2, p_3, p_4, p_5, p_6 ... p\_n\}$

Then we multiply all of our prime numbers together and add 1, getting another number, A:

$(p_1 * p_2 * p_3 * p_4 * p_5 * p_6 \ldots * p\_n) + 1 = A$

Either A is a prime number or it is not a prime number. If A is a prime number, then we are done. If A is not a prime number, then it must be divisible by another prime number. However, A cannot be divisible by any prime number already listed, because there would be a remainder of 1. In other words, if A is not a prime number then it must be divisible by another new prime number pa. Either way, given a finite number of primes, we can create a new prime. So if this process was repeated over and over, the result is infinite primes.

Filip Saidak's Proof:

Dr. Filip Saidak is a professor of mathematics at University of North Carolina at Greensboro.

Here is Filip Saidak's proof:

(continued at the top of the page)

Let n be an arbitrary positive integer greater than 1. Since n and n + 1 are consecutive integers, they must be coprime. Hence the number N2 = n(n + 1) must have at least two different prime factors. Similarly, since the integers n(n+1) and n(n+1)+1 are consecutive, and therefore coprime, the number N_3= n(n + 1)[n(n + 1) + 1] must have at least three different prime factors. This process can be continued indefinitely, so the number of primes must be infinite.
His argument that n(n+1) has two distinct prime factors relies on the fact that n and n+1 are coprime. All numbers have at least one prime factor. And since n and n+1 are coprime, and thus their GCF is one, their prime factors must be different. So when you multiply them you get a product with at least two distinct prime factors. When you repeat that process over and over, you keep getting numbers with more and more distinct prime factors. And since you can repeat that process over and over, there must be infinite prime numbers.

Kummer's Restatement of Euclid's Proof

Ernst Eduard Kummer was a German mathematician who was born on January 29, 1810.

Proof:

Let there be a finite list of all primes:
$p_1, p_2, p_3, p_4, p_5, p_6 ... p\_n$

Let $N = (p_1 * p_2 * p_3 * p_4 * p_5 * p_6 \ldots * p\_n) + 1$

Thus N-1 is a product of all of those prime numbers. All integers must have at least one prime factor, let one prime factor of N be pi. Since all of the prime numbers are a factor of N-1, pi must also be a factor of N-1. If pi is a factor of both N and N-1, then it must be a factor of N - (N-1), which is equal to 1. So pi must be a factor of one, which is impossible because 1 has no prime factors. Thus given our original premise we have reached a contradiction, and so our original assumption that there is a finite list of prime numbers is false: there must be infinite primes!

Algebraic Inequalities

By Ryan Rosenthal

An algebraic innequality is simply an equation where the relationship between the two or more sides isn't equality. The sides can either be less than, greater than, or less than or greater than and equal to each other (signified by these symbols respectively: <,>,≤, and ≥). Some exapmles of innequalities are $4 < 5$ and $3 \leq x \leq 8$. The first of these symbolizes the fact that 4 is less than 5. The second of these symbolizes a range in which x is contained; x represents a single value between 3 and 8. Solving an innequality is very similar to solving an equation, no matter how many expressions the innequality contains: Whatever you do to one side must be done to all others. The only difference is, whenever you multiply or divide an innequality by a negative, you have to reverse all of the signs. For example, if you divide $-2x < 6$ by -2, it becomes $x > -3$, not $x < -3$. A compound innequality is the combination of two or more innequalities joined by "and" or "or." If they are joined by an and, both conditions must be met. If they are joined by an or, only one condition may be met. For example, $x > -1$ and $x < 4$ states that x is between -1 and 4. $x < -1$ or $x > 4$ states that x is either less than -1 or greater than 4. An "and" compound innequality provides the same conditions as an innequality in three pieces: $x > -1$ and $x < 4$ is the same as $-1 < x < 4$. Solving a quadratic innequality is a little more complex. It begins like any other quadratic equation, by simply factoring out the equation. For an example, lets use $x^2 + 2x + 8 > 0$. By factoring, we find that $(x + 4)(x - 2) > 0$. Therefore, our roots are -4 and 2. We can draw a number line, and place open circles on -4 and 2 to signify those as our key values. Next, we pick any number less than -4, for example, -8. By plugging -8 back into the expression, we get 40. Since $40 > 0$, we know that -8 is a potential value for x. Thus, everything less than -4 is a potential value for x. We can do the same thing for the range from -4 to 2 and for numbers greater than 2. We will find that the numbers are negative and positive respectively. So, our final answer is $x < -4$ or $x > 2$. This will work for any number of intervals, with and degree of innequality (cubic, quartic, etc.). In the end, the positivity and negativity of the ranges on the number line will always alternate; once we find one, we know the rest.

Spherical Geometry by Danielle Paulson, continued from page 9.

When you add up the pairs of lunar areas, you get the surface area of the sphere with T counted three times and its antipodal duplicate counted three times as well. So, we have:

$$2LA + 2LB + 2LC = 4\pi r^2 + 4T$$

$$\rightarrow$$

$$T = 1/2(LA + LB + LC - 2\pi r^2)$$

$$\rightarrow$$

$$T = 1/2(A(2r^2) + B(2r^2) + C(2r^2) - 2\pi r^2)$$

$$\rightarrow$$

$$T = r^2(A + B + C - \pi)$$

... which is Girard's Theorem!

Math Book Recommendations!

College Geometry, by Nathan Altshiller Court (in re-print through Dover Publications)
"This is my favorite geometry book, a compendium of advanced Euclidean geometry discoveries that begins right at the boundaries of what we cover at Horace Mann in our geometry classes, and then travels through ridiculously surprising problems and properties, often with proofs whose elegance, cleverness, and aura of inevitability, will take your breath away. My first time through, I literally laughed out loud every few pages as I read through yet another golden geometric nugget. Admittedly, Altshiller-Court's style is old-fashioned, jargony, stilted and at time frustratingly brief, but once you get acclimated, you start to see that he's written a math book that brings to light what is most beautiful in the subject--and therefore what is beautiful about math itself."

Submitted by Mr. Worrall

In order of increasing difficulty:
Flatland by Abbott, *Taxicab Geometry* by Krause, and anything from the MAA *Proofs Without Words* Series
Chaotic Elections by Saari (what I am reading now!)

Submitted by Mr. Garcia

Pristine Landscapes in Elementary Mathematics by Titu Andreescu, Cristinel Mortici and Marian Tetiva
"Each chapter (there is a total of 14 chapters) introduces an interesting, accessable topic (such as geometric series, the pigeonhole principle or digital sums) and then challenges the reader with a problem set (solutions are in the book.) My personal favorite chapter is the dot product chapter - the problem set includes many geometry problems, that can be solved in beautiful ways using either synthetic geometry or the dot product. It's a lot of fun to see all the completely different ways the problems can be solved. Another great thing about the book is that it assumese very little prior knowledge, but gets to advanced topics very quickly; for example, there is a chapter dedicated to the greatest common divisor. It may seem basic at the beginning of the chapter, but then moves on to much more difficult, less widely known theorems and very challenging problems on the problem set."

Submitted by Dora Woodruff (11)

Fermat's Enigma, and *Prime Numbers and the Riemann Hypothesis.*

Submitted by Zachary Brooks (11)

Citations

The Brachistochrone Problem:

"Historical Activities for Calculus - Module 3: Optimization – Galileo and the Brachistochrone Problem." Iterative Methods for Solving Ax = b - Gauss-Seidel Method | Mathematical Association of America, www.maa.org/press/periodicals/convergence/historical-activities-for-calculus-module-3-optimization-galileo-and-the-brachistochrone-problem.
"Mathonline." Piecewise Smooth Curves in the Complex Plane - Mathonline, mathonline.wikidot.com/the-cycloid.

Brouwer's Fixed Point Theorem:

https://www.maths.ed.ac.uk/~v1ranick/jordan/tverberg.pdf
http://math.stmarys-ca.edu/wp-content/uploads/2017/07/Colin-Buxton.pdf
http://math.mit.edu/~fox/MAT307-lecture03.pdf
https://lmb.univ-fcomte.fr/IMG/pdf/A-_Prochazka_19-3-13.pdf

Quantum Computing:

https://quantumcomputingreport.com/our-take/how-to-explain-quantum-to-your-classical-friends/some-important-problems-can-never-be-solved-using-classical-computing/
https://www.codeproject.com/Articles/1182179/Quantum-Computing-for-Everyone-Part-I-Classical-vs
https://uwaterloo.ca/institute-for-quantum-computing/quantum-computing-101
https://www.youtube.com/watch?v=OWJCfOvochA
https://www.quantamagazine.org/quantum-computers-struggle-against-classical-algorithms-20180201/

Intro to the Fourier Transform

https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/

Challenge Problems!

Collected by Danielle Paulson, Dora Woodruff and Mandy Liu

1. After a meeting, everyone shook hands with everybody else. There were 66 handshakes in total. How many people attended the meeting?

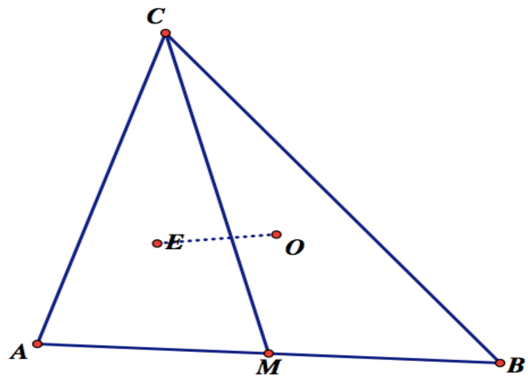
2. A unit square is divided into rectangles. The WL ratio is the ratio of a rectangle's width to its length. Prove that the sum of all the WL ratios of these rectangles is greater than or equal to 1.

3. 11 girls and n boys went mushroom picking. At the end of the day, they collected n^2+9n-2 mushrooms, and each person found the same number of mushrooms. Were there more boys or girls?

4. Given an isosceles right triangle (with legs of length one), how do you construct the shortest line segment that divides the triangle into two sections of equal area? What is the length of that segment?

5. Prove that cos(sin(x)) is greater than sin(cos(x)).

6. In triangle ABC, M is the midpoint of AB, E is the centroid of AMC and O is the circumcenter of ABC. Prove that OE is perpendicular to MC if and only if AB=AC.



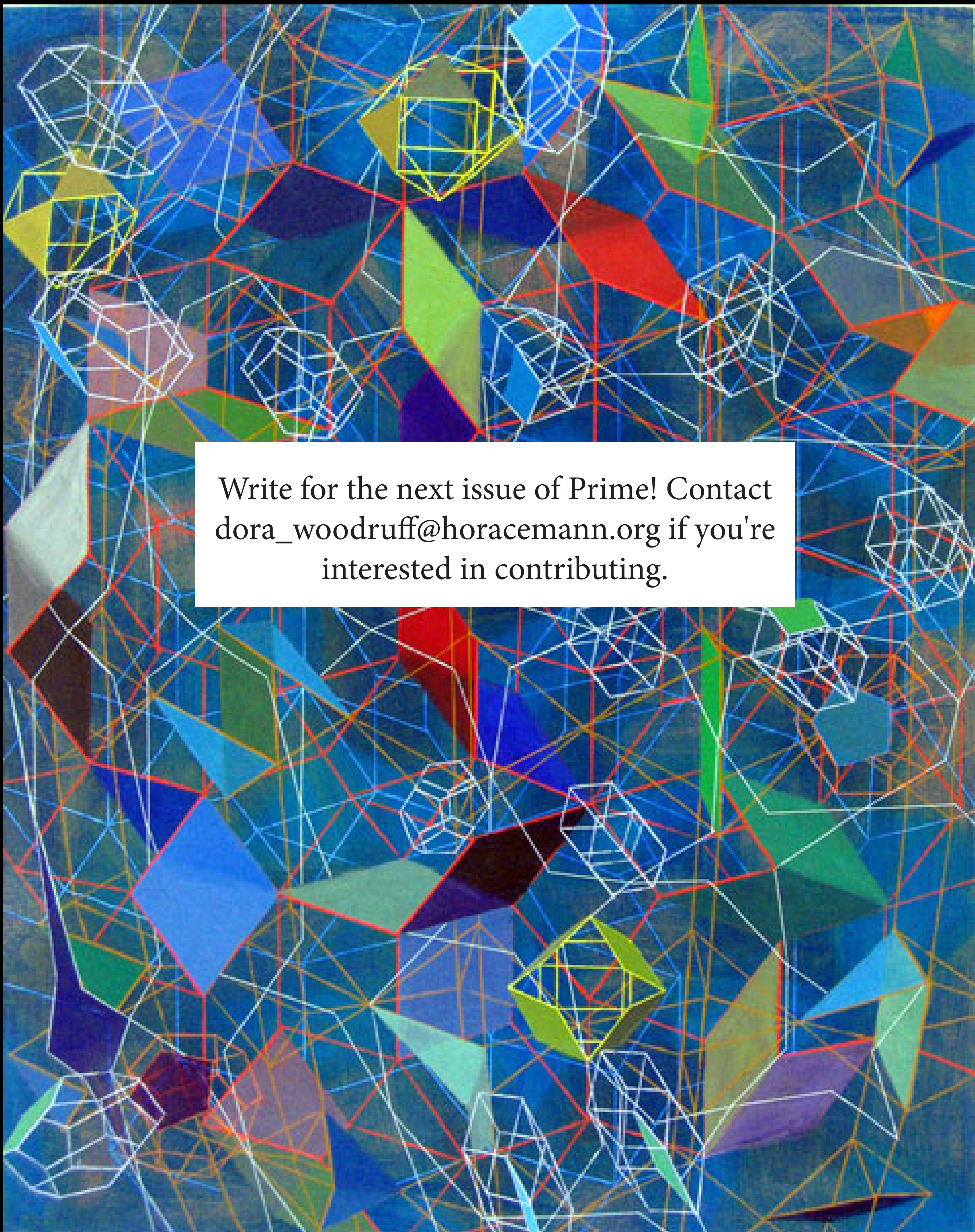
7. How many four digit numbers n are there such that multiplying n by four and adding three reverses the digits of n? (For example, 17 is a two digit number such that 17 multiplied by 4 plus 3 is 71, which is the number you get by reversing the digits of 17.)

8. How many n digit numbers are there whose digits, when read left to right, do not ever decrease? (Find a formula in terms of n. 125569 is an example of such a number, but 125469 is not.)

9. a, b, c, d, e and f are real numbers. Prove that at least one of ac + bd, ec + fd, ge + hf, ga + hb, ae + bf and cg + hd is nonnegative.

10. Four couples go to a party. How many ways are can each person shake hands with exactly one other person? How many ways can the people do this if they do not shake hands with their partner?

11. H is the orthocenter of triangle ABC. Let O be the circle with diameter AH, and let M be the midpoint of BC. Choose two points P, Q on O such that P, Q and M are collinear. Prove that the orthocenter of triangle APQ is on the circumcircle of ABC.



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