Living on the Number line: Development of fraction magnitude understanding in children at risk for learning difficulties in mathematics

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Broader Impacts

- According to the recent U.S. "Nation's Report Card" (NAEP, 2015), just 33% of 8th graders nationwide scored at or above proficiency in math, with the situation being most critical for non-Asian racial and ethnic minority students, low-income students, and students with disabilities.
  - Only 13% of Black children and 19% of Hispanic children scored proficient or above in eighth grade.
  - Among low-income students, only 13% reached proficiency.
  - Among students with diagnosed disabilities only 9% were proficient
- Fractions are foundational for learning algebra, thus representing a crucial component mathematics education in the intermediates grades.
- Even weaker students who might not major in math-related fields need to understand fractions and additional math to contribute to the future work force.
- Facility with fractions affects daily life functioning in many areas -- food preparation, personal finances, home repairs, and healthcare decisions.

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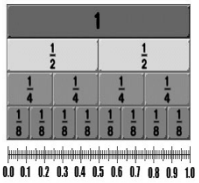
Fractions are Hard!

- Numbers of the same magnitude can look different 3/4 9/12
- Sometimes, when numerals get bigger, the fraction gets smaller 1/4 1/6 1/8
- Not always the case, however 2/4 < 6/7
- Infinite amount of numbers between 2 fractions.



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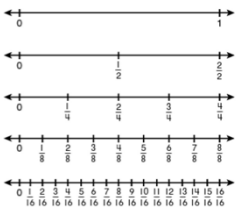
The number line

- Mathematician, Richard Askey notes: "Mathematicians spend most of their lives living on a number line."
- Mathematician Wu (2010) observes that the number line should be **central** in the teaching of fractions in the intermediate grades.



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Why is the number line important for magnitude understanding and fraction learning?

- Numerical development is a continuous process that broadens the class of numbers understood to possess **magnitudes** that can be represented on a number line (e.g., whole numbers, fractions, decimals; Siegler, 2016).
- Number line broadens the concepts of fractions in ways that traditional part-whole representations cannot.
- Over-reliance on part-whole interpretations of fractions can lead to narrow thinking about what fractions represent ("a pie" or "the shaded part").
- Number line sets the stage for higher mathematics learning.

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Delaware Longitudinal Project Aims:

- Identify predictors of fraction learning from 3rd through 6th grades, the period when primary fractions instruction takes place in school.
- Explain **why** some children struggle with fractions.
- Examine connections between math and reading fluency.
- Identify patterns of growth in fraction learning.
- Based on findings, develop and test an intervention to help children learn fractions.

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Overarching hypotheses:

- Accurate representations of numerical magnitudes on a number line are uniquely important for acquisition of fraction knowledge.
  - to test this premise, we examined the extent to which domain general and number specific cognitive processes predict fraction outcomes.
- An intervention focused on understanding of fraction magnitudes plus general learning principles will improve at-risk learners' knowledge of fractions.

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Longitudinal sample

- Students (*N* = 536) were followed from 3rd grade through 6th grade.
- Drawn from 9 elementary schools in 2 public school districts.
- Over-sampled children from low-income communities, who might be at most risk.

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Sample demographics

Characteristic	%
Gender	
Male	47.0
Female	53.0
Race	
White	51.9
Black	40.0
Asian/Pacific Island	5.7
American Indian/Alaskan Native	2.5
Hispanic	17.7
Low Income	60.9
English Learner	10.6
Special Education	10.6
Mean Age (at start of study, in months)	105.9

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Predictor Variables

General Predictors	Number-Related Predictors
Verbal Ability (PPVT) 3rd grade	**Whole Number Line Estimation** (0-1000) 3rd and 5th grades
Nonverbal Ability (WASI) 3rd grade	**Calculation Fluency** all grades
Attention (SWAN teacher survey) all grades	**Non-symbolic Proportional Reasoning** 5th grade
Working Memory (Counting Recall) 3rd and 5th grade	**Long division** 5th grade
Reading Fluency all grades	

Background variables (age, district membership, gender, special education status, income status, and EL status) were included in analyses.

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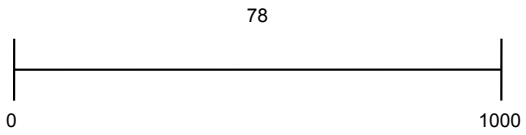
Whole Number Line Estimation

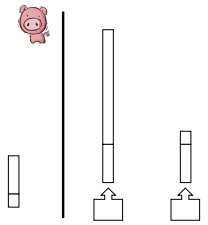
- 22 whole number estimations (0 to 1000 line)
  - 56, 606, 179, 122, 34, 78, 150, 938, 100, 163, 754, 5, 725, 18, 246, 722, 818, 738, 366, 2, 486, 147



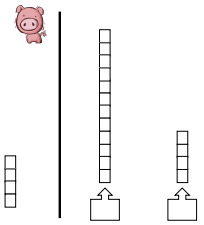
Non-symbolic proportional reasoning (Boyer & Levine, 2012)

Here is my friend Harry the Hog. Harry enjoys drinking all kinds of juice, and likes to mix the juice himself. Harry must be careful to have the correct mix of water and juice for each type of mix…Which of these two would taste just like Harry's juice

Continuous (1:4 scaled to 3:12)



Discrete



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Fractions Outcome Measures:

>Fraction concepts (mainly NAEP released items)
>Fraction number line estimation
>Fraction arithmetic

Given over multiple time points between 4th and 6th grades

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Fraction Concepts		
Area Models		

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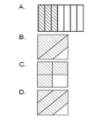
Questions Assessing Fraction Concepts		
Which shows 3/4 of the picture shaded? A. B. C. D.		



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Questions Assessing Fraction Concepts		
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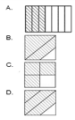

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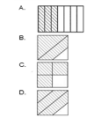
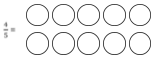
Questions Assessing Fraction Concepts		
Which shows 3/4 of the picture shaded? A. B. C. D.	Shade the set of figures the amount indicated by the fraction next to it.	





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Questions Assessing Fraction Concepts

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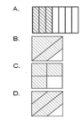

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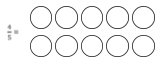

Number Lines

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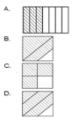
Questions Assessing Fraction Concepts

Which shows 3/4 of the picture shaded?



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Number Lines

20

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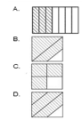

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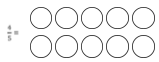

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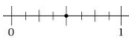

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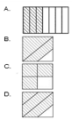

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Comparison/ Ordering

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Questions Assessing Fraction Concepts

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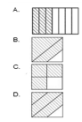

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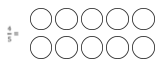

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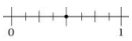

In which of the following are the three fractions arranged from least to greatest?

A. $\frac{2}{7}, \frac{1}{2}, \frac{5}{9}$

B. $\frac{1}{2}, \frac{2}{7}, \frac{5}{9}$

C. $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$

D. $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$

23

Questions Assessing Fraction Concepts

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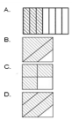

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Equivalence

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Questions Assessing Fraction Concepts

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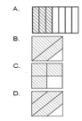

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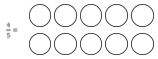

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C. $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$

D. $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$

These three fractions are equivalent. Write two more fractions that are equivalent to these.

$\frac{4}{8}$ $\frac{25}{50}$ $\frac{5}{10}$

Questions Assessing Fraction Concepts

Which shows 3/4 of the picture shaded?

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Estimation

Questions Assessing Fraction Concepts

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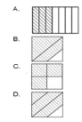

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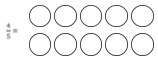

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These three fractions are equivalent. Write two more fractions that are equivalent to these.

$\frac{4}{8}$ $\frac{25}{50}$ $\frac{5}{10}$

Which fraction has a value closest to 1/2?

A. $\frac{5}{8}$

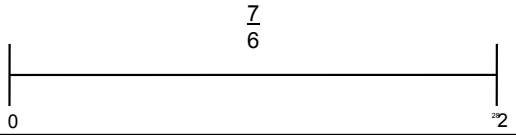
B. $\frac{1}{6}$

C. $\frac{2}{2}$

D. $\frac{1}{5}$

Fraction number line estimation task

- Students estimated the location of fractions on 0-1 and 0-2 number lines.
- Fractions as well as mixed numbers
- Percent absolute error (PAE) calculated by dividing the absolute difference between the estimated and actual magnitudes by the numerical range of the number line and then multiplying by one hundred for each estimate. **Thus the lower the score, the more accurate.**



Fraction arithmetic

1. $\frac{2}{5} + \frac{1}{5} =$
2. $\frac{3}{4} - \frac{1}{4} =$
3. $\frac{3}{6} + \frac{1}{6} =$
4. $\frac{5}{6} - \frac{2}{6} =$
5. $1\frac{3}{4} - \frac{1}{4} =$
6. $\frac{3}{4} + \frac{2}{4} =$
7. $3\frac{3}{8} + 1\frac{2}{8} =$
8. $2\frac{2}{3} - 1\frac{1}{3} =$
9. $\frac{5}{6} + \frac{2}{3} =$
10. $\frac{7}{8} - \frac{1}{2} =$

Multiple Regression Standardized Beta Coefficients of Predictors of Fraction Concepts and Arithmetic by Grade (Jordan et al. 2013; Hansen et al., 2015)

	Fourth grade		Sixth grade	
Variable	Concepts	Arithmetic	Concepts	Arithmetic
Whole number line estimation	-.271***	-.237***	-.361***	-.183***
Attention	.225***	.135*	.170***	.254***
Verbal ability	.196***	.081	---	---
Addition fluency	.169***	.126*	---	---
Nonverbal ability	.111**	.097	---	---
Reading fluency	.086*	-.037	.048	.011
Working memory	.044	.137**	.114**	.011
Non-symbolic proportional reasoning		---	.198***	.080
Multiplication fluency	---	---	.037	.171**
Long division	---	---	.113*	.171**

Note. **Predictors of fourth-grade outcomes were assessed in third grade; predictors of sixth-grade outcomes were assessed in fifth grade**. Empty cells indicate variables that were not included in the multiple regression analysis. * $p < .05$ ** $p < .01$ *** $p < .001$.

Summary of regression analyses

- Together, the predictors explained about **58%** of the **variance** in 6th grade **fraction concepts**. Independently important were: **whole number line estimation, non-symbolic proportional reasoning, attention, working memory**, and **long division**.
- The predictors explained about **40%** of the variance in performance in 6th grade **fraction procedures**, with **attention, number line estimation, multiplication fact fluency,** and **long division** all making independent contributions.
- 3rd grade **reading fluency** uniquely predicted fraction concepts but not arithmetic in 4th grade; 5th grade reading fluency did not predict fraction outcomes in 6th grade.

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Implications:

- A constellation of processes influences fraction learning, including numerical magnitude understanding, arithmetic fluency, attention and memory, and reading/language skills.
- Developing an accurate representation of whole number magnitudes on the number line provides organization for reasoning about fraction magnitudes.
- Unobserved variables explaining the remaining variance may include instructional and emotional influences.

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Examination of the relation between word reading fluency and calculation fluency between 3rd and 5th grades.

- We found significant effects of **reading fluency** on multiplication fluency, but **not** on addition or subtraction fluency. (No direct effects of multiplication fluency on reading growth)
  - There was a ***direct*** effect of early 3rd grade reading fluency on late 3rd grade multiplication fluency, the period when children are first learning multiplication facts.
  - Initial reading fluency also predicted the slope of growth in multiplication fluency between 3rd and 5th grade.
  - Unlike addition and subtraction, multiplication facts learned primarily through rote memory, leading to increased reliance on written materials.
- Number line estimation accuracy predicted both initial multiplication fluency skill and growth over time.
  - A good sense of numerical magnitudes makes it easier to learn multiplication facts.
  - Effects of reading fluency after controlling for number line estimation point to processes associated with rote memorization.

Growth in fraction magnitude skills (Resnick et al., 2016)

- Examined **fraction number line estimation** growth over multiple time points between 4th and 6th grades and whether this predicts math achievement.
- Over course of study, most students increased in estimation accuracy.
- However, latent class growth analyses revealed 3 empirically distinct growth trajectory classes.

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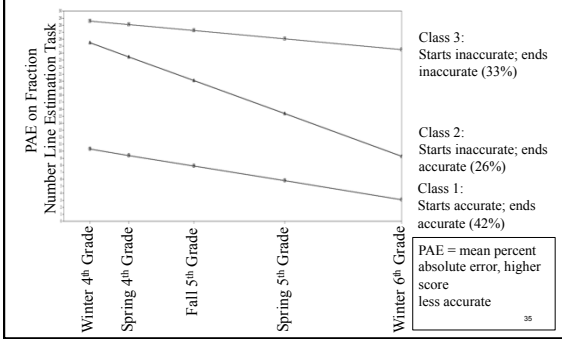
Fraction number line estimation trajectory classes



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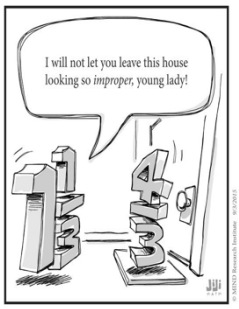
Different kinds of fractions

Proper
1/19, 1/3, 3/8, 1/2, 2/3

"Improper"
5/5, 7/6, 4/3, 3/2, 7/4

Mixed
1 1/5, 1 2/4, 1 5/8, 1 5/6



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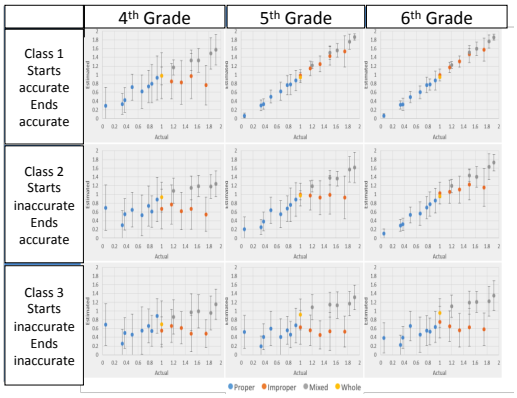

Performance on 0-2 number line

fa37

Observations

- Younger and lower-performing students tended to estimate both proper and improper fractions as being < 1, failing to base estimates on relation between numerator and denominator.
- Define fractions as "really small".
- May be related to emphasis on proper fractions in early fraction instruction.

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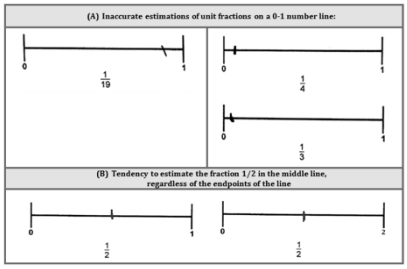
Examples of Misconceptions



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Growth class predicts end of 6th grade achievement, while controlling for cognitive and background variables

	Sixth grade Math Proficiency Level Groups (%)			
	1 well below standards	2 below standards	3 meeting standard	4 advanced
Class 1 Starts accurate; Ends accurate (n=108)	1	5	26	**68**
Class 2 Starts inaccurate; Ends accurate (n = 94)	3	14	**64**	19
Class 3 Starts inaccurate; Ends inaccurate (n=140)	**35**	31	33	1

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Implications:

- A substantial number of students enter middle school without a basic understanding of fraction magnitudes on the number line, ***despite*** several years of instruction.
- On the other hand, some students enter 4th grade with strong understanding of fraction magnitude. How did they develop these intuitions?

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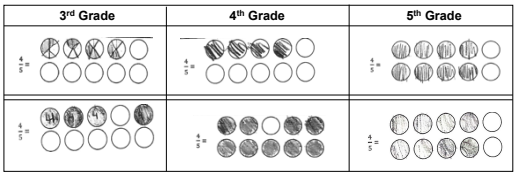
Developmental progression: Shaded fractions

Fall 3rd Grade	Spring 3rd Grade	Fall 4th Grade	Spring 4th Grade	Fall 5th Grade	Spring 5th Grade
16%	24%	27%	44%	47%	63%



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Developmental progression: Equivalence

$\frac{4}{8}$ $\frac{25}{50}$ $\frac{5}{10}$

These three fractions are equivalent. Write two more fractions that are equivalent to these.

2 Fractions Correct		
Fall 4th Grade	Spring 4th Grade	Spring 5th Grade
43%	65%	79%

Fall 4th grade	Spring 4th grade	Spring 5th grade
$\frac{4}{4}$, $\frac{10}{5}$	$\frac{20}{100}$, $\frac{8}{16}$	$\frac{3}{6}$, $\frac{50}{100}$

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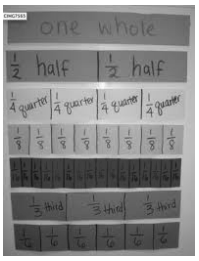
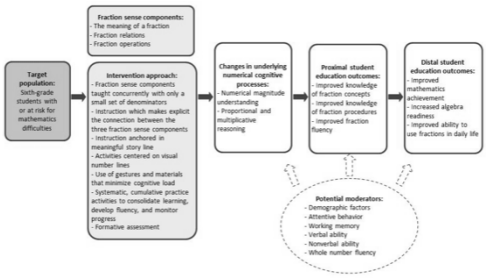


Developing fraction sense in children with or at risk for math difficulties

Our Theory of Change



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Intervention study goals

1. Design and evaluate the effectiveness of an intervention for 6th graders with MD developing fraction sense, including:
   i. General concepts
   ii. Numerical magnitudes
   iii. Fraction arithmetic
2. Consider **moderating** effects of cognitive and behavioral competencies
3. Incorporate principles from the science of learning

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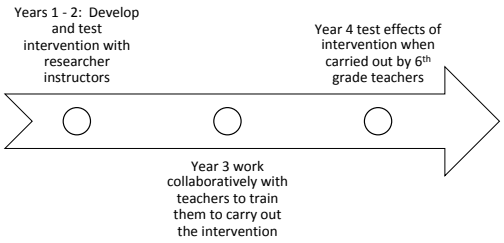
Multiple iterations:



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Participants:

- 6th graders in 2 public schools were administered a validated fractions screener.
- Struggling students (*N* = 51) randomly assigned at the individual level to intervention or BAU control
- **~30% had diagnosed reading learning disability**
- 43% male, 31% receiving special education, 10% EL
- Intervention group did not differ from control on demographics.

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Measures:

Fractions
- General fraction concepts
- Fraction comparisons
- Fraction number line estimation
- Fraction arithmetic

Potential Covariates
- Working Memory:
- Receptive Vocabulary
- Selective Visual Attention
- Non-symbolic Proportional Reasoning
- Classroom Attention

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Study Design

Procedure	
Pretest	Fraction NLE, Comparisons, Arithmetic; Cognitive covariates
Six-week Intervention Period: ***Intervention***: 27 40-minute scripted lessons ***Control***: School math intervention (computer adaptive software)	
Immediate Posttest	Fraction Concepts, NLE, Comparisons, Arithmetic
Delayed Posttest (7 weeks)	Fraction Concepts, NLE, Comparisons, Arithmetic

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Instructional Strategies

- Use fractions with only a few familiar denominators to teach many concepts (2, 4, 8, 3, 6)
- Emphasize meaning of numerator and denominator and how they work together to form a magnitude
- Emphasize equivalence
- When the numerator and the denominator are the same, the fraction always equals one (8/8 = 12/12)
- Always ask "of what" to find the whole
- Use benchmarks of 0, 1, ½ on the number line
- Build multiplicative reasoning and fact fluency through oral activities

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Overview of lesson structure

Activity	Description	Time
Warm-up	Individual worksheet practice of material from previous day.	3 minutes
Multiplication practice	Speeded practice of whole number multiplication facts using multiplicands that are aligned with denominators in the corresponding lesson.	3 minutes
Counting	Practice of oral counting of fractions with like denominators (e.g., "one-fourth, two-fourths, three-fourths…") using the number line as reference.	3 minutes
Targeted instruction	Explicit instruction targeting the lesson's learning goals and focused on the number line.	20 - 25min.
Games	Short, fast-paced card games targeting fraction magnitude judgements (e.g., comparing two fractions to each other, to one-half) and fraction equivalencies (e.g., 3 is the same as how many halves?; 3 halves is the same as how many fourths?).	3 minutes
Cool Down (Independent Practice and Formative Assessment)	Individual worksheet practice of material from that day's lesson and prior content.	3 minutes

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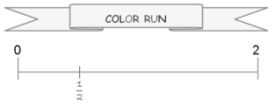
Color Run

- Each number line was presented as a "course" for a **color run race**.
- When shown a 0-2 number line, for example, students understood it was a **2-mile course** for the color run, and runners would be showered with colored powder at every 1/2 mile of the race.
- Activities required students to practice **flexibility** with different number lines.

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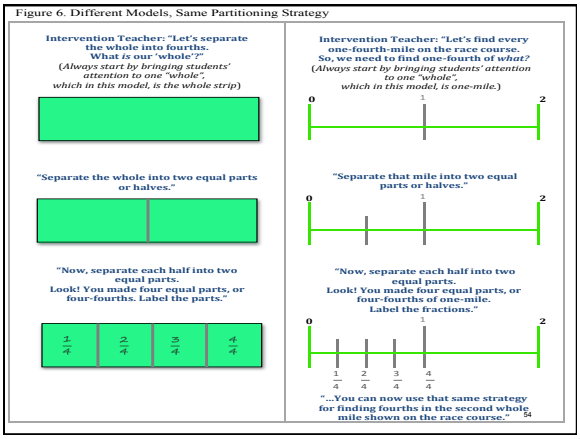
Figure 6. Different Models, Same Partitioning Strategy



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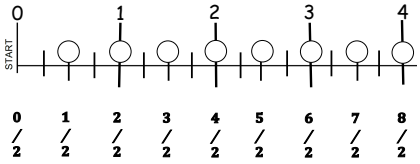
NL APPROACH: LABELING EQUIVALENT FRACTIONS



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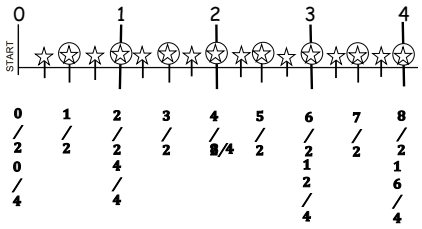

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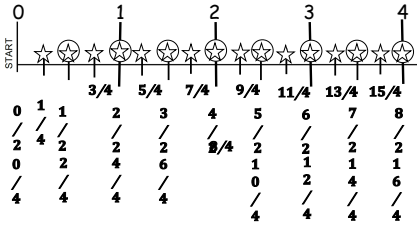
NL APPROACH: LABELING EQUIVALENT FRACTIONS



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Fraction bars

1/2 = 2/4 = 4/8

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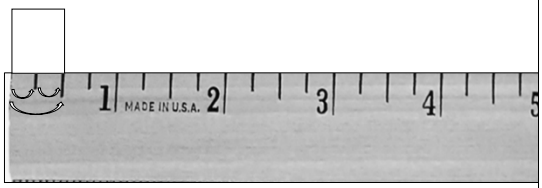

2/4 =1/2

Simplest Form

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Approach: Principles from the science of learning

Explicit Instruction[1]

- **Representational Gestures[2]**
- **Verbal & Visual: Dual Coding[3]**
- **Interleaved[5] and Spaced Practice**
- **Feedback[6-7]** (Immediate and scaffolded)

1. Gersten et al., 2009; 2. Ping & Goldin-Meadow, 2008; 3. Cuevas, 2016; 4. Rohrer & Taylor, 2006; 5. Rohrer et al., 2015; 6. Graesser et al., 2005; 7. Fyfe & Rittle-Johnson, 2016

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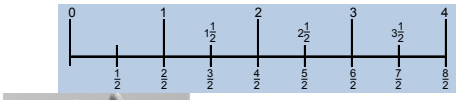
Representational gestures



"Chunking" one half mile

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Verbal & Visual Presentation

$\frac{1}{4}$

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Spaced & interleaved practice

Comparisons

$\frac{1}{4}$ or $\frac{1}{8}$ $\frac{3}{8}$ or $\frac{6}{8}$

Modeling Multiplication

$\frac{3}{4} \times 16 =$ ____

Measurement

Examples from Lesson 20 warm-up activity

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Results:

- Intervention and control were equivalent at pretest, except on receptive vocabulary
  - *Control > Intervention on receptive vocabulary; controlled for in subsequent analyses*
- Four 2 (Group) X 3 (Time) RMANCOVAs on fraction measures
- **Intervention > Control on immediate and delayed post test**: Fraction Concepts, FNLE, and Fraction Comparisons
- Improvements in fraction arithmetic scores did not vary by condition.

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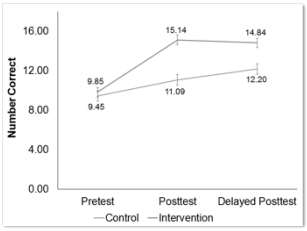
Fraction Concepts by Condition



Effect size (Hedges' g)	
Pre – Posttest	1.09
Pre – Delayed Posttest	.66

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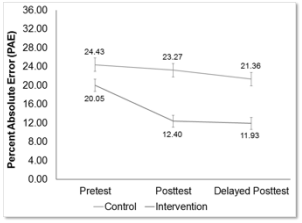
Fraction NLE by Condition



Effect size (Hedges' g)	
Pre – Posttest	-.85
Pre – Delayed Posttest	-.60

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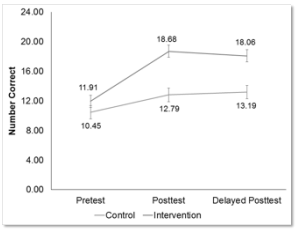
Fraction Comparisons by Condition

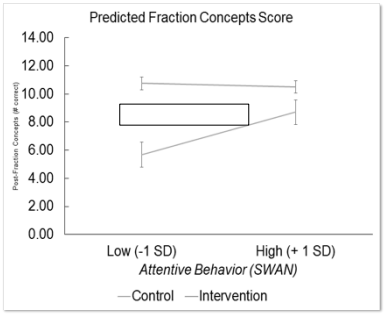


Effect size (Hedges' g)	
Pre – Posttest	.82
Pre – Delayed Posttest	.61

Moderators: exploratory analyses

- Assessed interaction between Intervention & cognitive and behavioral measures
- Significant interaction: Attentive behavior × Intervention (β = -.30)

Intervention × Attentive Behavior

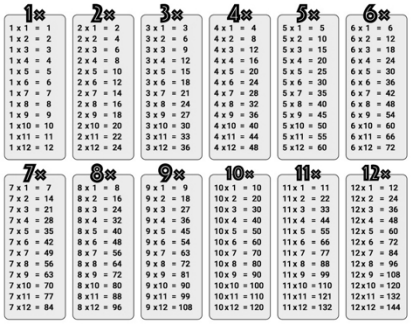


Intervention had a ***buffering effect*** on the normally negative impact of low attentive behavior on learning.

Discussion:

- Intervention yielded large effects on conceptual understanding of fractions at immediate posttest and **skills were retained even at a 7-week delay.**
  - Likely associated with principles shown to improve learning and retention.
- Even students at risk for math failure can make robust gains with carefully designed intervention.
- Future work is continuing to address instructional methods for using fraction concepts to teach fraction arithmetic.
- Also to what extent can we close the gap between MD and normal achievers?
- How do improvements in fraction understanding translate to later algebra outcomes?
- **Current Goal: Development of intervention feasible to use by teachers in whole class settings**

Example of common fact practice: Avoid for students with RD



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	Kelly Gueselli	Heather Suchanec	