## Team Selection Test 2006

1. A communications network consisting of some terminals is called a *3-connector* if among any three terminals, some two of them can directly communicate with each other. A communications network contains a *windmill* with $n$ blades if there exist $n$ pairs of terminals $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$ such that each $x_i$ can directly communicate with the corresponding $y_i$ and there is a *hub* terminal that can directly communicate with each of the $2n$ terminals $x_1, y_1, \ldots, x_n, y_n$. Determine the minimum value of $f(n)$, in terms of $n$, such that a 3-connector with $f(n)$ terminals always contains a windmill with $n$ blades.

**Solution:** The answer is

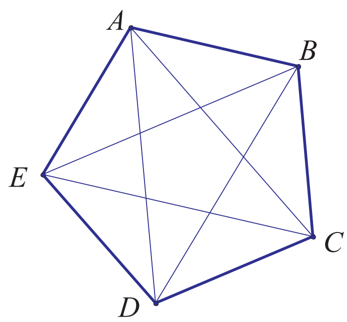
$$f(n) = \begin{cases} 6 & \text{if } n = 1; \\ 4n+1 & \text{if } n \geq 2. \end{cases}.$$

We will use *connected* as a synonym for directly communicating, call a set of $k$ terminals for which each of the $\binom{k}{2}$ pairs of terminals is connected *complete* and call a set of $2k$ terminals forming $k$ disjoint connected pairs a *k-matching*.

We first show that $f(n) = 4n+1$ for $n > 1$. The $4n$-terminal network consisting of two disconnected complete sets of $2n$ terminals clearly does not contain an $n$-bladed windmill (henceforth called an $n$-mill), since such a windmill requires a set of $2n+1$ connected terminals. So we need only demonstrate that $f(n) = 4n+1$ is sufficient.

Note that we can inductively create a $k$-matching in any subnetwork of $2k+1$ elements, as there is a connected pair in any set of three or more terminals. Also, the set of terminals that are not connected to a given terminal $x$ must be complete, as otherwise there would be a set of three mutually disconnected terminals. We now proceed by contradiction and assume that there is a $(4n+1)$-terminal network without an $n$-mill. Any terminal $x$ must then be connected to at least $2n$ terminals, for otherwise there would be a complete set of size at least $2n+1$, which includes an $n$-mill. In addition, $x$ cannot be directly connected to more than $2n$ terminals, for otherwise we could construct an $n$-matching among these, and therefore an $n$-mill. Therefore every terminal is connected to precisely $2n$ others.

If we take two terminals $u$ and $v$ that are not connected we can then note that at least one must be connected to the $2n-1$ remaining terminals, and therefore there must be exactly one, $w$, to which both are connected. The rest of the network now consists of two complete sets of terminals $A$ and $B$ of size $2n-1$, where every terminal in $A$ is connected to $u$ and not connected to $v$, and every terminal in $B$ is not connected to $u$ and connected to $v$. If $w$ were connected to any terminal in $A$ or $B$, it would form a blade with this element and hub $u$ or $v$ respectively, and we could fill out the rest of an $n$-mill with terminals in $A$ or $B$ respectively. Hence $w$ is only connected to two terminals, and therefore $n = 1$.



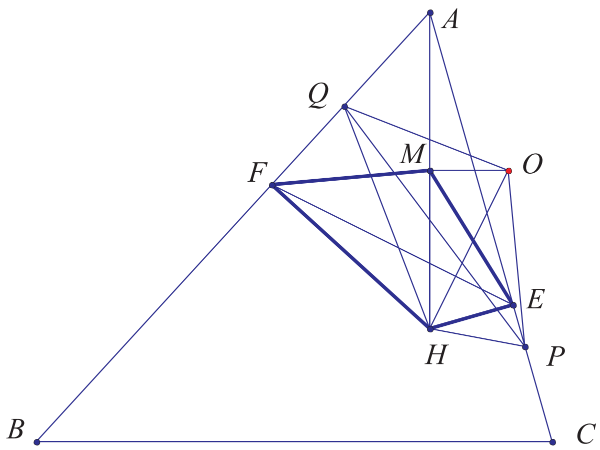
Examining the preceding proof, we can find the only 5-terminal network with no 1-mill: With terminals labeled $A, B, C, D$, and $E$, the connected pairs are $(A, B)$, $(B, C)$, $(C, D)$, $(D, E)$, and $(E, A)$. (As indicated in the figure above, a pair of terminals are connected if and only if the edge connecting them are darkened.) To show that any 6-terminal network has a 1-mill, we note that any complete set of three terminals is a 1-mill. We again work by contradiction. Any terminal $a$ would have to be connected to at least three others, $b$, $c$, and $d$, or the terminals not connected to $a$ would form a 1-mill. But then one of the pairs $(b, c)$, $(c, d)$, and $(b, d)$ must be connected, and this creates a 1-mill with that pair and $a$.

(This problem was proposed by Cecil C Rousseau.)

2. In acute triangle $ABC$, segments $AD, BE$, and $CF$ are its altitudes, and $H$ is its orthocenter. Circle $\omega$, centered at $O$, passes through $A$ and $H$ and intersects sides $AB$ and $AC$ again at $Q$ and $P$ (other than $A$), respectively. The circumcircle of triangle $OPQ$ is tangent to segment $BC$ at $R$. Prove that $CR/BR = ED/FD$.

**Note:** We present two solutions. We set $\angle CAB = x$, $\angle ABC = y$, and $\angle BCA = z$. Without loss of generality, we assume that $Q$ is in between $A$ and $F$. It is not difficult to show that $P$ is in between $C$ and $E$. (This is because $\angle FQH = \angle APH$.)

**First Solution:** (Based on work by Ryan Ko) Let $M$ be the midpoint of segment $AH$. Since $\angle AEH = \angle AFH = 90°$, quadrilateral $AEHF$ is cyclic with $M$ as its circumcenter. Hence triangle $EFM$ is isosceles with vertex angle $\angle EMF = 2\angle CAB = 2x$. Likewise, triangle $PQO$ is also an isosceles angle with vertex angle $\angle POQ = 2x$. Therefore, triangles $EFM$ and $PQO$ are similar.




Since $AEHF$ and $APHQ$ are cyclic, we have $\angle EFH = \angle EAH = \angle EQH$ and $\angle FEH = \angle FAH = \angle QPH$. Consequently, triangles $HEF$ and $HPQ$ are similar. It is not difficult to see that quadrilaterals $EHFM$ and $PHQO$ are similar. More precisely, if $\angle QHF = \theta$, there is a spiral similarity $\mathbf{S}$, centered at $H$ with clockwise rotation angle $\theta$ and ratio $QH/FH$, that sends $FMEH$ to $QOPH$. Let $R_1$ be the point in between $B$ and $D$ such that $\angle R_1HD = \theta$. Then triangles $QHF$ and $R_1HD$ are similar. Hence $\mathbf{S}(D) = R_1$. It follows that

$$\mathbf{S}(DFME) = R_1QOP.$$

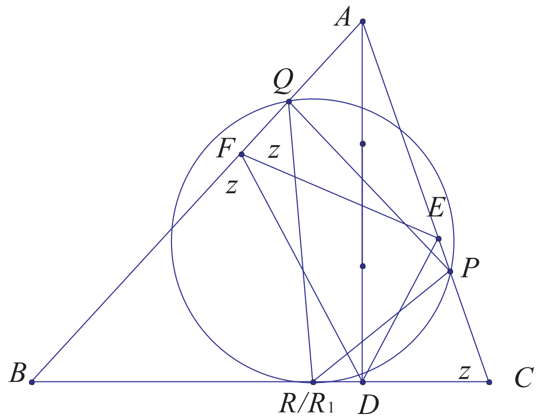
It is well known that points $D, E, F$, and $M$ lie on a circle (the **nine-point circle** of triangle $ABC$). (This fact can be established easily by noting that $ABDE$ and $ACDF$ are cyclic, implying that

$\angle FDB = \angle CAF = x$, $\angle EDC = \angle BAE = x$, and $\angle EDF = 180^\circ - 2x = 180^\circ - \angle EMF$.) Since $DFME$ is cyclic, $R_1QOP$ must also be cyclic. By the given conditions of the problem, we conclude that $R_1 = R$, implying that

$$\mathbf{S}(DEF) = RPQ,$$

or triangles $DEF$ and $RPQ$ are similar. It follows that

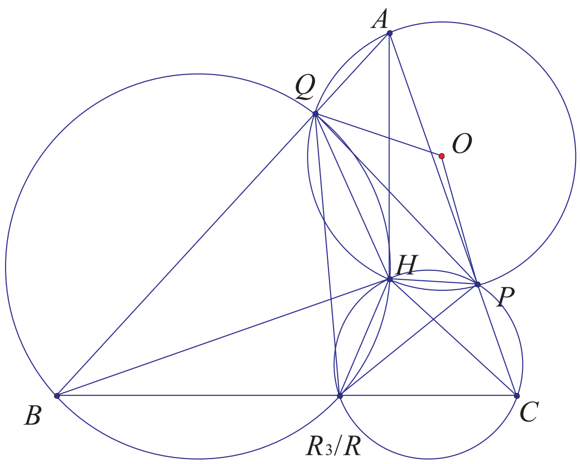
$$\frac{ED}{FD} = \frac{PR}{QR}.$$




Now we are ready to finish our proof. Since $ACDF$ and $ABDE$ are cyclic, $\angle BFD = \angle AFE = \angle ACB = z$. Thus $\angle DFE = 180^\circ - 2z$. Since triangles $DEF$ and $RPQ$ are similar, $\angle RQP = 180^\circ - 2z$. Because $CR$ is tangent to the circumcircle of triangle $PQR$, $\angle CRP = \angle RQP = 180^\circ - 2z$. Thus, in triangle $CPR$, $\angle CPR = z$, and so it is isosceles with $CR = PR$. Likewise, we have $BR = QR$. Therefore, we have

$$\frac{ED}{FD} = \frac{PR}{QR} = \frac{CR}{BR}.$$

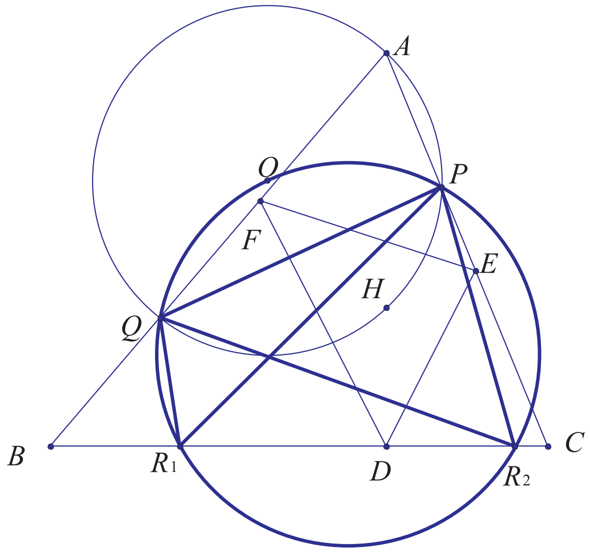
**Second Solution:** (Based on work by Brady Zarathustra) Let the circumcircle of triangle $BQH$ meet line $BC$ at $R_3$ (other than $B$).




Since $APHQ$ and $BQHR_3$ are cyclic, $\angle PHQ = 180^\circ - \angle PAQ$ and $\angle QHR_3 = 180^\circ - \angle QBR_3$, implying that $\angle PHR_3 = 360^\circ - \angle PHQ - \angle QHR_3 = 180^\circ - \angle ACB$. Hence $CPHR_3$ is also cyclic.

(We just established a special case of **Miquel's Theorem**.) Because $BQHR_3$ and $CR_3HP$ are cyclic, we have $\angle QR_3H = \angle QBH = 90^\circ - \angle BAC$ and $\angle HPR_3 = \angle HCP = 90^\circ - \angle BAC$. Hence $\angle QR_3P = 180^\circ - \angle BAC = 180^\circ - 2x$. Likewise, we have $\angle PQR = 180^\circ - 2z$ and $\angle R_3PQ = 180^\circ - 2y$. As we have shown in the first solution, triangle $DEF$ have the same angles. Hence triangle $R_3PQ$ is similar to triangle $DEF$. Also note that $\angle POQ + \angle PR_3Q = 2x + 180^\circ - 2x = 180^\circ$, implying that $R_3$ lies on the circumcircle of triangle $OPQ$. By the given condition, have $R_3 = R$. We can then finish our proof as we did in the first solution.

**Note:** As we have seen, the first solution is related to the 9-point circle of the triangle, and the second is related to the Miquel's Theorem. Indeed, it is the special case (for $R_1 = R_2$) of the following interesting facts:




In acute triangle $ABC$, segments $AD, BE$, and $CF$ are its altitudes, and $H$ is its orthocenter. Circle $\omega$, centered at $O$, passes through $A$ and $H$ and intersects sides $AB$ and $AC$ again at $Q$ and $P$ (other than $A$), respectively.

(a) The perpendicular bisectors of segments $BQ$ and $CP$ meet at a point $R_1$ lying on line $BC$.

(b) There is a point $R_2$ on line $BC$ such that triangle $PQR_2$ is similar to triangle $EFD$.

(c) Points $O, P, Q, R_1$, and $R_2$ are cyclic.

(This problem was proposed by Zuming Feng and Zhonghao Ye.)

3. Find the least real number $k$ with the following property: if the real numbers $x, y$, and $z$ are not all positive, then

$$k(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) \geq (xyz)^2 - xyz + 1.$$

**First Solution:** The answer is $k = \frac{16}{9}$.

We start with a lemma.

**Lemma 1.** *If real numbers $s$ and $t$ are not all positive, then*

$$\frac{4}{3}(s^2 - s + 1)(t^2 - t + 1) \geq (st)^2 - st + 1. \tag{$*$}$$

*Proof:* Without loss of generality, we assume that $s \geq t$.

We first assume that $s \geq 0 \geq t$. Setting $u = -t$, $(*)$ reads

$$\frac{4}{3}(s^2 - s + 1)(u^2 + u + 1) \geq (su)^2 + su + 1,$$

or

$$4(s^2 - s + 1)(u^2 + u + 1) \geq 3s^2u^2 + 3su + 3.$$

Expanding the left-hand side gives

$$4s^2u^2 + 4s^2u - 4su^2 - 4su + 4s^2 + 4u^2 - 4s + 4u + 4 \geq 3s^2u^2 + 3su + 3,$$

or

$$s^2u^2 + 4u^2 + 4s^2 + 1 + 4s^2u + 4u \geq 4su^2 + 4s + 7su$$

which is evident as $s^2u^2 + 4u^2 \geq 4su^2$, $4s^2 + 1 \geq 4s$, and $4s^2u + 4u \geq 8su \geq 7su$.

We second assume that $0 \geq s \geq t$. Let $v = -s$. By our previous argument, we have

$$\frac{4}{3}(v^2 - v + 1)(t^2 - t + 1) \geq (vt)^2 - vt + 1.$$

It is clear that $t^2 - t + 1 > 0$, $s^2 - s + 1 \geq v^2 - v + 1$, and $(vt)^2 - vt + 1 \geq (st)^2 - st + 1$. Combining the last four inequalities gives $(*)$, and this completes the proof of the lemma. ∎

Now we show that if $x, y, z$ are not all positive real numbers, then

$$\frac{16}{9}(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) \geq (xyz)^2 - xyz + 1. \tag{$**$}$$

We consider three cases.

(a) We assume that $y \geq 0$. Setting $(s,t) = (y,z)$ and then $(s,t) = (x, yz)$ in the lemma gives the desired result.

(b) We assume that $0 \geq y$. Setting $(s,t) = (x,y)$ and then $(s,t) = (xy, z)$ in the lemma gives the desired result.

Finally, we confirm that the minimum value of $k$ is $\frac{16}{9}$ by noting that the equality holds in $(**)$ when $(x,y,z) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

**Second Solution:** We establish $(**)$ by showing

$$g(z) = \frac{16}{9}(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) - (xyz)^2 + xyz - 1 \geq 0.$$

Note that $g(z)$ is a quadratic in $z$ whose axis of symmetry (found by comparing the linear and quadratic terms) is at

$$\begin{aligned} z &= \frac{1}{2} - \frac{9}{32} \cdot \frac{xy}{(x^2 - x + 1)(y^2 - y + 1)} \\ &= \frac{1}{2} - \frac{9}{32} \cdot \frac{1}{\left(x + \frac{1}{x} - 1\right)\left(y + \frac{1}{y} - 1\right)}. \end{aligned}$$

For any $t$, we have $\left|x + \frac{1}{x} - 1\right| \geq 1$, so the absolute value of the second quantity on the right-hand side of the above equation is at most $\frac{9}{32}$, which is less than $\frac{1}{2}$. That is, the axis of symmetry occurs to the right side of the $y$-axis, so we only decrease the difference between the sides by replacing $z$ by 0. But when $z = 0$, we only need to show

$$g(0) = \frac{16}{9}(x^2 - x + 1)(y^2 - y + 1) - 1 \geq 0,$$

which is evident as $t^2 - t + 1 = \left(t - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$.

**Third Solution:** This is the Calculus version of the second solution. We maintain the same notation as in the second solution. We have

$$\frac{dg}{dz} = \frac{16}{9}(2z - 1)(x^2 - x + 1)(y^2 - y + 1) - 2zx^2y^2 + xy$$

or

$$\frac{dg}{dz} = 2z\left[\frac{4}{3}(x^2 - x + 1)\frac{4}{3}(y^2 - y + 1) - x^2y^2\right] + \left[xy - \frac{4}{3}(x^2 - x + 1)\frac{4}{3}(y^2 - y + 1)\right]. \qquad (\dagger)$$

It is evident that

$$\frac{4}{3}(t^2 - t + 1) \geq t^2 \geq 0$$

as it is equivalent to $t^2 - 4t + 4 = (t - 2)^2 \geq 0$. It follows that

$$2z\left[\frac{4}{3}(x^2 - x + 1)\frac{4}{3}(y^2 - y + 1) - x^2y^2\right] \leq 0;$$

that is, the first summand on the right-hand side of ($\dagger$) is not positive. It is also evident that

$$\frac{4}{3}(t^2 - t + 1) \geq t$$

as it is equivalent to $4t^2 - 7t + 4 = 4\left(t - \frac{7}{8}\right)^2 + \frac{15}{16} > 0$. If $y \geq 0$, then multiplying the inequalities

$$\frac{4}{3}(x^2 - x + 1) \geq x \geq 0 \quad \text{and} \quad \frac{4}{3}(y^2 - y + 1) \geq y \geq 0$$

gives

$$\frac{4}{3}(x^2 - x + 1)\frac{4}{3}(y^2 - y + 1) - xy \geq 0.$$

If $y < 0$, then $xy < 0$, and so

$$\frac{4}{3}(x^2 - x + 1)\frac{4}{3}(y^2 - y + 1) \geq 0 \geq xy.$$

In either case, we have shown that the second summand in ($\dagger$) is also negative. We conclude that $\frac{dg}{dz} \leq 0$ for $z \leq 0$. Hence $g(z)$ reaches minimum when $z = 0$, and we can finish as we did in the second solution.

(This problem was proposed by Titu Andreescu and Gabriel Dospinescu.)

4. Let $n$ be a positive integer. Find, with proof, the least positive integer $d_n$ which cannot be expressed in the form
$$\sum_{i=1}^{n}(-1)^{a_i}2^{b_i},$$
where $a_i$ and $b_i$ are nonnegative integers for each $i$.

**Solution:** The answer is $d_n = (2^{2n+1})/3$. We first show that $d_n$ cannot be obtained. For any $p$ let $t(p)$ be the minimum $n$ required to express $p$ in the desired form and call any realization of this minimum a *minimal representation.* If $p$ is even, any sequence of $b_i$ that can produce $p$ must contain an even number of zeros. If this number is nonzero, then canceling one against another or replacing two with a $b_i = 1$ term would reduce the number of terms in the sum. Thus a minimal representation cannot contain a $b_i = 0$ term, and by dividing each term by two we see that $t(2m) = t(m)$. If $p$ is odd, there must be at least one $b_i = 0$ and removing it gives a sequence that produces either $p-1$ or $p+1$. Hence
$$t(2m-1) = 1 + \min(t(2m-2), t(2m) = 1 + \min(t(m-1), t(m)).$$
With $d_n$ as defined above and $c_n = (2^{2n}-1)/3$, we have $d_0 = c_1 = 1$, so $t(d_0) = t(c_1) = 1$ and
$$t(d_n) = 1 + \min(t(d_{n-1}), t(c_n)) \quad \text{and} \quad t(c_n) = 1 + \min(t(d_{n-1}), t(c_{n-1})).$$
Hence, by induction, $t(c_n) = n$ and $t(d_n) = n+1$ and $d_n$ cannot be obtained by a sum with $n$ terms.

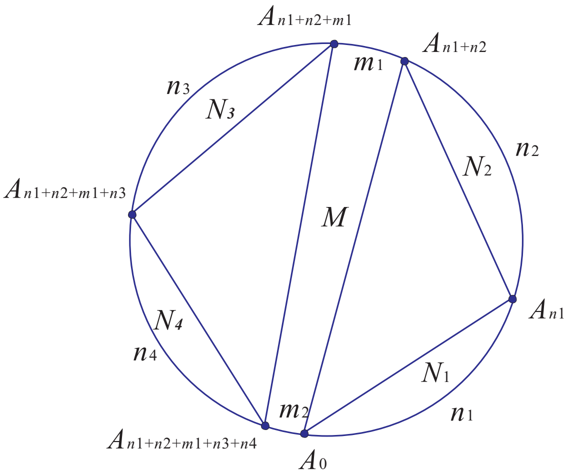
Next we show by induction on $n$ that any positive integer less than $d_n$ can be obtained with $n$ terms. By the inductive hypothesis and symmetry about zero, it suffices to show that by adding one summand we can reach every $p$ in the range $d_{n-1} \le p < d_n$ from an integer $q$ in the range $-d_{n-1} < q < d_{n-1}$. Suppose that $c_n + 1 \le p \le d_n - 1$. By using a term $2^{2n-1}$, we see that $t(p) \le 1 + t(|p - 2^{2n-1}|)$. Since $d_n - 1 - 2^{2n-1} = 2^{2n-1} - (c_n + 1) = d_{n-1} - 1$, it follows from the inductive hypothesis that $t(p) \le n$. Now suppose that $d_{n-1} \le p \le c_n$. By using a term $2^{2n-2}$, we see that $t(p) \le 1 + t(|p - 2^{2n-2}|)$. Since $c_n - 2^{2n-2} = 2^{2n-2} - d_{n-1} = c_{n-1} < d_{n-1}$, it again follows that $t(p) \le n$.

(This problem was proposed by Richard Stong.)

5. Let $n$ be a given integer with $n$ greater than 7, and let $\mathcal{P}$ be a convex polygon with $n$ sides. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. A triangle in the triangulation of $\mathcal{P}$ is an interior triangle if all of its sides are diagonals of $\mathcal{P}$.

Express, in terms of $n$, the number of triangulations of $\mathcal{P}$ with exactly two interior triangles, in closed form.

**Solution:** The answer is
$$n2^{n-9}\binom{n-4}{4}.$$
Denote the vertices of $P$ counter-clockwise by $A_0, A_1 \ldots, A_{n-1}$. We will count first the number of triangulations of $P$ with two interior triangles positioned as in the following figure. We say that such a triangulation starts at $A_0$.




The numbers $m_1$, $m_2$, $n_1$, $n_2$, $n_3$, $n_4$ in the figure denote the number of sides of $P$ determining the regions $N_1$, $N_2$, $N_3$, $N_4$ and $M$ that consist of exterior triangles (triangles that are not interior). The two interior triangles are

$$A_0 A_{n_1} A_{n_1+n_2} \quad \text{and} \quad A_{n_1+n_2+m_1} A_{n_1+n_2+m_1+n_3} A_{n_1+n_2+m_1+n_3+n_4},$$

respectively.

We will show that triangulations starting at $A_0$ are in bijective correspondence to 7-tuples

$$(m, n_1, n_2, n_3, n_4, w_M, w_N),$$

where $m \geq 0$, $n_1, n_2, n_3, n_4 \geq 2$ are integers,

$$m + n_1 + n_2 + n_3 + n_4 = n, \tag{†}$$

$w_M$ is a binary sequence (sequence of 0's and 1's) of length $m$ and $w_N$ is a binary sequence of length $n - m - 8$.

Indeed, given a triangulation as in the figure, the numbers $m = m_1 + m_2$ and $n_1, n_2, n_3, n_4$ satisfy (†) and the associated constraints.

Further, the triangulation of the outside region $N_1$ determines a binary sequence of length $n_1 - 2$ as follows. Denote the exterior triangle in $N_1$ using the diagonal $A_0 A_{n_1}$ by $T_1$. If $n_1 \geq 3$, $T_1$ has a unique neighboring exterior triangle in $N_1$, denoted $T_2$. If $n_1 \geq 4$, the triangle $T_2$ has another neighbor in $N_1$ denoted $T_3$, etc. Thus we have a sequence of $n_1 - 1$ exterior triangles in $N_1$. We encode this sequence as follows. If $T_1$ uses the vertex $A_1$ as its third vertex we encode this by 0 and if it uses $A_{n_1-1}$ we encode this by 1. In each case there are two possible choices for the third vertex in $T_2$. If the one with smaller index is used we encode this by 0 and if the one with larger index is used we encode this by 1. Eventually, a sequence of $n_1 - 2$ 0's and 1's is constructed describing the choice of the third vertex in the triangles $T_1, \ldots, T_{n_1-2}$. Finally, there is only one choice for the third vertex in the triangle $T_{n_1-1}$ (this triangle is uniquely determined by the previous one), so we get $2^{n_1-2}$ possible triangulations of $N_1$ encoded in a binary sequence of length $n_1 - 2$. Similarly, there are $2^{n_i-2}$ triangulations of the region $N_i$, $i = 1, 2, 3, 4$, encoded by binary sequences of length $n_i - 2$. Thus a binary sequence $w_N$ of length $n_1 - 2 + n_2 - 2 + n_3 - 2 + n_4 - 2 = n - m - 8$, uniquely determines the triangulations of the regions $N_1, N_2, N_3, N_4$ (once the regions are precisely determined within $P$, which is done once $m_1$,$m_2$,$n_1$,$n_2$,$n_3$ and $n_4$ are known).

It remains to uniquely encode the triangulation of the middle region $M$. Denote by $M_1$ the unique exterior triangle in $M$ using the diagonal $A_0A_{n_1+n_2}$. If $m \geq 2$. $M_1$ has a unique neighboring exterior triangle $M_2$ in $M$. If $m \geq 3$, the triangle $M_2$ has another neighbor in $M$ denoted $M_3$, etc. Thus we have a sequence of $m$ exterior triangles in $M$. We encode this sequence as follows. If $M_1$ uses the vertex $A_{n_1+n_2+1}$ as its third vertex we encode this by 0 and if it uses $A_{n-1}$ we encode this by 1. In each case there are two possible choices for the third vertex in $M_2$. If the one with smaller index is used we encode this by 0 and if the one with larger index is used we encode this by 1. Eventually, a sequence of $m$ 0's and 1's is constructed describing the choice of the third vertex in the triangles $M_1, \ldots, M_m$. Thus a binary sequence $w_M$ of length $m$ uniquely determines the triangulation of the region $M$. In addition such a sequence $w_M$ uniquely determines $m_1$ and $m_2$ as the number of 0's and 1's respectively in $w_M$ and therefore also the exact position of the middle region $M$ within $P$ (once $n_1$ and $n_2$ are known), which in turn then exactly determines the position of all the regions considered in the figure.

The number of solutions of the equation (†) subject to the given constraints is equal to the number of positive integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = n - 3,$$

which is $\binom{n-4}{4}$ (a sequence of $n-3$ objects is split into 5 nonempty groups by placing 4 separators in the $n-4$ available positions between the objects). Thus the number of 7-tuples $(m, n_1, n_2, n_3, n_4, w_M, w_N)$ describing triangulations as in the figure is

$$2^m \cdot 2^{n-m-8}\binom{n-4}{4} = 2^{n-8}\binom{n-4}{4}.$$

Finally, in order to get the total number of triangulations we multiply the above number by $n$ (since we could start building the triangulation at any vertex rather than at $A_0$) and divide by 2 (since every triangulation is now counted twice, once as starting at one of the interior triangles and once as starting at the other).

**Note:** The problem is more tricky than it might seem. In particular, the idea of choosing $m$ first and then letting the bits in $w_M$ split it into $m_1$ and $m_2$ while, in the same time, determining the triangulation of $M$ is not that obvious. If one does the "more natural thing" and chooses all the the numbers $m_1$, $m_2$, $n_1$, $n_2$, $n_3$, $n_4$ first and then tries to encode the triangulations of the obtained regions one gets into more complicated considerations involving the middle region $M$ (and most likely has to resort to messy summations over different pairs $m_1, m_2$).

As an quick exercise, one can compute number of triangulations of $P$ ($n \geq 6$) with exactly one interior region. This is much easier since there is no middle region $M$ to worry about and the number of triangulations is

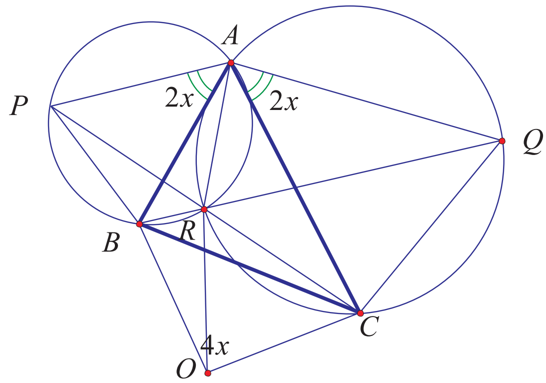
$$\frac{n}{3}2^{n-6}\binom{n-4}{2}.$$

(This problem was proposed by Zoran Sunik.)

6. Let $ABC$ be a triangle. Triangles $PAB$ and $QAC$ are constructed outside of triangle $ABC$ such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments $BQ$ and $CP$ meet at $R$. Let $O$ be the circumcenter of triangle $BCR$. Prove that $AO \perp PQ$.

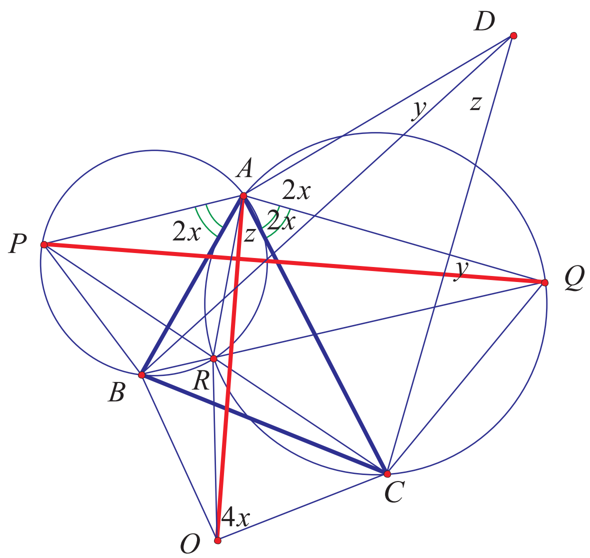
**Note:** We present five differen approaches. The first three synthetic solutions are all based on the following simple observation.

We first note that $APBR$ and $AQCR$ are cyclic quadrilaterals. It is easy to see that triangles $APC$ and $ABQ$ are congruent to each other, implying that $\angle APR = \angle APC = \angle ABQ = \angle ABR$. Thus, $APBR$ is a cyclic quadrilateral. Likewise, we can show that $AQCR$ is also cyclic.




Let $\angle PAB = 2x$. Then in isosceles triangle $APB$, $\angle APB = 90^\circ - x$. In cyclic quadrilateral $APBR$, $\angle ARB = 180^\circ - \angle APB = 90^\circ + x$. Likewise, $\angle ARC = 90^\circ + x$. Hence $\angle BRC = 360^\circ - \angle ARB - \angle ARC = 180^\circ - 2x$. It follows that $\angle BOC = 4x$.

**First Solution:** Reflect $C$ across line $AQ$ to $D$. Then $\angle BAD = 4x + \angle BAC = \angle BAQ$. It is easy to see that triangles $BAD$ and $PAQ$ are congruent, implying that $\angle ADB = \angle AQP = y$.




Note also that $CAD$ and $COB$ are two isosceles triangles with the same vertex angle, and so they are similar to each other. It follows that triangle $CAO$ and $CBD$ are similar by SAS (side-angle-side), implying that $\angle CAO = \angle CDB = z$.

The angle formed by lines $AO$ and $PQ$ is equal to

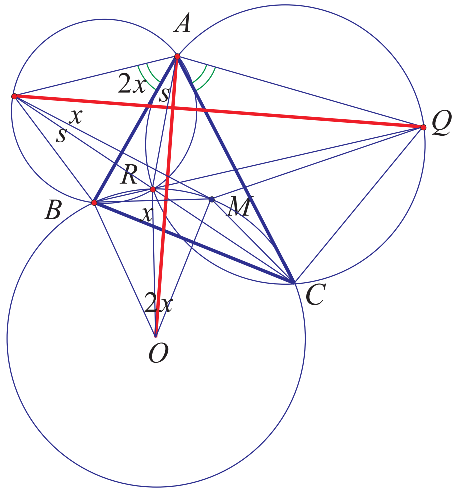
$$180^\circ - \angle OAQ - \angle AQP = 180^\circ - \angle OAC - \angle CAQ - \angle AQP = 180^\circ - z - 2x - y.$$

Since $AQ$ is perpendicular to the base $CD$ in isosceles triangle $ACD$, we have

$$90^\circ = \angle QAD + \angle CDA = \angle QAD + \angle ADB + \angle BDC = 2x + y + z.$$

Combining the last two equations yields that fact the angle formed by lines $AO$ and $PQ$ is equal to $90^\circ$; that is, $AO \perp PQ$.

**Second Solution:** We maintain the same notations as in the first solution. Let $M$ be the midpoint of arc $\widehat{BC}$ on the circumcircle of triangle $BOC$. Then $BM = CM$. Since triangles $APC$ and $ABQ$ are congruent, $PC = BQ$. Since $BRMC$ is cyclic, $\angle PCM = \angle RCM = \angle RBM = \angle QBM$. Hence triangles $BMQ$ and $CMP$ are congruent by SAS. It follows triangles $MPQ$ and $MBC$ are similar. Since $\angle BOC = 4x$, $\angle MBC = \angle MCB = x$, and so $\angle MPQ = x$.




Note that both triangles $PAB$ and $MOB$ are isosceles triangles with vertex angle $2x$; that is, they are similar to each other. Hence triangles $BMP$ and $BOA$ are also similar by SAS, implying that $\angle OAB = MPB = s$. We also note that in isosceles triangle $APB$,

$$90^\circ = \angle APB + \angle PAB/2 = \angle APQ + \angle QPM + \angle MPB + \angle PAB/2 = \angle APQ + 2x + s.$$

Putting the above together, we conclude that

$$\angle PAO + \angle APQ = \angle PAB + \angle BAO + \angle APQ = 2x + s + \angle APQ = 90^\circ,$$

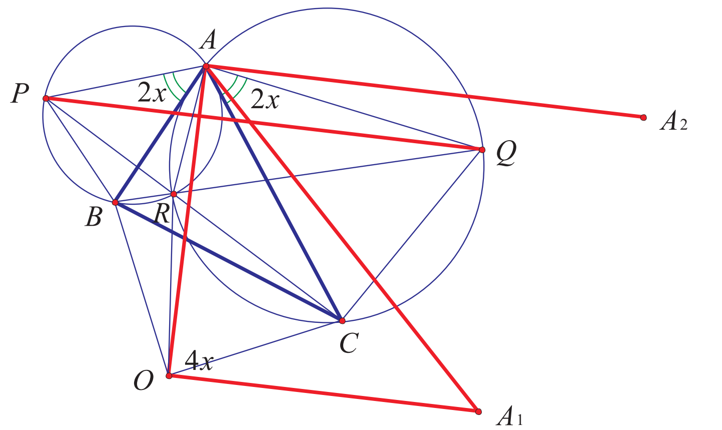
that is $AO \perp PQ$.

**Third Solution:** We consider two rotations:

$$\begin{aligned} \mathbf{R}_1: &\quad \text{a counterclockwise } 2x \text{ (degree) rotation centered at } A,\\ \mathbf{R}_2: &\quad \text{a clockwise } 4x \text{ (degree) rotation centered at } O. \end{aligned}$$

Let $\mathbf{T}$ denote the composition $\mathbf{R}_1\mathbf{R}_2\mathbf{R}_1$. Then $\mathbf{T}$ is a counterclockwise $2x - 4x + 2x = 0^\circ$ rotation; that is, $\mathbf{T}$ is translation. Note that

$$\mathbf{T}(P) = \mathbf{R}_1(\mathbf{R}_2(\mathbf{R}_1(P))) = \mathbf{R}_1(\mathbf{R}_2(B)) = \mathbf{R}_1(C) = PQ,$$




or, $\mathbf{T}$ is the vector translation $\overrightarrow{PQ}$.

Let $A_1 = \mathbf{R}_2(A)$ and $A_2 = \mathbf{R}_1(A_1)$. Then $\mathbf{T}(A) = A_2$; that is, $\overrightarrow{AA_2} = \overrightarrow{PQ}$, or $AA_2 \parallel PQ$.

By the definitions of $\mathbf{R}_2$ and $\mathbf{R}_1$, we know that triangles $OAA_1$ and $A_1AA_2$ are isosceles triangles with respect vertex angles $\angle AOA_1 = 4x$ and $\angle A_1AA_2 = 2x^\circ$. It is routine to compute that $\angle OAA_2 = 90^\circ$; that $AO \perp AA_2$, or $AO \perp PQ$.

**Fourth Solution:** (By Lan Le) In this solutions, let each lowercase letter denote the number assigned to the point labeled with the corresponding uppercase letter. We further assume that $A$ is origin; that is, let $a = 0$. Let $\omega = e^{2xi}$ (or $\omega = \cos(2x) + i\sin(2x)$, and $\omega^{-1} = \cos(2x) - i\sin(2x)$). Then because $O$ lies on the perpendicular bisector of $BC$ and $\angle BOC = 4x$,

$$o = c + \frac{(b-c)i}{2\omega\sin(2x)} = c + \frac{bi}{2\omega\sin(2x)} - \frac{ci}{2\omega\sin(2x)}.$$

Note that

$$c - \frac{ci}{2\omega\sin(2x)} = c + \frac{c\omega^{-1}}{2i\sin(2x)} = \frac{c(\omega^{-1} + 2i\sin(2x))}{2i\sin(2x)} = \frac{c\omega}{2i\sin(2x)},$$

Combining the last two equations gives

$$o = \frac{bi}{2\omega\sin(2x)} + \frac{c\omega}{2i\sin(2x)} = -\frac{b}{2i\omega\sin(2x)} + \frac{c\omega}{2i\sin(2x)} = \frac{1}{2i\sin(2x)}\left(c\omega - \frac{b}{\omega}\right).$$

Now we note that $p = \frac{b}{\omega}$ and $q = c\omega$. Consequently, we obtain

$$\frac{q-p}{o-a} = 2i\sin(2x),$$

which is clearly a pure imaginary number; that is, $OA \perp PQ$.

**Fifth Solution:** (By Lan Le) In this solutions, we set $BC = a, AB = c$, $CA = b$, $A = \angle BAC$, $B = \angle ABC$, and $C = \angle BCA$. We use the fact that

$$OA \perp PQ \quad \text{if and only if} \quad AP^2 - AQ^2 = OP^2 - OQ^2.$$

Clearly $AP^2 - AQ^2 = c^2 - b^2$. It remains to show that

$$OP^2 - OQ^2 = c^2 - b^2. \tag{$*$}$$

In isosceles triangles $APB$ and $BOC$, $BP = 2c \sin x$ and $BO = \frac{a}{2\sin(2x)}$. Note that $\angle PBA + \angle ABC + \angle CBO = 90^\circ - x + B + 90^\circ - 2x = 180^\circ + B - 3x$. Applying the **law of cosines** to triangle $PBO$ yields

$$OP^2 = 4c^2 \sin^2 x + \frac{a^2}{4\sin^2(2x)} + \frac{ac\cos(B-3x)}{\cos x}.$$

In exactly the same way, we can show that

$$OQ^2 = 4b^2 \sin^2 x + \frac{a^2}{4\sin^2(2x)} + \frac{ab\cos(C-3x)}{\cos x}.$$

Hence

$$OP^2 - OQ^2 = 4(c^2 - b^2)\sin^2 x + \frac{a}{\cos x}(c\cos(B-3x) - b\cos(C-3x)). \tag{†}$$

Using **Addition and Substraction formulas** and the **law of sines** (more precisely, $c \sin B = b \sin C$), we have

$$\begin{aligned} & c\cos(B-3x) - b\cos(C-3x) \\ = \ & c\cos(3x)\cos B + c\sin(3x)\sin B - b\cos(3x)\cos C - b\sin(3x)\sin C \\ = \ & \cos(3x)(c\cos B - b\cos C). \end{aligned}$$

Substituting the last equation into (†) gives

$$OP^2 - OQ^2 = 4(c^2 - b^2)\sin^2 x + \frac{\cos 3x}{\cos x}(ac\cos B - ab\cos C).$$

Note that

$$ac\cos B - ab\cos C = c(a\cos B + b\cos A) - b(a\cos C + c\cos A) = c^2 - b^2.$$

Combining the last equations gives

$$OP^2 - OQ^2 = (c^2 - b^2)\left(4\sin^2 x + \frac{\cos 3x}{\cos x}\right).$$

By the **Triple-angle formulas**, we have $\cos 3x = 4\cos^3 x - 3\cos x$, and so

$$OP^2 - OQ^2 = (c^2 - b^2)(4\sin^2 x + 4\cos^2 x - 3) = c^2 - b^2,$$

which is $(*)$.

(This problem was proposed by Zuming Feng and Zhonghao Ye.)