3^{rd} United States of America Junior Mathematical Olympiad

Day II   12:30 PM – 5 PM EDT

April 25, 2012

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

JMO 4. Let α be an irrational number with $0 < \alpha < 1$, and draw a circle in the plane whose circumference has length 1. Given any integer $n \geq 3$, define a sequence of points P_1, P_2, $\dots$, P_n as follows. First select any point P_1 on the circle, and for $2 \leq k \leq n$ define P_k as the point on the circle for which the length of arc $P_{k-1}P_k$ is α, when travelling counterclockwise around the circle from P_{k-1} to P_k. Suppose that P_a and P_b are the nearest adjacent points on either side of P_n. Prove that $a + b \leq n$.

JMO 5. For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \leq k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

JMO 6. Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB, respectively. Prove that A', B', C' are collinear.