3^{rd} United States of America Junior Mathematical Olympiad

Day I, II 12:30 PM – 5 PM EDT

April 24-25, 2012

JMO 1. **Solution 1** We use the following lemma.

Lemma. Given a triangle ABC, X, Y, Z are points on BC, CA, AB respectively. Then three perpendicular lines of BC, CA, AB which go through X, Y, Z respectively are concurrent if and only if $AY^2 + BZ^2 + CX^2 = AZ^2 + BX^2 + CY^2$.

Proof of Lemma. If the lines are concurrent, let P be the point on the three lines. From $BX^2 - CX^2 = (PB^2 - PX^2) - (PC^2 - PX^2) = PB^2 - PC^2$ and so on, we obtain the desired result. Conversely, if $AY^2 + BZ^2 + CX^2 = AZ^2 + BX^2 + CY^2$ holds, let Q be the intersection of perpendicular lines of BC, CA which go through X, Y respectively. Then as we have seen $BX^2 - CX^2 = QB^2 - QC^2$ and $CY^2 - AY^2 = QC^2 - QA^2$ holds. Summing up these equations, we have $AZ^2 - BZ^2 = QA^2 - QB^2$. This implies that QZ and AB are perpendicular, as desired. **End of the Proof**

Let M be the midpoint of SR. We show that $AP^2 + BM^2 + CQ^2 = AQ^2 + BP^2 + CM^2$. Since $AP = AQ$, $CQ^2 = CR \cdot CS$, $BP^2 = BS \cdot BR$, and $BM^2 - CM^2 = (BM + CM)(BM - CM) = BC(BS - RC)$, we have $(AP^2 + BM^2 + CQ^2) - (AQ^2 + BP^2 + CM^2) = BC(BS - RC) - BS \cdot BR + CR \cdot CS = BS \cdot CR - CR \cdot BC = 0$. Thus there exists a point O such that $OP \perp BC$, $OQ \perp AC$, $OM \perp BC$. Then O is the center of a circimcircle of PRS, since the circle is tangent to AB at P. Similarly, O is the center of a circumcircle of QRS, which implies that P, Q, R, S are on a circle.

Solution 2 By the given hypothesis, we have a circle Γ_1 which passes through S and R, and touches AB at P. Similarly, we have a circle Γ_2 which passes through S and R, and touches AC at Q. Suppose that the circles Γ_1 and Γ_2 are different from each other. Then the power of A onto Γ_1 is AP^2, and the power of A onto Γ_2 is AQ^2. This implies that A is on the radical axis of Γ_1 and Γ_2, namely the line BC, which is a contradiction. Hence, we have $\Gamma_1 = \Gamma_2$, so that P, Q, R, S are concyclic, as desired.

Solution 3 We use the same notations as in the Solution 2. Suppose again that $\Gamma_1 \neq \Gamma_2$. Let l be the perpendicular bisector of SR, and consider a circle γ passing through S and R whose center is moving on l. Suppose that initially the center of γ is on the half plane divided by BC in which A does not lie. Moving the center toward A, γ would touch AB and AC, not simultaneously by the hypothesis. Without loss of generality, suppose that γ touches AB at P first, and then touches AC at Q. Note that γ of these situations are Γ_1 and Γ_2 respectively.

We increases the radius of Γ_1, keeping the circle tangent to AB. Then it will touch AC eventually. Let Γ_1' be the circle, which is tangent to AB and AC at P and Q respectively and meets BC at two points S' and R'. Note that on BC, the points are ordered as B, S', S, R, R', C. We have $\angle BPS = \angle PRS$ and $\angle BPS' = \angle PR'S'$, which imply $\angle SPS' = \angle RPR'$. Similarly, we have $\angle SQS' = \angle RQR'$. Without loss of generality, suppose that on the circle Γ_1', the points are ordered as S', P, Q, R'. Let lines PS, PR, QS, QR meet Γ_1' again at T_1, U_1, T_2, U_2 respectively. Then the points on Γ_1' are ordered as $S', T_2, T_1, U_2, U_1, R'$. From $\angle SPS' = \angle RPR'$ we have $\overset{\frown}{S'T_1} = \overset{\frown}{U_1R'}$ and from $\angle SQS' = \angle RQR'$, we have $\overset{\frown}{S'T_2} = \overset{\frown}{U_2R'}$. However, we have $\overset{\frown}{S'T_2} < \overset{\frown}{S'T_1} = \overset{\frown}{U_1R'} < \overset{\frown}{U_2R'}$, which leads us to a contradiction. Hence, we have $\Gamma_1 = \Gamma_2$, as desired.

Solution 4 Let Γ_3 be the circle tangent to AB and AC at P and Q respectively. Inverse the plane around P. We denote by X' the image of any point or any set X via the inversion. A', P, B' are collinear in this order, and the image of AC is a circle $(AC)'$ passing through A' and P. Then Γ_3' is a line which is tangent to $(AC)'$ and parallel to $A'P$. Note that the tangency point is Q'. Γ_1' is a line parallel to $A'P$. Finally, B', S', R' are on a circle passing through P, and S', R' are on Γ_1'.

Suppose $\Gamma_1 \neq \Gamma_3$. Then clearly we have $\Gamma_1' \neq \Gamma_3'$. Note that Q' is on the perpendicular bisector l of $A'P$. Since $PB'R'S'$ is cyclic and PB' and $R'S'$ are parallel, it is an isosceles trapezoid. Now we consider Γ_2'. This circle should be tangent to Γ_1' at Q', so the center of Γ_2' must lie on l. However, Since Γ_2' passes through R' and S', the center must lie on the perpendicular bisector of $R'S'$ which is the same as the one of PB'. Since A' and B' lie on the different ray centered on P, this is impossible. Therefore, we have $\Gamma_1 = \Gamma_3$, on which P, Q, R, S lie.

Solution 5 In the case that $AB = AC$, suppose $\alpha = \angle BPS > \angle CQR = \beta$. Let R' be a point on BC such that $BS = R'C$. We then have that two triangles BPS and CQR' are congruent. Hence, $\angle CQR' = \alpha > \beta = \angle CQR$, so that R lies between R' and C. However, then we have $\beta = \angle QSC = \angle PR'S > \angle PRS = \alpha$, contradiction. Hence we have $\alpha = \beta$, so the trapezoid $PQRS$ is isosceles, as desired.

Now suppose $AB \neq AC$, and PQ and BC meet at X. Without loss of generality, suppose $B > C$ so that B lies between X and C. Let $AP = AQ = t, XB = x, BS = y, RC = z$. To deduce x, we apply Menelaus' theorem to the triangle ABC and a line XPQ to obtain $\frac{AQ}{QC}\frac{CX}{XB}\frac{BP}{PA} = 1$. This yields $x = \frac{c-t}{b-c}a$.

From the hypothesis, we have $(c-t)^2 = y(a-z)$ and $(b-t)^2 = z(a-y)$. From these results, we have $(c-t)^2 - (b-t)^2 = (y-z)a$, so that $y - z = \frac{(c-b)(b+c-2t)}{a}$. Hence, we obtain

$$\begin{aligned}
XS \cdot XR &= (x+y)(x+a-z) = x^2 + (a+y-z)x + (c-t)^2 \\
&= x^2 + (a + \frac{(c-b)(b+c-2t)}{a})x + (c-t)^2 \\
&= \frac{(c-t)^2}{(b-c)^2}a^2 + \frac{c-t}{b-c}a^2 + (t-c)(b+c-2t) + (c-t)^2 \\
&= \frac{(b-t)(c-t)}{(b-c)^2}a^2 + (t-c)(b-t) = \frac{(b-t)(c-t)}{(b-c)^2}(a^2 - (b-c)^2) \\
&= \frac{(b-t)(c-t)}{(b-c)^2}(a-b+c)(a+b-c).
\end{aligned}$$

On the other hand, since $\angle APQ = \frac{\pi - A}{2}$, we have $\angle PXB = \frac{B-C}{2}$. Applying the Sine theorem to the triangle XPB, we have $\frac{x}{\sin\frac{\pi-A}{2}} = \frac{XP}{\sin B} \Leftrightarrow XP = x\frac{\sin B}{\cos\frac{A}{2}}$. From Menelaus' theorem again, we have $\frac{QX}{XP}\frac{PB}{BA}\frac{AC}{CQ} = 1$, or equivalently $XQ = XP\frac{c}{c-t}\frac{b-t}{b}$. Hence, we have

$$\begin{aligned}
XP \cdot XQ &= x^2 \frac{\sin^2 B}{\cos^2\frac{A}{2}}\frac{c(b-t)}{b(c-t)} \\
&= \frac{(c-t)^2}{(b-c)^2}a^2\frac{(\frac{b}{2R})^2}{\frac{(a+b+c)(-a+b+c)}{4bc}}\frac{c(b-t)}{b(c-t)} \\
&= \frac{(b-t)(c-t)}{(b-c)^2}\frac{a^2b^2c^2}{R^2(a+b+c)(-a+b+c)} \\
&= \frac{(b-t)(c-t)}{(b-c)^2}\frac{16R^2S^2}{R^2(a+b+c)(-a+b+c)} \\
&= \frac{(b-t)(c-t)}{(b-c)^2}(a-b+c)(a+b-c),
\end{aligned}$$

where R is the circumradius of the triangle ABC and S is the area of the triangle ABC. Since we have now that $XP \cdot XQ = XS \cdot XR$, the four points are concyclic, as desired.

Comment. It is a degenerated version of the following statement: if $ABCDEF$ is a convex hexagon and $ABCD, CDEF$, and $EFAB$ are cyclic quadrilaterals, then $ABCDEF$ is a cyclic hexagon. This can be easily verified by the similar idea to the First and Second solution.

This problem and solution were suggested by Sungyoon Kim and Inseok Seo.

JMO 2. First we prove that any $n \geq 13$ is a solution of the problem. Suppose that $a_1, a_2, ..., a_n$ satisfy $\max(a_1, a_2, ..., a_n) \leq n \cdot \min(a_1, a_2, ..., a_n)$, and that we cannot find three that are the side-lengths of an acute triangle. We may assume that $a_1 \leq a_2 \leq ... \leq a_n$. Then $a_{i+2}^2 \geq a_i^2 + a_{i+1}^2$ for all $i \leq n-2$. Let (F_n) be the Fibonacci sequence, with $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. It is easy to check that $F_n < n^2$ for $n \leq 11$, $F_{12} = 12^2$ and $F_n > n^2$ for $n > 12$ (the last inequality follows by an immediate induction, while the first one can be checked by hand). The inequality $a_{i+2}^2 \geq a_i^2 + a_{i+1}^2$ and the fact that $a_1 \leq a_2 \leq ... \leq a_n$ imply that $a_i^2 \geq F_i \cdot a_1^2$ for all $i \leq n$. Hence, if $n \geq 13$, we obtain $a_n^2 > n^2 \cdot a_1^2$, contradicting the hypothesis. This shows that any $n \geq 13$ is a solution of the problem.

By taking $a_i = \sqrt{F_i}$ for $1 \leq i \leq n$, we have $\max(a_1, a_2, ..., a_n) \leq n \cdot \min(a_1, a_2, ..., a_n)$, for any $n < 13$, but it is easy to see that no three a_i's can be the side-lengths of an acute triangle. Hence the answer to the problem is: all $n \geq 13$.

This problem and solution were suggested by Titu Andreescu.

JMO 3. **Solution 1:** Recall the following form of Cauchy-Schwarz inequality,

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \ldots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{y_1 + y_2 + \ldots + y_n}.$$

It also follows from the Cauchy-Schwarz inequality that $x_1^2+x_2^2+x_3^2 \geq x_1x_2+x_2x_3+x_3x_1$. From these two inequalities, deduce that

$$\begin{aligned}
\frac{a^3}{5a+b}+\frac{b^3}{5b+c}+\frac{c^3}{5c+a} &= \frac{a^4}{5a^2+ab}+\frac{b^4}{5b^2+bc}+\frac{c^4}{5c^2+ca} \\
&\geq \frac{(a^2+b^2+c^2)^2}{5(a^2+b^2+c^2)+(ab+bc+ca)} \\
&\geq \frac{1}{6}(a^2+b^2+c^2).
\end{aligned}$$

The equality holds if and only if $a=b=c$.

This problem and solution were suggested by Titu Andreescu.

Solution 2: Note that

$$\begin{aligned}
0 &\leq (41a+83b)(a-b)^2 \\
&= 41a^3+a^2b-125ab^2+83b^3,
\end{aligned}$$

which is equivalent to

$$(5a+b)\left(-a^2+25b^2\right) \leq 36\left(a^3+3b^3\right).$$

Hence,

$$\frac{a^3+3b^3}{5a+b} \geq -\frac{1}{36}a^2+\frac{25}{36}b^2.$$

Adding this with two other analogous inequalities completes the proof.

Discovery: The solution can be discovered naturally. We start with guessing

$$\frac{a^3+3b^3}{5a+b} \geq ta^2+\left(\frac{2}{3}-t\right)b^2,$$

and rewrite it into

$$(1-5t)a^3-ta^2b-5\left(\frac{2}{3}-t\right)ab^2+\left(\frac{7}{3}+t\right)b^3 \geq 0.$$

Wishing $(a-b)^2$ to be a factor, we use synthetic division to write the left-hand side as

$$(a-b)^2\left[(1-5t)a+(2-11t)b\right]-\left(\frac{1}{3}+12t\right)b^3,$$

and get $t=-1/36$ by setting the remainder equal to 0.

This solution was suggested by Titu Andreescu and independently by Li Zhou, Polk State College, Winter Haven, FL.

Solution 3:

It is convenient to use the shorthand notation $\sum_{\text{cyc}} *$ to denote the sum of the three expressions obtained from $*$ by cyclically permuting the variables a, b, c. For instance,

$$\sum_{\text{cyc}} a^4 b = a^4 b + b^4 c + c^4 a.$$

In this notation, by clearing denominators, we may rewrite the desired inequality as

$$0 \leq \sum_{\text{cyc}} (190a^4 b + 35a^3 b^2 + 38ab^4 - 35a^2 b^3 - 168a^3 bc - 60a^2 b^2 c). \tag{1}$$

It is tempting to attempt to prove this using Muirhead's inequality, but this fails because we are working with cyclic sums rather than symmetric sums. For instance, it is <u>not true</u> that

$$\sum_{\text{cyc}} a^4 b \geq \sum_{\text{cyc}} a^3 b^2$$

(e.g., take $(a, b, c) = (10, 7, 1)$) even though Muirhead's inequality does imply the corresponding inequality for symmetric sums.

One must instead keep in mind not the statement of Muirhead's inequality but its underlying intuition: one should use "less mixed" monomials to dominate "more mixed" monomials. We will see two key techniques for realizing this intuition in the following argument. (Note that the breakdown we will give is in no way unique; there is some flexibility in the choice of how to separate (1) into tractable pieces.)

We first use what one might call a "sum of squares" argument: writing down cyclic sums of manifestly nonnegative expressions in order to match a few of the terms in (1). For instance, the following inequalities are all valid:

$$0 \leq \sum_{\text{cyc}} 84a^2 b(a-c)^2 = \sum_{\text{cyc}} (84a^4 b - 168a^3 bc + 84a^2 b^2 c), \tag{2}$$

$$0 \leq \sum_{\text{cyc}} \frac{35}{2} ab^2 (a-b)^2 = \sum_{\text{cyc}} \left(\frac{35}{2} a^3 b^2 - 35a^2 b^3 + \frac{35}{2} ab^4 \right), \tag{3}$$

$$0 \leq \sum_{\text{cyc}} \frac{35}{2} ab^2 (a-c)^2 = \sum_{\text{cyc}} \left(\frac{35}{2} a^3 b^2 - 35a^2 b^2 c + \frac{35}{2} ab^2 c^2 \right), \tag{4}$$

and these completely account for the summands $35a^3 b^2, -35a^2 b^3, -168a^3 bc$ in (1). We would like to add (2), (3), (4), and one more true inequality to get (1); that final inequality then would have to be

$$0 \leq \sum_{\text{cyc}} \left(\frac{177}{2} a^4 b + 38ab^4 - \frac{253}{2} a^2 b^2 c \right). \tag{5}$$

This inequality does not immediately present itself as a sum of squares, so we resort to a second technique: the weighted arithmetic-geometric mean inequality. This inequality implies that for any nonnegative real numbers u, v, w adding up to 1,

$$\sum_{\text{cyc}} a^4 b = \sum_{\text{cyc}} (ua^4 b + vb^4 c + wc^4 a) \geq \sum_{\text{cyc}} a^{4u+w} b^{u+4v} c^{v+4w}.$$

We may then deduce that

$$\sum_{\text{cyc}} a^4b \geq \sum_{\text{cyc}} a^2b^2c \tag{6}$$

by solving the linear equations

$$4u + w = 2, u + 4v = 2, v + 4w = 1$$

and discovering that the unique real solution

$$(u, v, w) = \left(\frac{6}{13}, \frac{5}{13}, \frac{2}{13}\right)$$

consists of nonnegative real numbers. (It is not necessary to check separately that the three numbers add up to 1, because adding the three given equations together gives $5(u+v+w) = 5$.) By switching a and b, we also obtain the valid inequality

$$\sum_{\text{cyc}} ab^4 \geq \sum_{\text{cyc}} a^2b^2c. \tag{7}$$

Adding 177/2 times (6) by 177/2 plus 38 times (7) then gives (5), so this inequality is also valid. As noted earlier, we may then add (5) to (2), (3), (4) to obtain the desired inequality (1).

This solution was adapted and refined by Kiran Kedlaya from several students' solutions.

JMO 4. Observe that since α is irrational no two of the points will coincide. It will be useful to define the auxiliary point P_0 such that the length of arc P_0P_1 is α, when travelling counter-clockwise around the circle from P_0 to P_1. We begin by noting that for any $n \geq 3$, if $a + b = n$ then P_0 lies on the arc from P_a to P_b containing P_n. For if we travel back (clockwise) around the circle through a distance of $b\alpha$ from P_n then we reach P_a. The same translation must map P_b to P_0, and since P_n is situated between P_a and P_b, we deduce that P_0 must be also.

The claim is clearly true for $n = 3$. Now suppose to the contrary that for some value of n we have $a + b > n$ and consider the minimal such counterexample. If in fact $a + b > n + 1$, then we may translate the three points P_a, P_b, and P_n clockwise around the circle through a distance α to find points P_{a-1} and P_{b-1} adjacent to P_{n-1} on either side. But then we would have $(a - 1) + (b - 1) > (n - 1)$ for this trio of points, which contradicts our assumption that n was the minimal counterexample.

Therefore we must have $a + b = n + 1$. Again we translate points P_a, P_b, and P_n clockwise around the circle through a distance α to obtain points P_{a-1} and P_{b-1} adjacent to P_{n-1} on either side with $(a - 1) + (b - 1) = (n - 1)$. By our earlier observation this implies that P_0 lies on the arc from P_{a-1} to P_{b-1} containing P_{n-1}. But now translating forward again, we conclude that P_1 lies on the arc from P_a to P_b containing P_n, contradicting the fact that P_a and P_b were the nearest adjacent points to P_n on either side. This completes the proof.

This problem and solution were suggested by Sam Vandervelde.

JMO 5. For simplicity, we will define $g(n)$ to be $n \pmod{2012}$. Note that $g(ak)+g(a(2012-k))$ is either 0 or 2012; it is 0 exactly when 2012 divides ak. This means that for $1 \le k \le 1005$, the number of elements i in $\{k, 2012-k\}$ such that $ai \pmod{2012} > bi \pmod{2012}$ is

$$\begin{cases} 0 & \text{if } g(ak) = 0 \text{ or } g(ak) = g(bk); \\ 2 & \text{if } g(bk) = 0 \text{ and } g(ak) \neq 0; \\ 1 & \text{otherwise.} \end{cases}$$

Let $T = \{1, 2, \ldots, 1005\}$. Note that the condition $g(ak) = g(bk)$ is equivalent to $g((a-b)k) = 0$. We will try to choose a, b so as to maximize the number of numbers k in T such that the first of the three cases occurs. From the prime factorization $2012 = 2 \cdot 2 \cdot 503$, the proper divisors of 2012 are 1, 2, 4, 503, and 1006. We shall choose a and $a-b$ to be multiples of some of these numbers. It is not hard to verify that we can choose a to be a multiple of 1006 and $a-b$ to be a multiple of 4. We will take $a = 1006$ and $b = 1002$.

With this choice of a and b, the second of the three cases (i.e. $g(bk) = 0$ and $g(ak) \neq 0$) never occurs, hence minimizing the number of elements i in $T - \{1006\}$ such that $ai \pmod{2012} > bi \pmod{2012}$. Moreover, $g(1006a) = 0$, meaning that $g(1006a) > g(1006b)$ does not hold. This means that our choice of a and b minimizes $f(a, b)$.

Note that $g(1006k) = 0$ occurs for 502 values in T, and $g(1006k) = g(1002k)$ occurs for 1 value in T. No value in T satisfies both condition. Hence $S = 1005 - 502 - 1 = 502$.

Note: Similarly, we can solve the problem in which 2012 is replaced by any positive integer $n \ge 3$. The answer is

$$\begin{cases} \dfrac{n}{2}\left(1 - \dfrac{1}{p}\right) & \text{if } n = p^k \text{ for some prime } p; \\ \dfrac{n}{2}\left(1 - \dfrac{1}{p_1}\right)\left(1 - \dfrac{1}{p_2}\right) & \text{otherwise, where } p_1 \text{ and } p_2 \text{ are the two smallest prime divisors of } n. \end{cases}$$

It is worth noting that the answer depends on no more than two prime divisors of n. Hence it might be interesting to ask the question for a value of n with at least three distinct prime divisors, or for all n.

This problem and solution were suggested by Warut Suksompong.

JMO 6. **Solution 1:** The proof is split into two cases.

Case 1: P is on the circumcircle of ABC. Then P is the Miquel point of A', B', C' with respect to ABC. Indeed, because $\angle A'B'C' = \angle CBA = \angle CPA = \angle A'PC'$, points P, A', B', C' are concyclic, and the same can be said for P, A, B', C' and P, A', B', C. Hence $\angle CA'B' = \angle CPB' = \angle BPC' = \angle BA'C'$, so $A'B'C'$ are collinear.

Case 2: P is not on the circumcircle of ABC. Let Q be isogonal conjugate of P with respect to ABC (which is not degenerate).

Claim. Let Q' be the isogonal conjugate of P with respect to $AB'C'$. Then $Q = Q'$.

Proof of the claim. Note that

$$\angle BQC = \angle BAC + \angle CPB \quad \text{(because } P \text{ and } Q \text{ are isogonal conjugates in } ABC\text{)}$$

$$
\begin{aligned}
&= \angle C'AB' + \angle B'PC' \\
&= \angle C'Q'B' \quad \text{(because } P \text{ and } Q \text{ are isogonal conjugates in } AB'C'\text{).}
\end{aligned}
$$

Let X, Y, Z denote the reflections of P in sides BC, CA, AB, respectively, and let X' denote P's reflection in side $B'C'$ of triangle $AB'C'$. Then $\angle ZXY = \angle BQC$ (because QC is orthogonal to XY and QB is orthogonal to XZ), whereas $\angle ZX'Y' = \angle C'Q'B'$ because $Q'B'$ is orthogonal to $X'Y$ and $Q'C'$ is orthogonal to $X'Z$ and $Q'C'$ is orthogonal to $X'Z$, so since $\angle C'Q'B' = \angle BQC$, we get $\angle ZXY = \angle ZX'Y$. It follows that X, Y, Z, X' are concyclic. The center of the XYZ-circle is Q while the center of the $X'Y'Z$-circle is Q'. Thus $Q = Q'$.

Note. We have made use of the well-known fact that the circumcenter of the triangle determined by the reflections of a point across the sidelines of another given triangle is precisely the isogonal conjugate of the point with respect to that triangle. For a proof see R. A. Johnson, *Advanced Euclidean Geometry*, 1929 ed., reprinted by Dover, 2007.

Similar arguments show that Q is also the isogonal point of P with respect to triangles $A'BC'$ and $A'B'C$. Therefore,

$$
\begin{aligned}
\angle BC'A' &= \angle AC'A' = \angle AC'P + \angle PC'Q + \angle QC'A' \\
&= \angle QC'B' + \angle PC'Q + \angle BC'P \\
&= \angle BC'B' = \angle AC'B'.
\end{aligned}
$$

This means that A', B', C' are collinear. ■

This problem and solution were suggested by Titu Andreescu and Cosmin Pohoata.

Solution 2: It's easy to see (say, by law of sines) that

$$
\frac{AC'}{BC'} = \frac{AP\sin\angle APC'}{BP\sin\angle BPC'}, \quad \frac{BA'}{CA'} = \frac{BP\sin\angle BPA'}{CP\sin\angle CPA'}, \quad \frac{CB'}{AB'} = \frac{CP\sin\angle CPB'}{AP\sin\angle APB'}.
$$

The construction of A', B', C' by reflections implies that

$$
\sin\angle APC' = \sin\angle CPA', \quad \sin\angle BPC' = \sin\angle CPB', \quad \sin\angle BPC' = \sin\angle CPB'.
$$

Hence,

$$
\frac{AC'}{BC'}\cdot\frac{BA'}{CA'}\cdot\frac{CB'}{AB'} = 1,
$$

and the proof is complete by Menelaus' theorem.

This second solution was suggested by Li Zhou, Polk State College, Winter Haven FL.