31st United States of America Mathematical Olympiad

Cambridge, Massachusetts

Part I 1 p.m. - 5:30 p.m.

May 3, 2002

1. Let S be a set with 2002 elements, and let N be an integer with $0 \le N \le 2^{2002}$. Prove that it is possible to color every subset of S either blue or red so that the following conditions hold:

   (a) the union of any two red subsets is white:

   (b) the union of any two blue subsets is black;

   (c) there are exactly N red subsets.

First Solution: We prove that this can be done for any n-element set, where n is an positive integer, $S_n = \{1, 2, \ldots, n\}$ and integer N with $0 \le N \le 2^n$.

We induct on n. The base case $n = 1$ is trivial. Assume that the desired coloring can be done to the subsets of set $S_n = \{1, 2, \ldots, n\}$ and integer N_n with $0 \le N \le 2^n$. We show that there is a desired coloring for set $S_{n+1} = \{1, 2, \ldots, n, n+1\}$ and integer N with $0 \le N_{n+1} \le 2^{n+1}$. We consider the following cases.

(i) $0 \le N_{n+1} \le 2^n$. Applying the induction hypothesis to S_n and $N_n = N_{n+1}$,we geet a coloring of all subsets of S_n satisfying conditions (a), (b), (c). All uncolored subsets of S_{n+1} contains element $n+1$, we color all of them blue. It is not hard to see that this coloring of all the subsets of S_{n+1} satisfying conditions (a), (b), (c).

(ii) $2^n + 1 \le N_{n+1} \le 2^{n+1}$. Applying the induction hypothesis to S_n and $N_n = 2^{n+1} - N_{n+1}$,we geet a coloring of all subsets of S_n satisfying conditions (a), (b), (c). All uncolored subsets of S_{n+1} contains element $n+1$, we color all of them blue. Finally, we switch the color of each subset: if it is blue now, we recolor it red: it is red now, we recolor it blue. It is not hard to see that this coloring of all the subsets of S_{n+1} satisfying conditions (a), (b), (c).

Thus our induction is complete.

Second Solution: If $N = 0$, we color every subset black; if $N = 2^{2002}$, we color every subset white. Now suppose neither of these holds. We may assume that $S = \{0, 1, 2, \ldots, 2001\}$. Write N in binary representation:

$$N = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k},$$

where the a_i are all distinct; then each a_i is an element of S. Color each a_i red, and color all the other elements of S blue. Now declare each nonempty subset of S to be the color of its largest element, and color the empty subset blue. If T, U are any two nonempty subsets of S, then the largest element of $T \cup U$ equals the largest element of T or the largest element of U, and if T is empty, then $T \cup U = U$. Statements (a) and (b) readily follow from this. To verify (c), notice that, for each i, there are 2^{a_i} subsets of S whose largest element is a_i (obtained by taking a_i in combination with any of the elements $0, 1, \ldots, a_i - 1$). If we sum over all i, each red subset is counted exactly once, and we get $2^{a_1} + 2^{a_2} + \cdots + 2^{a_k} = N$ red subsets.

2. Let ABC be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2\cot \frac{B}{2}\right)^2 + \left(3\cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisor and determine these integers.

First Solution: For simplification, let $u = \cot \frac{A}{2}$, $v = \cot \frac{B}{2}$, $w = \cot \frac{C}{2}$. We start with a few basic facts.

- *Fact 1.* Let $[\mathcal{R}]$ denote the area region $\mathcal{R}$. Then

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = rs.$$

The first equality is the **Heron's formula**. The second equality follows from $[ABC] = [AIB] + [BIC] + [CIA] = rs$, where I is the incenter of triangle ABC.

- *Fact 2.* We have

$$u = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.$$

Let ω be the **excircle** of triangle ABC opposite A, and let I_A be its center. Circle ω is tangent to side BC, rays AB and AC and X, Y, Z, respectively. By equal tangents, $AY = AZ$, $BX = BY$ and $CX = CZ$. Hence $AX = AY = s$. Then

$$[ABC] = [ABI_A] + [ACI_A] - [BCI_A] = \frac{r_a(b+c-a)}{2} = r_a(s-a).$$

where r_a is the radius of circle ω. Combining with the Heron's formula, we obtain

$$\sqrt{s(s-a)(s-b)(s-c)} = r_a(s-a),$$

or, $r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}$. On the other hand, in right triangle AI_AY,

$$u = \cot \frac{A}{2} = \frac{AY}{YI} = \frac{s}{r_a}.$$

Putting the above equalities together gives

$$u = \frac{s}{r_a} = \frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.$$

Likewise, we have

$$v = \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} \quad \text{and} \quad w = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}.$$

- *Fact 3.* From fact 2, we obtain

$$\begin{aligned} u+v+w &= \frac{\sqrt{s}[(s-a)+(s-b)+(s-c)]}{\sqrt{(s-a)(s-b)(s-c)}} \\ &= \frac{s\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}} = uvw \end{aligned}$$

From fact 1, we obtain

$$\begin{aligned} \frac{s\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}} &= \frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}} \\ &= \frac{s^2}{[ABC]} = \frac{s^2}{rs} = \frac{s}{r}. \end{aligned}$$

Hence,

$$uvw = u+v+w = \frac{s}{r}. \tag{1}$$

By (1), and by noticing that $2^2+3^2+6^2=7^2$, we can rewrite the given relation as

$$(6^2+3^2+2^2)[u^2+(2v)^2+(3w)^2] = (6u+6v+6w)^2.$$

This means that we have equality in the Chauchy-Schwartz inequality. It follows that

$$\frac{u}{6} = \frac{2v}{3} = \frac{3w}{2},$$

or,

$$u = 36k, \quad v = 9k, \quad w = 4k,$$

for some positive real number k. Pluging these back into (1) gives $k = \frac{7}{36}$, and consequently, $u = 7$, $v = \frac{7}{4}$, and $w = \frac{7}{9}$. Hence by the **Double angle formulas**, $\sin A = \frac{7}{25}$, $\sin B = \frac{56}{65}$, and $\sin C = \frac{63}{65}$, or,

$$\sin A = \frac{13}{\frac{325}{7}}, \quad \sin B = \frac{40}{\frac{325}{7}}, \quad \sin C = \frac{45}{\frac{325}{7}}.$$

By the **Extanded law of sines**, triangle ABC is similar to triangle T with the side lengths 13, 40, and 45. (The circumradius of T is $\frac{325}{7}$.)

Second Solution: Let D be the point of tangency of the incircle of triangle ABC and side AB. Then $AI = r$ and $AE = s - a$, where I is the incenter of triangle ABC. Hence $u = \frac{AE}{AI} = \frac{s-a}{r}$. Likewise, $v = \frac{s-b}{r}$ and $w = \frac{s-c}{r}$. Since

$$\frac{s}{r} = \frac{(s-a)+(s-b)+(s-c)}{r} = u + v + w,$$

we can rewrite the given relation as

$$49[u^2 + 4v^2 + 9w^2] = 36(u + v + w)^2.$$

Exapnding the last equality and cancelling the like terms, we obtain

$$13u^2 + 160v^2 + 405w^2 - 72(uv + vw + wu) = 0,$$

or

$$(3u - 12v)^2 + (4v - 9w)^2 + (18w - 2u)^2 = 0.$$

Therefore $u : v : w = 1 : 4 : 9$.

By the **Addition formula**, we obtain

$$\begin{aligned}
\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} &= \frac{\cot\frac{A}{2}\cot\frac{B}{2} - 1}{\cot\frac{A+B}{2}} + \cot\frac{C}{2} \\
&= \cot\frac{C}{2}\left(\cot\frac{A}{2}\cot\frac{B}{2} - 1\right) + \cot\frac{C}{2} \\
&= \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2},
\end{aligned}$$

or, $u + v + w = uvw$. The rest is the same as the last part of the first solution.

3. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Solution: Let $p(x)$ be monic real polynomial of degree n. If $n = 1$, then $p(x) = x + a$ for some real number a. It is easy to see that $p(x)$ is the average of $x + 2a$ and x, each of which has 1 real root. Now we assume that $n > 1$. Let polynomial

$$g(x) = (x-2)(x-4)\cdots(x-2(n-1)).$$

The degree of $g(x)$ is $n - 1$. Consider the polynomials

$$q(x) = x^n - kg(x) \quad \text{and} \quad r(x) = 2p(x) - q(x) = 2p(x) - x^n + kg(x).$$

We will show that for large enough k these two polynomials have n real roots. Since they are monic and their average is clearly $p(x)$, this will solve the problem.

Consider the values of polynomial $g(x)$ at n points $x = 1, 3, 5, \ldots, 2n-1$. These values alternate in sign and are at least 1 (since at most two of the factors have magnitude 1 and the others have magnitude at least 2). On the other hand, there is a constant $c > 0$ such that for $0 \leq x \leq n$, we have $|x^n| < c$ and $|2p(x) - x^n| < c$. Take $k > c$. Then we see that $q(x)$ and $r(x)$ evaluated at n points $x = 1, 3, 5, \ldots, 2n-1$ alternate in sign. Thus polynomials $p(x)$ and $r(x)$ each has at least $n-1$ real roots. How ever since they are polynomials of degree n, they must then each have n real roots, as desired.

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4. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

Solution: Setting $x = y = 0$ in the given condition yields $f(0) = 0$. Since

$$-xf(-x) - yf(y) = f[(-x)^2 - y^2] = f(x^2 - y^2) = xf(x) - yf(y),$$

we have $f(x) = -f(x)$ for $x \neq 0$. Hence $f(x)$ is odd. From now on, we assume $x, y \geq 0$. Setting $y = 0$ in the given condition yields $f(x^2) = xf(x)$. Hence $f(x^2 - y^2) = f(x^2) - f(y^2)$, or, $f(x^2) = f(x^2 - y^2) + f(y^2)$. Since for $x \geq 0$ there is a unique $t \geq 0$ such that $t^2 = x$, it follows that

$$f(x) = f(x - y) + f(y) \tag{1}$$

Setting $x = 2t$ and $y = t$ in (1) gives

$$f(2t) = 2f(t). \tag{2}$$

Setting $x = t + 1$ and $y = t$ in the given condition yields

$$f(2t + 1) = (t + 1)f(t + 1) - tf(t). \tag{3}$$

By (2) and by setting $x = 2t + 1$ and $y = 1$ in (1), the left-hand side of (3) becomes

$$f(2t + 1) = f(2t) + f(1) = 2f(t) + f(1). \tag{4}$$

On the other hand, by setting $x = t + 1$ and $y = 1$ in (1), the right-hand side of (3) reads

$$(t + 1)f(t + 1) - tf(t) = (t + 1)[f(t) + f(1)] - tf(t) = f(t) + (t + 1)f(1). \tag{5}$$

Putting (3), (4), and (5) together leads to $2f(t) + f(1) = f(t) + (t + 1)f(1)$, or,

$$f(t) = tf(1)$$

for $t \geq 0$. Recall that $f(x)$ is odd, we conclude that $f(-t) = -f(t) = -tf(1)$ for $t \geq 0$. Hence $f(x) = kx$ for all x, where $k = f(1)$ is a constant. It is not difficult to see that all such functions indeed satisfy the conditions of the problem.

5. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \le i < k)$.

First Solution: We write $a \leftrightarrow b$ if the desired sequence exists. Note that for positive integer n with $n \ge 3$, $n \leftrightarrow 2n$ as the sequence

$$n_1 = n,\ n_2 = n(n-1),\ n_3 = n(n-1)(n-2),\ n_4 = n(n-2),\ n_5 = 2n$$

satisfies the conditions of the problem. For positive integer $n \ge 4$, $n' = (n-1)(n-2) \ge 3$, hence $n' \leftrightarrow 2n'$ by the above argument. It follows that $n \leftrightarrow n-1$ for $n \ge 4$ by $n' \leftrightarrow 2n'$ and by the sequences

$$n_1 = n,\ n_2 = n(n-1),\ n_3 = n(n-1)(n-2),\ n_4 = n(n-1)(n-2)(n-3),$$
$$n_5 = 2(n-1)(n-2) = 2n'$$

and $n_1' = n' = (n-1)(n-2)$, $n_2' = n-1$. Iterating this, we connect all integers larger than 2.

Second Solution: We write $a \leftrightarrow b$ if the desired sequence exists. Note that this relation is symmetric ($a \leftrightarrow b$ implies $b \leftrightarrow a$) and transitive ($a \leftrightarrow b$, $b \leftrightarrow c$ imply $a \leftrightarrow c$). Our crucial observation will be the following: If $d > 2$ and n is a multiple of d, then $n \leftrightarrow (d-1)n$. Indeed, $n + (d-1)n = dn \mid n^2 \mid (d-1)n^2 = n \cdot (d-1)n$.

Let us call a positive integer k *safe* if $n \leftrightarrow kn$ for all $n > 2$. Notice by transitivity that any product of safe numbers is safe. Now, we claim that 2 is safe. To prove this, define $f(n)$, for $n > 2$, to be the smallest divisor of n which is greater than 2. We show that $n \leftrightarrow 2n$ by strong induction on $f(n)$. In case $f(n) = 3$, we immediately have $n \leftrightarrow 2n$ by our earlier observation. Otherwise, notice that $f(n) - 1$ is a divisor of $(f(n) - 1)n$ which is greater than 2 and less than $f(n)$; thus $f((f(n) - 1)n) < f(n)$, and the induction hypothesis gives $(f(n)-1)n \leftrightarrow 2(f(n)-1)n$. We also have $n \leftrightarrow (f(n)-1)n$ (by our earlier observation) and $2(f(n)-1)n \leftrightarrow 2n$ (by the same observation, since $f(n) \mid n \mid 2n$). Thus, by transitivity, $n \leftrightarrow 2n$. This completes the induction step and proves the claim.

Next, we show that any prime p is safe, again by strong induction. The base case $p = 2$ has already been done. If p is an odd prime, then $p + 1$ is a product of primes strictly less than p, which are safe by the induction hypothesis; hence, $p + 1$ is safe. Thus, for any $n > 2$,

$$n \leftrightarrow (p+1)n \leftrightarrow p(p+1)n \leftrightarrow pn.$$

This completes the induction step. Thus, all primes are safe, and hence every integer ≥ 2 is safe. In particular, our given numbers a, b are safe, so we have $a \leftrightarrow ab \leftrightarrow b$, as needed.

6. I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations

separating adjacent stamps, and each block must come out of a sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are constants c and d such that

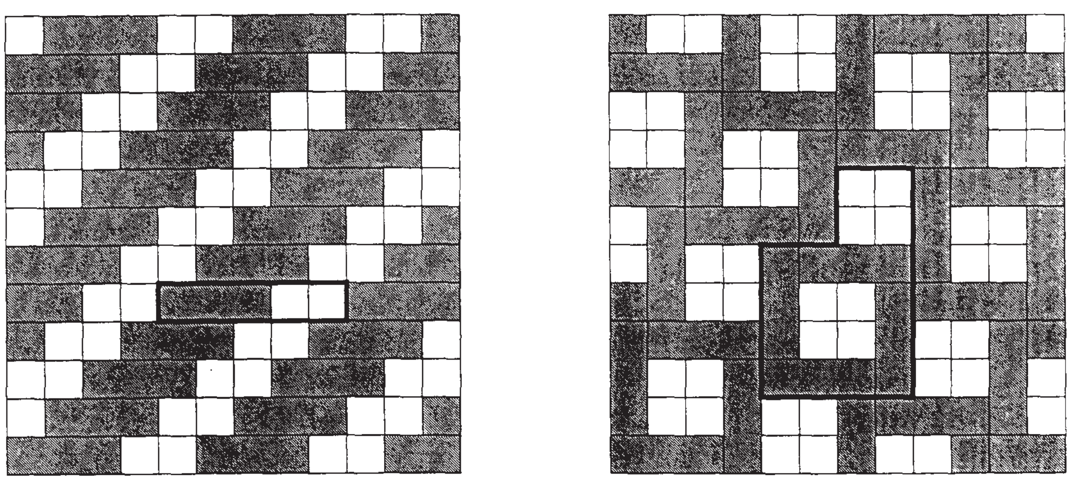
$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

for all $n > 0$.

Solution: The upper bound requires an example of a set of $\frac{1}{5}n^2 + dn$ blocks whose removal makes it impossible to remove any further blocks. It suffices to show that we can tile the plane by tiles containing one block for every five stamps so that no more blocks can be chosen. Two such tilings are shown below with one tile outlined in heavy lines. Given an $n \times n$ section of the tiling take all blocks lying entirely within that section and add as many additional blocks as possible. If the basic tile is contained in an $m+1 \times m+1$ square, then the $n \times n$ section is covered by tiles contained in a concentric $(n+2m) \times (n+2m)$ square. Hence there are at most $\frac{1}{5}(n+2m)^2$ blocks entirely within the section. For an $n \times n$ section of the tiling, there are at most $4n$ blocks which lie partially in and partially out of that section (hence these block contain at most $8n$ stamps in the $n \times n$ square) and each of the additional blocks must contain one of these stamps. Thus there are at most $8n$ additional blocks. Thus there are at most

$$\frac{1}{5}(n+2m)^2 + 8n \leq \frac{1}{5}n^2 + \frac{4m^2+4m+40}{5}n$$

blocks total.



The lower bound requires an argument. Suppose that we have a set of $b(n)$ blocks whose removal makes removing any further blocks impossible.

1) There are $2n(n-2)$ potential blocks of three consecutive stamps in a row or column. Each of these must meet at least one of the $b(n)$ blocks removed. Conversely, each of the $b(n)$ blocks removed meets at most 14 of these potential blocks

(5 oriented the same way, including itself, and 9 oriented the orthogonal way). Therefore $14b(n) \geq 2n(n-2)$ or

$$b(n) \geq \frac{1}{7}n^2 - \frac{2}{7}n.$$

2) Call a stamp used if it belongs to one of the $b(n)$ removed blocks. Consider the $(n-2)^2$ five-stamp crosses centered at each stamp not on an edge of the sheet. Each cross must contain two used stamps. (One stamp not in the center is not enough to prevent another block from being torn out, and it is impossible to use one stamp in the center and use no other stamps in the cross.) In addition, each block not lying along an edge of the sheet lies entirely inside one cross, which thus contains three used stamps. There are at most $4n/3$ of the $b(n)$ blocks lying along the edges, hence there are at least $b(n) - 4n/3$ crosses containing three used stamps.

Now count the number of pairs of a used stamp and a cross containing that stamp, in two ways. First counting block by block, we get $3b(n)$ used stamps, and each used stamp is contained in at most five crosses (exactly five if it is not on an edge), for a total of at most $15b(n)$ pairs. Next, counting cross by cross, each of the $(n-2)^2$ crosses contains at least two used stamps and we have at least $b(n) - 4n/3$ crosses containing three used stamps, for a total of at least $2(n-2)^2 + b(n) - 4n/3$ pairs. Therefore

$$15b(n) \geq 2(n-2)^2 + b(n) - \frac{4n}{3},$$

or

$$b(n) \geq \frac{1}{7}n^2 - \frac{16}{21}n.$$

3) Call a stamp used if it belongs to one of the $b(n)$ removed blocks. Count the number of pairs consisting of a used stamp and an adjacent unused stamp, in two ways.

There are at least $(n-2)^2 - 3b(n)$ unused stamps which are not on an edge. Since no more blocks can be torn out, either the stamp to the left or right and either the stamp above or below such an unused stamp must be used. Thus we have at least $2n^2 - 8n - 6b(n)$ such pairs.

Each block removed is adjacent to at most eight other stamps. However these eight stamps contain two blocks of three consecutive stamps. Hence at most six of these eight stamps can be unused. Thus each of the $b(n)$ block removed is involved in at most six pairs. Thus there are at most $6b(n)$ pairs.

Combining these we have

$$6b(n) \geq 2n^2 - 8n - 6b(n),$$

or

$$b(n) \geq \frac{1}{6}n^2 - \frac{2}{3}n.$$