45th United States of America Junior Mathematical Olympiad

Day II 12:30PM — 5PM EDT

April 20, 2016

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 4. Find, with proof, the least integer N such that if any 2016 elements are removed from the set $\{1, 2, \ldots, N\}$, one can still find 2016 distinct numbers among the remaining elements with sum N.

USAJMO 5. Let $\triangle ABC$ be an acute triangle, with O as its circumcenter. Point H is the foot of the perpendicular from A to line $\overleftrightarrow{BC}$, and points P and Q are the feet of the perpendiculars from H to the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, respectively.

Given that

$$AH^2 = 2 \cdot AO^2,$$

prove that the points O, P, and Q are collinear.

USAJMO 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2 .$$