

44th United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 28, 2015

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in a 1-point automatic deduction.

USAMO 1. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1 \right)^3.$$

USAMO 2. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment \overline{PQ} . Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that \overline{XT} is perpendicular to \overline{AX} . Let M denote the midpoint of chord \overline{ST} . As X varies on segment \overline{PQ} , show that M moves along a circle.

USAMO 3. Let $S = \{1, 2, \dots, n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S ,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$