1^{st} United States of America Junior Mathematical Olympiad 2010

Day I 12:30 PM – 5 PM EDT

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1. A *permutation* of the set of positive integers $[n] = \{1, 2, \ldots, n\}$ is a sequence $(a_1, a_2, \ldots, a_n)$ such that each element of $[n]$ appears precisely one time as a term of the sequence. For example, $(3, 5, 1, 2, 4)$ is a permutation of $[5]$. Let $P(n)$ be the number of permutations of $[n]$ for which ka_k is a perfect square for all $1 \le k \le n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

2. Let $n > 1$ be an integer. Find, with proof, all sequences $x_1, x_2, \ldots, x_{n-1}$ of positive integers with the following three properties:

   (a) $x_1 < x_2 < \cdots < x_{n-1}$;

   (b) $x_i + x_{n-i} = 2n$ for all $i = 1, 2, \ldots, n-1$;

   (c) given any two indices i and j (not necessarily distinct) for which $x_i + x_j < 2n$, there is an index k such that $x_i + x_j = x_k$.

3. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.