**$4^{\text{th}}$ United States of America Junior Mathematical Olympiad**

**Day I, II 12:30 PM – 5 PM EDT**

**April 29 - April 30, 2014**

JMO 1. We start by observing that the denominators of the fractions involved in the statement of the problem are positive. Next, we argue by contradiction and assume that

$$10a^2 - 5a + 1 > abc(b^2 - 5b + 10)$$

and similar inequalities obtained by cyclic permutations. Multiplying these inequalities yields

$$\prod[a^3(a^2 - 5a + 10)] < \prod(10a^2 - 5a + 1).$$

This is impossible, since

$$a^3(a^2 - 5a + 10) - (10a^2 - 5a + 1) = (a - 1)^5 \geq 0$$

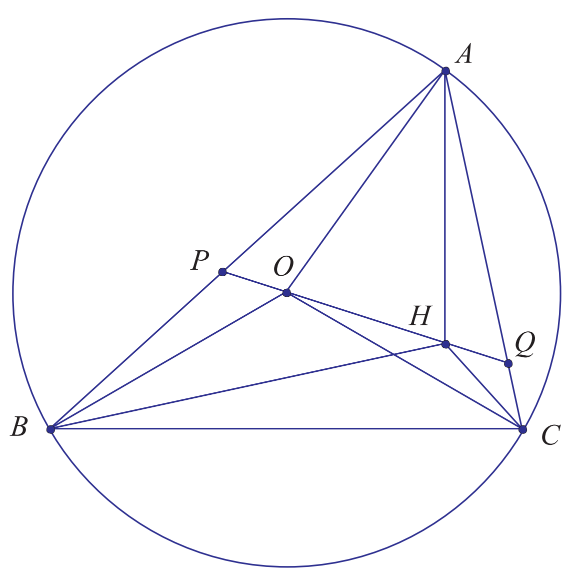
and similarly for $b$ and $c$.

This problem and solution was suggested by Titu Andreescu.

JMO 2. **(a):** Without loss of generality, we assume that $AB > AC$. Set $\beta = \angle ABC$ and $\gamma = \angle ACB$. We have $\beta < 60^\circ < \gamma$ and $\beta + \gamma = 120^\circ$.

Note that $\angle BAO = 90^\circ - \angle ACB = 90^\circ - \gamma < 90^\circ - \beta = 90^\circ - \angle ABC = \angle BAH$, and so $AO$ lies inside $\angle BAH$. Similarly, $\angle ABO = 90^\circ - \gamma < 30^\circ = \angle ABH$, and so $BO$ lies inside $\angle ABH$. Hence $O$ lies inside $\triangle ABH$, and line $OH$ intersects side $AB$. In the same way, $\angle CAH = 90^\circ - \gamma < 90^\circ - \beta = \angle CAO$ and $\angle ACH = 30^\circ < 90^\circ - \beta = \angle ACO$; hence $H$ lies inside $\triangle ACO$, and line $OH$ intersects side $AC$.

**(b):** The range of $s/t$ is the open interval $(4/5, 1)$.



Based on (a), we may consider the configuration shown above. Note that $\angle BOC = 2\angle BAC = 120^\circ$ and $\angle BHC = 180^\circ - \angle HBC - \angle HCB = 180^\circ - (90^\circ - \gamma) - (90^\circ - \beta) = 120^\circ$, from which it follows that $BOHC$ is cyclic. In particular, $\angle POB = 180^\circ - \angle HOB = \angle HCB = 90^\circ - \beta$, and it follows that

$$\angle APQ = \angle ABO + \angle POB = (90^\circ - \gamma) + (90^\circ - \beta) = 60^\circ.$$

Since $\angle PAQ = 60^\circ$ as well, we see that $\triangle APQ$ is equilateral.

Next note that $\angle POB = 90^\circ - \beta = \angle ACO = \angle QCO$ and $\angle PBO = 90^\circ - \gamma = \angle HBC = \angle HOC = \angle QOC$; since $BO = OC$, we have congruent triangles $\triangle BPO \cong \triangle OQC$. Thus

$$AB + AC = AP + PB + CQ + QA = AP + QO + OP + QA = AP + PQ + QA$$

and so $AP = PQ = QA = \frac{b+c}{3}$, where we write $b = AC$ and $c = AB$. Therefore we have

$$\frac{s}{s+t} = \frac{\text{Area}(\triangle APQ)}{\text{Area}(\triangle ABC)} = \frac{AP}{AB}\frac{AQ}{AC} = \frac{\left(\frac{b+c}{3}\right)^2}{bc} = \frac{2 + m + 1/m}{9},$$

where $m = c/b$.

By our assumptions that $b < c$ and $\triangle ABC$ is acute, it follows that the range of $m$ is $1 \le m < 2$. (One can see this, for instance, by having $A$ move along the major arc $\overset{\frown}{\text{BC}}$ from one extreme, where $ABC$ is equilateral and $c/b = 1$, to the other, where $\angle ACB = 90^\circ$ and $c/b = 2$, and noting that $c$ increases and $b$ decreases during this motion.) For $m \in (1, 2)$, the function $f(m) = m + 1/m$ is continuous and increasing: if $1 < m < m' < 2$, then $f(m') - f(m) = \frac{(m'-m)(mm'-1)}{mm'} > 0$. Thus the range of $f(m)$ for $m \in (1, 2)$ is $(f(1), f(2)) = (2, \frac{5}{2})$. It follows that the range of $\frac{s}{s+t} = \frac{2+f(m)}{9}$ is $(\frac{4}{9}, \frac{1}{2})$, and the range of $\frac{s}{t}$ is $(\frac{4}{5}, 1)$.

This problem and the first solution was suggested by Zuming Feng.

**OR**

**(b):** We use complex numbers. Let $O = 0$, $B = 1$, $C = \omega = e^{2\pi i/3}$, and $A = a$ with $|a| = 1$. Then $H = 1 + \omega + a = a - \omega^2$. Bearing in mind that the equation for the line through complex numbers $w_1$ and $w_2$ is $\frac{z-w_1}{w_2-w_1} = \frac{\overline{z}-\overline{w_1}}{\overline{w_2}-\overline{w_1}}$ (i.e., the quotient $\frac{z-w_1}{w_2-w_1}$ is purely real), we see that $P$, which is the intersection of $AB$ and $OH$, lies at the point $z$ satisfying

$$\frac{z-1}{a-1} = \frac{\overline{z}-1}{\overline{a}-1} \quad \text{and} \quad \frac{z}{a-\omega^2} = \frac{\overline{z}}{\overline{a}-\omega}.$$

Substituting $\overline{a} = 1/a$, eliminating $\overline{z}$, and solving for $z$ yields $z = \frac{a+1}{1-\omega}$. Thus the vector $\overrightarrow{AP}$ is given by the complex number $\frac{a+1}{1-\omega} - a = \frac{a\omega+1}{1-\omega}$. Similarly $Q$ lies at the point $\frac{a\omega+\omega^2}{\omega-1}$ and the vector $\overrightarrow{AQ}$ is $\frac{a+\omega^2}{\omega-1}$. It follows that $AP = \frac{1}{\sqrt{3}}|\omega a + 1| = \frac{1}{\sqrt{3}}|a + \omega^2| = AQ$.

Now $\overrightarrow{AB} = 1 - a$ is collinear with $\overrightarrow{AP} = \frac{a\omega+1}{1-\omega}$, and the ratio of the lengths of these vectors is $\frac{AB}{AP} = (1-a)/\left(\frac{a\omega+1}{1-\omega}\right) = \frac{(1-a)(1-\omega)}{a\omega+1}$; similarly $\overrightarrow{AC} = \omega - a$ is collinear with $\overrightarrow{AQ} = \frac{a+\omega^2}{\omega-1}$, and $\frac{AC}{AQ} = \frac{(\omega-a)(\omega-1)}{a+\omega^2} = \frac{(\omega-a)(\omega^2-\omega)}{a\omega+1}$. Thus

$$\frac{AB+AC}{AP} = \frac{AB}{AP} + \frac{AC}{AQ} = \frac{(1-a)(1-\omega)+(\omega-a)(\omega^2-\omega)}{a\omega+1} = \frac{3a\omega+3}{a\omega+1} = 3,$$

and so

$$\frac{AP}{AB}\frac{AQ}{AC} = \frac{(AB+AC)^2}{9(AB)(AC)}.$$

The second solution was suggested by Razvan Gelca.

JMO 3. Let $f$ be a solution of the problem. Let $p$ be a prime. Since $p$ divides $f(p)^2$, $p$ divides $f(p)$ and so $p$ divides $\frac{f(p)^2}{p}$. Taking $y=0$ and $x=p$, we deduce that $p$ divides $f(0)$. As $p$ is arbitrary, we must have $f(0)=0$. Next, take $y=0$ to obtain $xf(-x) = \frac{f(x)^2}{x}$. Replacing $x$ by $-x$, and combining the two relations yields $f(x)=0$ or $f(x)=x^2$ for all $x$.

Suppose now that there exists $x_0 \neq 0$ such that $f(x_0)=0$. Taking $y=x_0$, we obtain $xf(-x)+x_0^2 f(2x) = \frac{f(x)^2}{x}$, yielding $x_0^2 f(2x)=0$ for all $x$ and so $f$ vanishes on even numbers. Assume that there exists an odd number $y_0$ such that $f(y_0)\neq 0$, so $f(y_0)=y_0^2$. Taking $y=y_0$, we obtain

$$xf(2y_0^2-x)+y_0^2 f(2x-y_0^2) = \frac{f(x)^2}{x} + f(y_0^3).$$

Choosing $x$ even, we deduce that $y_0^2 f(2x-y_0^2) = f(y_0^3)$. This forces $f(y_0^3)=0$, as otherwise we would have $f(2x-y_0^2)=(2x-y_0^2)^2$ for all even $x$ and so $y_0^2(2x-y_0^2)^2 = f(y_0^3)$ for all such $x$, obviously impossible. Thus $f(2x-y_0^2)=0$ for all even numbers $x$, that is $f$ vanishes on numbers of the form $4k+3$. But since $x^2 f(-x) = f(x)^2$, $f$ also vanishes on all $x$ such that $-x \equiv -1 \pmod 4$, that is on $4\mathbb{Z}+1$. Thus $f$ also vanishes on all odd numbers, contradicting the choice of $y_0$. Hence, if $f$ is not the zero map, then $f$ does not vanish outside 0 and so $f(x)=x^2$ for all $x$.

In conclusion, $f(x)=0$ for all $x\in\mathbb{Z}$ and $f(x)=x^2$ for all $x\in\mathbb{Z}$ are the only possible solutions. The first function clearly satisfies the given relation, while the second also satisfies the Sophie Germaine identity

$$x(2y^2-x)^2 + y^2(2x-y^2)^2 = x^3+y^6$$

for all $x, y \in \mathbb{Z}$.

**OR**

$f(0)=0$: If $f(0)\neq 0$, set $x=2f(0)$ to obtain

$$2(f(0))^2 = \frac{(f(2f(0)))^2}{2f(0)} + f(0)$$

that is

$$2(f(0))^2(2f(0)-1) = f(2f(0))^2.$$

But $2(2f(0)-1)$ cannot be a perfect square since it is of the form $4k+2$. So $f(0)=0$.

This problem and the solutions were suggested by Titu Andreescu and Gabriel Dospinescu.

JMO 4. Let $f(n) = n + s_b(n)$. For a positive integer $m$, let $k = \lfloor \log_b(m/2) \rfloor$, so that $m \geq 2b^k$. Note that if $b^m - b^k \leq n < b^m$, then the base $b$ expansion of $n$ begins with $m-k$ digits equal to $b-1$, and therefore

$$f(n) > b^m - b^k + (m-k)(b-1) \geq b^m - b^k + (2b^k - k)(b-1) \geq b^m. \tag{1}$$

Now consider the set $\{f(1), f(2), \ldots, f(b^m)\}$. Any number that is $\leq b^m$ and in the range of $f$ is in this set. However, we see from (1) that $f(n) > b^m$ whenever $b^m - b^k \leq n < b^m$. Therefore, there are at least $b^k$ numbers from 1 to $b^m$ that are not in the range of $f$. Since $k$ goes to infinity as $m$ goes to infinity, the desired result follows.

This problem and solution was suggested by Palmer Mebane.

**OR**

We first show that there exist infinitely many pairs $(n_1, m_1), (n_2, m_2), \ldots$ such that $n_i + s_b(n_i) = m_i + s_b(m_i)$ for all $i$.

- **Case 1** $b = 2$. Let $i$ be a positive integer, and set $j = 2^i + 3$; note $j > i$. Then for $n_i = 2^j - 1$, we have $s_2(n_i) = j$. If we then consider $m_i = 2^j + j - 3$, we have by the definition of $j$ that $m_x = 2^j + 2^i$, so $s_2(m_i) = 2$. It is easy to see that $n_i + s_2(n_i) = m_i + s_2(m_i)$.
- **Case 2** $b > 2$. Let $i$ be a positive integer, and set $j = \frac{b^i+b-2}{b-1} + 1$; note $j > i$. Then for $n_i = b^j - b + 2$, we have $s_b(n_i) = (b-1)(j-1) + 2$. If we then consider $m_i = b^j - b + (b-1)(j-1) + 2$, plugging in our definition for $j$ in the third term gives
$$m_i = b^j - b + (b-1)\left(\frac{b^i + b - 2}{b-1}\right) + 2 = b^j + b^i,$$
so $s_b(m_i) = 2$. We can easily compute that $n_i + s_b(n_i) = m_i + s_b(m_i)$.

In both cases, since $j$ grows exponentially with $i$, it is easy to check that $n_i < m_i < n_{i+1} < m_{i+1}$, so all of the constructed pairs contain pairwise distinct positive integers.

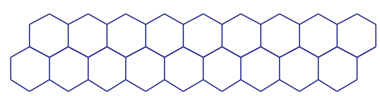
Now we will show at least $k$ positive integers cannot be represented in the form $n+s_b(n)$ for any $k$. Take $(n_1, m_1), \ldots (n_k, m_k)$ and let $A$ be a number greater than any of the $2k$ numbers in these pairs. For a positive integer $x$ with $x \leq A$, if we have $x = n + s_b(n)$ then we must have $n \leq x \leq A$. So in finding ways to represent the numbers $1, 2, \ldots A$ in the form $n + s_b(n)$, all of them require $n \leq A$. However, among numbers at most $A$ there are at least $k$ pairs $n_i, m_i$ with $n_i + s_b(n_i) = m_i + s_b(m_i)$. Therefore the set

$\{n + s_b(n) \mid n = 1, 2, \ldots A\}$ has at most $A - k$ elements, and so at least $k$ of the numbers $1, 2, \ldots A$ are not members of this set and thus have no representation in the form $n + s_b(n)$. This proves our original claim. Since $k$ is arbitrary there cannot be a finite amount of positive integers with no representation, so there are infinitely many as desired.

The second solution was suggested by Palmer Mebane.

JMO 5. The answer is $k = 6$. First we show that $A$ cannot win for $k \geq 6$. Color the grid in three colors so that no two adjacent spaces have the same color, and arbitrarily pick one color $C$. $B$ will play by always removing a counter from a space colored $C$ that $A$ just played. If there is no such counter, $B$ plays arbitrarily. Because $A$ cannot cover two spaces colored $C$ simultaneously, it is possible for $B$ to play in this fashion. Now note that any line of six consecutive squares contains two spaces colored $C$. For $A$ to win he must cover both, but $B$'s strategy ensures at most one space colored $C$ will have a counter at any time.

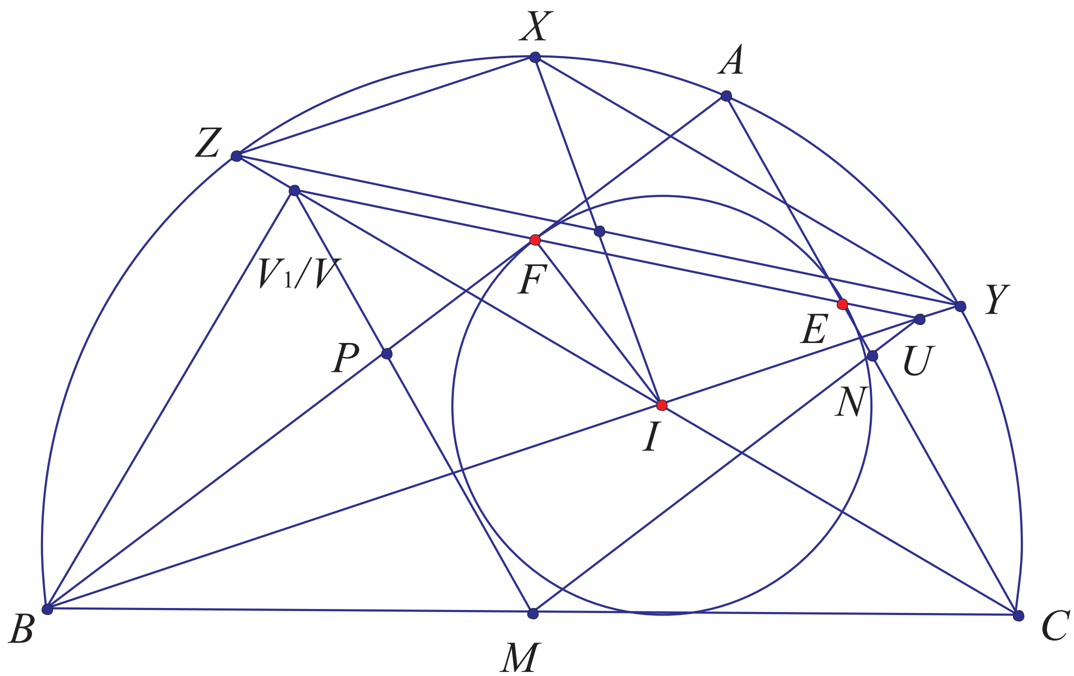
Now we show that $A$ can obtain 5 counters in a row. Take a set of cells in the grid forming the shape shown below. We will have $A$ play counters only in this set of grid cells until this is no longer possible. Since $B$ only removes one counter for every two $A$ places, the number of counters in this set will increase each turn, so at some point it will be impossible for $A$ to play in this set anymore. At that point any two adjacent grid spaces in the set have at least one counter between them.



Consider only the top row of cells in the set, and take the lengths of each consecutive run of cells. If there are two adjacent runs that have a combined length of at least 4, then $A$ gets 5 counters in a row by filling the space in between. Otherwise, a bit of case analysis shows that there exists a run of 1 counter which is neither the first nor last run. This single counter has an empty space on either side of it on the first row. As a result, the four spaces of the second row touching these two empty spaces all must have counters. Then $A$ can play in the 5th cell on either side of these 4 to get 5 counters in a row. So in all cases $A$ can win with $k \leq 5$.

This problem and solution was suggested by Palmer Mebane.

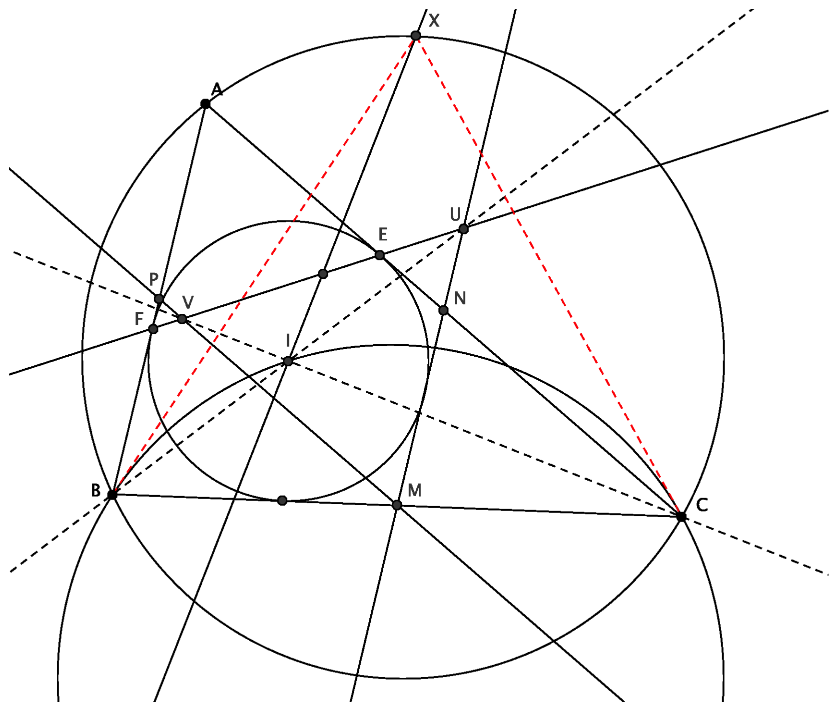
JMO 6. Set $\angle ABC = 2y$ and $\angle BCA = 2z$. First, we start with a known fact that $I$ lies on ray $CV$. Let $V_1$ be the foot of the perpendicular from $B$ to ray $CI$. Then in right triangle $BV_1C$, $V_1M = MB = MC$ and $\angle MV_1C = \angle MCV_1 = z = \angle V_1CA$, implying that $\overline{MV_1} \| \overline{CA}$; in particular, $V_1$ lies on line $MP$. Because $\angle BV_1I = \angle BFI = 90°$, $BIFV_1$ is cyclic, from which it follows that $\angle V_1FB = \angle V_1IB = y + z = \angle AEF = \angle AFE$; in particular, $V_1$ lies on $\overline{EF}$. Because $V_1$ lies on both line $MP$ and line $EF$, $V = V_1$ and $V$ lies on line $CI$. Likewise we can prove that $U$ lies on line $BI$.




Rays $BI$ and $CI$ intersect again at $Y$ and $Z$. Note that $\angle UVC = \angle EVC = \angle AEV - \angle ECV = \angle AEF - \angle ECV = y$. Because $BCYZ$ is cyclic, we have $\angle YZC = \angle YBC = y$. Therefore, $\overline{UV} \| \overline{YZ}$. It suffices to show that $IX$ bisects segment $\overline{YZ}$, which is clearly true because $IYXZ$ is a parallelogram. (Indeed, $\angle YZX = \overset{\frown}{\text{XAY}} = \angle XBC - \angle YBC = y + z - y = z = \angle ZYB$, from which it follows that $\overline{ZX} \| \overline{IY}$ . Likewise, we can show that $\overline{IZ} \| \overline{XY}$.)

**OR**

First, note that $U$ and $V$ lie on the bisectors $BI$ and $CI$, respectively. Indeed, let $D$ be the tangency point of $\gamma$ with $BC$ and let $U'$ be the intersection of $BI$ with $EF$. Note that triangles $BFU'$ and $BDU'$ are congruent (by SAS), so $\angle BU'F = \angle BU'D$. In addition, the pencil $(U'F, U'B, U'D, U'C)$ is harmonic; thus, it follows that $U'B \perp U'C$, so, in particular, $U'M = MB$, which gives $\angle MU'B = \angle MBU' = \frac{1}{2}\angle B = \angle ABU'$; thus, $MU' \| AB$; hence $U' = U$, which proves the claim that $U$ lies on $BI$. Similarly, we get that $V$ is on $CI$. Also, remember the perpendicularities $IB \perp CU$ and $IC \perp VB$, which we will use soon.




Next, note that the lines $XB$ and $XC$ are tangent to the circumcircle of triangle $IBC$; indeed, observe that

$$\begin{aligned}
\angle XBI &= \angle ABI - \angle ABX \\
&= \frac{1}{2}\angle B - (\angle BCX - \angle C) \\
&= \frac{1}{2}\angle B - \frac{1}{2}(180^\circ - \angle A) + \angle C \\
&= \frac{1}{2}\angle C \\
&= \angle BCI.
\end{aligned}$$

Similarly, $\angle XCI = \angle IBC$. This means that $X$ is the intersection of the tangents at $B$ and $C$ to the circumcircle of $IBC$; hence, $IX$ is the $I-$symmedian of triangle $IBC$.

But we proved before that $U$ and $V$ are on $IB$ and $IC$, respectively and that $IB \perp CU$ and $IC \perp VB$. In other words, we showed that $U$ and $V$ are the feet of the altitudes from $C$ and $B$ in triangle $IBC$ - so, in particular, we have that $BCUV$ is cyclic and that $UV$ is an antiparallel to $BC$ in triangle $IBC$. This yields the conclusion, since we know that the $I-$symmedian of $IBC$ is the locus of the midpoints of the antiparallels to $BC$ in triangle $IBC$; hence we showed that $IX$ bisects $UV$, as claimed. ■

This problem and and the second solution were suggested by Titu Andreescu and Cosmin Pohoata. The first solution was suggested by Zuming Feng.