

44th United States of America Mathematical Olympiad Solutions

Day I, II 12:30 PM – 5 PM EDT

April 28 - April 29, 2015

USAMO 1. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1 \right)^3.$$

Solution: Let $x + y = 3k$, with $k \in \mathbb{Z}$. Then $x^2 + x(3k - x) + (3k - x)^2 = (k + 1)^3$, which reduces to

$$x^2 - (3k)x - (k^3 - 6k^2 + 3k + 1) = 0.$$

Its discriminant Δ is

$$9k^2 + 4(k^3 - 6k^2 + 3k + 1) = 4k^3 - 15k^2 + 12k + 4.$$

We notice the (double) root $k = 2$, so $\Delta = (4k + 1)(k - 2)^2$. It follows that $4k + 1 = (2t + 1)^2$ for some nonnegative integer t , hence $k = t^2 + t$ and

$$x = \frac{1}{2}(3(t^2 + t) \pm (2t + 1)(t^2 + t - 2)).$$

We obtain $(x, y) = (t^3 + 3t^2 - 1, -t^3 + 3t + 1)$ and $(x, y) = (-t^3 + 3t + 1, t^3 + 3t^2 - 1)$, $t \in \{0, 1, 2, \dots\}$.

OR

One can also try to simplify the original equation as much as possible. First with $k = \frac{x+y}{3} + 1$ we get

$$x^2 - 3xk + 3x = k^3 - 9k^2 + 18k - 9.$$

But then we recognize terms from the expansion of $(k - 3)^3$ so we use $s = k - 3$ and obtain

$$x^2 - 3xs - 6x = s^3 - 9s - 9.$$

So again it becomes natural to use $x - 3 = u$. The equation becomes

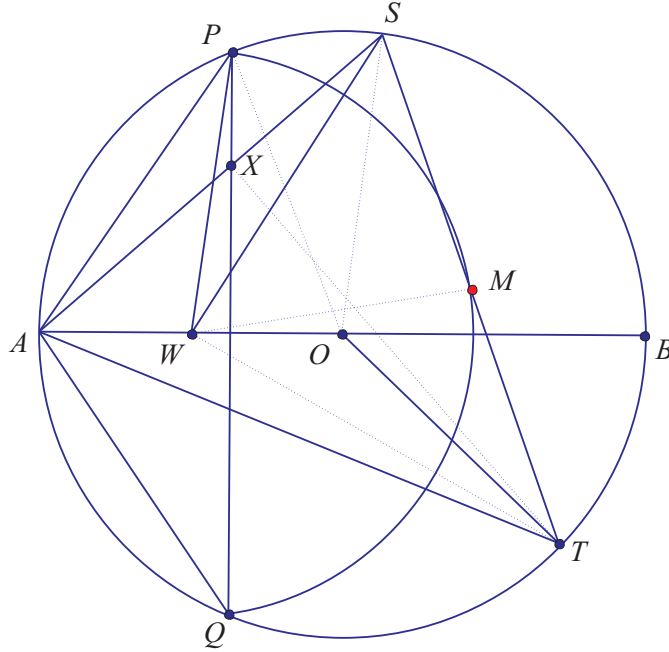
$$u^2 - 3su - s^3 = 0.$$

We view this as a quadratic in u , whose discriminant is $s^2(9 + 4s)$, and so $9 + 4s$ must be a perfect square, and because it is odd, it must be of the form $(2t + 1)^2$. It follows that $s = t^2 + t - 2$, and so $k = t^2 + t + 1$. We obtain the same family of solutions.

USAMO 2. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment \overline{PQ} . Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that \overline{XT} is perpendicular to \overline{AX} . Let M denote the midpoint of chord \overline{ST} . As X varies on segment \overline{PQ} , show that M moves along a circle.

Solution: Let O denote the center of ω , and let W denote the midpoint of segment \overline{AO} . Denote by Ω the circle centered at W with radius WP . We will show that $WM = WP$, which will imply that M always lies on Ω and so solve the problem.

We present two solutions. The first solution is more computational (in particular, with extensive applications of the formula for a median of a triangle); the second is more synthetic.



Set r to be the radius of circle ω . Applying the median formula in triangles APO , SWT , ASO , ATO gives

$$\begin{aligned} 4WP^2 &= 2AP^2 + 2OP^2 - AO^2 = 2AP^2 + r^2, \\ 4WM^2 &= 2WS^2 + 2WT^2 - ST^2, \\ 2WS^2 &= AS^2 + OS^2 - AO^2/2 = AS^2 + r^2/2, \\ 2WT^2 &= AT^2 + OT^2 - AO^2/2 = AT^2 + r^2/2. \end{aligned}$$

Adding the last three equations yields $4WM^2 = AS^2 + AT^2 - ST^2 + r^2$. It suffices to show that

$$4WP^2 = 4WM^2 \quad \text{or} \quad AS^2 + AT^2 - ST^2 = 2AP^2. \quad (1)$$

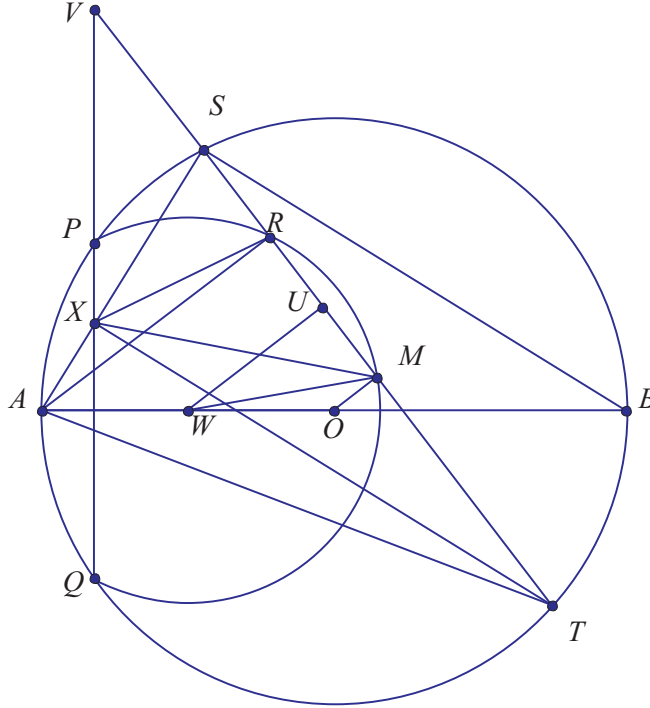
Because $\overline{XT} \perp \overline{AS}$,

$$AT^2 - ST^2 = (AX^2 + XT^2) - (SX^2 + XT^2)$$

$$\begin{aligned}
&= AX^2 - SX^2 \\
&= (AX + XS)(AX - XS) \\
&= AS(AX - XS).
\end{aligned}$$

It follows that $AS^2 + AT^2 - ST^2 = AS^2 + AS \cdot (AX - XS) = AS^2 + AS(2AX - AS) = 2AS \cdot AX$, and (1) reduces to $AP^2 = AS \cdot AX$, which is true because triangle APX is similar to triangle ASP (as $\angle PAX = \angle SAP$ and $\angle APX = \text{arc}(AQ)/2 = \text{arc}(AP)/2 = \angle ASP$).

OR



In the following solution, we use directed distances and directed angles in order to avoid issues with configuration (segments \overline{ST} and \overline{PQ} may intersect, or may not as depicted in the figure.)

Let R be the foot of the perpendicular from A to line ST . Note that $OM \perp ST$, and so $ARMO$ is a right trapezoid. Let U be the midpoint of segment \overline{RM} . Then \overline{WU} is the midline of the trapezoid. In particular, $\overline{WU} \perp \overline{RM}$. Hence line WU is the perpendicular bisector of segment \overline{RM} . It is also clear that AW is the perpendicular bisector of segment \overline{PQ} . Therefore, W is the intersection of the perpendicular bisectors of segments \overline{RM} and \overline{PQ} . It suffices to show that quadrilateral $PQMR$ is cyclic, since then W must be its circumcenter, and so $WP = WM$.

(To be precise, this argument fails when ST and PQ are parallel, because then $R = M$ and the perpendicular bisector of \overline{RM} is not defined. However, it is easy to see that this can happen for only one position of X . Because the argument works for all other X , continuity then implies that M lies on Ω for this exceptional case as well.)

Let lines PQ and ST meet in V . By the converse of the power-of-a-point theorem, it suffices to show that $VP \cdot VQ = VR \cdot VM$. On the other hand, because $PQTS$ is cyclic, by the power-of-a-point theorem, we have $VP \cdot VQ = VS \cdot VT$. Therefore, we only need to show that

$$VS \cdot VT = VR \cdot VM. \quad (2)$$

Note that M is the midpoint of segment \overline{ST} . Then (2) is equivalent to

$$2VS \cdot VT = VR \cdot (2VM) = VR \cdot (VS + VT)$$

or

$$VS \cdot VT - VS \cdot VR = VT \cdot VR - VT \cdot VS$$

or equivalently

$$VS \cdot RT = VT \cdot SR \quad \text{or} \quad \frac{VS}{SR} = \frac{VT}{RT}. \quad (3)$$

We claim that XS bisects $\angle VXR$. Indeed, because AB is the symmetry line of the kite $APBQ$, $AB \perp PQ$, and so $\angle VXS = \angle QXA = 90^\circ - \angle XAO = 90^\circ - \angle SAO$. Because O is the circumcenter of triangle AST ,

$$\angle VXS = 90^\circ - \angle SAO = \angle ATS.$$

On the other hand, because $\angle AXT$ and $\angle ART$ are both right angles, quadrilateral $AXRT$ is cyclic, implying that $\angle SXR = \angle ATR = \angle ATS$. Our claim follows from the last two equations.

Combining our claim and the fact that $XS \perp XT$, we know that XS and XT are the interior and exterior bisectors of $\angle VXR$, from which (3) follows, by the angle-bisector theorem. We saw that (3) was equivalent to (2) and that this was enough to show that $PQMR$ is cyclic, which completes the solution, so we are done.

USAMO 3. Let $S = \{1, 2, \dots, n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S ,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

Solution: The answer is $3^n + 1$.

Specifically, the colorings we want are of the following forms: either there are no blue sets; or for each element $x \in S$ we define one of three types of restriction — either x must be in T , x can't be in T , or x is unrestricted — and the blue sets T are exactly the ones that satisfy every restriction. It's easy to check such a coloring meets the condition, using the formula

$$f(T) = \prod_{x \in T} a_x \prod_{x \notin T} b_x,$$

where $a_x = 2$ if x is unrestricted and 1 otherwise, and $b_x = 0$ if x is required to be present and 1 otherwise.

We want to show that if there's at least one blue set, then the class of blue sets is of this form.

If some element of S is in every blue set, take it out and induct. If some element of S is not in any blue set, take it out and induct. Otherwise, every element x has some blue set containing it and some blue set not containing it. In this case we'll show that all sets are blue (i.e. every element is unrestricted).

First show \emptyset is blue. To show this, let T be a minimal blue set. If nonempty, take $x \in T$; by assumption there's blue T' not containing x . Then the condition is violated with T and T' , since $f(T \cap T') = 0$. Next, show any singleton is blue. Otherwise, let U be a minimal blue set containing x , and let $T = \{x\}$ and $T' = U \setminus \{x\}$. We get $1 \cdot m = 1 \cdot (1 + m)$ (where $m = f(T')$), a contradiction. Finally, any set is blue. Otherwise, let U be a minimal non-blue set and x, y two different elements. Taking $T = U \setminus \{x\}, T' = U \setminus \{y\}$ gives a contradiction.

USAMO 4. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he is finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i, k), (i, l), (j, k), (j, l)$ for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k) respectively, or removing one stone from each of (i, l) and (j, k) and moving them to (i, k) and (j, l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

Solution: We think of the pilings as assigning a positive integer to each square on the grid. Now, we restrict ourselves to the types of moves in which we take a lower left and upper right stone and move them to the upper left and lower right of our chosen rectangle. Call this a Type 1 stone move. We claim that we can perform a sequence of Type 1 stone moves on any piling to obtain an equivalent piling for which we cannot perform any Type 1 move, i.e. in which no square that has stones is above and to the right of any other square that has stones. We call such a piling a “down-right” piling.

To prove that any piling is equivalent to a down-right piling, first consider the squares in the leftmost column and topmost row of the grid. Let a be the entry (number of stones) in the upper left corner, and let b and c be the sum of the remaining entries in the leftmost column and topmost row respectively. If $b < c$, we can perform a sequence of Type 1 stone moves to remove all the stones from the leftmost column except for the top entry, and if $c < b$ we can similarly clear all squares in the top row except for the top left square. In the former case, we can now ignore the leftmost column and repeat the process on the second-to-leftmost column and the top row; similarly, in the latter case, we can ignore the

top row and proceed as before. Since the corner square a cannot be part of any Type 1 move at each step in the process, it follows that we end up with a down-right piling.

We next show that down-right pilings in any size grid (not necessarily $n \times n$) are uniquely determined by their row-sums and column-sums, given that the row sums and column sums are nonnegative integers which sum to m both along the rows and the columns. Let the topmost row sum be R_1 and the leftmost column sum be C_1 . Then the upper left square must contain $\min(R_1, C_1)$ stones, since otherwise there would be stones both in the first row and first column that are not in the upper left square. Whichever is smaller indicates that either the row or the column respectively is empty save for the upper left square; then we can remove this row or column and are reduced to a smaller grid in which we know all the row and column sums. Since one-row and one-column pilings are clearly uniquely determined by their column and row sums, it follows by induction that down-right pilings are determined uniquely by their row-sums and column sums.

Finally, notice that row sums and column sums are both invariant under stone moves. Therefore every piling is equivalent to a *unique* down-right piling. It therefore suffices to count the number of down-right pilings, which is also equivalent to counting the number of possibilities for the row-sums and column-sums. As stated above, the row sums and column sums can be the sums of any two n -tuples of nonnegative integers that each sum to m . The number of such tuples is $\binom{n+m-1}{m}$, and so the total number of non-equivalent pilings is the number of pairs of these tuples, i.e. $\left(\binom{n+m-1}{m}\right)^2$.

USAMO 5. Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

Solution: We approach indirectly by assuming that $p = ac + bd$ is a prime. By symmetry, we may assume that $\max\{a, b, c, d\} = a$, then because $a^4 + b^4 = c^4 + d^4$, we infer that $\min\{a, b, c, d\} = b$. Note that $ac \equiv -bd \pmod{p}$, implying that $a^4c^4 \equiv b^4d^4 \pmod{p}$. Consequently, we have

$$b^4d^4 + b^4c^4 \equiv a^4c^4 + b^4c^4 = c^8 + c^4d^4 \pmod{p},$$

from which it follows that $(c^4 + d^4)(b^4 - c^4) \equiv 0 \pmod{p}$. Thus, p divides at least one of $b - c, b + c, b^2 + c^2, c^4 + d^4$. Because $p = ac + bd > c^2 + b^2$, and $-(b^2 + c^2) < b - c < 0$ (because b and c are distinct), p must divide $c^4 + d^4 = e^5$. Thus $p^5 = (ac + bd)^5$ divides $c^4 + d^4$, which is clearly impossible because it is evident that $(ac + bd)^5 > c^4 + d^4$.

OR

A stronger result is possible:

Claim. Suppose a, b , and e are positive integers such that $a^4 + b^4 = e^5$. Then a and b have a common prime factor.

Proof. Suppose on the contrary that $\gcd(a, b) = 1$. If e is even, then this forces a and b to both be odd, so $a^4 + b^4 \equiv 2 \pmod{8}$ and $e^5 \equiv 0 \pmod{8}$, a contradiction. Thus e is odd. Note for use below that 5 cannot divide both a and b , so we may assume without loss that 5 does not divide a (swapping the roles of a and b if necessary).

Factoring over the Gaussian integers we find $a^4 + b^4 = (a^2 + ib^2)(a^2 - ib^2)$ and $\gcd(a^2 + ib^2, a^2 - ib^2) = \gcd(a^2 + ib^2, 2a^2)$. But $\gcd(a, b) = 1$ implies no prime factor of a can divide $a^2 + ib^2$ and e odd implies no prime factor of 2 divides $a^2 + ib^2$. Thus these factors are relatively prime, and hence both are a unit multiplied by a fifth power. Since every unit in the Gaussian integers is a fifth power, that means both factors are fifth powers, or

$$a^2 + ib^2 = (r + is)^5 = r^5 + 5ir^4s - 10r^3s^2 - 10ir^2s^3 + 5rs^4 + is^5.$$

Thus

$$\begin{aligned} a^2 &= r(r^4 - 10r^2s^2 + 5s^4), \quad \text{and} \\ b^2 &= s(s^4 - 10r^2s^2 + 5r^4). \end{aligned}$$

Note that since $\gcd(a, b) = 1$, $\gcd(r, s) = 1$. Also since 5 does not divide a , it also does not divide r . Since

$$\gcd(r, r^4 - 10r^2s^2 + 5s^4) = \gcd(r, 5s^4) = \gcd(r, 5) = 1,$$

r must be a perfect square and we have found an integer solution $(x, y, z) = (r, a/r, s)$ to

$$y^2 = x^4 - 10x^2z^2 + 5z^4$$

with $\gcd(x, z) = 1$. The following Lemma will then complete the proof of the claim.

Lemma. There are no nontrivial ($z \neq 0$) integer solutions to

$$y^2 = x^4 - 10x^2z^2 + 5z^4.$$

Proof. Suppose (x, y, z) is a solution in the positive integers with minimal z . Note that this implies that x, y, z are pairwise relatively prime. (The only case that takes a little work is that if a prime p divides x and y , then p^2 divides $5z^4$, hence p also divides z . But then p^4 divides x^2 so p^2 divides x and $(x/p^2, y/p, z/p)$ is a smaller solution.) Rewrite this as

$$20z^4 = (x^4 - 5z^2)^2 - y^2 = (x^2 - 5z^2 + y)(x^2 - 5z^2 - y).$$

The two factors on the right have the same parity and their product is even. Hence both are even. Any common factor p of $\frac{x^2 - 5z^2 + y}{2}$ and $\frac{x^2 - 5z^2 - y}{2}$ would have $p^2 | 5z^4$, hence $p | z$, and $p | \frac{x^2 - 5z^2 + y}{2} - \frac{x^2 - 5z^2 - y}{2} = y$, a contradiction. Thus these factors of $5z^4$ are relatively prime. Hence they must be $\pm v^4$ and $\pm 5w^4$ for some relatively prime v and w with $vw = z$. Then

$$x^2 - 5v^2w^2 = x^2 - 5z^2 = \frac{x^2 - 5z^2 + y}{2} + \frac{x^2 - 5z^2 - y}{2} = \pm v^4 \pm 5w^4$$

or

$$x^2 = \pm v^4 + 5v^2w^2 \pm 5w^4.$$

If v and w both odd, then the right hand side is either $1+5+5 \equiv 3 \pmod{8}$ or $-1+5-5 \equiv 7 \pmod{8}$, neither of which is possible for a square like the left hand side. Hence one of v

and w is even, and in either case we get $x^2 \equiv \pm 1 \pmod{4}$. Thus we must have the plus sign and

$$x^2 = v^4 + 5v^2w^2 + 5w^4.$$

This is not the equation we started with, so we repeat the argument above (with a few changes). Rewrite this new equation as

$$5w^4 = (2v^2 + 5w^2)^2 - 4x^2 = (2v^2 + 5w^2 + 2x)(2v^2 + 5w^2 - 2x).$$

There are two very similar cases depending on whether w is odd or even. These cases can be forced together, but we prefer to be more clear and keep them separate.

If w is odd, then the two factors on the right are both odd and any common (odd) prime factor p would have $p^2 | 5w^4$, hence $p | w$, and $p | (2v^2 + 5w^2 + 2x) - (2v^2 + 5w^2 - 2x) = 4x$, hence $p | x$. But then p also divides v and we get a contradiction. Thus these factors of $5w^4$ are relatively prime and so must be $\pm t^4$ and $\pm 5u^4$ for some relatively prime t and u with $tu = w$. Then

$$4v^2 + 10t^2u^2 = 4v^2 + 10w^2 = (2v^2 + 5w^2 + 2x) + (2v^2 + 5w^2 - 2x) = \pm(t^4 + 5u^4).$$

The left hand side is positive, so we must have the plus sign, and hence

$$(2v)^2 = t^4 - 10t^2u^2 + 5u^4.$$

Thus $(t, 2v, u)$ is a solution to the original equation. Since $u | w$ and $w | z$, we either have $u < z$ (contradicting the minimality of z) or $u = z$ and hence $t = v = 1$ (giving nonsense $4 = 1 - 10u^2 + 5u^4 \equiv 1 \pmod{5}$). Thus this case cannot occur.

If w is even, then the two factors are even, congruent mod 4, and their product is divisible by 16. Hence both are multiples of 4. Any common prime factor p of $\frac{2v^2+5w^2+2x}{4}$ and $\frac{2v^2+5w^2-2x}{4}$ would have $p^2 | 5(w/2)^4$, hence $p | w$, and $p | \frac{2v^2+5w^2+2x}{4} - \frac{2v^2+5w^2-2x}{4} = x$. But this would mean $p | v$, a contradiction. Thus $\frac{2v^2+5w^2+2x}{4}$ and $\frac{2v^2+5w^2-2x}{4}$ must be $\pm t^4$ and $\pm 5u^4$ for some relatively prime t and u with $2tu = w$. Then

$$v^2 + 10t^2u^2 = v^2 + \frac{5}{2}w^2 = \frac{2v^2 + 5w^2 + 2x}{4} + \frac{2v^2 + 5w^2 - 2x}{4} = \pm(t^4 + 5u^4).$$

Again, the left hand side is positive, so we must have the plus sign, and hence

$$v^2 = t^4 - 10t^2u^2 + 5u^4.$$

Thus (t, v, u) is a solution to the original equation and since $2u | w$ and $w | z$, we have $u < z$. This contradicts the minimality of z and completes the proof of the lemma.

Remark. One can use essentially the same argument to show that any nontrivial integer solution to $x^2 + y^4 = z^5$ has $\gcd(x, y) > 1$. In this case one cannot assume 5 does not divide r so there is a second case where $r = 5q^2$. Then $(x, y, z) = (s, a/(5q), q^2)$ is a solution to

$$y^2 = x^4 - 50x^2z^2 + 125z^4.$$

This Diophantine equation also has no nontrivial integer solutions and the proof is nearly identical to the proof of the Lemma above. This stronger result was (apparently) first proven by Nils Bruin (1999). This result is at least tangentially related to Beal's conjecture.

The more general solution is due to Richard Stong.

USAMO 6. Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \leq n\}$. Assume that for every $n \in \mathbb{N}$, the set A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$. (A multiset is a set-like collection of elements in which order is ignored, but repetition of elements is allowed and multiplicity of elements is significant. For example, multisets $\{1, 2, 3\}$ and $\{2, 1, 3\}$ are equivalent, but $\{1, 1, 2, 3\}$ and $\{1, 2, 3\}$ differ.)

Solution: Set $b_n = |A_n|$, $a_n = n\lambda - b_n \geq 0$. There are $b_i - b_{i-1}$ elements equal to i . Therefore the sum of elements in A_n is

$$\sum_{i=1}^n i(b_i - b_{i-1}) = nb_n - \sum_{i=1}^n b_i.$$

Now $b_n = n\lambda - a_n$, so the sum of elements in A_n may be written as

$$\Sigma_n = \lambda \frac{n(n+1)}{2} - na_n + \sum_{i=1}^{n-1} a_i.$$

Assume, by way of contradiction, that for all $n \geq n_0$, the sum of elements in A_n is greater than $\lambda \frac{n(n+1)}{2}$. Then

$$na_n < a_{n-1} + a_{n-2} + \dots + a_1,$$

so

$$a_n < \frac{a_{n-1} + a_{n-2} + \dots + a_1}{n} \leq \frac{M_n \cdot (n-1)}{n} \quad (4)$$

where $M_n = \max\{a_1, a_2, \dots, a_{n-1}\}$. We deduce that $a_n < \frac{(n-1)M_n}{n}$, so $M_{n+1} = M_n = M$, where $M = M_{n_0}$.

Let $\{x\}$ denote the fractional part of x ; i.e., $\{x\} = x - \lfloor x \rfloor$. We note that $\{a_{k+1} - a_k\} = \lambda$, so $\{(M - a_k) - (M - a_{k+1})\} = \lambda$. We claim that

$$(M - a_k) + (M - a_{k+1}) \geq \min(\lambda, 1 - \lambda). \quad (5)$$

To see this, we first note that $M - a_k \geq 0$ and $M - a_{k+1} \geq 0$. If either $M - a_k \geq 1$ or $M - a_{k+1} \geq 1$, then we are done. Assume that $0 < M - a_k, M - a_{k+1} < 1$. Then $-1 < (M - a_k) - (M - a_{k+1}) < 1$, so either $(M - a_k) - (M - a_{k+1}) = \lambda - 1$ or $(M - a_k) - (M - a_{k+1}) = \lambda$. In the former case, we get $M - a_{k+1} > 1 - \lambda$, and in the latter case we get $M - a_k > \lambda$. In either case, (5) follows.

We deduce from (5) that $a_k + a_{k+1} \leq 2M - \mu$, where $\mu = \min(\lambda, 1 - \lambda)$. From this and (4), we see that

$$a_n \leq M - \frac{\mu}{2} \quad (6)$$

for $n \geq n_1 = n_0 + 1$.

Let $\delta = \mu/3$. We will use induction to prove that for any integer $k \geq 1$ and $n \geq n_k$,

$$a_n \leq M - k\delta. \quad (7)$$

We have already proved the base case. Assume that (7) is true for a given fixed k . Using (6), we see that $a_k + a_{k+1} \leq 2M - 2k\delta - \mu = 2M - (2k + 3)\delta$. (Note that $\delta \leq 1/6$, so $\min(\delta, 1 - \delta) = \delta$.) Now if we take $n > (2k + 3)n_k$, we deduce that

$$a_n \leq \frac{n_k M + (n - n_k)(M - (k + \frac{3}{2})\delta)}{n} \leq M - (k + 1)\delta.$$

Statement (7) then follows by induction. However, it then follows that $a_n < 0$ when $k > M/\delta$, and this is a contradiction.