Solutions to USA(J)MO 2016

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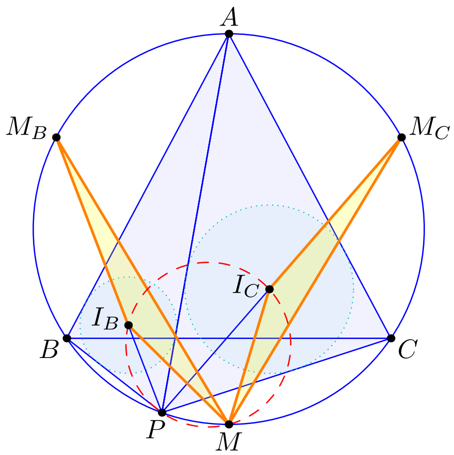
§1 Solution to JMO1

Let M be the midpoint of arc BC not containing A. We claim M is the desired fixed point.



Since $\angle MPA = 90^\circ$ and ray PA bisects $\angle I_B P I_C$, it suffices to show that $MI_B = MI_C$. Let M_B, M_C be the second intersections of PI_B and PI_C with circumcircle. Now $M_B I_B = M_B B = M_C C = M_C I_C$, and moreover $MM_B = MM_C$, and $\angle I_B M_B M = \frac{1}{2}\widehat{PB} = \angle I_C M_C M$, so triangles $\triangle I_B M_B M \cong \triangle I_C M_C M$, done.

§2 Solution to JMO2

One answer is $n = 20 + 2^{19} = 524308$.

First, observe that

$$\begin{aligned} 5^n &\equiv 5^{20} \pmod{5^{20}} \\ 5^n &\equiv 5^{20} \pmod{2^{20}} \end{aligned}$$

the former being immediate and the latter since $\varphi(2^{20}) = 2^{19}$. Hence $5^n \equiv 5^{20} \pmod{10^{20}}$. Moreover, we have

$$5^{20} = \frac{1}{2^{20}} \cdot 10^{20} < \frac{1}{1000^2} \cdot 10^{20} = 10^{-6} \cdot 10^{20}.$$

Thus the last 20 digits of 5^n will begin with six zeros. This completes the proof.

§3 Solution to JMO3 / USAMO1

The answer is that $|S| \geq 8$.

First, we provide a inductive construction for $S = \{1, \dots, 8\}$. Actually, for $n \geq 4$ we will provide a construction for $S = \{1, \dots, n\}$ which has $2^{n-1} + 1$ elements in a line. (This is sufficient, since we then get 129 for $n = 8$.) The idea is to start with the following construction for $|S| = 4$:

$$34 \quad 1 \quad 23 \quad 4 \quad 12 \quad 3 \quad 14 \quad 2 \quad 13 \ .$$

Then inductively, we do the following procedure to move from n to $n+1$: take the chain for n elements, delete an element, and make two copies of the chain (which now has even length). Glue the two copies together, joined by $\varnothing$ in between. Then place the element $n+1$ in alternating positions starting with the first (in particular, this hits $n+1$).

Explicitly, when $n = 8$ this construction gives

345678	1	235678	4	125678	3	145678	2	5678
34	15678	23	45678	12	35678	14	678	
345	1678	235	4678	125	3678	145	2678	5
34678	15	23678	45	12678	35	78		
3456	178	2356	478	1256	378	1456	278	56
3478	156	2378	456	1278	356	1478	6	
34578	16	23578	46	12578	36	14578	26	578
346	1578	236	4578	126	8			
34567	18	23567	48	12567	38	14567	28	567
348	1567	238	4567	128	3567	148	67	
3458	167	2358	467	1258	367	1458	267	58
3467	158	2367	458	1267	358	7		
34568	17	23568	47	12568	37	14568	27	568
347	1568	237	4568	127	3568	147	68	
3457	168	2357	468	1257	368	1457	268	57
3468	157	2368	457	1268				

Now let's check $|S| \geq 8$ is sufficient. Consider a chain on a set of size $|S| = 7$. (We need $|S| \geq 7$ else $2^{|S|} < 100$.) Observe that there are sets of size ≥ 4 can only be neighbored by sets of size ≤ 2, of which there are $\binom{7}{1} + \binom{7}{2} = 28$. So there are ≤ 30 sets of size ≥ 4. Also, there are $\binom{7}{3} = 35$ sets of size 3. So the total number of sets in a chain can be at most $30 + 28 + 35 = 93 < 100$.

§4 Solution to USAMO2

We show the exponent of any given prime p is nonnegative in the expression. Recall that the exponent of p in $n!$ is equal to $\sum_{i \geq 1} \lfloor n/p^i \rfloor$. In light of this, it suffices to show that for any prime power P, we have

$$\left\lfloor \frac{k^2}{P} \right\rfloor \geq \sum_{j=0}^{k-1} \left(\left\lfloor \frac{j+k}{P} \right\rfloor - \left\lfloor \frac{j}{P} \right\rfloor \right).$$

Since both sides are integers, we it is equivalent to show:

$$\left\lfloor \frac{k^2}{P} \right\rfloor > -1 + \sum_{j=0}^{k-1} \left(\left\lfloor \frac{j+k}{P} \right\rfloor - \left\lfloor \frac{j}{P} \right\rfloor \right).$$

Suppose we denote by $\{x\}$ the fractional part of x. Since $\lfloor x \rfloor = x - \{x\}$, it suffices to prove that

$$\left\{\frac{k^2}{P}\right\} + \sum_{j=0}^{k-1} \left\{\frac{j}{P}\right\} < 1 + \sum_{j=0}^{k-1} \left\{\frac{j+k}{P}\right\}$$

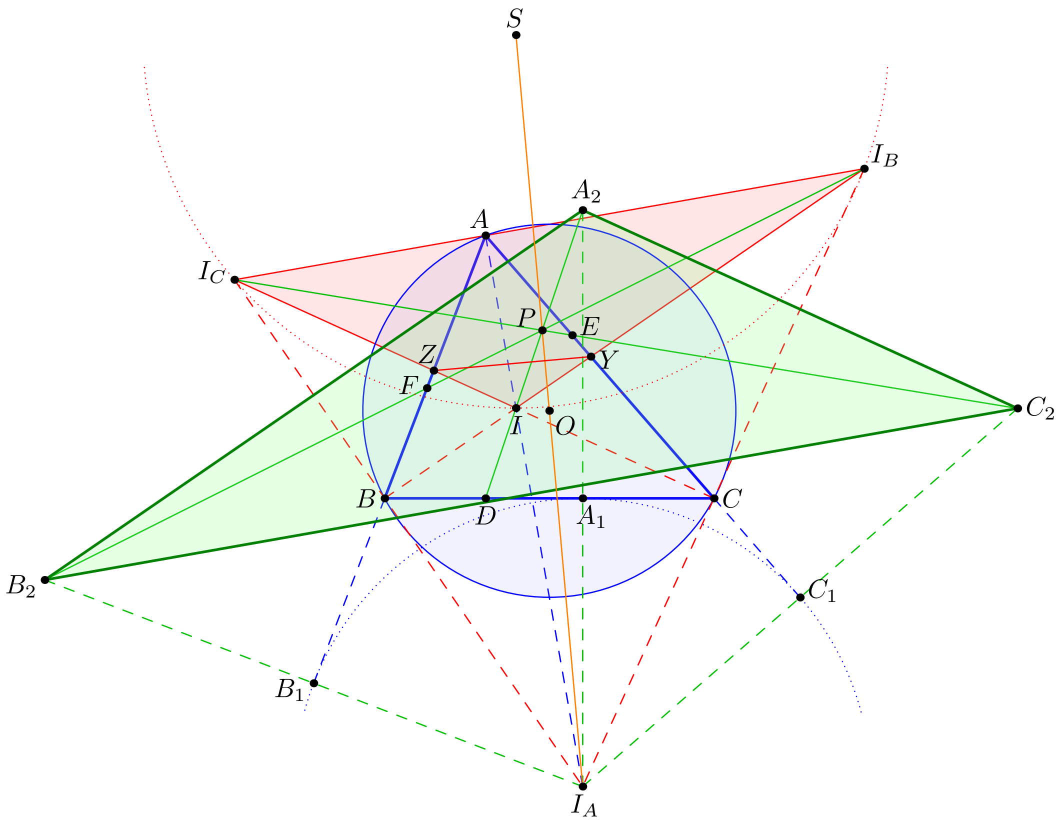
However, the sum of remainders when $(0, 1, \dots, k-1)$ is taken modulo P is easily seen to be less than the sum of remainders when $(k, k+1, \dots, 2k-1)$ is taken modulo P. So

$$\sum_{j=0}^{k-1} \left\{\frac{j}{P}\right\} \le \sum_{j=0}^{k-1} \left\{\frac{j+k}{P}\right\}$$

follows, and we are done upon noting $\left\{k^2/P\right\} < 1$.

§5 Solution to USAMO3

Let I_A denote the A-excenter and I the incenter. Then let D denote the foot of the altitude from A. Suppose the A-excircle is tangent to $\overline{BC}$, $\overline{AB}$, $\overline{AC}$ at A_1, B_1, C_1 and let A_2, B_2, C_2 denote the reflections of I_A across these points. Let S denote the circumcenter of $\triangle II_BI_C$.



We begin with the following observation: points D, I, A_2 are collinear, as are points E, I_C, C_2 are collinear and points F, I_B, B_2 are collinear. This follows from the "midpoints of altitudes" lemma.

Observe that $\overline{B_2C_2} \parallel \overline{B_1C_1} \parallel \overline{I_BI_C}$. Proceeding similarly on the other sides, we discover $\triangle II_BI_C$ and $\triangle A_2B_2C_2$ are homothetic. Hence P is the center of this homothety (in particular, D, I, P, A_2 are collinear). Moreover, P lies on the line joining I_A to S, which is the Euler line of $\triangle II_BI_C$, so it passes through the nine-point center of $\triangle II_BI_C$, which is O. Consequently, P, O, I_A are collinear as well.

To finish, we need only prove that $\overline{OS} \perp \overline{YZ}$. In fact, we claim that $\overline{YZ}$ is the radical axis of the circumcircles of $\triangle ABC$ and $\triangle II_BI_C$. Actually, Y is the radical center of these two circumcircles and the circle with diameter $\overline{II_B}$ (which passes through A and C). Analogously Z is the radical center of the circumcircles and the circle with diameter $\overline{II_C}$, and the proof is complete.

§6 Solution to JMO4

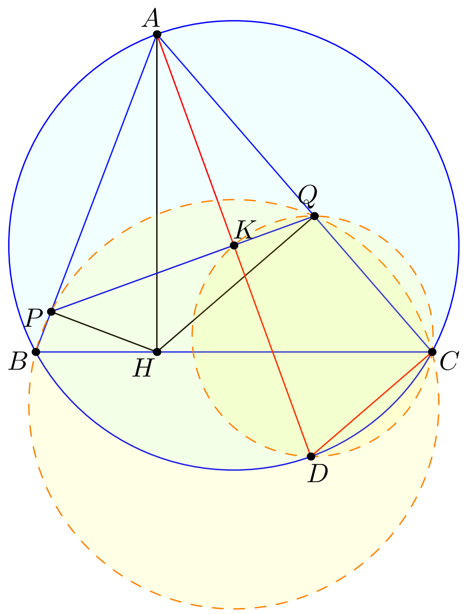
The answer is

$$N = 2017 + 2018 + \cdots + 4032 = 1008 \cdot 6049 = 6097392.$$

To see that N must be at least this large, simply consider the situation when 1, 2, ..., 2016 are removed. Then among the remaining elements, any sum of 2016 elements is certainly at least $2017 + 2018 + \cdots + 6049$.

Now we show this value of N works. Consider the 3024 pairs of numbers $(1, 6048)$, $(2, 6047)$, ..., $(3024, 3025)$. After the elements of $\{1, 2, \dots, N\}$ are deleted, at least $3024 - 2016 = 1008$ of these pairs have both elements remaining. Since each pair has sum 6049, we can take these pairs to be the desired numbers.

§7 Solution to JMO5



First, since $AP \cdot AB = AH^2 = AQ \cdot AC$, it follows that $PQCB$ is cyclic. Consequently, we have $AO \perp PQ$. Let K be the foot of A onto PQ, and let D be the point diametrically opposite A. Thus A, K, O, D are collinear.

Since quadrilateral $KQCD$ is cyclic ($\angle QKD = \angle QCD = 90^\circ$), we have

$$AK \cdot AD = AQ \cdot QC = AH^2 \implies AK = \frac{AH^2}{AD} = \frac{AH^2}{2AO} = AO$$

so $K = O$.

§8 Solution to JMO6 / USAMO4

First, taking $x = y = 0$ in the given yields $f(0) = 0$, and then taking $x = 0$ gives $f(y)f(-y) = f(y)^2$. So also $f(-y)^2 = f(y)f(-y)$, from which we conclude f is even. Then taking $x = -y$ gives

$$\forall x \in \mathbb{R}: \qquad f(x) = x^2 \qquad \text{or} \qquad f(4x) = 0 \qquad (\star)$$

for all x.

Next, we claim that

$$\forall x \in \mathbb{R}: \qquad f(x) = x^2 \qquad \text{or} \qquad f(x) = 0 \qquad (\heartsuit)$$

To see this assume $f(t) \neq 0$ (hence $t \neq 0$). By $(\star)$ we get $f(t/4) = t^2/16$. Now take $(x, y) = (3t/4, t/4)$ to get

$$\frac{t^2}{4} f(2t) = f(t^2) \implies f(2t) \neq 0.$$

If we apply $(\star)$ again we actually also get $f(t/2) \neq 0$. Together these imply

$$f(t) \neq 0 \iff f(2t) \neq 0 \qquad (\spadesuit).$$

Repeat $(\spadesuit)$ to get $f(4t) \neq 0$, hence $f(t) = t^2$, proving $(\heartsuit)$.

We are now ready to show the claimed solutions are the only ones. Assume there's an $a \neq 0$ for which $f(a) = 0$; we show that $f \equiv 0$. There are two approaches from here, by using inequalities or polynomials.

First approach

Pick $b \in \mathbb{R}$, we show directly $f(b) = 0$.

First, note that $\boxed{f \geq 0}$ always holds by $(\heartsuit)$. By using $(\spadesuit)$ we can generate $c > 100b$ such that $f(c) = 0$ (by taking $c = 2^n a$ for n large). Now, select $x, y > 0$ such that $x - 3y = b$ and $x + y = c$ id est

$$(x, y) = \left(\frac{3c + b}{4}, \frac{c - b}{4} \right).$$

Substitution into the original equation gives

$$0 = (f(x) + xy) f(b) + (f(y) + xy) f(3x - y).$$

But everything on the right-hand side is nonnegative. Thus it follows that $f(b) = f(3x - y) = 0$ as desired.

Second approach

First, observe that for all $x \in \mathbb{R}$

$$f(4x - a) \neq 0 \implies (f(x) + x(3x - a)) f(3a - 8x) = f(4x - a)^2 \neq 0$$

by taking $y = 3x - a$ in the original equation. Finally, consider the equations

$$0 = (4x - a)^4 - (x(3x - a)) (3a - 8x)^2$$
$$0 = (4x - a)^4 - \left(x^2 + x(3x - a)\right) (3a - 8x)^2$$

Each right-hand side is a nonzero polynomial in x. Thus there are finitely many roots in x, hence there are only finitely many values of x with $f(4x - a) \neq 0$. But $(\spadesuit)$ then implies there cannot be any values of x at all, i.e. we conclude that $f \equiv 0$.

§9 Solution to USAMO5

First solution

In fact, we show that we only need $AM = AQ = NP$ and $MN = QP$.

We use complex numbers with ABC the unit circle, assuming WLOG that A, B, C are labeled counterclockwise. Let x, y, z be the complex numbers corresponding to the arc midpoints of BC, CA, AB, respectively; thus $x + y + z$ is the incenter of $\triangle ABC$. Finally, let $s > 0$ be the side length of $AM = AQ = NP$.

Then, since $MA = s$ and $MA \perp OX$, it follows that

$$m - a = i \cdot sx.$$

Similarly, $n - p = i \cdot sy$ and $a - q = i \cdot sz$, so summing these up gives

$$i \cdot s(x + y + z) = (p - q) + (m - n) = (m - n) - (q - p).$$

Since $MN = PQ$, the argument of $(m - n) - (q - p)$ is along the external angle bisector of the angle formed, which is perpendicular to ℓ. On the other hand, $x + y + z$ is oriented in the same direction as OI, as desired.

Second solution

Let δ and ϵ denote $\angle MNB$ and $\angle CPQ$. Also, assume $AMNPQ$ has side length 1.

In what follows, assume $AB < AC$. First, we note that

$$\begin{aligned} BN &= (c-1)\cos B + \cos\delta \\ CP &= (b-1)\cos C + \cos\epsilon \\ \implies a &= 1 + BN + CP \\ \implies \cos\delta + \cos\epsilon &= \cos B + \cos C - 1. \end{aligned}$$

Also, by Law of Sines, we have $\frac{c-1}{\sin\delta} = \frac{1}{\sin B}$ and similarly on triangle CPQ, and from this we deduce

$$\sin\epsilon - \sin\delta = \sin B - \sin C.$$

Using sum-to-product formulas on our relations implies that

$$\tan\left(\frac{\epsilon - \delta}{2}\right) = \frac{\sin B - \sin C}{\cos B - \cos C + 1}.$$

Now note that ℓ makes an angle of $\frac{1}{2}(\pi + \epsilon - \delta)$ with line BC. Moreover, if line OI intersects line BC with angle φ then

$$\tan\varphi = \frac{r - R\cos A}{\frac{1}{2}(b-c)}.$$

So in order to prove the result, we only need to check that

$$\frac{r - R\cos A}{\frac{1}{2}(b-c)} = \frac{\cos B - \cos C + 1}{\sin B - \sin C}.$$

Using the fact that $b = 2R\sin B$, $c = 2R\sin C$, this just reduces to the fact that $r/R + 1 = \cos A + \cos B + \cos C$, which is the so-called Carnot theorem.

§10 Solution to USAMO6

The game is winnable if and only if $n \neq k$.

First suppose $2 \leq k < n$. Query the cards in positions $\{1, \dots, k\}$, then $\{2, \dots, k+1\}$, and so on, up to $\{2n-k+1, 2n\}$. By taking the diff of any two adjacent queries, we can deduce for certain the values on cards $1, 2, \dots, 2n-k$. If $k \leq n$, this is more than n cards, so we can find a matching pair.

For $k = n$ we remark the following: at each turn after the first, assuming one has not won, there are n cards representing each of the n values exactly once, such that the player has no information about the order of those n cards. We claim that consequently the player cannot guarantee victory. Indeed, let S denote this set of n cards, and $\overline{S}$ the other n cards. The player will never win by picking only cards in S or $\overline{S}$. Also, if the player selects some cards in S and some cards in $\overline{S}$, then it is possible that the choice of cards in S is exactly the complement of those selected from $\overline{S}$; the strategy cannot prevent this since the player has no information on S. This implies the result.