36th United States of America Mathematical Olympiad

1. Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k - 1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \ldots$. Prove that for any n the sequence $a_1, a_2, a_3, \ldots$ eventually becomes constant.

First Solution: For $k \ge 1$, let

$$s_k = a_1 + a_2 + \cdots + a_k.$$

We have

$$\frac{s_{k+1}}{k+1} < \frac{s_{k+1}}{k} = \frac{s_k + a_{k+1}}{k} \le \frac{s_k + k}{k} = \frac{s_k}{k} + 1.$$

On the other hand, for each k, s_k/k is a positive integer. Therefore

$$\frac{s_{k+1}}{k+1} \le \frac{s_k}{k},$$

and the sequence of quotients s_k/k is eventually constant. If $s_{k+1}/(k+1) = s_k/k$, then

$$a_{k+1} = s_{k+1} - s_k = \frac{(k+1)s_k}{k} - s_k = \frac{s_k}{k},$$

showing that the sequence a_k is eventually constant as well.

Second Solution: For $k \ge 1$, let

$$s_k = a_1 + a_2 + \cdots + a_k \qquad \text{and} \qquad \frac{s_k}{k} = q_k.$$

Since $a_k \le k - 1$, for $k \ge 2$, we have

$$s_k = a_1 + a_2 + a_3 + \cdots + a_k \le n + 1 + 2 + \cdots + (k-1) = n + \frac{k(k-1)}{2}.$$

Let m be a positive integer such that $n \le \frac{m(m+1)}{2}$ (such an integer clearly exists). Then

$$q_m = \frac{s_m}{m} \le \frac{n}{m} + \frac{m-1}{2} \le \frac{m+1}{2} + \frac{m-1}{2} = m.$$

We claim that

$$q_m = a_{m+1} = a_{m+2} = a_{m+3} = a_{m+4} = \ldots \ .$$

This follows from the fact that the sequence $a_1, a_2, a_3, \ldots$ is uniquely determined and choosing $a_{m+i} = q_m$, for $i \geq 1$, satisfies the range condition

$$0 \leq a_{m+i} = q_m \leq m \leq m + i - 1,$$

and yields

$$s_{m+i} = s_m + iq_m = mq_m + iq_m = (m+i)q_m.$$

Third Solution: For $k \geq 1$, let

$$s_k = a_1 + a_2 + \cdots + a_k.$$

We claim that for some m we have $s_m = m(m-1)$. To this end, consider the sequence which computes the differences between s_k and $k(k-1)$, i.e., whose k-th term is $s_k - k(k-1)$. Note that the first term of this sequence is positive (it is equal to n) and that its terms are strictly decreasing since

$$(s_k - k(k-1)) - (s_{k+1} - (k+1)k) = 2k - a_{k+1} \geq 2k - k = k \geq 1.$$

Further, a negative term cannot immediately follow a positive term. Suppose otherwise, namely that $s_k > k(k-1)$ and $s_{k+1} < (k+1)k$. Since s_k and s_{k+1} are divisible by k and $k+1$, respectively, we can tighten the above inequalities to $s_k \geq k^2$ and $s_{k+1} \leq (k+1)(k-1) = k^2 - 1$. But this would imply that $s_k > s_{k+1}$, a contradiction. We conclude that the sequence of differences must eventually include a term equal to zero.

Let m be a positive integer such that $s_m = m(m-1)$. We claim that

$$m - 1 = a_{m+1} = a_{m+2} = a_{m+3} = a_{m+4} = \ldots\ .$$

This follows from the fact that the sequence $a_1, a_2, a_3, \ldots$ is uniquely determined and choosing $a_{m+i} = m - 1$, for $i \geq 1$, satisfies the range condition

$$0 \leq a_{m+i} = m - 1 \leq m + i - 1,$$

and yields

$$s_{m+i} = s_m + i(m-1) = m(m-1) + i(m-1) = (m+i)(m-1).$$

This problem was suggested by Sam Vandervelde.

2. A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

Solution: It is not possible. The proof is by contradiction. Suppose that such a covering family $\mathcal{F}$ exists. Let $D(P, \rho)$ denote the disc with center P and radius ρ. Start with an arbitrary disc $D(O, r)$ that does not overlap any member of $\mathcal{F}$. Then $D(O, r)$ covers no grid point. Take the disc $D(O, r)$ to be maximal in the sense that any further enlargement would cause it to violate the non-overlap condition. Then $D(O, r)$ is tangent to at least three discs in $\mathcal{F}$. Observe that there must be two of the three tangent discs, say $D(A, a)$ and $D(B, b)$, such that $\angle AOB \leq 120°$. By the Law of Cosines applied to triangle ABO,

$$(a+b)^2 \leq (a+r)^2 + (b+r)^2 + (a+r)(b+r),$$

which yields

$$ab \leq 3(a+b)r + 3r^2, \qquad \text{and thus} \qquad 12r^2 \geq (a-3r)(b-3r).$$

Note that $r < 1/\sqrt{2}$ because $D(O, r)$ covers no grid point, and $(a-3r)(b-3r) \geq (5-3r)^2$ because each disc in $\mathcal{F}$ has radius at least 5. Hence $2\sqrt{3}r \geq (5-3r)$, which gives $5 \leq (3+2\sqrt{3})r < (3+2\sqrt{3})/\sqrt{2}$ and thus $5\sqrt{2} < 3+2\sqrt{3}$. Squaring both sides of this inequality yields $50 < 21 + 12\sqrt{3} < 21 + 12 \cdot 2 = 45$. This contradiction completes the proof.

Remark: The above argument shows that no covering family exists where each disc has radius greater than $(3 + 2\sqrt{3})/\sqrt{2} \approx 4.571$. In the other direction, there exists a covering family in which each disc has radius $\sqrt{13}/2 \approx 1.802$. Take discs with this radius centered at points of the form $(2m + 4n + \frac{1}{2}, 3m + \frac{1}{2})$, where m and n are integers. Then any grid point is within $\sqrt{13}/2$ of one of the centers and the distance between any two centers is at least $\sqrt{13}$. The extremal radius of a covering family is unknown.

This problem was suggested by Gregory Galperin.

3. Let S be a set containing $n^2 + n - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

Solution: In order to apply induction, we generalize the result to be proved so that it reads as follows:

Proposition. If the n-element subsets of a set S with $(n+1)m-1$ elements are partitioned into two classes, then there are at least m pairwise disjoint sets in the same class.

Proof. Fix n and proceed by induction on m. The case of $m = 1$ is trivial. Assume $m > 1$ and that the proposition is true for $m - 1$. Let $\mathcal{P}$ be the partition of the n-element subsets into two classes. If all the n-element subsets belong to the same class, the result is obvious. Otherwise select two n-element subsets A and B from different classes so that their intersection has maximal size. It is easy to see that $|A \cap B| = n - 1$. (If $|A \cap B| = k < n-1$, then build C from B by replacing some element not in $A \cap B$ with an element of A not already in B. Then $|A \cap C| = k+1$ and $|B \cap C| = n-1$ and either A and C or B and C are in different classes.) Removing $A \cup B$ from S, there are $(n+1)(m-1)-1$ elements left. On this set the partition induced by $\mathcal{P}$ has, by the inductive hypothesis, $m-1$ pairwise disjoint sets in the same class. Adding either A or B as appropriate gives m pairwise disjoint sets in the same class. $\square$

Remark: The value $n^2 + n - 1$ is sharp. A set S with $n^2 + n - 2$ elements can be split into a set A with $n^2 - 1$ elements and a set B of $n - 1$ elements. Let one class consist of all n-element subsets of A and the other consist of all n-element subsets that intersect B. Then neither class contains n pairwise disjoint sets.

This problem was suggested by András Gyárfás.

4. An *animal* with n *cells* is a connected figure consisting of n equal-sized square cells.[1] The figure below shows an 8-cell animal.

A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* if its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

[1]Animals are also called *polyominoes*. They can be defined inductively. Two cells are *adjacent* if they share a complete edge. A single cell is an animal, and given an animal with n-cells, one with $n + 1$ cells is obtained by adjoining a new cell by making it adjacent to one or more existing cells.

Solution: Let s denote the minimum number of cells in a dinosaur; the number this year is $s = 2007$.

Claim: The maximum number of cells in a primitive dinosaur is $4(s-1)+1$.

First, a primitive dinosaur can contain up to $4(s-1)+1$ cells. To see this, consider a dinosaur in the form of a cross consisting of a central cell and four arms with $s-1$ cells apiece. No connected figure with at least s cells can be removed without disconnecting the dinosaur.

The proof that no dinosaur with at least $4(s-1)+2$ cells is primitive relies on the following result.

Lemma. *Let D be a dinosaur having at least $4(s-1)+2$ cells, and let R (red) and B (black) be two complementary animals in D, i.e., $R \cap B = \varnothing$ and $R \cup B = D$. Suppose $|R| \leq s-1$. Then R can be augmented to produce animals $\tilde{R} \supset R$ and $\tilde{B} = D \setminus \tilde{R}$ such that at least one of the following holds:*

(i) $|\tilde{R}| \geq s$ *and* $|\tilde{B}| \geq s$,

(ii) $|\tilde{R}| = |R| + 1$,

(iii) $|R| < |\tilde{R}| \leq s-1$.

Proof. If there is a black cell adjacent to R that can be made red without disconnecting B, then (ii) holds. Otherwise, there is a black cell c adjacent to R whose removal disconnects B. Of the squares adjacent to c, at least one is red, and at least one is black, otherwise B would be disconnected. Then there are at most three resulting components $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of B after the removal of c. Without loss of generality, $\mathcal{C}_3$ is the largest of the remaining components. (Note that $\mathcal{C}_1$ or $\mathcal{C}_2$ may be empty.) Now $\mathcal{C}_3$ has at least $\lceil (3s-2)/3 \rceil = s$ cells. Let $\tilde{B} = \mathcal{C}_3$. Then $|\tilde{R}| = |R| + |\mathcal{C}_1| + |\mathcal{C}_2| + 1$. If $|\tilde{B}| \leq 3s-2$, then $|\tilde{R}| \geq s$ and (i) holds. If $|\tilde{B}| \geq 3s-1$ then either (ii) or (iii) holds, depending on whether $|\tilde{R}| \geq s$ or not. $\square$

Starting with $|R| = 1$, repeatedly apply the Lemma. Because in alternatives (ii) and (iii) $|R|$ increases but remains less than s, alternative (i) eventually must occur. This shows that no dinosaur with at least $4(s-1)+2$ cells is primitive.

This problem was suggested by Reid Barton.

5. Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n+3$ (not necessarily distinct) primes.

Solution: The proof is by induction. The base is provided by the $n = 0$ case, where $7^{7^0} + 1 = 7^1 + 1 = 2^3$. To prove the inductive step, it suffices to show that if $x = 7^{2m-1}$ for some positive integer m then $(x^7+1)/(x+1)$ is composite. As a consequence, $x^7 + 1$ has at least two more prime factors than does $x + 1$. To confirm that $(x^7 + 1)/(x + 1)$ is composite, observe that

$$\begin{aligned}
\frac{x^7+1}{x+1} &= \frac{(x+1)^7 - ((x+1)^7 - (x^7+1))}{x+1} \\
&= (x+1)^6 - \frac{7x(x^5+3x^4+5x^3+5x^2+3x+1)}{x+1} \\
&= (x+1)^6 - 7x(x^4+2x^3+3x^2+2x+1) \\
&= (x+1)^6 - 7^{2m}(x^2+x+1)^2 \\
&= \{(x+1)^3 - 7^m(x^2+x+1)\}\{(x+1)^3 + 7^m(x^2+x+1)\}
\end{aligned}$$

Also each factor exceeds 1. It suffices to check the smaller one; $\sqrt{7x} \leq x$ gives

$$\begin{aligned}
(x+1)^3 - 7^m(x^2+x+1) &= (x+1)^3 - \sqrt{7x}(x^2+x+1) \\
&\geq x^3 + 3x^2 + 3x + 1 - x(x^2+x+1) \\
&= 2x^2 + 2x + 1 \geq 113 > 1.
\end{aligned}$$

Hence $(x^7 + 1)/(x + 1)$ is composite and the proof is complete.

This problem was suggested by Titu Andreescu.

6. Let ABC be an acute triangle with ω, Ω, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to Ω at A and tangent externally to ω. Circle Ω_A is tangent internally to Ω at A and tangent internally to ω. Let P_A and Q_A denote the centers of ω_A and Ω_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

$$8 P_A Q_A \cdot P_B Q_B \cdot P_C Q_C \leq R^3,$$

with equality if and only if triangle ABC is equilateral.

Solution: Let the incircle touch the sides AB, BC, and CA at C_1, A_1, and B_1, respectively. Set $AB = c$, $BC = a$, $CA = b$. By equal tangents, we may assume that $AB_1 = AC_1 = x$, $BC_1 = BA_1 = y$, and $CA_1 = CB_1 = z$. Then $a = y + z$, $b = z + x$, $c = x + y$. By the AM-GM inequality, we have $a \geq 2\sqrt{yz}$, $b \geq 2\sqrt{zx}$, and $c \geq 2\sqrt{xy}$. Multiplying the last three inequalities yields

$$abc \geq 8xyz, \qquad (\dagger),$$

with equality if and only if $x = y = z$; that is, triangle ABC is equilateral.

Let k denote the area of triangle ABC. By the Extended Law of Sines, $c = 2R\sin\angle C$. Hence

$$k = \frac{ab\sin\angle C}{2} = \frac{abc}{4R} \quad \text{or} \quad R = \frac{abc}{4k}. \qquad (\ddagger)$$

We are going to show that

$$P_AQ_A = \frac{xa^2}{4k}. \qquad (*)$$

In exactly the same way, we can also establish its cyclic analogous forms

$$P_BQ_B = \frac{yb^2}{4k} \quad \text{and} \quad P_CQ_C = \frac{zc^2}{4k}.$$

Multiplying the last three equations together gives

$$P_AQ_A \cdot P_BQ_B \cdot P_CQ_C = \frac{xyza^2b^2c^2}{64k^3}.$$

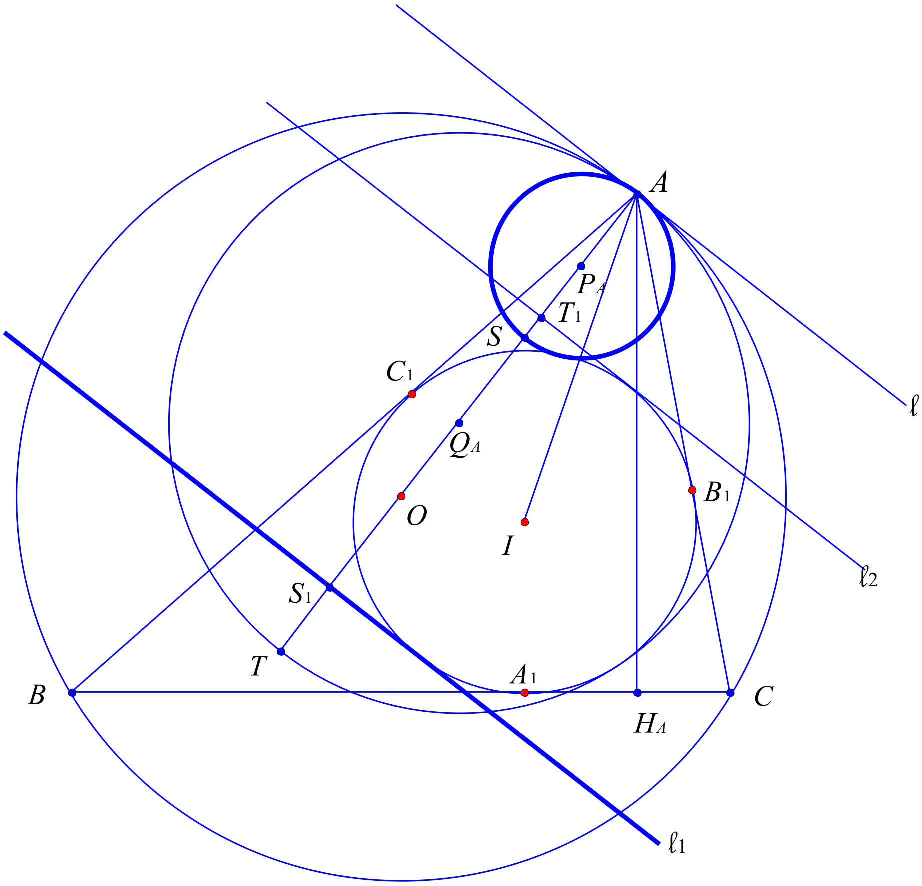
Further considering $(\dagger)$ and $(\ddagger)$, we have

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQC = \frac{8xyza^2b^2c^2}{64k^3} \leq \frac{a^3b^3c^3}{64k^3} = R^3,$$

with equality if and only if triangle ABC is equilateral.

Hence it suffices to show $(*)$. Let r, r_A, r'_A denote the radii of $\omega, \omega_A, \Omega_A$, respectively. We consider the inversion $\mathbf{I}$ with center A and radius x. Clearly, $\mathbf{I}(B_1) = B_1$, $\mathbf{I}(C_1) = C_1$, and $\mathbf{I}(\omega) = \omega$. Let ray AO intersect ω_A and Ω_A at S and T, respectively. It is not difficult to see that $AT > AS$, because ω is tangent to ω_A and Ω_A externally and internally, respectively. Set $S_1 = \mathbf{I}(S)$ and $T_1 = \mathbf{I}(T)$. Let ℓ denote the line tangent to Ω at A. Then the image of ω_A (under the inversion) is the line (denoted by ℓ_1) passing through S_1 and parallel to ℓ, and the image of Ω_A is the line (denoted by ℓ_2) passing through T_1 and parallel to

ℓ. Furthermore, since ω is tangent to both ω_A and Ω_A, ℓ_1 and ℓ_2 are also tangent to the image of ω, which is ω itself. Thus the distance between these two lines is $2r$; that is, $S_1T_1 = 2r$. Hence we can consider the following configuration. (The darkened circle is ω_A, and its image is the darkened line ℓ_1.)



By the definition of inversion, we have $AS_1 \cdot AS = AT_1 \cdot AT = x^2$. Note that $AS = 2r_A$, $AT = 2r'_A$, and $S_1T_1 = 2r$. We have

$$r_A = \frac{x^2}{2AS_1}. \quad \text{and} \quad r'_A = \frac{x^2}{2AT_1} = \frac{x^2}{2(AS_1 - 2r)}.$$

Hence

$$P_AQ_A = AQ_A - AP_A = r'_A - r_A = \frac{x^2}{2}\left(\frac{1}{AS_1 - 2r} + \frac{1}{AS_1}\right).$$

Let H_A be the foot of the perpendicular from A to side BC. It is well known that $\angle BAS_1 = \angle BAO = 90^\circ - \angle C = \angle CAH_A$. Since ray AI bisects $\angle BAC$, it follows that rays AS_1 and AH_A are symmetric with respect to ray AI. Further note that both line ℓ_1

(passing through S_1) and line BC (passing through H_A) are tangent to ω. We conclude that $AS_1 = AH_A$. In light of this observation and using the fact $2k = AH_A \cdot BC = (AB + BC + CA)r$, we can compute P_AQ_A as follows:

$$\begin{aligned}
P_AQ_A &= \frac{x^2}{2}\left(\frac{1}{AH_A - 2r} - \frac{1}{AH_A}\right) = \frac{x^2}{4k}\left(\frac{2k}{AH_A - 2r} - \frac{2k}{AH_A}\right) \\
&= \frac{x^2}{4k}\left(\frac{1}{\frac{1}{BC} - \frac{2}{AB+BC+CA}} - BC\right) = \frac{x^2}{4k}\left(\frac{1}{\frac{1}{y+z} - \frac{1}{x+y+z}} - (y+z)\right) \\
&= \frac{x^2}{4k}\left(\frac{(y+z)(x+y+z)}{x} - (y+z)\right) \\
&= \frac{x(y+z)^2}{4k} = \frac{xa^2}{4k},
\end{aligned}$$

establishing $(*)$. Our proof is complete.

Note: Trigonometric solutions of $(*)$ are also possible.
Query: For a given triangle, how can one construct ω_A and Ω_A by ruler and compass?

This problem was suggested by Kiran Kedlaya and Sungyoon Kim.