8^{th} United States of America Junior Mathematical Olympiad Solutions

USAJMO 1. (Proposed by Gregory Galperin)

Let n be an odd positive integer, and take $a = 2n-1$, $b = 2n+1$. Then $a^b + b^a \equiv 1 + 3 \equiv 0 \pmod 4$, and $a^b + b^a \equiv -1 + 1 \equiv 0 \pmod n$. Therefore $a + b = 4n$ divides $a^b + b^a$.

Alternate solution: Let $p > 5$ be a prime and let $p \not\equiv 1 \pmod 5$. For each such prime p we construct a pair of relatively prime numbers (a, b) that satisfy the conclusion of the problem. Thus, we will get infinitely many distinct pairs (a, b) as required.

Let $a = 3p + 2$, $b = 7p - 2$. Then $a + b = 10p$. We have $\varphi(10p) = 4(p-1) = b - a$, where φ is Euler's function.

Obviously, a and b are odd and not divisible by p. They are not divisible by 5 because $p \not\equiv 1 \pmod 5$. Thus, a and b are relatively prime to $10p = a + b$, and therefore relatively prime to each other.

Therefore, using Euler's theorem,

$$a^b = a^{a+\varphi(10p)} = a^a \cdot a^{\varphi(10p)} \equiv a^a \pmod{10p},$$

and since $10p = a + b$,

$$a^b + b^a \equiv a^a + b^a \pmod{a+b}.$$

However, since a is odd, $a^a + b^a$ is divisible by $a + b$. Hence, $a^b + b^a$ is divisible by $a + b$.

USAJMO 2. (Proposed by Titu Andreescu)

For $x > 0$ and $y > 0$, the left-hand side of the equation is positive, implying that $x > y$.

(a) Set $\dfrac{x}{y} = k + 1$, for some positive rational number k. Then the equation is equivalent to

$$(k+1)(3k^2 + 6k + 4)(k^2 + 2k + 4) = (k^7)y.$$

Take any positive integer n. Letting $k = \frac{1}{n}$ yields an infinite family of solutions

$$(x, y) = (n(n+1)^2(4n^2 + 6n + 3)(4n^2 + 2n + 1), n^2(n+1)(4n^2 + 6n + 3)(4n^2 + 2n + 1))$$

to the given equation.

(b) Write the equation as

$$x(3x^2 + y^2)y(x^2 + 3y^2) = (x-y)^7,$$

which is equivalent to

$$(x^3 + 3xy^2)(3x^2y + y^3) = (x-y)^7.$$

Let $x^3+3xy^2=a$ and $3x^2y+y^3=b$. Then $a+b=(x+y)^3$, $a-b=(x-y)^3$ and the equation becomes

$$(ab)^3=(a-b)^7.$$

Let $d=\gcd(a,b)$. Then $a=du$ and $b=dv$ for some relatively prime positive integers u and v. Hence

$$(uv)^3=d(u-v)^7.$$

Because $\gcd(u,v)=1$, we have $\gcd(u-v,u)=1$, $\gcd(u-v,v)=1$, hence $\gcd(u-v,uv)=1$. It follows that $u-v=1$ and $d=(uv)^3$. Hence $u=k+1$ and $v=k$, where k is a positive integer, and so $a=(k+1)^4k^3$ and $b=k^4(k+1)^3$. Then

$$(x-y)^3=a-b=[k(k+1)]^3$$

and

$$(x+y)^3=a+b=[k(k+1)]^3(2k+1).$$

It follows that $2k+1=n^3$ for some odd integer $n>1$ and that $x+y=nk(k+1)$ and $x-y=k(k+1)$. Hence

$$(x,y)=\left(\frac{(n+1)k(k+1)}{2},\ \frac{(n-1)k(k+1)}{2}\right)$$

where $k=\frac{n^3-1}{2}$. Thus

$$(x,y)=\left(\frac{(n+1)(n^6-1)}{8},\ \frac{(n-1)(n^6-1)}{8}\right)$$

where n is an odd integer greater than 1, and it is easy to check that these are solutions to the given equation. Hence these pairs describe all the solutions to the equation.

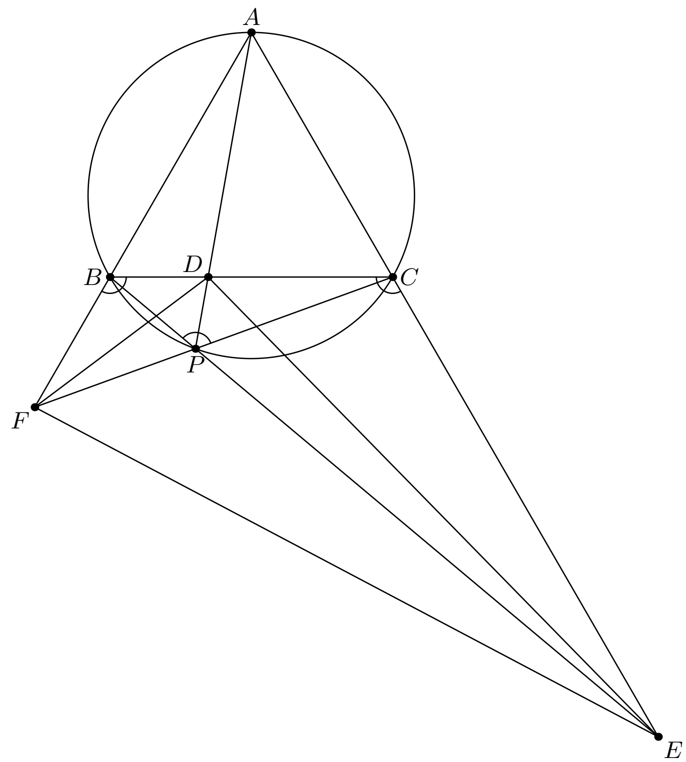
USAJMO 3. (Proposed by Titu Andreescu, Luis Gonzalez, and Cosmin Pohoata)

We offer several solutions. Throughout, we use bracket notation for areas: for example, $[ABC]$ means the area of triangle ABC.

We first present three down-to-earth approaches. One of them is a coordinate geometry approach. The other two approaches utilize the fact of many pairs of similar triangles in this configuration:

- BPC, FPA, FBC, APE, and BCE;
- FBP and FCA;
- ECP and EBA.

In these solutions, we assume the points are configured so that P is on minor arc $\widehat{BC}$ of the circle, as shown in the figure.



Solution 1. (By USA(J)MO packet reviewers.) We may assume that $AB = 1$. Then $[ABC] = \sqrt{3}/4$. Set $b = PB$, $c = PC$, $e = PE$, and $f = PF$. Note that $\angle FBD = \angle ECD = \angle BPC = 120°$. Hence

$$[DEF] = [BCEF] - [FBD] - [ECD] = \frac{1}{2}\sin 120°(BE \cdot CF - BF \cdot BD - CE \cdot CD).$$

It suffices to show that $[DEF] = \sqrt{3}/2$ or

$$2 = (BE \cdot CF - BF \cdot BD - CE \cdot CD) = (b+e)(c+f) - BF \cdot BD - CE \cdot CD.$$

Because $\angle FBC = \angle BPC$ and $\angle FCB = \angle PCB$, triangles FCB and BCP are similar to each other, implying that

$$\frac{FC}{BC} = \frac{CB}{CP} = \frac{BF}{PB} \quad \text{or} \quad \frac{c+f}{1} = \frac{1}{c} = \frac{BF}{b}.$$

Thus, $c + f = 1/c$ and $BF = b/c$. Analogously, $b + e = 1/b$ and $CE = c/b$. It remains to show that

$$2 = (b+e)(c+f) - BF \cdot BD - CE \cdot CD = \frac{1}{bc} - \frac{b}{c} \cdot BD - \frac{c}{b} \cdot CD.$$

Note that $\angle BPD = \angle CPD = 60°$, so we have $BD/CD = BP/CP$ by the Angle-Bisector theorem. Consequently, we have $BD = b/(b+c)$ and $CD = c/(b+c)$. Thus, we want to show that

$$2 \quad = \quad \frac{1}{bc} - \frac{b}{c} \cdot BD - \frac{c}{b} \cdot CD = \frac{1}{bc} - \frac{b^2}{c(b+c)} - \frac{c^2}{b(b+c)}$$

$$= \quad \frac{1}{bc} - \frac{b^3+c^3}{bc(b+c)} = \frac{1-b^2-c^2+bc}{bc},$$

or $b^2+c^2+bc=1$, which is true by applying the Law of Cosines in triangle BPC.

Solution 2. (By USA(J)MO packet reviewers.) Note that $\angle DPF = \angle DPE = \angle EPF = 120^\circ$. We have

$$[DEF] = \frac{1}{2} \cdot \sin 120^\circ \left(PD \cdot PE + PE \cdot PF + PF \cdot PD \right).$$

To show that $[DEF] = 2[ABC]$, it suffices to show that

$$PD \cdot PE + PE \cdot PF + PF \cdot PD = 2BC^2.$$

Set $b = PB$ and $c = PC$. We will express the lengths of BC, PD, PE, and PF in terms of b and c. Note that $\angle BPC = 120^\circ$. Applying the Law of Cosines in triangle BPC gives $BC^2 = b^2+bc+c^2$. Applying Ptolemy's theorem to cyclic quadrilateral $ABCP$ yields $AP \cdot BC = BP \cdot AC + CP \cdot AB$ or $AP = b+c$. Because $\angle ACB = \angle ABC = \angle APC = 60^\circ$, triangles ACD and APC are similar, and so

$$\frac{AC}{AP} = \frac{CD}{PC} = \frac{DA}{CA},$$

or $b^2+bc+c^2 = AC^2 = AP \cdot AD = (b+c) \cdot AD$. We conclude that

$$AD = \frac{b^2+bc+c^2}{b+c} \quad \text{and} \quad PD = AP - AD = b + c - \frac{b^2+bc+c^2}{b+c} = \frac{bc}{b+c}.$$

Finally, because $\angle FBP = 180^\circ - \angle ABP = \angle ACP$ and $\angle BPF = \angle APC = 60^\circ$, triangles FBP and ACP are similar. Hence

$$\frac{FB}{AC} = \frac{BP}{CP} = \frac{PF}{PA},$$

from which it follows that $PF = AP \cdot BP/CP = b(b+c)/c$. In exactly the same way, we get $PE = c(b+c)/b$. It follows that

$$\begin{aligned} PD \cdot PE + PE \cdot PF + PF \cdot PD &= \frac{bc}{b+c}\left(\frac{c(b+c)}{b} + \frac{b(b+c)}{c}\right) + \frac{c(b+c)}{b} \cdot \frac{b(b+c)}{c} \\ &= 2(b^2+bc+c^2), \end{aligned}$$

as desired.

Solution 3. (By USA(J)MO packet reviewers.) Without loss of generality, we may assume that $A = (0,2)$, $B = (-\sqrt{3}, -1)$, and $C = (\sqrt{3}, -1)$. Set $P = (a,b)$ with $a^2+b^2 = 4$.

Solving for line equations $y = -1$ and $y = \dfrac{(b-2)}{a} \cdot x + 2$ gives $D = \left(-\dfrac{3a}{b-2}, -1\right)$.

Solving for line equations $y = \sqrt{3}x + 2$ and $y = \dfrac{(b+1)}{a-\sqrt{3}} \cdot (x - \sqrt{3}) - 1$ gives

$$F = \left(\frac{3a + \sqrt{3}b - 2\sqrt{3}}{b + 4 - \sqrt{3}a}, \ \frac{\sqrt{3}a + 5b + 2}{b + 4 - \sqrt{3}a} \right).$$

Solving for line equations $y = -\sqrt{3}x + 2$ and $y = \dfrac{(b+1)}{a+\sqrt{3}} \cdot (x + \sqrt{3}) - 1$ gives

$$E = \left(\frac{3a - \sqrt{3}b + 2\sqrt{3}}{b + 4 + \sqrt{3}a}, \ \frac{-\sqrt{3}a + 5b + 2}{b + 4 + \sqrt{3}a} \right).$$

Hence

$$\overrightarrow{DF} = \left[\frac{3a + \sqrt{3}b - 2\sqrt{3}}{b + 4 - \sqrt{3}a} + \frac{3a}{b-2}, \ \frac{6(b+1)}{b + 4 - \sqrt{3}a} \right]$$

and

$$\overrightarrow{DE} = \left[\frac{3a - \sqrt{3}b + 2\sqrt{3}}{b + 4 + \sqrt{3}a} + \frac{3a}{b-2}, \ \frac{6(b+1)}{b + 4 + \sqrt{3}a} \right].$$

Therefore,

$$\begin{aligned}
2[DEF] &= \frac{6(b+1)}{b+4+\sqrt{3}a} \cdot \left(\frac{3a + \sqrt{3}b - 2\sqrt{3}}{b+4-\sqrt{3}a} + \frac{3a}{b-2} \right) - \frac{6(b+1)}{b+4-\sqrt{3}a} \cdot \left(\frac{3a - \sqrt{3}b + 2\sqrt{3}}{b+4+\sqrt{3}a} + \frac{3a}{b-2} \right) \\
&= \frac{12\sqrt{3}(b+1)(b-2)}{(b+4)^2 - 3a^2} + \frac{18a(b+1)}{b-2} \cdot \left(\frac{1}{b+4+\sqrt{3}a} - \frac{1}{b+4-\sqrt{3}a} \right) \\
&= \frac{12\sqrt{3}(b+1)(b-2)}{(b+4)^2 - 3a^2} - \frac{36\sqrt{3}a^2(b+1)}{(b-2)((b+4)^2 - 3a^2)} \\
&= \frac{12\sqrt{3}(b+1)(b-2)}{(b+4)^2 - 3(4-b^2)} - \frac{36\sqrt{3}(4-b^2)(b+1)}{(b-2)((b+4)^2 - 3(4-b^2))} \\
&= \frac{12\sqrt{3}(b+1)(b-2)}{4b^2 + 8b + 4} - \frac{36\sqrt{3}(2-b)(2+b)(b+1)}{(b-2)(4b^2 + 8b + 4)} \\
&= \frac{3\sqrt{3}(b-2)}{b+1} + \frac{9\sqrt{3}(2+b)}{b+1} = \frac{3\sqrt{3}(b - 2 + 6 + 3b)}{b+1} = 12\sqrt{3},
\end{aligned}$$

implying that $[DEF] = 6\sqrt{3} = 2[ABC]$, as desired.

The next solution is by the problem authors. It uses more advanced tools that USAJMO participants are not expected to know, but offers some additional insight into the origins of the problem.

Solution 4. (By the posers.) Without loss of generality, let us assume that P lies on the arc AC, which does not contain vertex B. Because P is on the circumcircle, its isogonal conjugate, say Q, is a point at infinity. Furthermore, the intersections D', E', F' of lines QA, QB, QC with lines BC, CA, AB, respectively, are the reflections of D, E, F across the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. This essentially follows from the fact that $\triangle ABC$ is equilateral: isogonal conjugates with respect to it are also isotomic conjugates. We are thus led to the following lemma.

Lemma 1. *Let ABC be a triangle with D, E, F points lying on the lines BC, CA, AB, respectively. Let D', E', F' be the reflections of D, E, F with respect to the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Then, triangles DEF and $D'E'F'$ have the same area.*

Proof. The statement holds regardless of the position of points D, E, F on lines BC, CA, AB, so, for convenience, in the computations below we shall assume that these all lie close enough to the midpoints of the sides so that all points D, E, F, D', E', F' lie on the sides of $\triangle ABC$. The proof for the other scenarios is similar.

We begin by writing

$$[CD'E'] = [AD'E] = [AD'C] - [CD'E].$$

Analogously, $[AE'F'] = [BE'A] - [AE'F]$ and $[BF'D'] = [CF'B] - [BF'D]$. Adding these three together, we get

$$\begin{aligned} & [CD'E'] + [AE'F'] + [BF'D'] \\ = \; & [AD'C] + [BE'A] + [CF'B] - [CD'E] - [AE'F] - [BF'D]. \end{aligned}$$

Furthermore,

$$[CDE] = [BD'E] = [BEC] - [CD'E],$$

and similarly $[AEF] = [CFA] - [AE'F]$ and $[BFD] = [ADB] - [BF'D]$. Therefore,

$$\begin{aligned} & [CDE] + [AEF] + [BFD] \\ = \; & [BEC] + [CFA] + [ADB] - [CD'E] - [AE'F] - [BF'D]. \end{aligned}$$

But $D'C = DB$, $E'A = EC$, $F'B = FA$, so $[AD'C] = [ADB]$, $[BE'A] = [BEC]$, $[CF'B] = [CFA]$. Using all of the above, we get

$$[CD'E'] + [AE'F'] + [BF'D'] = [CDE] + [AEF] + [BFD],$$

and so $[ABC] - [D'E'F'] = [ABC] - [DEF]$, i.e., $[DEF] = [D'E'F']$, establishing the lemma. $\square$

Assuming Lemma 1, we just have to check that $[D'E'F'] = 2[ABC]$. Because P lies on the small arc AC, points D and F lie on the extensions of segments BC and AB, respectively, and so D' and F' do too. Furthermore, B lies in the interior of triangle $D'E'F'$, therefore

$$[D'E'F'] = [D'BF'] + [F'BE'] + [E'BD'].$$

On the other hand, $AD' \| CF'$ implies $[D'CF'] = [ACF']$, which, after subtracting $[BCF']$ from both sides, gives $[D'BF'] = [ABC]$. Likewise, $BE' \| CF'$ gives $[F'BE'] = [CBE']$ and $AD' \| BE'$ gives $[E'BD'] = [E'BA]$. Hence, it follows that

$$[D'E'F'] = [ABC] + [CBE'] + [E'BA] = 2[ABC],$$

as claimed.

Note: One can also establish the lemma using barycentric coordinates. Suppose points D, E, F are dividing the sides BC, CA, AB in the ratios

$$BD : DC = x : 1-x, \quad CE : EA = y : 1-y, \quad AF : FB = z : 1-z.$$

In terms of barycentric coordinates with respect to triangle ABC, we have

$$D = (1-x)B + xC, \quad E = (1-y)C + yA, \quad F = (1-z)A + zB.$$

Consequently, by definition, points D', E', F' satisfy

$$D' = xB + (1-x)C, \quad E' = yC + (1-y)A, \quad F' = zA + (1-z)B.$$

Now, without loss of generality, rescale so that $[ABC] = 1$. It can then be easily checked that

$$\begin{aligned}
[DEF] &= [ABC] - ([AEF] + [BFD] + [CDE]) \\
&= (1 - ((1-y)z + (1-z)x + (1-x)y)) \\
&= (1 - (x+y+z) + (xy+yz+zx)) \\
&= (1 - (y(1-z) + z(1-x) + x(1-y))) \\
&= [ABC] - ([AE'F'] + [BF'D'] + [CD'E']) \\
&= [D'E'F'].
\end{aligned}$$

This proves Lemma 1. The rest of the solution is as before.

USAJMO 4. (Proposed by Titu Andreescu)
Suppose (a, b, c) is such a triple. The prime $(a-2)(b-2)(c-2) + 12$ also divides

$$\begin{aligned}
&a^2 + b^2 + c^2 + abc - 2017 - (a-2)(b-2)(c-2) - 12 \\
&= (a+b+c)^2 - 4(a+b+c) + 4 - 2025 \\
&= (a+b+c-2)^2 - 45^2 \\
&= (a+b+c-47)(a+b+c+43).
\end{aligned}$$

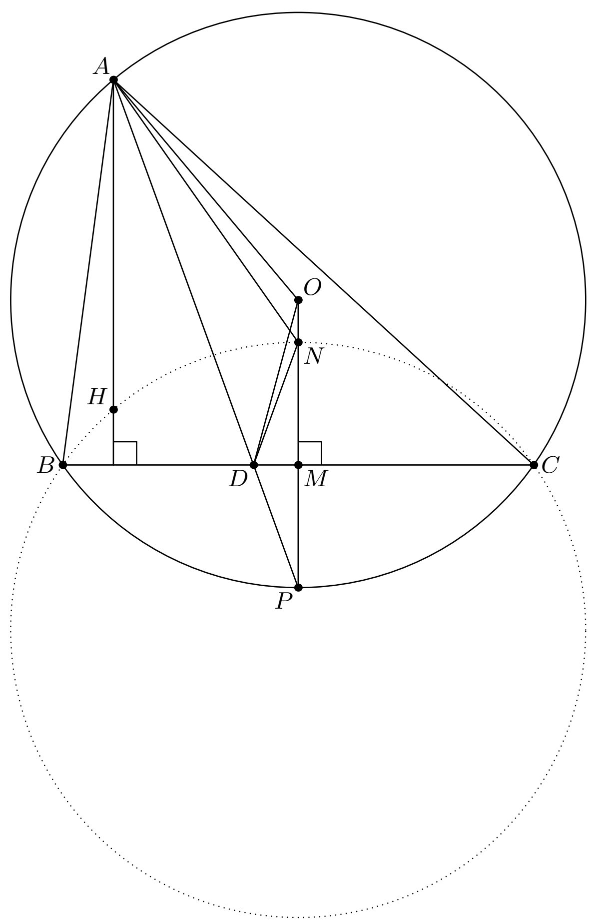
We may assume without loss of generality that $a \le b \le c$. If $a = b = 1$, $c + 10$ must be a prime that properly divides $c^2 + c - 2015$, implying $c + 10$ divides $1925 = 5^2 \cdot 7 \cdot 11$. So $c + 10 = 11$, and we obtain the triple $(1, 1, 1)$. However, this does not make $a^2 + b^2 + c^2 + abc - 2017$ positive.

If $a = 1$ and $b = 2$, then $(a-2)(b-2)(c-2) + 12 = 12$ is not prime. If $a = 1$ and $b = 3$, $14 - c$ must be a prime. The allowable choices for c are $3, 7, 9, 11$ and 12, but none of these work. If $a = 1$ and $b = 4$, the prime is even, so must be 2 and hence $c = 7$, but this doesn't work either. If $a = 1$ and $b \ge 5$ then $c \ge 5$ also, so $(a-2)(b-2)(c-2) + 12 \le 12 - 9 = 3$, and the only possibility is $b = c = 5$, but this also doesn't work. This rules out the cases with $a = 1$. Also $a = 2$ is impossible, again because 12 is not prime.

Now let $x = a - 2$, $y = b - 2$, $z = c - 2$. We now know that $1 \le x \le y \le z$ and $(x+2) + (y+2) + (z+2) > 47$. So $x + y + z \ge 41$, and therefore $z \ge 14$. The prime $xyz + 12$ cannot divide $(x+2) + (y+2) + (z+2) - 47$ since $xyz - 4 > x + y + z - 41$. Indeed, this latter inequality reduces to $x(yz - 1) > y + z - 37$, which will follow if we can prove that $yz - 1 > y + z - 37$ (since $x \ge 1$). The last statement is equivalent to $(y-1)(z-1) > -36$, which is evidently true.

Hence $xyz+12$ divides $(x+2)+(y+2)+(z+2)+43$. They cannot be equal: x,y,z must all be odd, otherwise $xyz+12$ is not prime, but then $(x+2)+(y+2)+(z+2)+43$ is even and so not equal to $xyz+12$. Thus $2(xyz+12) \le x+y+z+49$, implying $2yz-1 \le x(2yz-1) \le y+z+25$. It follows that $(2y-1)(2z-1) \le 53$. Earlier we proved that $z \ge 14$; since z is odd, we must in fact have $z \ge 15$. Moreover, $2y-1 \le 53/(2z-1) \le 53/29 < 2$. Therefore $x = y = 1$. It follows that $z+12$ is prime and $15 \le z \le 27$; therefore $z = 17, 19$, or 25. Also, $z+12$ divides $(x+2)+(y+2)+(z+2)+43 = z+51$. However, this is false for $z = 17, 19$, or 25. Consequently, the answer is negative; i.e., the requested triples (a,b,c) do not exist.

USAJMO 5. (Proposed by Ivan Borsenco)



Set $\angle CAB = A$, $\angle ABC = B$, and $\angle BCA = C$. Because H is the orthocenter, we have $\angle HBC = 90^\circ - C$ and $\angle HCB = 90^\circ - B$. In triangle BHC, we have $\angle BHC = 180^\circ - \angle HBC - \angle HCB = B + C$. Because $BHNC$ is cyclic, we have $\angle BNC = \angle BHC = B + C$. Extend segment AD through D to meet the circumcircle (denoted by ω) of triangle ABC at P. It is clear that P is the midpoint of minor arc $\widehat{BC}$ (of ω) and O, M, P all lie on the perpendicular bisector of segment

BC. In particular, $BPCN$ is a kite with symmetry axis PN. Because $ABPC$ is cyclic, we have $\angle BPC = 180^\circ - \angle BAC = B + C = \angle BNC$. We can further conclude that $BPCN$ is a rhombus, implying that line BC is the perpendicular bisector of segment NP, and so $DN = NP$ and $\angle DPN = \angle DNP$.

Set $x = \angle HAP$. Because $AH \parallel OP$, we have $\angle DNP = \angle DPN = \angle HAP = x$. Because O is the circumcenter of triangle ABC, we have $\angle AOC = 2B$ and $\angle CAO = \angle ACO = 90^\circ - B$. Because H is the orthocenter of triangle ABC, we have $\angle BAH = 90^\circ - B$. Because $\angle BAH = 90^\circ - B = \angle CAO$, $\angle BAC$ and $\angle HAO$ share common angle bisector AD; that is,

$$\angle DNP = \angle DPN = \angle HAP = \angle OAP = \angle OAD = x.$$

Consequently, we have

$$\angle ADO = \angle ADN - \angle ODN = \angle DNP + \angle DPN - \angle ODN = 2x - \angle ODN$$

and

$$\angle HAN = \angle HAO - \angle OAN = \angle HAP + \angle OAP - \angle OAN = 2x - \angle OAN.$$

It suffices to show that $\angle ODN = \angle OAN$, which is clearly true because $ADNO$ is cyclic as $\angle DNP = \angle OAD = x$.

Alternate solution (by Titu Andreescu and Cosmin Pohoata). The key idea is to prove that $ADNO$ is cyclic. Once this is proven, the problem follows by noticing that $\angle ADO = \angle ANO = \angle HAN$, where the latter holds due to the fact that $ON \| AH$.

To prove the concyclicity, one can simply use Power of a Point. First, one has to construct P as in the first solution, and notice that M is the midpoint of segment $\overline{PN}$. This follows from the fact that the reflection of H across line BC lies on the circumcircle Ω of $\triangle ABC$. This implies that the circumcircle of $\triangle BHC$ is the reflection of Ω across line BC, so line BC must indeed bisect $\overline{PN}$ by symmetry. Next, let O' denote the orthogonal projection of O on AD. Clearly $OO'DM$ is cyclic, so Power of a Point yields $PM \cdot PO = PD \cdot PO'$. But O' is the midpoint of PA, so $PO' = PA/2$. Since $PM = PN/2$, this yields

$$PN \cdot PO = PD \cdot PA,$$

which by Power of a Point gives the concyclity of $ADNO$. This completes the proof.

USAJMO 6. (Proposed by Maria Monks Gillespie)

We may assume the points have been labeled as $P_1, P_2, \ldots, P_{2n}$ in order, going counterclockwise from $(1,0)$. Now, write out the color of each point in order, and replace each R with a $+1$ and each B with a -1, to get a list $p_1, \ldots, p_{2n}$ of $+1$'s and -1's. Consider the partial sums $p_1 + \cdots + p_k$ of this sequence, and choose the index k such that the kth partial sum has as small a value as possible; if several partial sums are tied for smallest, let k be the lowest index among them. Now, rotate the circle clockwise so that points $P_1, \ldots, P_k$ are moved past $(1,0)$; the resulting sequence of $+1$'s and -1's from the new orientation now has all nonnegative partial sums, and the total sum is 0.

Consider any red point in the rotated diagram and label it R_1. The arc $R_1 \to B_1$ does not cross $(1,0)$, for otherwise the sequence ends with a string of $+1$'s and the partial sums before those $+1$'s would be negative. Furthermore, the sequence of entries from R_1 to B_1 looks like $+1, +1, +1, \ldots, +1, -1$, and so removing R_1 and B_1 is equivalent to removing a consecutive pair of a $+1$ and -1, so the partial sums remain all nonnegative. It follows that the next pairing also doesn't cross $(1, 0)$, and so on, so no matter which way we pick the ordering of the red points in the rotated circle, there are no counterclockwise arcs $R_i \to B_i$ containing $(1, 0)$.

Finally, note that in any ordering of the red points, the blue points among $P_1, \ldots, P_k$ are all paired with red points, and those red points among $P_1, \ldots, P_k$ are paired with blue points in this same subsequence since there are no crossings in the rotated picture. Let m be the difference between the number of blue and red points among $P_1, \ldots, P_k$. Then it follows that exactly m blue points in $P_1, \ldots, P_k$ were matched with red points from $P_{k+1}, \ldots, P_{2n}$. Therefore, when we rotate the circle back to its original position, there are exactly m crossings, no matter which ordering we pick for the red points. Since m is independent of the ordering, the proof is complete.