6^{th} United States of America Junior Mathematical Olympiad Solutions

Day I, II   12:30 PM – 5 PM EDT

April 28 - April 29, 2015

JMO 1. Given a sequence of real numbers, a move consists of choosing two terms and replacing each by their arithmetic mean. Show that there exists a sequence of 2015 distinct real numbers such that after one initial move is applied to the sequence – no matter what move – there is always a way to continue with a finite sequence of moves so as to obtain in the end a constant sequence.

Solution: The sequence $(x_1,\, x_2,\, \ldots,\, x_{2015}) = (1,\, 2,\, \ldots,\, 2015)$ satisfies the required property (as does any arithmetic sequence).

Assume that $(x_m,\, x_n) = (m,\, n)$ is replaced by $\left(\frac{m+n}{2},\, \frac{m+n}{2}\right)$ in the first move. We consider two cases.

In the first case, we assume that none of m and n is equal to 1008. In the second move, we replace $(x_{2016-m},\, x_{2016-n}) = (2016-m,\, 2016-n)$ by $\left(2016 - \frac{m+n}{2},\, 2016 - \frac{m+n}{2}\right)$. Let all the subsequent moves be applied to the pairs $(x_j,\, x_{2016-j})$, $j = 1, 2, \ldots, 1008$. This yields the constant sequence $(1008,\, 1008,\, \ldots,\, 1008)$.

In the second case, we assume that one of m and n, say, n is equal to 1008. After the first move we have $x_m = x_{1008} = \frac{1008+m}{2}$. Choose k different from 1008, m, and $2016-m$. We illustrate our next four moves in the following table. (In each move, we operate on the the numbers in bold.)

$$
\begin{aligned}
& (x_k,\, x_m,\, x_{1008},\, x_{2016-m},\, x_{2016-k}) \\
= \;& \left(\boldsymbol{k},\, \frac{1008+m}{2},\, \frac{1008+m}{2},\, 2016-m,\, \mathbf{2016}-\boldsymbol{k}\right) \\
\to \;& \left(\mathbf{1008},\, \frac{1008+m}{2},\, \frac{1008+m}{2},\, \mathbf{2016}-\boldsymbol{m},\, 1008\right) \\
\to \;& \left(\boldsymbol{\frac{3024-m}{2}},\, \boldsymbol{\frac{1008+m}{2}},\, \frac{1008+m}{2},\, \frac{3024-m}{2},\, 1008\right) \\
\to \;& \left(1008,\, 1008,\, \boldsymbol{\frac{1008+m}{2}},\, \boldsymbol{\frac{3024-m}{2}},\, 1008\right) \\
\to \;& (1008,\, 1008,\, 1008,\, 1008,\, 1008)
\end{aligned}
$$

Finally apply the move to all the pairs $(x_j,\, x_{2016-j})$ (with $j \neq m, k, 2016-m, 2016-k$) to obtain the constant sequence $(1008,\, 1008,\, \ldots,\, 1008)$.

Query: If the initial sequence is $(1,\, 2,\, 3,\, \ldots,\, 2013,\, 2014,\, 2016)$, where "2015" is replaced by "2016", is it possible to obtain a constant sequence after a finite sequence of moves?

JMO 2. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

Solution: Let $x+y=3k$, with $k \in \mathbb{Z}$. Then $x^2+x(3k-x)+(3k-x)^2=(k+1)^3$, which reduces to

$$x^2-(3k)x-(k^3-6k^2+3k+1)=0.$$

Its discriminant Δ is

$$9k^2+4(k^3-6k^2+3k+1)=4k^3-15k^2+12k+4.$$

We notice the (double) root $k=2$, so $\Delta=(4k+1)(k-2)^2$. It follows that $4k+1=(2t+1)^2$ for some nonnegative integer t, hence $k=t^2+t$ and

$$x=\frac{1}{2}(3(t^2+t)\pm(2t+1)(t^2+t-2)).$$

We obtain $(x,y)=(t^3+3t^2-1,-t^3+3t+1)$ and $(x,y)=(-t^3+3t+1,t^3+3t^2-1)$, $t \in \{0,1,2,...\}$.

OR

One can also try to simplify the original equation as much as possible. First with $k=\frac{x+y}{3}+1$ we get

$$x^2-3xk+3x=k^3-9k^2+18k-9.$$

But then we recognize terms from the expansion of $(k-3)^3$ so we use $s=k-3$ and obtain

$$x^2-3xs-6x=s^3-9s-9.$$

So again it becomes natural to use $x-3=u$. The equation becomes

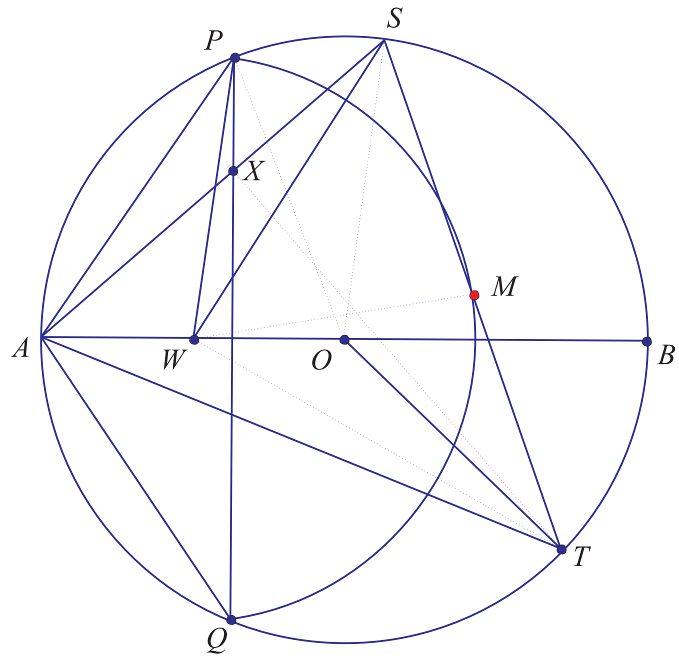
$$u^2-3su-s^3=0.$$

We view this as a quadratic in u, whose discriminant is $s^2(9+4s)$, and so $9+4s$ must be a perfect square, and because it is odd, it must be of the form $(2t+1)^2$. It follows that $s=t^2+t-2$, and so $k=t^2+t+1$. We obtain the same family of solutions.

JMO 3. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P=\angle Q=90^\circ$ and $AP=AQ<BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

Solution: Let O denote the center of ω, and let W denote the midpoint of segment $\overline{AO}$. Denote by Ω the circle centered at W with radius WP. We will show that $WM=WP$, which will imply that M always lies on Ω and so solve the problem.

We present two solutions. The first solution is more computational (in particular, with extensive applications of the formula for a median of a triangle); the second is more synthetic.



Set r to be the radius of circle ω. Applying the median formula in triangles APO, SWT, ASO, ATO gives

$$\begin{aligned}
4WP^2 &= 2AP^2 + 2OP^2 - AO^2 = 2AP^2 + r^2, \\
4WM^2 &= 2WS^2 + 2WT^2 - ST^2, \\
2WS^2 &= AS^2 + OS^2 - AO^2/2 = AS^2 + r^2/2, \\
2WT^2 &= AT^2 + OT^2 - AO^2/2 = AT^2 + r^2/2.
\end{aligned}$$

Adding the last three equations yields $4WM^2 = AS^2 + AT^2 - ST^2 + r^2$. It suffices to show that

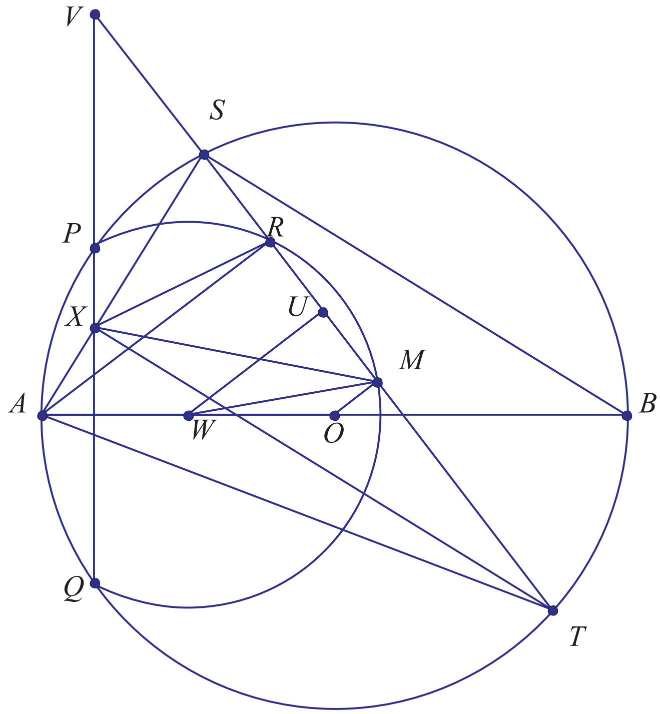
$$4WP^2 = 4WM^2 \quad \text{or} \quad AS^2 + AT^2 - ST^2 = 2AP^2. \tag{1}$$

Because $\overline{XT} \perp \overline{AS}$,

$$\begin{aligned}
AT^2 - ST^2 &= (AX^2 + XT^2) - (SX^2 + XT^2) \\
&= AX^2 - SX^2 \\
&= (AX + XS)(AX - XS) \\
&= AS(AX - XS).
\end{aligned}$$

It follows that $AS^2 + AT^2 - ST^2 = AS^2 + AS \cdot (AX - XS) = AS^2 + AS(2AX - AS) = 2AS \cdot AX$, and (1) reduces to $AP^2 = AS \cdot AX$, which is true because triangle APX is similar to triangle ASP (as $\angle PAX = \angle SAP$ and $\angle APX = \text{arc}(AQ)/2 = \text{arc}(AP)/2 = \angle ASP$).

OR



In the following solution, we use directed distances and directed angles in order to avoid issues with configuration (segments $\overline{ST}$ and $\overline{PQ}$ may intersect, or may not as depicted in the figure.)

Let R be the foot of the perpendicular from A to line ST. Note that $OM \perp ST$, and so $ARMO$ is a right trapezoid. Let U be the midpoint of segment $\overline{RM}$. Then $\overline{WU}$ is the midline of the trapezoid. In particular, $\overline{WU} \perp \overline{RM}$. Hence line WU is the perpendicular bisector of segment $\overline{RM}$. It is also clear that AW is the perpendicular bisector of segment $\overline{PQ}$. Therefore, W is the intersection of the perpendicular bisectors of segments $\overline{RM}$ and $\overline{PQ}$. It suffices to show that quadrilateral $PQMR$ is cyclic, since then W must be its circumcenter, and so $WP = WM$.

(To be precise, this argument fails when ST and PQ are parallel, because then $R = M$ and the perpendicular bisector of $\overline{RM}$ is not defined. However, it is easy to see that this can happen for only one position of X. Because the argument works for all other X, continuity then implies that M lies on Ω for this exceptional case as well.)

Let lines PQ and ST meet in V. By the converse of the power-of-a-point theorem, it suffices to show that $VP \cdot VQ = VR \cdot VM$. On the other hand, because $PQTS$ is cyclic, by the power-of-a-point theorem, we have $VP \cdot VQ = VS \cdot VT$. Therefore, we only need to show that

$$VS \cdot VT = VR \cdot VM. \tag{2}$$

Note that M is the midpoint of segment $\overline{ST}$. Then (2) is equivalent to

$$2VS \cdot VT = VR \cdot (2VM) = VR \cdot (VS + VT)$$

or

$$VS \cdot VT - VS \cdot VR = VT \cdot VR - VT \cdot VS$$

or equivalently

$$VS \cdot RT = VT \cdot SR \quad \text{or} \quad \frac{VS}{SR} = \frac{VT}{RT}. \tag{3}$$

We claim that XS bisects $\angle VXR$. Indeed, because AB is the symmetry line of the kite $APBQ$, $AB \perp PQ$, and so $\angle VXS = \angle QXA = 90^\circ - \angle XAO = 90^\circ - \angle SAO$. Because O is the circumcenter of triangle AST,

$$\angle VXS = 90^\circ - \angle SAO = \angle ATS.$$

On the other hand, because $\angle AXT$ and $\angle ART$ are both right angles, quadrilateral $AXRT$ is cyclic, implying that $\angle SXR = \angle ATR = \angle ATS$. Our claim follows from the last two equations.

Combining our claim and the fact that $XS \perp XT$, we know that XS and XT are the interior and exterior bisectors of $\angle VXR$, from which (3) follows, by the angle-bisector theorem. We saw that (3) was equivalent to (2) and that this was enough to show that $PQMR$ is cyclic, which completes the solution, so we are done.

JMO 4. Find all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x) + f(t) = f(y) + f(z)$$

for all rational numbers $x < y < z < t$ that form an arithmetic progression. ($\mathbb{Q}$ is the set of all rational numbers.)

Solution: Choose any $n \in \mathbb{Z}$, $t \in \mathbb{Q}$. Applying the condition for $nt, (n{+}1)t, (n{+}2)t, (n{+}3)t$ yields

$$f((n+3)t) - f((n+2)t) = f((n+1)t) - f(nt)$$

and similarly

$$f((n+4)t) - f((n+3)t) = f((n+2)t) - f((n+1)t).$$

Adding the two yields

$$f((n+4)t) - f((n+2)t) = f((n+2)t) - f(nt),$$

in particular $f(2kt + 2t) - f(2kt)$ is the same for all $k \in \mathbb{Z}$, which means f is linear on $2t \cdot \mathbb{Z}$. Since $\mathbb{Q}$ is a nested union of such sets, f is linear and all linear functions work.

JMO 5. Let $ABCD$ be a cyclic quadrilateral. Prove that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if there exists a point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$ and $\angle CDB = \angle YDA$.

Solution. By the symmetry, it suffices to show the "only if" part by assuming that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$.

Because $ABCD$ is cyclic, we have $\angle XAD = \angle BAC = \angle BDC = \angle XDC$ and $\angle XDA = \angle BDA = \angle BCA = \angle XCD$. Hence triangles AXD and DXC (and ABC) are similar to each other. In particular,

$$\frac{AX}{DX} = \frac{DX}{XC} \quad \text{or} \quad DX^2 = AX \cdot CX$$

Because $\angle BAC = \angle XAD$, we have $\angle BAX = \angle CAD$. Because $ABCD$ is cyclic, we have $\angle CAD = \angle CBD = \angle CBX$. Consequently, $\angle BAX = \angle CBX$. Note that

$$\angle AXB = \angle XAD + \angle ADX = \angle BAC + \angle ACB = \angle BDC + \angle DCX = \angle CXB.$$

From the above facts, we conclude that triangles ABX and BCX (and ACD) are similar to each other and so we have $BX^2 = AX \cdot CX$. Thus, $BX^2 = AX \cdot CX = DX^2$; that is, X is the midpoint of the segment $\overline{BD}$. Therefore

$$\frac{AB}{BC} = \frac{DX}{XC} = \frac{BX}{XC} = \frac{AD}{DC} \quad \text{or} \quad \frac{BC}{CD} = \frac{BA}{AD}.$$

Construct point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$. From $\angle CBD = \angle YBA$ and $\angle BAY = \angle BAC = \angle BDC$, we conclude that triangles BAY and BDC are similar to each other, from which it follow that

$$\frac{BY}{YA} = \frac{BC}{CD} = \frac{BA}{AD} \quad \text{or} \quad \frac{BY}{BA} = \frac{AY}{AD}.$$

Note also that $\angle YBA = \angle CBD = \angle CAD = \angle YAD$. We conclude that triangles BYA and AYD are similar to each other, implying that $\angle CDB = \angle YAB = \angle YDA$. This is the desired point Y.

OR

By symmetry, it suffices to show that there exists X on the segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if $AB \cdot CD = AD \cdot BC$.

There is a unique point X_1 on segment $\overline{BD}$ such that $\angle X_1AD = \angle BAC$. There is a unique point X_2 on segment $\overline{BD}$ such that $\angle BCA = \angle X_2CD$. Because $ABCD$ is cyclic, $\angle BCA = \angle BDA = \angle X_1DA$. Hence triangles ABC and AX_1D are similar to each other, implying that

$$\frac{AC}{BC} = \frac{AD}{X_1D}.$$

Likewise, we can show that ABC and DX_2C are similar to each other and $\frac{AB}{AC} = \frac{DX_2}{DC}$. Multiplying the last two equations together gives

$$\frac{AB}{BC} = \frac{AB}{AC} \cdot \frac{AC}{BC} = \frac{DX_2}{DC} \cdot \frac{AD}{X_1D},$$

from which it follows that

$$\frac{AB \cdot CD}{AD \cdot BC} = \frac{DX_2}{DX_1}.$$

Note that point X exists if and only if $X_1 = X_2$, or $DX_2 = DX_1$; that is, $AB \cdot CD = AD \cdot BC$.

JMO 6. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he is finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i,k), (i,l), (j,k), (j,l) for some $1 \leq i,j,k,l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

Solution: We think of the pilings as assigning a positive integer to each square on the grid. Now, we restrict ourselves to the types of moves in which we take a lower left and upper right stone and move them to the upper left and lower right of our chosen rectangle. Call this a Type 1 stone move. We claim that we can perform a sequence of Type 1 stone moves on any piling to obtain an equivalent piling for which we cannot perform any Type 1 move, i.e. in which no square that has stones is above and to the right of any other square that has stones. We call such a piling a "down-right" piling.

To prove that any piling is equivalent to a down-right piling, first consider the squares in the leftmost column and topmost row of the grid. Let a be the entry (number of stones) in the upper left corner, and let b and c be the sum of the remaining entries in the leftmost column and topmost row respectively. If $b < c$, we can perform a sequence of Type 1 stone moves to remove all the stones from the leftmost column except for the top entry, and if $c < b$ we can similarly clear all squares in the top row except for the top left square. In the former case, we can now ignore the leftmost column and repeat the process on the second-to-leftmost column and the top row; similarly, in the latter case, we can ignore the top row and proceed as before. Since the corner square a cannot be part of any Type 1 move at each step in the process, it follows that we end up with a down-right piling.

We next show that down-right pilings in any size grid (not necessarily $n \times n$) are uniquely determined by their row-sums and column-sums, given that the row sums and column sums are nonnegative integers which sum to m both along the rows and the columns. Let the topmost row sum be R_1 and the leftmost column sum be C_1. Then the upper left square must contain $\min(R_1, C_1)$ stones, since otherwise there would be stones both in the first row and first column that are not in the upper left square. Whichever is smaller indicates that either the row or the column respectively is empty save for the upper left square; then we can remove this row or column and are reduced to a smaller grid in which we know all the row and column sums. Since one-row and one-column pilings are clearly uniquely determined by their column and row sums, it follows by induction that down-right pilings are determined uniquely by their row-sums and column sums.

Finally, notice that row sums and column sums are both invariant under stone moves. Therefore every piling is equivalent to a *unique* down-right piling. It therefore suffices to count the number of down-right pilings, which is also equivalent to counting the number of possibilities for the row-sums and column-sums. As stated above, the row sums and column sums can be the sums of any two n-tuples of nonnegative integers that each sum to m. The number of such tuples is $\binom{n+m-1}{m}$, and so the total number of non-equivalent pilings is the number of pairs of these tuples, i.e. $\left(\binom{n+m-1}{m}\right)^2$.