4^{th} United States of America Junior Mathematical Olympiad

Day I, II 12:30 PM – 5 PM EDT

April 30 - May 1, 2013

JMO 1. The answer is negative. Modulo 9, a cube is 0 or ± 1. Assuming that one of a^5b+3 and ab^5+3 is 0 mod 9, it follows that at least one of the numbers a and b, say a, is divisible by 3, hence a^5b+3 is 3 mod 27, not a perfect cube. If a^5b+3 and ab^5+3 are both perfect cubes of the form ± 1 mod 9, then a^5b and ab^5 are both 7 or 5 mod 9, and so their product, $(ab)^6$, is -1, -2, or 4 mod 9. But $(ab)^6$ is the square of a perfect cube not divisible by 3, so is precisely 1 mod 9, a contradiction.

This problem and solution were suggested by Titu Andreescu.

JMO 2. Answer: $2^{mn}-1$.

First note that if $m=n=1$, then condition (ii) is vacuously satisfied, so the one cell must contain 0. Henceforth, we assume that $m>1$ or $n>1$, so that every cell has at least one adjacent cell.

We define the distance between two cells to be $|x_1-x_2|+|y_1-y_2|$, where (x_1,y_1), (x_2,y_2) are the centers of the respective cells. In particular, two cells are adjacent if and only if the distance between them is 1.

By condition (ii), the smallest value among the cells of any given garden must be 0. In particular, a garden has at least one zero.

We construct an explicit bijection between the set of nonempty subsets of the mn cells in the array filled with 0 and the set of all possible gardens. Given a subset of the mn cells filled with zeroes, fill every cell in the array with the value of the distance to the nearest cell filled with a zero. This filling of the cells is well-defined and satisfies both properties (i) and (ii). Given two different subsets of cells filled with zeroes, the filling of all cells with minimum distances must necessarily be different, so the function is injective (or one-to-one).

Let an arbitrary garden be given and suppose that a cell in that garden contains an integer $k\geq 1$. By condition (ii), it has an adjacent cell with a smaller integer. Since the difference is either 0 or 1, the difference must be 1. Thus, a cell assigned k will have an adjacent cell assigned $k-1$. We draw a line segment between the two center points of these two cells. Repeating this procedure, we can find a path from k to a 0-cell. We call such a path a *garden path.* There may be more than one garden path from a given cell, but all such paths will have length k.

Suppose that for some cell C assigned k there is a path of length $n<k$ from C to a 0-cell D. Let the numbers in the cells the path goes through be $a_0=k, a_1, \ldots, a_n=0$. Now $a_i-a_{i+1}\leq 1$, so

$$k=\sum_{i=0}^{n-1}(a_i-a_{i+1})\leq n<k,$$

a contradiction. Thus, the nearest 0-cell to C has distance $\geq k$ from C. By the previous paragraph, there exists a path from C to a 0-cell with distance k. Therefore, the distance to the nearest 0-cell is exactly k. The mapping is surjective (or onto).

Therefore, each garden is uniquely determined by the position of zeros. Consequently, we just need to count the number of ways to put zeros in mn cells, subject to the condition that there is at least one zero. This is clearly $2^{mn} - 1$.

This problem and solution were suggested by Sungyoon Kim.

JMO 3. **First Solution:** Assume that ω_B and ω_C intersect again at another point S (other than P). (The degenerate case of ω_B and ω_C being tangent at P can be dealt similarly.) Because $BPSR$ and $CPSQ$ are cyclic, we have $\angle RSP = 180° - \angle PBR$ and $\angle PSQ = 180° - \angle QCP$. Hence, we obtain

$$\angle QSR = 360° - \angle RSP - \angle PSQ = \angle PBR + \angle QCP = \angle CBA + \angle ACB = 180° - \angle BAC;$$

from which it follows that $ARSQ$ is cyclic; that is, $\omega_A, \omega_B, \omega_C$ meet at S. (This is Miquel's theorem.)

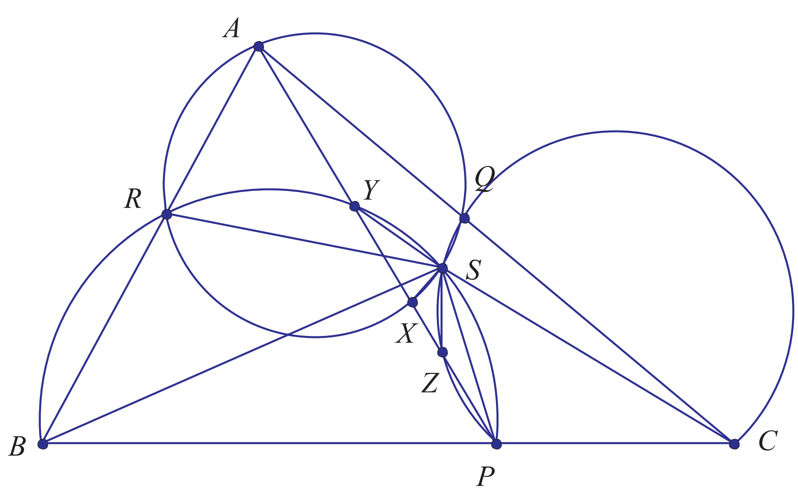
Because $BPSY$ is inscribed in ω_B, $\angle XYS = \angle PYS = \angle PBS$. Because $ARXS$ is inscribed in ω_A, $\angle SXY = \angle SXA = \angle SRA$. Because $BPSR$ is inscribed in ω_B, $\angle SRA = \angle SPB$. Thus, we have $\angle SXY = \angle SRA = \angle SPB$. In triangles SYX and SBP, we have $\angle XYS = \angle PBS$ and $\angle SXY = \angle SPB$. Therefore, triangles SYX and SBP are similar to each other, and, in particular,

$$\frac{YX}{BP} = \frac{SX}{SP}.$$

Similar, we can show that triangles SXZ and SPC are similar to each other and that

$$\frac{SX}{SP} = \frac{XZ}{PC}.$$

Combining the last two equations yields the desired result.



This problem and solution were suggested by Zuming Feng.

Second Solution: Assume that ω_B and ω_C intersect again at another point S (other than P). (The degenerate case of ω_B and ω_C being tangent at P can be dealt with similarly.) Because $BPSR$ and $CPSQ$ are cyclic, we have $\angle RSP = 180^\circ - \angle PBR$ and $\angle PSQ = 180^\circ - \angle QCP$. Hence, we obtain

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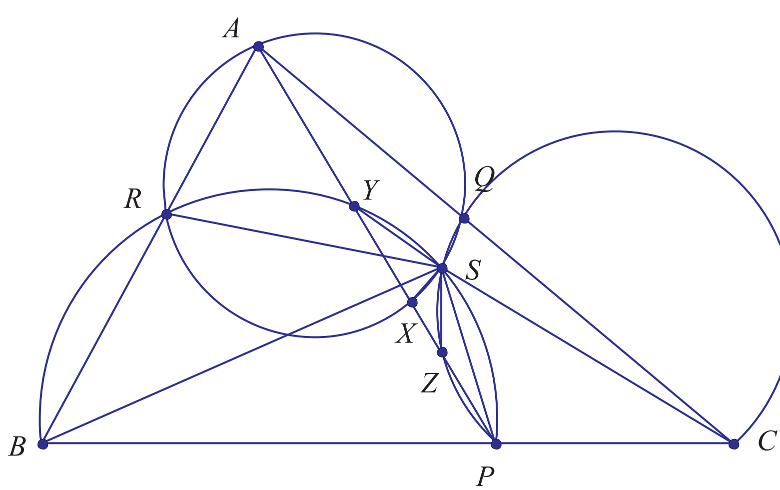


We consider the configuration shown in the above diagram. (We can adjust the proof below easily for other configurations. In particular, our proof is carried with directed angles modulo 180°.)

Line RY intersects ω_A again at T_Y (other than R). Because $BPYR$ is cyclic, $\angle T_Y YX = \angle T_Y YP = \angle RBP = \angle ABP$. Because $ARXT_Y$ is cyclic, $\angle XT_Y Y = \angle XAR = \angle PAB$. Hence triangles $T_Y YX$ and ABP are similar to each other. In particular,

$$\angle YXT_Y = \angle BPA \quad \text{and} \quad \frac{YX}{BP} = \frac{XT_Y}{PA}. \tag{1}$$

Likewise, if line QZ intersect ω_A again at T_Z (other than R), we can show that triangles $T_Z ZX$ and ACP are similar to each other and that

$$\angle T_Z XZ = \angle APC \quad \text{and} \quad \frac{XT_Z}{PA} = \frac{XZ}{PC}. \tag{2}$$

In the light of the second equations (on lengths proportions) in (1) and (2), it suffices to show that $T_Z = T_Y$. On the other hand, the first equations (on angles) in (1) and (2) imply that X, T_Y, T_Z lie on a line. But this line can only intersect ω_A twice with X being one of them. Hence we must have $T_Y = T_Z$, completing our proof.

Comment: The result remains to be true if segment AP is replaced by line AP. The current statement is given to simplify the configuration issue. Also, a very common mistake in attempts following the second solution is assuming line RY and QZ meet at a point on ω_A.

This solution was suggested by Zuming Feng.

JMO 4. **Solution 1.** The answer is 2047. We shall prove that $f(n)$ is odd iff $n = 2^k - 1$ for $k \geq 1$. It is easy to see that $f(1) = 1$, $f(2) = 2$, and $f(3) = 3$. Assume that the statement holds true for $k \leq m$. We will show that the statement is true for $k = m + 1$.

Let $m \geq 2$ be an integer such that $2^m \leq n \leq 2^{m+1} - 1$.

If $n = 2^m$ we write $n = 2^s + (n - 2^s)$ for $0 \leq s \leq m$. We see that $f(2^m) = f(2^m - 1) + f(2^m - 2) + \ldots + f(2^m - 2^{m-1}) + 1$. By induction hypothesis each of $f(2^m - 2), \ldots, f(2^m - 2^{m-1})$ is even, but $f(2^m - 1)$ is odd, so $f(2^m)$ is even.

If $2^m < n \leq 2^{m+1} - 1$ we have $f(n) = f(n-1) + f(n-2) + \ldots + f(n - 2^m)$.

By induction hypothesis each term on the right hand side is odd iff $n - 2^s = 2^r - 1$ for some positive integer r. For each n of the form $n = 2^s + 2^r - 1$ these odd summands appear in pairs: $n - 2^s$ and $n - 2^r$. Therefore $f(n)$ is odd iff $s = r$, that is iff $n = 2^{s+1} - 1 = 2^{m+1} - 1$.

Solution 2. The answer is 2047. We show that $f(n)$ is odd if and only if n is of the form $2^k - 1$.

We use the method of generating functions. Define the formal power series $b(x) = \sum_{j=0}^{\infty} x^{2^j}$. The desired statement can be interpreted as

$$1/(1 - b(x)) \equiv b(x)/x \pmod{2},$$

where the congruence means that the difference between the two sides has all coefficients divisible by 2. It is equivalent to prove the same thing after clearing denominators, in other words,

$$b(x)^2 - b(x) \equiv x \pmod{2}.$$

But this holds because $b(x)^2 \equiv b(x^2) \pmod{2}$ (all the cross terms in the expansion of $b(x)^2$ being even), so

$$b(x)^2 - b(x) \equiv b(x^2) - b(x) \equiv x \pmod{2}.$$

This problem and solution were suggested by Kiran Kedlaya and David Speyer.

Solution 3. Consider the operation of reversing the order of the sums. Call a sum a palindrome if it is invariant under this symmetry and let $g(n)$ be the number of palindromic decompositions of n. Since non-palindromic sums are paired under reversing order we have

$$f(n) \equiv g(n) \pmod 2.$$

Now suppose $n = 2m + 1$ is odd. By parity a palindromic decomposition of n must have an odd central term (and in particular cannot have even length). Hence the central term must be 1. Thus any palindromic decomposition of $n = 2m + 1$ starts with an arbitrary decomposition of m, followed by a 1 and the reverse of the starting decomposition. Thus

$$g(2m + 1) = f(m).$$

Hence $f(2m + 1) \equiv f(m) \pmod 2$.

Now suppose $n = 2m$ is even and positive. Then there are two kinds of palindromic decompositions of n. The first kind have even length. The second kind have odd length and a central element that is even, hence 2^k for some $k \geq 1$. These two kinds occur equally often since we can add together the two equal terms of a palindrome of equal length into two equal halves to reverse this operation. Thus $f(2m)$ and $g(2m)$ are even.

These two cases easily imply $f(n)$ is odd if and only if n is 1 less than a power of 2. One way to see this is to write n in binary. The first rule $f(2m + 1) \equiv f(m) \pmod 2$ says the parity of $f(n)$ is unchanged if we delete a least significant digit of 1. The second rule says $f(n)$ is even if its least significant digit is zero. Iterating these we see $f(n)$ is odd if and only if its binary representation is all 1s, that is, n is 1 less than a power of 2.

This solution was suggested by Steven Blasberg and Richard Stong.

JMO 5. **First Solution:** Note that $\angle XAY = \angle XBY = \angle XCY = \angle PZX = \angle PZY = 90^\circ$. In right triangles BXY, AXY, AXP, we have

$$BY = XY \cos\angle BYX, \quad AX = XY \cos\angle AXY, \quad XP = \frac{AX}{\cos\angle AXP} = \frac{XY\cos\angle AXY}{\cos\angle AXP},$$

from which it follows that

$$\frac{BY}{XP} = \frac{\cos\angle BYX \cos\angle AXP}{\cos\angle AXY}.$$

Likewise, we have

$$\frac{CY}{XQ} = \frac{\cos\angle CYX \cos\angle AXQ}{\cos\angle AXY}.$$

Adding the last two equations yields

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\cos\angle BYX \cos\angle AXP + \cos\angle CYX \cos\angle AXQ}{\cos\angle AXY}. \tag{3}$$

Because both CY and AZ are perpendicular to XC, $\angle CYX = \angle AZX$. Because $\angle XAP = \angle XZP = 90^\circ$, quadrilateral $AXZP$ is cyclic, from which it follows that $\angle AZX = \angle APX$. Therefore, we have $\angle CYX = \angle AZX = \angle APX = 90^\circ - \angle AXP$ or $\angle CYX + \angle AXP =$

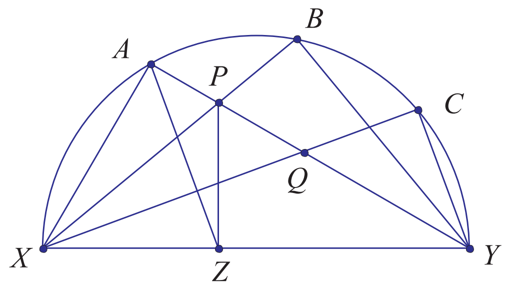
90°. Likewise, we can show that $\angle BYX + \angle AXQ = 90^\circ$. Consequently, we conclude that $\cos\angle BYX = \sin\angle AXQ$ and $\sin\angle CYX = \cos\angle AXP$. Thus, by the addition and substraction formula, (4) becomes

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\sin\angle AXQ \sin\angle CYX + \cos\angle CYX \cos\angle AXQ}{\cos\angle AXY} = \frac{\cos(\angle CYX - \angle AXQ)}{\cos\angle AXY}.$$

Because $ACYX$ is cyclic, $\angle AXQ = \angle AXC = \angle CYA$, implying that $\angle CYX - \angle AXQ = \angle CYX - \angle CYA = \angle AYX$. Therefore,

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\cos(\angle CYX - \angle AXQ)}{\cos\angle AXY} = \frac{\cos\angle AYX}{\cos\angle AXY} = \frac{\sin\angle AXY}{\cos\angle AXY} = \tan\angle AXY = \frac{AY}{AX},$$

as desired.



This problem and solution were suggested by Zuming Feng.

Second Solution: Note that $\angle XAY = \angle XBY = \angle XCY = \angle PZX = \angle PZY = 90^\circ$. In right triangles BXY, AXY, AXP, we have

$$BY = XY\cos(\angle BYX), \quad AX = XY\cos(\angle AXY), \quad XP = \frac{AX}{\cos(\angle AXP)} = \frac{XY\cos(\angle AXY)}{\cos(\angle AXP)},$$

from which it follows that

$$\frac{BY}{XP} = \frac{\cos(\angle BYX)\cos(\angle AXP)}{\cos(\angle AXY)}.$$

Likewise, we have

$$\frac{CY}{XQ} = \frac{\cos(\angle CYX)\cos(\angle AXQ)}{\cos(\angle AXY)}.$$

Adding the last two equations yields

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\cos(\angle BYX)\cos(\angle AXP) + \cos(\angle CYX)\cos(\angle AXQ)}{\cos(\angle AXY)}. \qquad (4)$$

Because both CY and AZ are perpendicular to XC, $\angle CYX = \angle AZX$. Because $\angle XAP = \angle XZP = 90^\circ$, quadrilateral $AXZP$ is cyclic, from which it follows that $\angle AZX = \angle APX$. Therefore, we have $\angle CYX = \angle AZX = \angle APX = 90^\circ - \angle AXP$ or $\angle CYX + \angle AXP = 90^\circ$. Likewise, we can show that $\angle BYX + \angle AXQ = 90^\circ$. Consequently, we conclude that

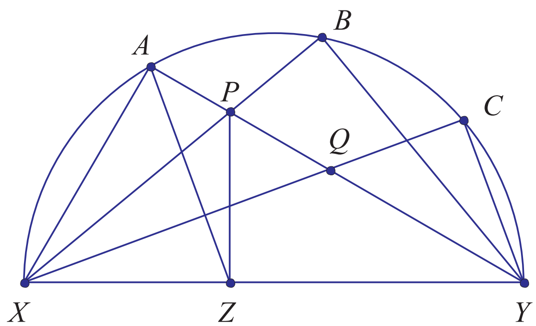
$\cos(\angle BYX) = \sin(\angle AXQ)$ and $\sin(\angle CYX) = \cos(\angle AXP)$. Thus, by the addition and substraction formula, (4) becomes

$$\frac{BY}{XP}+\frac{CY}{XQ} = \frac{\sin(\angle AXQ)\sin(\angle CYX) + \cos(\angle CYX)\cos(\angle AXQ)}{\cos(\angle AXY)} = \frac{\cos(\angle CYX - \angle AXQ)}{\cos(\angle AXY)}.$$

Because $ACYX$ is cyclic, $\angle AXQ = \angle AXC = \angle CYA$, implying that $\angle CYX - \angle AXQ = \angle CYX - \angle CYA = \angle AYX$. Therefore,

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\cos(\angle CYX - \angle AXQ)}{\cos\angle AXY} = \frac{\cos\angle AYX}{\cos\angle AXY} = \frac{\sin\angle AXY}{\cos\angle AXY} = \tan\angle AXY = \frac{AY}{AX},$$

as desired.



Rays YB and YC meet ray XA at B_1 and C_1 respectively. Because $\angle PAB_1 = \angle PBB_1 = 90°$, $APBB_1$ is cyclic, in particular, $\angle XB_1Y = \angle AB_1B = \angle APX$. Because $\angle PAX = \angle PZX = 90°$, $APZX$ is cyclic, in particular, $\angle APX = \angle AZX$. Note that both AC and CY are perpendicular to XC, $AZ \parallel CY$ and so $\angle AZX = \angle CYX = \angle C_1YX$. Therefore, we have $\angle XB_1Y = \angle APX = \angle AZX = \angle C_1YX$. It follows that triangles XYB_1 and XC_1Y are similar to each other, with XB and XC being corresponding altitudes. Hence

$$\frac{BY}{XP} = \frac{CC_1}{XQ} \quad \text{and} \quad \frac{BY}{XP} + \frac{CY}{XQ} = \frac{CC_1}{XQ} + \frac{CY}{XQ} = \frac{C_1Y}{XQ}.$$

It remains to show that

$$\frac{C_1Y}{XQ} = \frac{AY}{AX},$$

which is true because triangles AYC_1 and AXQ are similar to each other ($\angle C_1AY = \angle QAX = 90°$ and $\angle AYC_1 = \angle AYC = \angle AXC = \angle AXQ$.)

This solution was suggested by Zuming Feng.

JMO 6. **First Solution:** Let a, b, c be nonnegative real numbers such that $x = 1 + a^2$, $y = 1 + b^2$ and $z = 1+c^2$. We may assume that $c \le a, b$, so that the condition of the problem becomes

$$(1 + c^2)(1 + (1 + a^2)(1 + b^2)) = (a + b + c)^2.$$

The Cauchy-Schwarz inequality yields

$$(a + b + c)^2 \le (1 + (a + b)^2)(c^2 + 1).$$

Combined with the previous relation, this shows that

$$(1+a^2)(1+b^2) \leq (a+b)^2,$$

which can also be written $(ab-1)^2 \leq 0$. Hence $ab = 1$ and the Cauchy-Schwarz inequality must be an equality, that is, $c(a+b) = 1$. Conversely, if $ab = 1$ and $c(a+b) = 1$, then the relation in the statement of the problem holds, since $c = \frac{1}{a+b} < \frac{1}{b} = a$ and similarly $c < b$.

Thus the solutions of the problem are

$$x = 1 + a^2, \quad y = 1 + \frac{1}{a^2}, \quad z = 1 + \left(\frac{a}{a^2+1}\right)^2$$

for some $a > 0$, as well as permutations of this. (Note that we can actually assume $a \geq 1$ by switching x and y if necessary.)

This problem and solution were suggested by Titu Andreescu.

Second Solution: We maintain the notations in the first solution and again consider the equation

$$(a+b+c)^2 = 1 + c^2 + (1+a^2)(1+b^2)(1+c^2).$$

Expanding both sides of the equation yields

$$a^2+b^2+c^2+2ab+2bc+2ca = 1+c^2+1+a^2+b^2+c^2+a^2b^2+b^2c^2+c^2a^2+a^2b^2c^2$$

or

$$a^2b^2c^2 + a^2b^2 + b^2c^2 + c^2a^2 - 2ab - 2bc - 2ca + c^2 + 2 = 2(ab+bc+ca).$$

Setting $(u, v, w) = (ab, bc, ca)$, we can write the above equation as

$$uvw + u^2 + v^2 + w^2 - 2u - 2v - 2w + \frac{vw}{u} + 2 = 2(u+v+w).$$

which is the equality case of the sum of the following three special cases of the AM-GM inequality:

$$uvw + \frac{vw}{u} \geq 2vw, v^2 + w^2 + 2vw + 1 = 2(v+w) \geq 0, \quad u^2 + 1 \geq 2u.$$

Hence we must have the equality cases these AM-GM inequalities; that is, $ab = u = 1$ and $a(b+c) = v + w = 1$. We can then complete our solution as we did in the first solution.

This solution was suggested by Zuming Feng.