

42<sup>nd</sup> United States of America Mathematical Olympiad

Day I      12:30 PM – 5 PM EDT

April 30, 2013

**Note:** For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

- USAMO 1. In triangle  $ABC$ , points  $P, Q, R$  lie on sides  $BC, CA, AB$ , respectively. Let  $\omega_A, \omega_B, \omega_C$  denote the circumcircles of triangles  $AQR, BRP, CPQ$ , respectively. Given the fact that segment  $AP$  intersects  $\omega_A, \omega_B, \omega_C$  again at  $X, Y, Z$  respectively, prove that  $YX/XZ = BP/PC$ .
- USAMO 2. For a positive integer  $n \geq 3$  plot  $n$  equally spaced points around a circle. Label one of them  $A$ , and place a marker at  $A$ . One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of  $2n$  distinct moves available; two from each point. Let  $a_n$  count the number the number of ways to advance around the circle exactly twice, beginning and ending at  $A$ , without repeating a move. Prove that  $a_{n-1} + a_n = 2^n$  for all  $n \geq 4$ .
- USAMO 3. Let  $n$  be a positive integer. There are  $\frac{n(n+1)}{2}$  marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing  $n$  marks. Initially, each mark has the black side up. An *operation* is to choose a line parallel to one of the sides of the triangle, and flipping all the marks on that line. A configuration is called *admissible* if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration  $C$ , let  $f(C)$  denote the smallest number of operations required to obtain  $C$  from the initial configuration. Find the maximum value of  $f(C)$ , where  $C$  varies over all admissible configurations.