**40th United States of America Mathematical Olympiad**

1. The given condition is equivalent to $a^2 + b^2 + c^2 + ab + bc + ca \le 2$. We will prove that

$$\frac{2ab+2}{(a+b)^2} + \frac{2bc+2}{(b+c)^2} + \frac{2ca+2}{(c+a)^2} \ge 6\,.$$

Indeed, we have

$$\frac{2ab+2}{(a+b)^2} \ge \frac{2ab+a^2+b^2+c^2+ab+bc+ca}{(a+b)^2} = 1 + \frac{(c+a)(c+b)}{(a+b)^2}.$$

Adding the last inequality with its cyclic analogous forms yields

$$\frac{2ab+2}{(a+b)^2} + \frac{2bc+2}{(b+c)^2} + \frac{2ca+2}{(c+a)^2} \ge 3 + \frac{(c+a)(c+b)}{(a+b)^2} + \frac{(a+b)(a+c)}{(b+c)^2} + \frac{(b+c)(b+a)}{(c+a)^2}$$

Hence it remains to prove that

$$\frac{(c+a)(c+b)}{(a+b)^2} + \frac{(a+b)(a+c)}{(b+c)^2} + \frac{(b+c)(b+a)}{(c+a)^2} \ge 3.$$

But this follows directly from the AM–GM inequality. Equality holds if and only if $a+b = b+c = c+a$, which together with the given condition, shows that it occurs if and only if $a = b = c = \frac{1}{\sqrt{3}}$.

**OR**

Set $2x = a+b$, $2y = b+c$, and $2z = c+a$; that is, $a = z+x-y$, $b = x+y-z$, and $c = y+z-x$. Hence

$$\frac{ab+1}{(a+b)^2} = \frac{(z+x-y)(x+y-z)+1}{4x^2} = \frac{x^2-(y-z)^2+1}{4x^2} = \frac{x^2+2yz+1-y^2-z^2}{4x^2}.$$

On the other hand, the given condition is equivalent to $2a^2+2b^2+2c^2+2ab+2bc+2ca \le 4$ or $(a+b)^2+(b+c)^2+(c+a)^2 \le 4$; that is, $x^2+y^2+z^2 \le 1$ or $1-y^2-z^2 \ge x^2$. It follows that

$$\frac{ab+1}{(a+b)^2} = \frac{x^2+2yz+1-y^2-z^2}{4x^2} \ge \frac{x^2+2yz+x^2}{4x^2} = \frac{1}{2} + \frac{yz}{2x^2}.$$

Likewise, we have

$$\frac{bc+1}{(b+c)^2} = \frac{1}{2} + \frac{zx}{2y^2} \quad \text{and} \quad \frac{ca+1}{(c+a)^2} = \frac{1}{2} + \frac{xy}{2z^2}.$$

Adding the last three inequalities gives

$$\frac{ab+1}{(a+b)^2}+\frac{bc+1}{(b+c)^2}+\frac{ca+1}{(c+a)^2}\geq\frac{3}{2}+\frac{yz}{2x^2}+\frac{zx}{2y^2}+\frac{xy}{2z^2}\geq 3,$$

by the AM–GM inequality. Equality holds if and only if $x=y=z$ or $a=b=c=\frac{1}{\sqrt{3}}$.

2. Let $a_1$, $a_2$, $a_3$, $a_4$ and $a_5$ represent the integers at vertices $v_1$ to $v_5$ (in order around the pentagon) at the start of the game. We will first show that the game can be won at only one of the vertices. Observe that the quantity $a_1+2a_2+3a_3+4a_4$ mod 5 is an invariant of the game. For instance, one move involves replacing $a_1$, $a_3$ and $a_5$ by $a_1-m$, $a_3+2m$ and $a_5-m$. Thus the quantity $a_1+2a_2+3a_3+4a_4$ becomes

$$(a_1-m)+2a_2+3(a_3+2m)+4a_4=a_1+2a_2+3a_3+4a_4+5m,$$

which is unchanged mod 5. The other moves may be checked similarly. Now suppose that the game may be won at vertex $v_j$. The value of the invariant at the winning position is $2011j$. If the initial value of the invariant is $n$, then we must have $2011j\equiv n$ mod 5, or $j\equiv n$ mod 5. Hence the game may only be won at vertex $v_j$, where $j$ is the least positive residue of $n$ mod 5.

By renumbering the vertices, we may assume without loss of generality that the winning vertex is $v_5$. We will show that the game can be won in four moves by adding a suitable amount $2m_j$ at vertex $v_j$ (and subtracting $m_j$ from the opposite vertices) on the $j$th turn for $j=1,2,3,4$. The net change at vertex $v_1$ after these four moves is $2m_1-m_3-m_4$, which must equal $-a_1$ if we are to finish with 0 at $v_1$. In this fashion we find that

$$\begin{aligned}
2m_1-m_3-m_4 &= -a_1\\
2m_2-m_4 &= -a_2\\
2m_3\qquad -m_1 &= -a_3\\
2m_4-m_1-m_2 &= -a_4\\
-m_2-m_3 &= -a_5+2011.
\end{aligned}$$

The sum of the first four equations is the negative of the fifth equation, so it is redundant. Multiplying the first four equations by $-1$, 3, $-3$, 1 and adding them yields $5m_2-5m_3=a_1-3a_2+3a_3-a_4$. But

$$a_1-3a_2+3a_3-a_4\equiv a_1+2a_2+3a_3+4a_4\equiv n\equiv 5\equiv 0 \text{ mod } 5,$$

since we are assuming $v_5$ is the winning vertex. Therefore we may divide by 5 to obtain $m_2 - m_3 = \frac{1}{5}(a_1 - 3a_2 + 3a_3 - a_4)$. We also know that $m_2 + m_3 = a_1 + a_2 + a_3 + a_4$, and one easily confirms that the right-hand sides of these equations are integers with the same parity. Hence the system admits an integral solution for $m_2$ and $m_3$. The second and third equations then quickly give integer values for $m_1$ and $m_4$ as well, so it is indeed possible to win the game at vertex $v_5$.

3. We first give a recipe for constructing hexagons as in the problem statement. Let $ACE$ be a triangle, with all angles less than $2\pi/3$. Let $D$ be the reflection of $A$ across $CE$; let $F$ be the reflection of $C$ across $EA$; let $B$ be the reflection of $E$ across $AC$. Then, $\angle BAF = \angle BAC + \angle CAE + \angle EAF = 3\angle CAE = 3\angle CDE$, and similarly for the other angle equalities. Also, $AB = AE = DE$, and similarly for the other side equalities. Thus, the hexagon satisfies the equations in the problem statement. The diagonals $AD, BE, CF$ are simply the altitudes of the triangle $ACE$, so they are concurrent at the orthocenter.

   Now we show that the only possible hexagons meeting the conditions of the problem statement are the ones constructed in this manner. This will suffice to complete the solution.

   Given the hexagon $ABCDEF$ as in the problem statement, let $\beta, \delta, \phi$ be the measures of its angles $B, D, F$. Since $4(\angle B + \angle D + \angle F) = \angle A + \angle B + \angle C + \angle D + \angle E + \angle F = 4\pi$, we must have $\beta + \delta + \phi = \pi$. Also, the fact that opposite sides are not parallel implies that $\pi + 2\beta = \angle D + \angle E + \angle F \neq 2\pi$, so $\beta \neq \pi/2$; likewise $\delta, \phi \neq \pi/2$.

   We can construct a hexagon $A_1B_1C_1D_1E_1F_1$ meeting the angle and side equality conditions, with angles $\angle B_1 = \beta, \angle D_1 = \delta, \angle F_1 = \phi$, by taking $A_1C_1E_1$ to be a triangle with angles $\beta, \delta, \phi$, and reflecting each vertex across the opposite site as above. We wish to show that $ABCDEF \sim A_1B_1C_1D_1E_1F_1$.

   Treat the positions of $A, B$ as fixed, and treat $\beta, \delta, \phi$ as fixed; these are enough to uniquely determine the orientation of each edge of the hexagon, given the known angles. Let $x = AB = DE$, $y = BC = EF$, $z = CD = FA$. Our goal is to show that these lengths are uniquely determined (up to scale) by the given angles.

   Let $a, b, c, d, e, f$ be unit vectors in the directions of the edges from $A$ to $B$, $B$ to $C$, $C$ to $D$, $D$ to $E$, $E$ to $F$, and $F$ to $A$, respectively. Then the vector identity

$$x(a + d) + y(b + e) + z(c + f) = 0 \tag{1}$$

holds. Without loss of generality, assume the vertices of $ABCDEF$ are labeled in counterclockwise order. The respective orientations of vectors $b, c, d, e, f$, measured counterclockwise relative to $a$, are

$$\begin{aligned} b:&\quad \pi-\beta \\ c:&\quad -\beta-3\phi \\ d:&\quad -2\phi \\ e:&\quad \pi+2\delta-\beta \\ f:&\quad 2\delta-\phi-\beta \end{aligned}$$

(These angles are given modulo $2\pi$; we have made liberal use of the identity $\beta+\delta+\phi=\pi$.)

Now, whenever two unit vectors point in directions $\theta$ and $\psi$, which do not differ by $\pi$, then their sum is a nonzero vector pointing in direction $(\theta+\psi)/2$ or $(\theta+\psi)/2+\pi$. It follows that vectors $a+d, b+e, c+f$ are all nonzero and point in the following directions (modulo $\pi$):

$$\begin{aligned} a+d:&\quad -\phi \\ b+e:&\quad \delta-\beta \\ c+f:&\quad \delta-2\phi-\beta \end{aligned}$$

None of the differences between these angles are multiples of $\pi$. (This follows from the fact that $\beta, \delta, \phi \neq \pi/2$.) Thus, $a+d, b+e, c+f$ are not collinear, nonzero vectors. Consequently, the equation (1) determines the coefficients $x, y, z$ uniquely up to scale, as required.

It follows that $ABCDEF$ and $A_1B_1C_1D_1E_1F_1$ are similar to each other, as required, and this completes the proof.

4. The assertion is false, and the smallest $n$ for which it fails is $n=25$. Given $n \geq 2$, let $r$ be the remainder when $2^n$ is divided by $n$. Then $2^n = kn+r$ where $k$ is a positive integer and $0 \leq r < n$. It follows that

$$2^{2^n} = 2^{kn+r} \equiv 2^r \mod 2^n-1,$$

and $2^r < 2^n-1$ so $2^r$ is the remainder when $2^{2^n}$ is divided by $2^n-1$. If $r$ is even then $2^r$ is power of 4. Hence to disprove the assertion, it is enough to find an $n$ for which the corresponding $r$ is odd.

If $n$ is even then so is $r = 2^n - kn$.

If $n$ is an odd prime then $2^n \equiv 2 \pmod n$ by Fermat's Little Theorem; hence $r \equiv 2^n \equiv 2 \mod n$ and $r=2$.

There remains the case in which $n$ is odd and composite. In the first three instances $n = 9$, 15, 21 there is no contradiction to the assertion:

$$
\begin{aligned}
n = 9 :& 2^6 \equiv 1 \mod 9 \Rightarrow 2^9 \equiv 2^6 \cdot 2^3 \equiv 8 \mod 9 \\
n = 15 :& 2^4 \equiv 1 \mod 15 \Rightarrow 2^{15} \equiv (2^4)^3 \cdot 2^3 \equiv 8 \mod 15 \\
n = 21 :& 2^6 \equiv 1 \mod 21 \Rightarrow 2^{21} \equiv (2^6)^3 \cdot 2^3 \equiv 8 \mod 21
\end{aligned}
$$

However,

$$2^{10} = 1024 \equiv -1 \Rightarrow 2^{20} \equiv 1 \Rightarrow 2^{25} \equiv 2^5 \equiv 7 \mod 25,$$

so 7 is the remainder when $2^{25}$ is divided by 25 and $2^7$ is the remainder when $2^{2^{25}}$ is divided by $2^{25} - 1$.

5. We will prove that the lines $\overline{AB}$, $\overline{CD}$, and $\overline{Q_1Q_2}$ are either concurrent or all parallel. Let $X$ and $Y$ denote the reflections of $P$ across the lines $\overline{AB}$ and $\overline{CD}$. We first claim that $XQ_1 = YQ_1$ and $XQ_2 = YQ_2$. Indeed, let $Z$ be the reflection of $Q_1$ across $BC$. Then $XB = PB$, $BQ_1 = BZ$, and

$$\angle XBQ_1 = \angle XBA + \angle ABQ_1 = \angle ABC = \angle PBC + \angle CBZ = \angle PBZ,$$

whence $\triangle XBQ_1 \cong \triangle PBZ$ and thus $XQ_1 = PZ$. Similarly $YQ_1 = PZ$, and so $XQ_1 = YQ_1$. In exactly the same way, we see that $XQ_2 = YQ_2$, establishing the claim.

We conclude that the line $\overline{Q_1Q_2}$ is the perpendicular bisector of the segment $\overline{XY}$. If $\overline{AB} \parallel \overline{CD}$, then $\overline{XY} \perp \overline{AB}$ and it follows that $\overline{Q_1Q_2} \parallel \overline{AB}$, as desired. If the lines $\overline{AB}$ and $\overline{CD}$ are not parallel, then let $R$ denote their intersection. Since $RX = RP = RY$, $R$ lies on the perpendicular bisector of $\overline{XY}$ and thus $R$, $Q_1$, $Q_2$ are collinear, as desired.

**OR**

This solution uses isogonal conjugates. Recall that two points $S$, $T$ are isogonal conjugates with respect to $\triangle ABC$ if $\angle SAB = \angle CAT$, $\angle SBC = \angle ABT$, and $\angle SCA = \angle BCT$, with any two of these equalities implying the third.

If $\overline{AB} \parallel \overline{CD}$, then there is nothing to prove; thus we assume $\overline{AB}$ intersects $\overline{CD}$ in a point $R$. Then $Q_1$ and $P$ are isogonal conjugates with respect to $\triangle RBC$, whence $\angle Q_1RB = \angle CRP$, and $Q_2$ and $P$ are isogonal conjugates with respect to $\triangle RAD$, whence $\angle Q_2RA = \angle DRP$. Therefore $\angle Q_1RB = \angle Q_2RA = \angle Q_2RB$ and the lines $\overline{AB}$, $\overline{CD}$, $\overline{Q_1Q_2}$ all intersect at $R$.

Remark: Although not needed for the problem as stated, here is an alternate proof that if $\overline{AB} \parallel \overline{CD}$, then $\overline{Q_1Q_2}$ is parallel to both. Extend $\overline{BQ_1}$ and $\overline{BP}$ to meet $\overline{CD}$ at points $E$ and $F$, respectively. Then $\angle BCP = \angle Q_1CE$ and $\angle PBC = \angle ABQ_1 = \angle CEQ_1$, and so $\triangle PBC \sim \triangle Q_1EC$, whence $PC/PB = Q_1C/Q_1E$. Similarly $\triangle Q_1BC \sim \triangle PFC$ and $PC/PF = Q_1C/Q_1B$. We conclude that $Q_1B/Q_1E = PF/PB$. Similarly, extend $\overline{AQ_2}$ and $\overline{AP}$ to meet $\overline{CD}$ at $G$ and $H$; then $Q_2A/Q_2G = PH/PA = PF/PB = Q_1B/Q_1E$, and it follows that $\overline{Q_1Q_2} \parallel \overline{AB} \parallel \overline{CD}$.

6. Let $S$ be the complement of $A_1 \cup A_2 \cup \cdots \cup A_{11}$ in $A$; we wish to prove that $|S| \leq 60$. For $\ell \geq 0$, define

$$\theta(\ell) = \left(1 - \frac{\ell}{2}\right)\left(1 - \frac{\ell}{3}\right) = 1 - \frac{2}{3}\ell + \frac{1}{3}\binom{\ell}{2}.$$

Note that $\theta(0) = 1$ and $\theta(\ell) \geq 0$ for any integer $\ell > 0$. Therefore, since $S$ is the intersection of the complements of the $A_i$,

$$|S| \leq \sum_{n \in A} \theta(\ell(n)).$$

On the other hand,

$$\sum_{n \in A} \theta(\ell(n)) = \sum_{n \in A} \left(1 - \frac{2}{3}\ell(n) + \frac{1}{3}\binom{\ell(n)}{2}\right) = |A| - \frac{2}{3}\sum_i |A_i| + \frac{1}{3}\sum_{i<j} |A_i \cap A_j|.$$

Consequently,

$$|S| \leq 225 - \frac{2}{3} \cdot 11 \cdot 45 + \frac{1}{3} \cdot \binom{11}{2} \cdot 5 = 60,$$

and therefore $|A_1 \cup A_2 \cup \cdots \cup A_{11}| \geq 165$.

We construct an example to show that this lower bound is best possible. Let $p_1, p_2, \ldots, p_{11}$ be a set of 11 distinct primes, and let $A'$ denote the set of all products of three of these primes. Furthermore, let $A'' = \{q_1, q_2, q_3, \ldots, q_{60}\}$ be a set of 60 distinct positive integers that are all prime to $p_1, \ldots, p_{11}$. Set $A = A' \cup A''$, and define

$$A_i = \{n \in A' : p_i \,|\, n\}.$$

Then $|A_i| = \binom{10}{2} = 45$, $|A_i \cap A_j| = \binom{9}{1} = 9$, and

$$|A_1 \cup A_2 \cup \cdots \cup A_{11}| = |A'| = \binom{11}{3} = 165.$$

Finally, $|A| = |A'| + |A''| = 165 + 60 = 225$.

Remark: To get an upper bound for $|S|$, one could replace $\theta(\ell)$ by any function of the form

$$\left(1 - \frac{\ell}{r}\right)\left(1 - \frac{\ell}{r+1}\right)$$

for any positive integer $r$. The choice $r = 2$ is optimal for the stated problem. The choice $r = 1$ yields

$$|S| \leq |A| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j|,$$

which is a familiar truncated inclusion-exclusion inequality, known in number theory as "Brun's Pure Sieve" and in probability as "Bonferroni's Inequality."