2^{nd} United States of America Junior Mathematical Olympiad

1. The answer is $n = 1$. Clearly, $n = 1$ is a solution because $2 + 12 + 2011 = 45^2$. Next we show that there is no other solutions.

   Assume that $n \geq 2$. If n is odd, then $2^n + 12^n + 2011^n$ cannot be a perfect square because it is congruent to 3 modulo 4. If n is even, we can complete our solution in two ways.

   - $2^n + 12^n + 2011^n$ cannot be a perfect square because it is congruent to 2 modulo 3.
   - $2^n + 12^n + 2011^n$ cannot be a perfect square because it is in between two consecutive perfect squares. Indeed, say $n = 2k$, then

     $$(2011^k)^2 < 2^{2k}+12^{2k}+2011^{2k} = 4^k+144^k+2011^{2k} < 1+2\cdot 2011^k+2011^{2k} = (2011^k+1)^2.$$

2. The given condition is equivalent to $a^2 + b^2 + c^2 + ab + bc + ca \leq 2$. We will prove that

   $$\frac{2ab+2}{(a+b)^2} + \frac{2bc+2}{(b+c)^2} + \frac{2ca+2}{(c+a)^2} \geq 6\,.$$

   Indeed, we have

   $$\frac{2ab+2}{(a+b)^2} \geq \frac{2ab + a^2 + b^2 + c^2 + ab + bc + ca}{(a+b)^2} = 1 + \frac{(c+a)(c+b)}{(a+b)^2}.$$

   Adding the last inequality with its cyclic analogous forms yields

   $$\frac{2ab+2}{(a+b)^2} + \frac{2bc+2}{(b+c)^2} + \frac{2ca+2}{(c+a)^2} \geq 3 + \frac{(c+a)(c+b)}{(a+b)^2} + \frac{(a+b)(a+c)}{(b+c)^2} + \frac{(b+c)(b+a)}{(c+a)^2}$$

   Hence it remains to prove that

   $$\frac{(c+a)(c+b)}{(a+b)^2} + \frac{(a+b)(a+c)}{(b+c)^2} + \frac{(b+c)(b+a)}{(c+a)^2} \geq 3.$$

   But this follows directly from the AM–GM inequality. Equality holds if and only if $a+b = b+c = c+a$, which together with the given condition, shows that it occurs if and only if $a = b = c = \frac{1}{\sqrt{3}}$.

OR

Set $2x = a + b$, $2y = b + c$, and $2z = c + a$; that is, $a = z + x - y$, $b = x + y - z$, and $c = y + z - x$. Hence

$$\frac{ab+1}{(a+b)^2} = \frac{(z+x-y)(x+y-z)+1}{4x^2} = \frac{x^2-(y-z)^2+1}{4x^2} = \frac{x^2+2yz+1-y^2-z^2}{4x^2}.$$

On the other hand, the given condition is equivalent to $2a^2+2b^2+2c^2+2ab+2bc+2ca \le 4$ or $(a+b)^2+(b+c)^2+(c+a)^2 \le 4$; that is, $x^2+y^2+z^2 \le 1$ or $1-y^2-z^2 \ge x^2$. It follows that

$$\frac{ab+1}{(a+b)^2} = \frac{x^2+2yz+1-y^2-z^2}{4x^2} \ge \frac{x^2+2yz+x^2}{4x^2} = \frac{1}{2} + \frac{yz}{2x^2}.$$

Likewise, we have

$$\frac{bc+1}{(b+c)^2} = \frac{1}{2} + \frac{zx}{2y^2} \quad \text{and} \quad \frac{ca+1}{(c+a)^2} = \frac{1}{2} + \frac{xy}{2z^2}.$$

Adding the last three inequalities gives

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge \frac{3}{2} + \frac{yz}{2x^2} + \frac{zx}{2y^2} + \frac{xy}{2z^2} \ge 3,$$

by the AM–GM inequality. Equality holds if and only if $x = y = z$ or $a = b = c = \frac{1}{\sqrt{3}}$.

3. For $1 \le i < j \le 3$, solving the system $y = 2x_i x - x_i^2 = 2x_j x - x_j^2$ yields the intersection $\left(\frac{x_i+x_j}{2}, x_i x_j\right)$ of lines ℓ_i and ℓ_j. Hence the center of the equilateral triangle is

$$O = (O_x, O_y) = \left(\frac{x_1+x_2+x_3}{3}, \frac{x_1x_2+x_2x_3+x_3x_1}{3}\right).$$

Let $0^\circ \le \alpha_i < 180^\circ$ be the standard angle formed by lines ℓ_i and the positive x-axis. Without loss of generality, we may assume that $\alpha_1 < \alpha_2 < \alpha_3$. By the given condition, we have $\alpha_2 - \alpha_1 = \alpha_3 - \alpha_2 = 60^\circ$. By the subtraction formulas, we have

$$\tan 60^\circ = \frac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1\tan\alpha_2} = \frac{\tan\alpha_3 - \tan\alpha_2}{1+\tan\alpha_2\tan\alpha_3} \quad \text{and} \quad \tan 120^\circ = \frac{\tan\alpha_3 - \tan\alpha_1}{1+\tan\alpha_3\tan\alpha_1}$$

or

$$\sqrt{3} = \frac{2x_2-2x_1}{1+4x_1x_2} = \frac{2x_3-2x_2}{1+4x_2x_3} \quad \text{and} \quad -\sqrt{3} = \frac{2x_3-2x_1}{1+4x_3x_1}.$$

Therefore,

$$1+4x_1x_2 = \frac{2(x_2-x_1)}{\sqrt{3}}, \quad 1+4x_2x_3 = \frac{2(x_3-x_2)}{\sqrt{3}}, \quad 1+4x_3x_1 = \frac{2(x_1-x_3)}{\sqrt{3}}. \qquad (1)$$

Adding these equations gives $3+4(x_1x_2+x_2x_3+x_3x_1)=0$, implying that $O_y=-\frac{1}{4}$; that is, O always lie on the directrix ℓ of the parabola $y=x^2$.

Next we show that G can be any point on ℓ. Solving the first and the equations in (1) for x_2 and x_3 in terms of x_1 gives

$$x_2=\frac{2x_1+\sqrt{3}}{2-4\sqrt{3}x_1} \quad \text{and} \quad x_3=\frac{2x_1-\sqrt{3}}{2+4\sqrt{3}x_1},$$

implying that

$$\begin{aligned} x_1+x_2+x_3 &= x_1+\frac{(2x_1+\sqrt{3})(2+4\sqrt{3}x_1)+(2x_1-\sqrt{3})(2-4\sqrt{3}x_1)}{4-48x_1^2} \\ &= x_1+\frac{8x_1}{1-12x_1^2}=\frac{12x_1^3-9x_1}{12x_1^2-1}. \end{aligned}$$

Because lines ℓ_1,ℓ_2,ℓ_3 are evenly spaced with $60°$ between each other, slopes $2x_1,2x_2,2x_3$ are symmetric with each other; that is,

$$x_1+x_2+x_3=\frac{12x_i^3-9x_i}{12x_i^2-1} \quad \text{for } i=1,2,3.$$

Therefore,

$$O_x=\frac{x_1+x_2+x_3}{3}=\frac{4x^3-3x}{12x^2-1},$$

where $-\infty<x<\infty$, because $x=x_i$ for some $i=1,2,3$, and the combined ranges of slopes $2x_i$ are the interval $(-\infty,\infty)$. Because $4x^3-3x=O_x(12x^2-1)$ is a cubic equation, it has a real root in x for every real number O_x; that is, the range of O_x is the interval $(-\infty,\infty)$. We conclude that the locus of O is line $y=-\frac{1}{4}$.

4. According to the statement of the problem we have

$$W_0=a, \quad W_1=b, \quad W_2=ab, \quad W_3=bab, \quad W_4=abbab,$$

and so forth. Let $V_n=W_1W_2\cdots W_n$, where we place two or more words next to one another to denote the single word obtained by writing all their letters in succession. We find that

$$V_1=b, \quad V_2=bab, \quad V_3=babbab, \quad V_4=babbababbab.$$

We wish to show that V_n is a palindrome for all positive integers n. The above list shows this to be true for $1\le n\le 4$; these cases will serve as the base cases for a proof by strong induction.

We use a bar over a word to indicate writing its letters in the reverse order. Thus $\overline{W_4} =$ *babba* and $\overline{V_3} = V_3$ since V_3 is a palindrome. Now assume that the words V_1 through V_n are all palindromes; we will show that V_{n+1} is also a palindrome. By the definition of V_{n+1} and W_{n+1} we have

$$V_{n+1} = V_n W_{n+1} = \overline{V_n}\, W_{n-1} W_n,$$

using the fact that $\overline{V_n} = V_n$ since V_n is a palindrome. But we know that $V_n = V_{n-2} W_{n-1} W_n$, so we may write

$$\overline{V_n}\, W_{n-1} W_n = \overline{W_n}\ \overline{W_{n-1}}\ \overline{V_{n-2}}\, W_{n-1} W_n.$$

The latter word is clearly a palindrome since V_{n-2} reads the same forward as backwards. Hence V_{n+1} is a palindrome, thus completing the proof.

5. Let O be the center of circle ω and let M be the midpoint of $\overline{AC}$. It is clear that $\overline{DE}$ bisects $\overline{AC}$ if and only if E, M, B are collinear. Consequently, it suffices to show that

$$\angle MED = \angle BED. \tag{2}$$

The proof is divided into four parts.

1. Triangle MED is isosceles with $\angle MED = \angle MDE$. (Note that $ACDE$ is an isosceles trapezoid and M is midpoint of the base $\overline{AC}$. The fact that triangle MED is isosceles then follows by the Pythagorean Theorem if nothing more elegant comes to mind.) This fact together with Alternate Interior Angles gives

$$\angle AME = \angle MED = \angle MDE = \angle PMD.$$

2. *Claim.* The circle ω' with diameter $\overline{OP}$ contains points B, D, and M.

   Proof. For each of the cases $X = B$, D, M, it is straightforward to verify that $\overline{OX}$ is perpendicular to $\overline{PX}$. For $X = B$ it is true that $\overline{OBP}$ is a right angle because $\overline{PB}$ is tangent to the circle at B. The same is true for $X = D$. For $X = M$, simply use the fact that if M is the midpoint of any given chord, then $\overline{OM}$ is perpendicular to the chord.

3. Referring to the circle ω', the Inscribed Angle Theorem gives $\angle PBD = \angle PMD$.

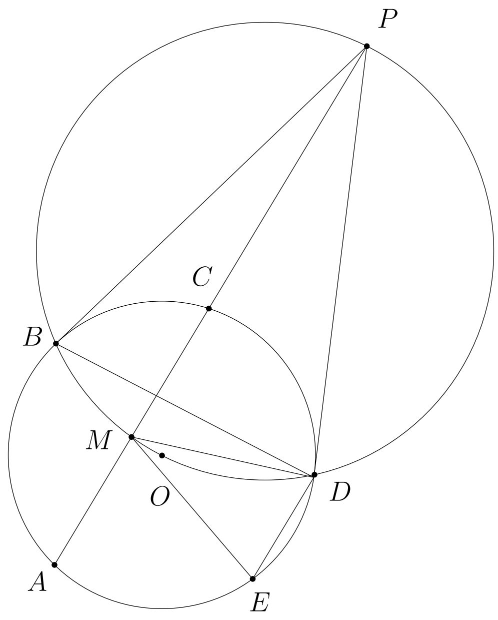
4. Because $\overline{BP}$ is tangent to ω at B,

$$\angle BED = \frac{1}{2}\,\overset{\frown}{BD} = \angle PBD.$$

Results from step 1 yield

$$\angle BED = \angle PBD = \angle PMD = \angle MED,$$

establishing 2 and completing the proof.



6. The assertion is false, and the smallest n for which it fails is $n = 25$. Given $n \geq 2$, let r be the remainder when 2^n is divided by n. Then $2^n = kn + r$ where k is a positive integer and $0 \leq r < n$. It follows that

$$2^{2^n} = 2^{kn+r} \equiv 2^r \mod 2^n - 1,$$

and $2^r < 2^n - 1$ so 2^r is the remainder when 2^{2^n} is divided by $2^n - 1$. If r is even then 2^r is power of 4. Hence to disprove the assertion, it is enough to find an n for which the corresponding r is odd.

If n is even then so is $r = 2^n - kn$.

If n is an odd prime then $2^n \equiv 2 \pmod{n}$ by Fermat's Little Theorem; hence $r \equiv 2^n \equiv 2$ mod n and $r = 2$.

There remains the case in which n is odd and composite. In the first three instances $n = 9$, 15, 21 there is no contradiction to the assertion:

$$\begin{aligned} n = 9 : & 2^6 \equiv 1 \mod 9 \Rightarrow 2^9 \equiv 2^6 \cdot 2^3 \equiv 8 \mod 9 \\ n = 15 : & 2^4 \equiv 1 \mod 15 \Rightarrow 2^{15} \equiv (2^4)^3 \cdot 2^3 \equiv 8 \mod 15 \\ n = 21 : & 2^6 \equiv 1 \mod 21 \Rightarrow 2^{21} \equiv (2^6)^3 \cdot 2^3 \equiv 8 \mod 21 \end{aligned}$$

However,

$$2^{10} = 1024 \equiv -1 \Rightarrow 2^{20} \equiv 1 \Rightarrow 2^{25} \equiv 2^5 \equiv 7 \mod 25,$$

so 7 is the remainder when 2^{25} is divided by 25 and 2^7 is the remainder when $2^{2^{25}}$ is divided by $2^{25} - 1$.