

45th United States of America Mathematical Olympiad

Day II 12:30PM — 5PM EDT

April 20, 2016

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in an automatic 1-point deduction.

USAMO 4. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all real numbers x and y ,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

USAMO 5. An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of \overleftrightarrow{MN} and \overleftrightarrow{PQ} . Denote by ℓ the angle bisector of $\angle MSQ$.

Prove that \overline{OI} is parallel to ℓ , where O is the circumcenter of triangle ABC , and I is the incenter of triangle ABC .

USAMO 6. Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard.

The wizard has $2n$ cards; for each $i = 1, \dots, n$, there are two cards labeled i . Initially, the wizard places all cards face down in a row, in unknown order.

You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds.

For which values of n and k is the game winnable?