

39th United States of America Mathematical Olympiad 2010

Day I 12:30 PM – 5 PM EDT

April 27, 2010

1. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB . Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ , respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB .
2. There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \dots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.
3. The 2010 positive numbers $a_1, a_2, \dots, a_{2010}$ satisfy the inequality $a_i a_j \leq i + j$ for all distinct indices i, j . Determine, with proof, the largest possible value of the product $a_1 a_2 \cdots a_{2010}$.