35$^{\text{th}}$ United States of America Mathematical Olympiad

1. Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p}$$

if and only if s is not a divisor of $p-1$.

(For x a real number, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x.)

First Solution. First suppose that s is a divisor of $p-1$; write $d = (p-1)/s$. As x varies among $1, 2, \ldots, p-1$, $\{sx/p\}$ takes the values $1/p, 2/p, \ldots, (p-1)/p$ once each in some order. The possible values with $\{sx/p\} < s/p$ are precisely $1/p, \ldots, (s-1)/p$. From the fact that $\{sd/p\} = (p-1)/p$, we realize that the values $\{sx/p\} = (p-1)/p, (p-2)/p, \ldots, (p-s+1)/p$ occur for

$$x = d, 2d, \ldots, (s-1)d$$

(which are all between 0 and p), and so the values $\{sx/p\} = 1/p, 2/p, \ldots, (s-1)/p$ occur for

$$x = p-d, p-2d, \ldots, p-(s-1)d,$$

respectively. From this it is clear that m and n cannot exist as requested.

Conversely, suppose that s is not a divisor of $p-1$. Put $m = \lceil p/s \rceil$; then m is the smallest positive integer such that $\{ms/p\} < s/p$, and in fact $\{ms/p\} = (ms-p)/p$. However, we cannot have $\{ms/p\} = (s-1)/p$ or else we would have $(m-1)s = p-1$, contradicting our hypothesis that s does not divide $p-1$. Hence the unique $n \in \{1, \ldots, p-1\}$ for which $\{nx/p\} = (s-1)/p$ has the desired properties (since the fact that $\{nx/p\} < s/p$ forces $n \geq m$, but $m \neq n$).

Second Solution. We prove the contrapositive statement:

> Let p be a prime number and let s be an integer with $0 < s < p$. Prove that the following statements are equivalent:
>
> (a) s is a divisor of $p-1$;

(b) if integers m and n are such that $0 < m < p$, $0 < n < p$, and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p},$$

then $0 < n < m < p$.

Since p is prime and $0 < s < p$, s is relatively prime to p and

$$S = \{s, 2s, \ldots, (p-1)s, ps\}$$

is a set of complete residues classes modulo p. In particular,

(1) there is an unique integer d with $0 < d < p$ such that $sd \equiv -1 \pmod{p}$; and

(2) for every k with $0 < k < p$, there exists a unique pair of integers (m_k, a_k) with $0 < m_k < p$ such that $m_k s + a_k p = k$.

Now we consider the equations

$$m_1 s + a_1 p = 1,\ m_2 s + a_2 p = 2,\ \ldots\ , m_s s + a_s p = s.$$

Hence $\{m_k s/p\} = k/p$ for $1 \le k \le s$.

Statement (b) holds if and only $0 < m_s < m_{s-1} < \cdots < m_1 < p$. For $1 \le k \le s-1$, $m_k s - m_{k+1} s = (a_{k+1} - a_k)p - 1$, or $(m_k - m_{k+1})s \equiv -1 \pmod{p}$. Since $0 < m_{k+1} < m_k < p$, by (1), we have $m_k - m_{k+1} = d$. We conclude that (b) holds if and only if $m_s, m_{s-1}, \ldots, m_1$ form an arithmetic progression with common difference $-d$. Clearly $m_s = 1$, so $m_1 = 1 + (s-1)d = jp - d + 1$ for some j. Then $j = 1$ because m_1 and d are both positive and less than p, so $sd = p - 1$. This proves (a).

Conversely, if (a) holds, then $sd = p - 1$ and $m_k \equiv -dsm_k \equiv -dk \pmod{p}$. Hence $m_k = p - dk$ for $1 \le k \le s$. Thus $m_s, m_{s-1}, \ldots, m_1$ form an arithmetic progression with common difference $-d$. Hence (b) holds.

This problem was proposed by Kiran Kedlaya.

2. For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $N/2$.

Solution. The minimum is $N = 2k^3 + 3k^2 + 3k$. The set

$$\{k^2+1, k^2+2, \ldots, k^2+2k+1\}$$

has sum $2k^3+3k^2+3k+1 = N+1$ which exceeds N, but the sum of the k largest elements is only $(2k^3+3k^2+3k)/2 = N/2$. Thus this N is such a value.

Suppose $N < 2k^3+3k^2+3k$ and there are positive integers $a_1 < a_2 < \cdots < a_{2k+1}$ with $a_1 + a_2 + \cdots + a_{2k+1} > N$ and $a_{k+2} + \cdots + a_{2k+1} \le N/2$. Then

$$(a_{k+1}+1) + (a_{k+1}+2) + \cdots + (a_{k+1}+k) \le a_{k+2} + \cdots + a_{2k+1} \le N/2 < \frac{2k^3+3k^2+3k}{2}.$$

This rearranges to give $2ka_{k+1} \le N - k^2 - k$ and $a_{k+1} < k^2+k+1$. Hence $a_{k+1} \le k^2+k$. Combining these we get

$$2(k+1)a_{k+1} \le N + k^2 + k.$$

We also have

$$(a_{k+1}-k) + \cdots + (a_{k+1}-1) + a_{k+1} \ge a_1 + \cdots + a_{k+1} > N/2$$

or $2(k+1)a_{k+1} > N + k^2 + k$. This contradicts the previous inequality, hence no such set exists for $N < 2k^3+3k^2+3k$ and the stated value is the minimum.

This problem was proposed by Dick Gibbs.

3. For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence $\{p(f(n^2)) - 2n\}_{n\ge 0}$ is bounded above. (In particular, this requires $f(n^2) \ne 0$ for $n \ge 0$.)

Solution. The polynomial f has the required properties if and only if

$$f(x) = c(4x-a_1^2)(4x-a_2^2)\cdots(4x-a_k^2), \qquad (*)$$

where $a_1, a_2, \ldots, a_k$ are odd positive integers and c is a nonzero integer. It is straightforward to verify that polynomials given by $(*)$ have the required property. If p is a prime divisor of $f(n^2)$ but not of c, then $p|(2n-a_j)$ or $p|(2n+a_j)$ for some $j \le k$. Hence $p - 2n \le \max\{a_1, a_2, \ldots, a_k\}$. The prime divisors of c form a finite set and do affect whether or not the given sequence is bounded above. The rest of the proof is devoted to showing that any f for which $\{p(f(n^2)) - 2n\}_{n\ge 0}$ is bounded above is given by $(*)$.

Let $\mathbb{Z}[x]$ denote the set of all polynomials with integral coefficients. Given $f \in \mathbb{Z}[x]$, let $\mathcal{P}(f)$ denote the set of those primes that divide at least one of the numbers in the sequence $\{f(n)\}_{n\ge 0}$. The solution is based on the following lemma.

Lemma. *If $f \in \mathbb{Z}[x]$ is a nonconstant polynomial then $\mathcal{P}(f)$ is infinite.*

Proof. Repeated use will be made of the following basic fact: if a and b are distinct integers and $f \in \mathbb{Z}[x]$, then $a - b$ divides $f(a) - f(b)$. If $f(0) = 0$, then p divides $f(p)$ for every prime p, so $\mathcal{P}(f)$ is infinite. If $f(0) = 1$, then every prime divisor p of $f(n!)$ satisfies $p > n$. Otherwise p divides $n!$, which in turn divides $f(n!) - f(0) = f(n!) - 1$. This yields $p|1$, which is false. Hence $f(0) = 1$ implies that $\mathcal{P}(f)$ is infinite. To complete the proof, set $g(x) = f(f(0)x)/f(0)$ and observe that $g \in \mathbb{Z}[x]$ and $g(0) = 1$. The preceding argument shows that $\mathcal{P}(g)$ is infinite, and it follows that $\mathcal{P}(f)$ is infinite. $\square$

Suppose $f \in \mathbb{Z}[x]$ is nonconstant and there exists a number M such that $p(f(n^2)) - 2n \leq M$ for all $n \geq 0$. Application of the lemma to $f(x^2)$ shows that there is an infinite sequence of distinct primes $\{p_j\}$ and a corresponding infinite sequence of nonnegative integers $\{k_j\}$ such that $p_j|f(k_j^2)$ for all $j \geq 1$. Consider the sequence $\{r_j\}$ where $r_j = \min\{k_j \pmod{p_j}, p_j - k_j \pmod{p_j}\}$. Then $0 \leq r_j \leq (p_j - 1)/2$ and $p_j|f(r_j^2)$. Hence $2r_j + 1 \leq p_j \leq p(f(r_j^2)) \leq M + 2r_j$, so $1 \leq p_j - 2r_j \leq M$ for all $j \geq 1$. It follows that there is an integer a_1 such that $1 \leq a_1 \leq M$ and $a_1 = p_j - 2r_j$ for infinitely many j. Let $m = \deg f$. Then $p_j|4^m f((p_j - a_1)/2)^2)$ and $4^m f((x - a_1)/2)^2) \in \mathbb{Z}[x]$. Consequently, $p_j|f((a_1/2)^2)$ for infinitely many j, which shows that $(a_1/2)^2$ is a zero of f. Since $f(n^2) \neq 0$ for $n \geq 0$, a_1 must be odd. Then $f(x) = (4x - a_1^2)g(x)$ where $g \in \mathbb{Z}[x]$. (See the note below.) Observe that $\{p(g(n^2)) - 2n\}_{n \geq 0}$ must be bounded above. If g is constant, we are done. If g is nonconstant, the argument can be repeated to show that f is given by $(*)$.

Note. The step that gives $f(x) = (4x - a_1^2)g(x)$ where $g \in \mathbb{Z}[x]$ follows immediately using a lemma of Gauss. The use of such an advanced result can be avoided by first writing $f(x) = r(4x - a_1^2)g(x)$ where r is rational and $g \in \mathbb{Z}[x]$. Then continuation gives $f(x) = c(4x - a_1^2) \cdots (4x - a_k^2)$ where c is rational and the a_i are odd. Consideration of the leading coefficient shows that the denominator of c is 2^s for some $s \geq 0$ and consideration of the constant term shows that the denominator is odd. Hence c is an integer.

This problem was proposed by Titu Andreescu and Gabriel Dospinescu.

4. Find all positive integers n such that there are $k \geq 2$ positive rational numbers $a_1, a_2, \ldots, a_k$ satisfying $a_1 + a_2 + \cdots + a_k = a_1 \cdot a_2 \cdots a_k = n$.

Solution. The answer is $n = 4$ or $n \geq 6$.

I. First, we prove that each $n \in \{4, 6, 7, 8, 9, \ldots\}$ <u>*satisfies*</u> the condition.

(1). If $n = 2k \geq 4$ is <u>*even*</u>, we set $(a_1,\ a_2,\ \ldots,\ a_k) = (k,\ 2,\ 1,\ \ldots,\ 1)$:

$$a_1 + a_2 + \ \ldots\ + a_k = k + 2 + 1 \cdot (k-2) = 2k = n,$$

and

$$a_1 \cdot a_2 \cdot \ \ldots\ \cdot a_k = 2k = n \ .$$

(2). If $n = 2k+3 \geq 9$ is <u>*odd*</u>, we set $(a_1,\ a_2,\ \ldots,\ a_k) = \left(k + \frac{3}{2},\ \frac{1}{2},\ 4,\ 1,\ \ldots,\ 1\right)$:

$$a_1 + a_2 + \ \ldots\ + a_k = k + \frac{3}{2} + \frac{1}{2} + 4 + (k-3) = 2k+3 = n,$$

and

$$a_1 \cdot a_2 \cdot \ \ldots\ \cdot a_k = \left(k + \frac{3}{2}\right) \cdot \frac{1}{2} \cdot 4 = 2k+3 = n \ .$$

(3). A very special case is <u>$n = 7$</u>, in which we set $(a_1,\ a_2,\ a_3) = \left(\frac{4}{3},\ \frac{7}{6},\ \frac{9}{2}\right)$. It is also easy to check that

$$a_1 + a_2 + a_3 = a_1 \cdot a_2 \cdot a_3 = 7 = n.$$

II. Second, we prove by contradiction that each $n \in \{1, 2, 3, 5\}$ <u>*fails to satisfy*</u> the condition.

Suppose, on the contrary, that there is a set of $k \geq 2$ positive rational numbers whose sum and product are both $n \in \{1, 2, 3, 5\}$. By the Arithmetic-Geometric Mean inequality, we have

$$n^{1/k} = \sqrt[k]{a_1 \cdot a_2 \cdot \ \ldots\ \cdot a_k} \leq \frac{a_1 + a_2 + \ \ldots\ + a_k}{k} = \frac{n}{k} \ ,$$

which gives

$$n \geq k^{\frac{k}{k-1}} = k^{1 + \frac{1}{k-1}} \ .$$

Note that $n > 5$ whenever $k = 3,\ 4$, or $k \geq 5$:

$$\begin{aligned} k = 3 \quad &\Rightarrow \quad n \geq 3\sqrt{3} = 5.196... > 5; \\ k = 4 \quad &\Rightarrow \quad n \geq 4\sqrt[3]{4} = 6.349... > 5; \\ k \geq 5 \quad &\Rightarrow \quad n \geq 5^{1 + \frac{1}{k-1}} > 5 \ . \end{aligned}$$

This proves that none of the integers 1, 2, 3, or 5 can be represented as the sum and, at the same time, as the product of three or more positive numbers $a_1,\ a_2,\ \ldots,\ a_k$, rational or irrational.

The remaining case $k = 2$ also goes to a contradiction. Indeed, $a_1 + a_2 = a_1 a_2 = n$ implies that $n = a_1^2/(a_1 - 1)$ and thus a_1 satisfies the quadratic

$$a_1^2 - na_1 + n = 0 \ .$$

Since a_1 is supposed to be *rational*, the discriminant $n^2 - 4n$ must be a perfect square (a square of a positive integer). However, it can be easily checked that this is not the case for any $n \in \{1, \ 2, \ 3, \ 5\}$. This completes the proof.

Remark. Actually, among all positive integers only $n = 4$ can be represented both as the sum and product of the same two rational numbers. Indeed, $(n-3)^2 < n^2 - 4n = (n-2)^2 - 4 < (n-2)^2$ whenever $n \geq 5$; and $n^2 - 4n < 0$ for $n = 1, \ 2, \ 3$.

This problem was proposed by Ricky Liu.

5. A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n + 2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

   First Solution. For $i \geq 0$ and $k \geq 1$, let $x_{i,k}$ denote the minimum number of jumps needed to reach the integer $n_{i,\,k} = 2^i k$. We must prove that

$$x_{i,k} > x_{i,1} \tag{1}$$

   for all $i \geq 0$ and $k \geq 2$. We prove this using the method of descent.

   First note that (1) holds for $i = 0$ and all $k \geq 2$, because it takes 0 jumps to reach the starting value $n_{0,\,1} = 1$, and at least one jump to reach $n_{0,k} = k \geq 2$. Now assume that that (1) is not true for all choices of i and k. Let i_0 be the minimal value of i for which (1) fails for some k, let k_0 be the minimal value of $k > 1$ for which $x_{i_0,k} \leq x_{i_0,1}$. Then it must be the case that $i_0 \geq 1$ and $k_0 \geq 2$.

   Let J_{i_0,k_0} be a shortest sequence of $x_{i_0,\,k_0}+1$ integers that the frog occupies in jumping from 1 to $2^{i_0}k_0$. The length of each jump, that is, the difference between consecutive integers in J_{i_0,k_0}, is either 1 or a positive integer power of 2. The sequence J_{i_0,k_0} cannot contain 2^{i_0} because it takes more jumps to reach $2^{i_0}k_0$ than it does to reach 2^{i_0}. Let 2^{M+1}, $M \geq 0$

be the length of the longest jump made in generating J_{i_0,k_0}. Such a jump can only be made from a number that is divisible by 2^M (and by no higher power of 2). Thus we must have $M < i_0$, since otherwise a number divisible by 2^{i_0} is visited before $2^{i_0}k_0$ is reached, contradicting the definition of k_0.

Let 2^{m+1} be the length of the jump when the frog jumps over 2^{i_0}. If this jump starts at $2^m(2t-1)$ for some positive integer t, then it will end at $2^m(2t-1)+2^{m+1} = 2^m(2t+1)$. Since it goes over 2^{i_0} we see $2^m(2t-1) < 2^{i_0} < 2^m(2t+1)$ or $(2^{i_0-m}-1)/2 < t < (2^{i_0-m}+1)/2$. Thus $t = 2^{i_0-m-1}$ and the jump over 2^{i_0} is from $2^m(2^{i_0-m}-1) = 2^{i_0} - 2^m$ to $2^m(2^{i_0-m}+1) = 2^{i_0}+2^m$.

Considering the jumps that generate J_{i_0,k_0}, let N_1 be the number of jumps from 1 to $2^{i_0}+2^m$, and let N_2 be the number of jumps from $= 2^{i_0}+2^m$ to $2^{i_0}k$. By definition of i_0, it follows that 2^m can be reached from 1 in less than N_1 jumps. On the other hand, because $m < i_0$, the number $2^{i_0}(k_0-1)$ can be reached from 2^m in exactly N_2 jumps by using the same jump length sequence as in jumping from $2^m+2^{i_0}$ to $2^{i_0}k_0 = 2^{i_0}(k_0-1)+2^i_0$. The key point here is that the shift by 2^{i_0} does not affect any of divisibility conditions needed to make jumps of the same length. In particular, with the exception of the last entry, $2^{i_0}k_0$, all of the elements of J_{i_0,k_0} are of the form $2^p(2t+1)$ with $p < i_0$, again because of the definition of k_0. Because $2^p(2t+1)-2^{i_0} = 2^p(2t-2^{i_0-p}+1)$ and the number $2t+2^{i_0-p}+1$ is odd, a jump of size 2^{p+1} can be made from $2^p(2t+1)-2^{i_0}$ just as it can be made from $2^p(2t+1)$.

Thus the frog can reach 2^m from 1 in less than N_1 jumps, and can then reach $2^{i_0}(k_0-1)$ from 2^m in N_2 jumps. Hence the frog can reach $2^{i_0}(k_0-1)$ from 1 in less than N_1+N_2 jumps, that is, in fewer jumps than needed to get to $2^{i_0}k_0$ and hence in fewer jumps than required to get to 2^{i_0}. This contradicts the definition of k_0.

Second Solution. Suppose $x_0 = 1, x_1, \ldots, x_t = 2^i k$ are the integers visited by the frog on his trip from 1 to $2^i k$, $k \geq 2$. Let $s_j = x_j - x_{j-1}$ be the jump sizes. Define a reduced path y_j inductively by

$$y_j = \begin{cases} y_{j-1}+s_j & \text{if } y_{j-1}+s_j \leq 2^i, \\ y_{j-1} & \text{otherwise.} \end{cases}$$

Say a jump s_j is deleted in the second case. We will show that the distinct integers among the y_j give a shorter path from 1 to 2^i. Clearly $y_j \leq 2^i$ for all j. Suppose $2^i - 2^{r+1} < y_j \leq 2^i - 2^r$ for some $0 \leq r \leq i-1$. Then every deleted jump before y_j must have length greater than 2^r, hence must be a multiple of 2^{r+1}. Thus $y_j \equiv x_j \pmod{2^{r+1}}$. If $y_{j+1} > y_j$, then either $s_{j+1} = 1$ (in which case this is a valid jump) or $s_{j+1}/2 = 2^m$ is the exact power of 2 dividing x_j. In the second case, since $2^r \geq s_{j+1} > 2^m$, the congruence says 2^m is also the exact power of 2 dividing y_j, thus again this is a valid jump. Thus the distinct y_j form a valid path for the frog. If $j = t$ the congruence gives $y_t \equiv x_t \equiv 0 \pmod{2^{r+1}}$, but this is impossible for $2^i - 2^{r+1} < y_t \leq 2^i - 2^r$. Hence we see $y_t = 2^i$, that is, the reduced path ends at 2^i. Finally since the reduced path ends at $2^i < 2^i k$ at least one jump must have been deleted and it is strictly shorter than the original path.

This problem was proposed by Zoran Sunik.

6. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

First Solution. Let P be the second intersection of the circumcircles of triangles TCF and TDE. Because the quadrilateral $PEDT$ is cyclic, $\angle PET = \angle PDT$, or

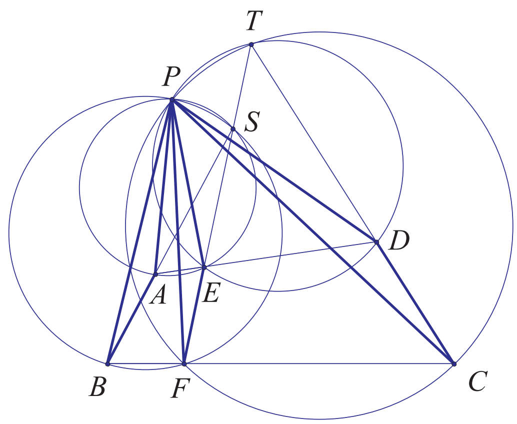
$$\angle PEF = \angle PDC. \tag{$*$}$$

Because the quadrilateral $PFCT$ is cyclic,

$$\angle PFE = \angle PFT = \angle PCT = \angle PCD. \tag{$**$}$$

By equations $(*)$ and $(**)$, it follows that triangle PEF is similar to triangle PDC. Hence $\angle FPE = \angle CPD$ and $PF/PE = PC/PD$. Note also that $\angle FPC = \angle FPE + \angle EPC = \angle CPD + \angle EPC = \angle EPD$. Thus, triangle EPD is similar to triangle FPC. Another way to say this is that there is a spiral similarity centered at P that sends triangle PFE to triangle PCD, which implies that there is also a spiral similarity, centered at P, that sends triangle PFC to triangle PED, and vice versa. In terms of complex numbers, this amounts to saying that

$$\frac{D-P}{E-P} = \frac{C-P}{F-P} \Longrightarrow \frac{E-P}{F-P} = \frac{D-P}{C-P}.$$



Because $AE/ED = BF/FC$, points A and B are obtained by extending corresponding segments of two similar triangles PED and PFC, namely, DE and CF, by the identical proportion. We conclude that triangle PDA is similar to triangle PCB, implying that triangle PAE is similar to triangle PBF. Therefore, as shown before, we can establish the similarity between triangles PBA and PFE, implying that

$$\angle PBS = \angle PBA = \angle PFE = \angle PFS \quad \text{and} \quad \angle PAB = \angle PEF.$$

The first equation above shows that $PBFS$ is cyclic. The second equation shows that $\angle PAS = 180^\circ - \angle BAP = 180^\circ - \angle FEP = \angle PES$; that is, $PAES$ is cyclic. We conclude that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through point P.

Note. There are two spiral similarities that send segment EF to segment CD. One of them sends E and F to D and C, respectively; the point P is the center of this spiral similarity. The other sends E and F to C and D, respectively; the center of this spiral similarity is the second intersection (other than T) of the circumcircles of triangles TFD and TEC.

Second Solution. We will give a solution using complex coordinates. The first step is the following lemma.

Lemma. *Suppose s and t are real numbers and x, y and z are complex. The circle in the complex plane passing through x, $x + ty$ and $x + (s + t)z$ also passes through the point $x + syz/(y - z)$, independent of t.*

Proof. Four points z_1, z_2, z_3 and z_4 in the complex plane lie on a circle if and only if the cross-ratio

$$cr(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

is real. Since we compute

$$cr(x, x + ty, x + (s + t)z, x + syz/(y - z)) = \frac{s + t}{s}$$

the given points are on a circle. $\square$

Lay down complex coordinates with $S = 0$ and E and F on the positive real axis. Then there are real r_1, r_2 and R with $B = r_1 A$, $F = r_2 E$ and $D = E + R(A - E)$ and hence $AE/ED = BF/FC$ gives

$$C = F + R(B - F) = r_2(1 - R)E + r_1 RA.$$

The line CD consists of all points of the form $sC + (1 - s)D$ for real s. Since T lies on this line and has zero imaginary part, we see from $\mathrm{Im}(sC + (1 - s)D) = (sr_1 R + (1 - s)R)\mathrm{Im}(A)$ that it corresponds to $s = -1/(r_1 - 1)$. Thus

$$T = \frac{r_1 D - C}{r_1 - 1} = \frac{(r_2 - r_1)(R - 1)E}{r_1 - 1}.$$

Apply the lemma with $x = E$, $y = A - E$, $z = (r_2 - r_1)E/(r_1 - 1)$, and $s = (r_2 - 1)(r_1 - r_2)$. Setting $t = 1$ gives

$$(x, x + y, x + (s + 1)z) = (E, A, S = 0)$$

and setting $t = R$ gives

$$(x, x + Ry, x + (s + R)z) = (E, D, T).$$

Therefore the circumcircles to SAE and TDE meet at

$$x + \frac{syz}{y - z} = \frac{AE(r_1 - r_2)}{(1 - r_1)E - (1 - r_2)A} = \frac{AF - BE}{A + F - B - E}.$$

This last expression is invariant under simultaneously interchanging A and B and interchanging E and F. Therefore it is also the intersection of the circumcircles of SBF and TCF.

This problem was proposed by Zuming Feng and Zhonghao Ye.