

41st United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 24, 2012

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

USAMO 1. Find all integers $n \geq 3$ such that among any n positive real numbers a_1, a_2, \dots, a_n with

$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$

there exist three that are the side lengths of an acute triangle.

USAMO 2. A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

USAMO 3. Determine which integers $n > 1$ have the property that there exists an infinite sequence a_1, a_2, a_3, \dots of nonzero integers such that the equality

$$a_k + 2a_{2k} + \dots + na_{nk} = 0$$

holds for every positive integer k .