# $38^{\text{th}}$ United States of America Mathematical Olympiad

## Day II 12:30 PM – 5 PM EDT

## April 29, 2009

4. For $n \geq 2$ let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \leq 4\min(a_1, a_2, \ldots, a_n)$.

5. Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in circle $\omega$ and point $G$ lies inside triangle $BCD$. Rays $AG$ and $BG$ meet $\omega$ again at points $P$ and $Q$, respectively. Let the line through $G$ parallel to $\overline{AB}$ intersect $\overline{BD}$ and $\overline{BC}$ at points $R$ and $S$, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

6. Let $s_1, s_2, s_3, \ldots$ be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that $s_1 = s_2 = s_3 = \cdots$. Suppose that $t_1, t_2, t_3, \ldots$ is also an infinite, nonconstant sequence of rational numbers with the property that $(s_i - s_j)(t_i - t_j)$ is an integer for all $i$ and $j$. Prove that there exists a rational number $r$ such that $(s_i - s_j)r$ and $(t_i - t_j)/r$ are integers for all $i$ and $j$.