45th United States of America Mathematical Olympiad

Day I 12:30PM — 5PM EDT

April 19, 2016

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in an automatic 1-point deduction.

USAMO 1. Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct non-empty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

USAMO 2. Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

USAMO 3. Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overleftrightarrow{I_B F}$ and $\overleftrightarrow{I_C E}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.