41^{st} United States of America Mathematical Olympiad

Day I, II 12:30 PM – 5 PM EDT

April 24-25, 2012

USAMO 1. First we prove that any $n \geq 13$ is a solution of the problem. Suppose that $a_1, a_2, ..., a_n$ satisfy $\max(a_1, a_2, ..., a_n) \leq n \cdot \min(a_1, a_2, ..., a_n)$, and that we cannot find three that are the side-lengths of an acute triangle. We may assume that $a_1 \leq a_2 \leq ... \leq a_n$. Then $a_{i+2}^2 \geq a_i^2 + a_{i+1}^2$ for all $i \leq n-2$. Let (F_n) be the Fibonacci sequence, with $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. It is easy to check that $F_n < n^2$ for $n \leq 11$, $F_{12} = 12^2$ and $F_n > n^2$ for $n > 12$ (the last inequality follows by an immediate induction, while the first one can be checked by hand). The inequality $a_{i+2}^2 \geq a_i^2 + a_{i+1}^2$ and the fact that $a_1 \leq a_2 \leq ... \leq a_n$ imply that $a_i^2 \geq F_i \cdot a_1^2$ for all $i \leq n$. Hence, if $n \geq 13$, we obtain $a_n^2 > n^2 \cdot a_1^2$, contradicting the hypothesis. This shows that any $n \geq 13$ is a solution of the problem.

By taking $a_i = \sqrt{F_i}$ for $1 \leq i \leq n$, we have $\max(a_1, a_2, ..., a_n) \leq n \cdot \min(a_1, a_2, ..., a_n)$, for any $n < 13$, but it is easy to see that no three a_i's can be the side-lengths of an acute triangle. Hence the answer to the problem is: all $n \geq 13$.

This problem and solution were suggested by Titu Andreescu.

USAMO 2. Let R, G, B, Y denote the sets of Red, Green, Blue, Yellow points, respectively, and let r, g, b, y denote a generic Red, Green, Blue, Yellow point, respectively. For integers $0 \leq k \leq 431$, let $\mathcal{T}_k$ denote the counterclockwise rotation of $\left(\frac{360k}{432}\right)$ degree around the center of the circle. Furthermore, for a set S, let $|S|$ denote the number of elements in S.

First, we claim that there is some index i_1 such that $|\mathcal{T}_{i_1}(R) \cap G| \geq 28$. Indeed, for each k, set $\mathcal{T}_k(R) \cap G$ consists of all Green points that are the images of Red points under the rotation $\mathcal{T}_k$. Hence the sum

$$s_1 = |\mathcal{T}_0(R) \cap G| + |\mathcal{T}_1(R) \cap G| + \cdots + |\mathcal{T}_{431}(R) \cap G|$$

is equal to the number of pairs of points (r, g) such that $g = \mathcal{T}_k(r)$ for some k. On the other hand, for each r and each g, there is a unique rotation $\mathcal{T}_k$ with $\mathcal{T}_k(r) = g$, form which it follows that $s_1 = 108^2 = 11664$. Clearly, $|\mathcal{T}_0(R) \cap G| = |R \cap G| = 0$ (because $R \cap G = \emptyset$). By the Pigeonhole principle, there is some index i_1 such that

$$|\mathcal{T}_{i_1}(R) \cap G| \geq \left\lceil \frac{s_1}{431} \right\rceil = \left\lceil \frac{11664}{431} \right\rceil = \lceil 27.06\ldots \rceil = 28,$$

establishing our claim. Let RG denote the set $\mathcal{T}_{i_1}(R) \cap G$, and let rg denote a generic point in RG.

Second, we claim that there is some index i_2 such that $|\mathcal{T}_{i_2}(RG) \cap B| \geq 8$. Again, for each k, set $\mathcal{T}_k(RG) \cap B$ consists of all Blue points that are the images of the points in RG under the rotation $\mathcal{T}_k$. Hence the sum

$$s_2 = |\mathcal{T}_0(RG) \cap B| + |\mathcal{T}_1(RG) \cap B| + \cdots + |\mathcal{T}_{431}(RG) \cap B|$$

is equal to the number of pairs of points (rg, b) such that $b = \mathcal{T}_k(rg)$ for some k. On the other hand, for each rg and each b, there is a unique rotation $\mathcal{T}_k$ with $\mathcal{T}_k(rg) = b$, form which it follows that $s_2 \geq 28 \cdot 108 = 3024$. Clearly, RG is a subset of B, which is disjoint with B, so $|\mathcal{T}_0(RG) \cap B| = 0$. Furthermore, $\mathcal{T}_{432-i_1}(\mathcal{T}_{i_1})$ is the identity transformation, implying that $\mathcal{T}_{432-i_1}(\mathcal{T}_{i_1}(R)) = R$ and $\mathcal{T}_{432-i_1}(RG)$ is a subset of R which is disjoint with B. In particular, $|\mathcal{T}_{432-i_1}(RG) \cap B| = 0$. By the Pigeonhole principle, there is some index i_2 such that

$$|\mathcal{T}_{i_2}(RG) \cap B| \geq \left\lceil \frac{s_2}{430} \right\rceil \geq \left\lceil \frac{3024}{430} \right\rceil = \lceil 7.0325\ldots \rceil = 8,$$

establishing our claim. Let RGB denote the set $\mathcal{T}_{i_2}(RG) \cap B$, and let rgb denote a generic point in RGB.

Third, we claim that there is some index i_3 such that $|\mathcal{T}_{i_3}(RGB) \cap Y| \geq 3$. We repeated our previous process one more time. We note that

$$s_3 = |\mathcal{T}_0(RGB) \cap Y| + |\mathcal{T}_1(RGB) \cap Y| + \cdots + |\mathcal{T}_{431}(RGB) \cap Y| \geq 8 \cdot 108 = 864$$

and

$$|\mathcal{T}_0(RGB) \cap Y| = |\mathcal{T}_{432-i_2}(RGB) \cap Y| = |\mathcal{T}_{432-i_2-i_1}(RGB) \cap Y| = 0.$$

By the Pigeonhole principle, there is some index i_3 such that

$$|\mathcal{T}_{i_3}(RGB) \cap Y| \geq \left\lceil \frac{s_3}{429} \right\rceil \geq \left\lceil \frac{864}{429} \right\rceil = \lceil 2.01\ldots \rceil = 3,$$

establishing our claim.

Let y_1, y_2, y_3 be three points in $\mathcal{T}_{i_2}(RGB) \cap Y$. Then

$$\begin{aligned}(y_1, y_2, y_3), (b_1, b_2, b_3) &= \mathcal{T}_{432-i_3}(y_1, y_2, y_3), (g_1, g_2, g_3) \\ &= \mathcal{T}_{432-i_3-i_2}(y_1, y_2, y_3), (r_1, r_2, r_3) \\ &= \mathcal{T}_{432-i_3-i_2-i_1}(y_1, y_2, y_3)\end{aligned}$$

are twelve points that we are looking for.

This problem and solution were suggested by Gregory Galperin.

USAMO 3. We will show that the sequence exists for all $n \geq 3$.

For $n = 2$, the sequence cannot exist: If it existed, we would have $a_k = -2a_{2k}$ for all k, from which $a_1 = (-2)^r a_{2^r}$ for all r by induction. Then a_1 would have to be divisible by 2^r for all r, which is impossible for $a_1 \neq 0$.

Now fix $n \geq 3$. We will show that the desired sequence exists. The construction is basically a repeated application of the Chinese Remainder Theorem, but the details require substantial care.

First we prove two lemmas.

Lemma 1 It is possible to partition the positive integers into subsets $S_1, S_2, S_3, \ldots$ so that for every positive integer k,

(i) the numbers $(n-1)k$ and nk are in the same subset, and

(ii) the numbers $k, 2k, \ldots, (n-2)k$ are all in strictly earlier subsets than $(n-1)k$.

Proof To show this, define a function f from positive integers to positive reals as follows. Let P_n be the set of primes dividing n. No element of P_n divides $n-1$. For any number k, write its prime factorization $k = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, and then define

$$f(k) = \prod_{p_i \notin P_n} p_i^{e_i} \cdot \prod_{p_i \in P_n} (p_i^{e_i})^{\log_n(n-1)}.$$

Notice that for every positive integer k,

$$f((n-1)k) = (n-1)f(k) = f(nk) \tag{1}$$

whereas for each $t = 1, 2, \ldots, n-2$,

$$f(tk) \leq tf(k) < f((n-1)k). \tag{2}$$

Also notice that for each k, $f(k) \geq k^{\log_n(n-1)}$, which implies that for any fixed C, there can only be finitely many values of k with $f(k) < C$. The latter fact means that the elements of the image of f can be arranged in increasing order, $x_1 < x_2 < x_3 < \cdots$. Now just let $S_i = f^{-1}(x_i)$ for each i. The sets S_i are a partition of the positive integers, and (1) and (2) ensure that they satisfy (i) and (ii) respectively.

Lemma 2 Let p, q be relatively prime positive integers and $t_1, t_2, \ldots, t_r$ arbitrary integers. Then it is possible to choose nonzero integers $b_1, b_2, \ldots, b_{r+1}$ such that

$$pb_i + qb_{i+1} = t_i \qquad \text{for } i = 1, 2, \ldots, r. \tag{3}$$

Proof We first prove existence of a sequence of integers satisfying (3) for each i, by induction on r. If $r = 1$, then since p, q are relatively prime, we can find c, d such that $pc + qd = 1$. Then, $b_1 = ct_1$ and $b_2 = dt_1$ satisfy (3). Now suppose we have $b_1, \ldots, b_r$ satisfying (3) for $i = 1, 2, \ldots, r-1$. If we choose any integer k, and replace each b_i with $b_i' = b_i + (-1)^i p^{i-1} q^{r-i} k$, then (3) still holds for $i = 1, 2, \ldots, r-1$, and $pb_r' = pb_r + (-1)^r p^{r-1} k$. Since p, q are relatively prime, we can choose k so as to make pb_r' congruent to t_r modulo q, and then we take $b_{r+1} = (t_r - pb_r')/q$. Then the numbers $b_1', b_2', \ldots, b_r', b_{r+1}$ satisfy (3) for $i = 1, 2, \ldots, r$.

This shows that we can find $b_1, b_2, \ldots, b_{r+1}$ satisfying (3), but they may not all be nonzero. However, once again, we can make the replacements $b_i' = b_i + (-1)^i p^{i-1} q^{r+1-i} k$ for any integer k, and the new sequence still satisfies (3). By an appropriate choice of k, we can ensure each b_i' is nonzero.

Now both lemmas are proven, and we resume the main proof. We will construct terms of the sequence inductively, but not in the order $a_1, a_2, \ldots$.

Suppose S is any set of positive integers, and we have chosen nonzero integers a_k for each $k \in S$. Say that there is a <u>conflict</u> in S if there exists some k such that $k, 2k, \ldots, nk$ are all in S, and

$$a_k + 2a_{2k} + \cdots + na_{nk} \neq 0.$$

Let $S_1, S_2, \ldots$ be as given by Lemma 1 We will inductively define our sequence as follows:

(a) Step 1: Choose nonzero values a_k for all $k \in S_1$ simultaneously, without creating a conflict in S_1.

(b) Step $t \geq 1$: Given the values of a_k for $k \in S_1 \cup \cdots \cup S_{t-1}$ chosen at previous steps, choose nonzero integers a_k for all $k \in S_t$ simultaneously, without creating a conflict in $S_1 \cup \cdots \cup S_t$.

If we can show that each step of this process can indeed be carried out, then it will eventually define a_k for all positive integers k, meeting the required condition

$$a_k + 2a_{2k} + \cdots + na_{nk} = 0 \tag{4}$$

for all k (since no conflicts are created).

For step 1, Lemma 1 implies we can choose a_k arbitrarily for $k \in S_1$ without creating any conflicts, since $(n-1)k, nk \notin S_1$ for all k. Now for step $t \geq 1$, suppose a_k have been assigned already for all $k \in S_1 \cup S_2 \cup \cdots \cup S_{t-1}$. We need to assign a_k for $k \in S_t$ to avoid creating any new conflicts. This just requires that the new assignments satisfy (4) for all integers k such that $(n-1)k$ and nk are in S_t: for any other value k, either $\{k, 2k, \ldots, nk\} \not\subseteq S_1 \cup \cdots \cup S_t$ so no conflict can be created, or else Lemma 1 implies $\{k, 2k, \ldots, nk\} \subseteq S_1 \cup \cdots \cup S_{t-1}$ so that the corresponding constraint (4) has been dealt with at an earlier step.

Thus for each k such that $(n-1)k, nk \in S_t$, we have a constraint

$$(n-1)a_{(n-1)k} + na_{nk} = X_k, \tag{5}$$

where $X_k = -(a_k + \cdots + (n-2)a_{(n-2)k})$ is determined by the assignments made at previous steps. We just need to show that it is possible to choose a_k for all $k \in S_t$ such that all these constraints are satisfied.

Form a directed graph whose vertices are the elements of S_t, with an edge leading from $(n-1)k$ to nk whenever both numbers are in S_t. Then every component of this graph is either a single vertex or a (directed) path. We wish to show that nonzero integer values can be assigned to elements of S_t so that for each edge, the corresponding constraint (5) is satisfied. It suffices to show this for each component of the graph. If the component is a single vertex, any nonzero value works. Otherwise, it is a path $k_1, k_2, \ldots, k_{r+1}$, and Lemma 2 ensures that we can choose nonzero integer values for $a_{k_1}, a_{k_2}, \ldots, a_{k_{r+1}}$ so as to satisfy (5) for each edge.

This shows that each step of our constructive process can indeed be performed successfully, and iterating eventually constructs every term of the sequence.

This problem and solution were suggested by Gabriel Carroll.

USAMO 4. There are three solutions: the constant functions $1, 2$ and the identity function. Let us prove that these are the only ones. Consider such a function f and suppose first of all that there exists $a > 2$ such that $f(a) = a$. Then $a!, (a!)!, \ldots$ are all fixed points of f. So there is an increasing sequence $(a_n)_{n \geq 0}$ of fixed points. If n is any positive integer, $a_k - n$ divides $a_k - f(n) = f(a_k) - f(n)$ for all k, and so it also divides $f(n) - n$ for all k. Thus $f(n) = n$ and since it holds for any n, we are done in this case.

Now suppose that f has no fixed points greater than 2. Let $p > 3$ be a prime and observe that $(p-2)! \equiv 1 \pmod{p}$ (by Wilson's theorem), thus $f(p-2)! - f(1) = f((p-2)!) - f(1)$ is a multiple of p. Clearly $f(1)$ is 1 or 2. As $p > 3$, the fact that p divides $f(p-2)! - f(1)$ implies that $f(p-2) < p$. Since $(p-1)! - f(1)$ is not a multiple of p (again by Wilson), we deduce that actually $f(p-2) \le p-2$. On the other hand, $p-3$ divides $f(p-2) - f(1) \le f(p-2) - 1$. Thus either $f(p-2) = f(1)$ or $f(p-2) = p-2$. As $p-2 > 2$, the last case is excluded and so $f(p-2) = f(1)$ and this for all primes $p > 3$. Taking n any positive integer, we deduce that $p-2-n$ divides $f(1) - f(n)$ and this holds for all large primes p. Thus $f(n) = f(1)$ and f is constant. The conclusion is now clear.

This problem and solution were suggested by Gabriel Dospinescu.

USAMO 5. **Solution 1:** The proof is split into two cases.

Case 1: P is on the circumcircle of ABC. Then P is the Miquel point of A', B', C' with respect to ABC. Indeed, because $\angle A'B'C' = \angle CBA = \angle CPA = \angle A'PC'$, points P, A', B', C' are concyclic, and the same can be said for P, A, B', C' and P, A', B', C. Hence $\angle CA'B' = \angle CPB' = \angle BPC' = \angle BA'C'$, so $A'B'C'$ are collinear.

Case 2: P is not on the circumcircle of ABC. Let Q be isogonal conjugate of P with respect to ABC (which is not degenerate).

Claim. Let Q' be the isogonal conjugate of P with respect to $AB'C'$. Then $Q = Q'$.

Proof of the claim. Note that

$$\begin{aligned}
\angle BQC &= \angle BAC + \angle CPB \quad \text{(because } P \text{ and } Q \text{ are isogonal conjugates in } ABC\text{)} \\
&= \angle C'AB' + \angle B'PC' \\
&= \angle C'Q'B' \quad \text{(because } P \text{ and } Q \text{ are isogonal conjugates in } AB'C'\text{)}.
\end{aligned}$$

Let X, Y, Z denote the reflections of P in sides BC, CA, AB, respectively, and let X' denote P's reflection in side $B'C'$ of triangle $AB'C'$. Then $\angle ZXY = \angle BQC$ (because QC is orthogonal to XY and QB is orthogonal to XZ), whereas $\angle ZX'Y' = \angle C'Q'B'$ because $Q'B'$ is orthogonal to $X'Y$ and $Q'C'$ is orthogonal to $X'Z$ and $Q'C'$ is orthogonal to $X'Z$, so since $\angle C'Q'B' = \angle BQC$, we get $\angle ZXY = \angle ZX'Y$. It follows that X, Y, Z, X' are concyclic. The center of the XYZ-circle is Q while the center of the $X'Y'Z$-circle is Q'. Thus $Q = Q'$.

Note. We have made use of the well-known fact that the circumcenter of the triangle determined by the reflections of a point across the sidelines of another given triangle is precisely the isogonal conjugate of the point with respect to that triangle. For a proof see R. A. Johnson, *Advanced Euclidean Geometry*, 1929 ed., reprinted by Dover, 2007.

Similar arguments show that Q is also the isogonal point of P with respect to triangles $A'BC'$ and $A'B'C$. Therefore,

$$\begin{aligned}
\angle BC'A' &= \angle AC'A' = \angle AC'P + \angle PC'Q + \angle QC'A' \\
&= \angle QC'B' + \angle PC'Q + \angle BC'P \\
&= \angle BC'B' = \angle AC'B'.
\end{aligned}$$

This means that A', B', C' are collinear. ■

This problem and solution were suggested by Titu Andreescu and Cosmin Pohoata.

Solution 2: It's easy to see (say, by law of sines) that

$$\frac{AC'}{BC'} = \frac{AP\sin\angle APC'}{BP\sin\angle BPC'}, \quad \frac{BA'}{CA'} = \frac{BP\sin\angle BPA'}{CP\sin\angle CPA'}, \quad \frac{CB'}{AB'} = \frac{CP\sin\angle CPB'}{AP\sin\angle APB'}.$$

The construction of A', B', C' by reflections implies that

$$\sin\angle APC' = \sin\angle CPA', \quad \sin\angle BPC' = \sin\angle CPB', \quad \sin\angle BPC' = \sin\angle CPB'.$$

Hence,

$$\frac{AC'}{BC'}\cdot\frac{BA'}{CA'}\cdot\frac{CB'}{AB'} = 1,$$

and the proof is complete by Menelaus' theorem.

This second solution was suggested by Li Zhou, Polk State College, Winter Haven FL.

USAMO 6. This problem is a form of Chebyshev's inequality for random variables. For each set $A \subseteq \{1, 2, \ldots, n\}$, define

$$\Delta_A = 2S_A = \sum_{i\in A} x_i \quad - \sum_{i\in\{1,2,\ldots,n\}\setminus A} x_i \quad = \quad \sum_{i=1}^{n} \epsilon_A(i)x_i,$$

where $\epsilon_A(i) = 1$ if $i \in A$ and -1 otherwise. Squaring, we have

$$\Delta_A^2 = \sum_{i=1}^{n} x_i^2 + \sum_{\substack{i,j\in\{1,\ldots,n\}\\ i\neq j}} \epsilon_A(i)\epsilon_A(j)x_ix_j. \tag{6}$$

Now sum the Δ_A^2's over all 2^n possible choices of A. For each pair $i \neq j$, there are 2^{n-2} sets A with $i, j \in A$, and another 2^{n-2} sets with $i, j \notin A$; these sets each contributes a term of $+x_ix_j$ to the sum in (6). There are also 2^{n-2} sets A with $i \in A$, $j \notin A$, and 2^{n-2} sets with $i \notin A$, $j \in A$. Each of these sets each contributes a term of $-x_ix_j$ to (6). Hence, x_ix_j appears 2^{n-1} times with a $+$ sign and 2^{n-1} times with a $-$ sign. Therefore all of these terms cancel, and we find

$$\sum_{A\subseteq\{1,2,\ldots,n\}} \Delta_A^2 = 2^n(x_1^2 + \cdots + x_n^2) = 2^n. \tag{7}$$

Now let $\lambda > 0$. There cannot be more than $2^{n-2}/\lambda^2$ terms Δ_A^2 whose value greater than or equal to $4\lambda^2$. If this were not the case, then the sum of these terms would be greater than 2^n, so the sum in (7) would exceed 2^n. Hence, there can be at most $2^{n-2}/\lambda^2$ sets A such that $|S_A| \geq \lambda$. (Recall that $\Delta_A = 2S_A$). Moreover, these sets can be arranged into complementary pairs because $S_A = -S_{\{1,\ldots,n\}\setminus A}$. In each of these pairs, exactly one of the two members is positive. Therefore there are at most $2^{n-3}/\lambda^2$ sets A with $S_A \geq \lambda$.

For equality to hold, it must be the case that all positive values of Δ_A^2 are equal to $4\lambda^2$; otherwise we would again have a contradiction because the sum of all Δ_A^2 would exceed 2^n. In particular, all positive values of Δ_A^2 must be the same. Thus all positive values of x_A must be the same. This will be the case only if at most one of the x_i is positive and at most one of the x_i is negative. Because we must have at least one of each, there must be exactly one positive term and one negative term. Thus it must be the case that one $x_k = \sqrt{2}/2$ for some k, one is $x_j = -\sqrt{2}/2$ for some $j \neq k$, and all other $x_i = 0$. Then the assumption that every positive $\Delta_A^2 = 4\lambda^2$ yields $\lambda = \sqrt{2}/2$.

Conversely, with the x_i and λ as described, we have exactly $2^{n-2} = 2^{n-3}/\lambda^2$ sets A such that $x_A \geq \lambda$ (namely, those sets A that contain the $\sqrt{2}/2$ term and do not contain the $-\sqrt{2}/2$ term.) Thus this is indeed the equality case.

This problem and solution were suggested by Gabriel Carroll.