5^{th} United States of America Junior Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 29, 2014

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

JMO 1. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2-5a+1}{b^2-5b+10}, \frac{10b^2-5b+1}{c^2-5c+10}, \frac{10c^2-5c+1}{a^2-5a+10}\right) \leq abc.$$

JMO 2. Let ΔABC be a non-equilateral, acute triangle with $\angle A = 60^\circ$, and let O and H denote the circumcenter and orthocenter of ΔABC, respectively.

(a) Prove that line OH intersects both segments AB and AC.

(b) Line OH intersects segments AB and AC at P and Q, respectively. Denote by s and t the respective areas of triangle APQ and quadrilateral $BPQC$. Determine the range of possible values for s/t.

JMO 3. Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.