46^{th} United States of America Mathematical Olympiad Solutions

USAMO 1. (Proposed by Gregory Galperin)

Let n be an odd positive integer, and take $a = 2n - 1$, $b = 2n + 1$. Then $a^b + b^a \equiv 1 + 3 \equiv 0 \pmod 4$, and $a^b + b^a \equiv -1 + 1 \equiv 0 \pmod n$. Therefore $a + b = 4n$ divides $a^b + b^a$.

Alternate solution: Let $p > 5$ be a prime and let $p \not\equiv 1 \pmod 5$. For each such prime p we construct a pair of relatively prime numbers (a, b) that satisfy the conclusion of the problem. Thus, we will get infinitely many distinct pairs (a, b) as required.

Let $a = 3p + 2$, $b = 7p - 2$. Then $a + b = 10p$. We have $\varphi(10p) = 4(p - 1) = b - a$, where φ is Euler's function.

Obviously, a and b are odd and not divisible by p. They are not divisible by 5 because $p \not\equiv 1 \pmod 5$. Thus, a and b are relatively prime to $10p = a + b$, and therefore relatively prime to each other.

Therefore, using Euler's theorem,

$$a^b = a^{a+\varphi(10p)} = a^a \cdot a^{\varphi(10p)} \equiv a^a \pmod{10p} ,$$

and since $10p = a + b$,

$$a^b + b^a \equiv a^a + b^a \pmod{a + b} .$$

However, since a is odd, $a^a + b^a$ is divisible by $a + b$. Hence, $a^b + b^a$ is divisible by $a + b$.

USAMO 2. (Proposed by Maria Monks Gillespie)

It suffices to show the result for $B = (0, 0, \ldots, 0)$, since then any sequence is equivalent to any other sequence via B. We first show that the result holds for all sequences of the form $A = (a, a, \ldots, a)$ for some a.

For each positive integer i define the ith **lifting map** B_i on the permutations of $m_1, \ldots, m_n$ by $B_i(w_1, \ldots, w_n) = v_1, \ldots, v_n$ where $v_j = i$ if and only if $w_{n+1-j} = i$, and where the subsequence of v consisting of all entries not equal to i (taken in order) is equal to the subsequence of w consisting of all entries not equal to i.

Lemma 1. *Let $A_{i-1} = (i-1, i-1, \ldots, i-1)$ and $A_i = (i, i, \ldots, i)$. Then the number of A_{i-1}-inversions of w equals the number of A_i-inversions of $B_i(w)$. Moreover, B_i is a bijection on the permutations of w, showing the result in this case.*

Proof. It is easy to see that B_i is a bijection for any i, since we can reverse the map.

Now, note that any A_{i-1}-inversions between entries not equal to i in w are still A_i-inversions in $B_i(w)$, and vice-versa. Notice also that there are no A_{i-1}-inversions in w having i as the left entry. Similarly there are no A_i-inversions having i as the *right* entry in $B_i(w)$.

On the other hand, in w, any non-i entry to the *left* of an i forms an A_{i-1}-inversion with that i. And in $B_i(w)$, any non-i entry to the *right* of an i forms an A_i-inversion with that i. Since the positions of the i's are reversed from w to $B_i(w)$, the number of inversions involving an i are equal in each case, and the result follows. $\square$

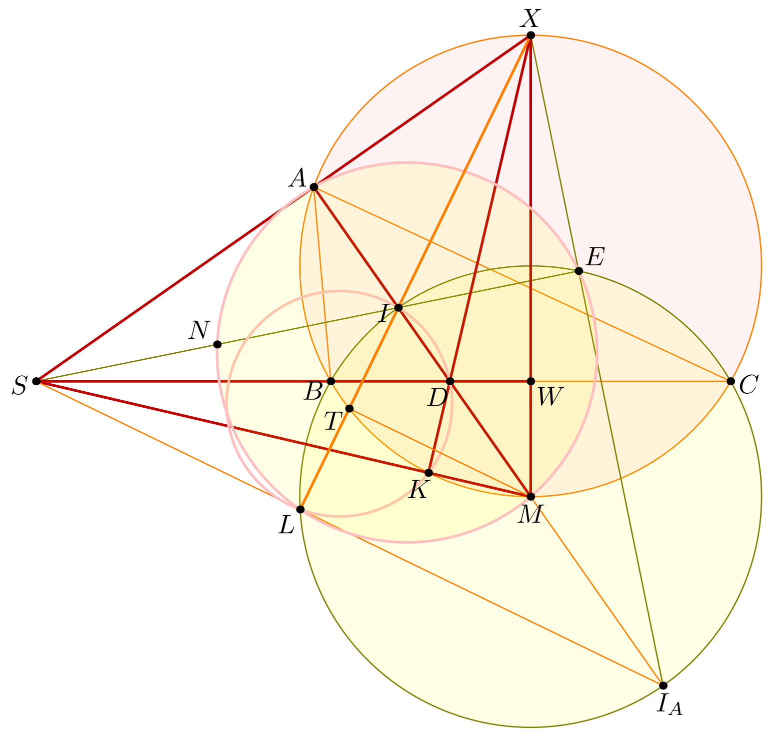
For $j > i$, we denote $B_{i \to j} := B_j \circ B_{j-1} \circ \cdots \circ B_{i+2} \circ B_{i+1}$. Also, for $j > i$, we denote $B_{j \to i} := B_{i \to j}^{-1} = B_{i+1}^{-1} \circ B_{i+2}^{-1} \circ \cdots \circ B_j^{-1}$. And we let $B_{i \to i}$ be the identity permutation.
Additionally, for $A = (a_1, \dots, a_n)$ and for a permutation w of $m_1, \dots, m_n$ we define $\phi_A(w)$ as follows. Let $w^{(1)} = B_{0 \to a_1}(w)$ and, inductively, for $i > 1$ let $w^{(i)}$ be the result of applying $B_{a_{i-1} \to a_i}$ to the last $n - i + 1$ terms of $w^{(i-1)}$ and leaving the first $i - 1$ terms unchanged. Finally let $\phi_A(w) = w^{(n)}$.

Lemma 2. *The number of A-inversions of $\phi_A(w)$ is equal to the number of B-inversions of w where $B = (0, 0, \dots, 0)$.*

Proof. This is a consequence of the definition of ϕ_A: At any step $w^{(i)}$ in the process of computing $\phi_A(w)$, we consider the sequence $A^{(i)}$ formed by changing the last $n - i + 1$ terms of the previous sequence $A^{(i-1)}$ (starting at $A^{(0)} = (0, 0, \dots, 0)$) from a_{i-1} to a_i. Then we have $A^{(n)} = A$, and at each step the number of $A^{(i)}$-inversions of $w^{(i)}$ is equal to the number of $A^{(i-1)}$-inversions of $w^{(i-1)}$ by Lemma 1. (More precisely, the lemma applies to the number of such inversions among the last $n - i + 1$ terms, but note that the number of inversions involving any of the first $i - 1$ terms is also unchanged at each step.) The result follows. $\square$

And since ϕ_A is a bijection, being a composition of bijections, we are done.

USAMO 3. (Proposed by Evan Chen)



Let W be the midpoint of $\overline{BC}$, and let X be the point on Ω opposite M. Observe that line KD passes through X, and thus lines BC, MK, XA concur at the orthocenter of $\triangle DMX$, which is S. Denote by I_A the A-excenter of ABC.

Next, let E be the foot of the altitude from I to $\overline{XI_A}$; observe that E lies on the circle ω centered at M through I, B, C, I_A. Then, S is the radical center of ω, Ω, and the circle with diameter $\overline{IX}$; hence line SI passes through E; accordingly I is the orthocenter of $\triangle XSI_A$; denote by L the foot of the altitude from X to $\overline{I_AS}$.

We claim that this L lies on both the circumcircle of $\triangle KID$ and $\triangle MAN$. It lies on the circumcircle of $\triangle MAN$ since this circle is the nine-point circle of $\triangle XSI_A$. For the other, note that $\triangle MWI \sim \triangle MIX$, since they share the same angle at M and $MW \cdot MX = MB^2 = MI^2$. Consequently, $\angle IWM = \angle MIX = 180^\circ - \angle LIM = 180^\circ - \angle MLI$, enough to imply that quadrilateral $MWIL$ is cyclic. But lines IL, DK, and WM meet at X, so Power of a Point in cyclic quadrilaterals $DKMW$ and $MWIL$ gives $XD \cdot XK = XM \cdot XW = XI \cdot XL$, hence $KDIL$ is cyclic as needed.

All that remains to show is that the midpoint T of $\overline{IL}$ lies on Ω. But this follows from the fact that $\overline{TM} \parallel \overline{I_AL} \implies \angle MTX = 90^\circ$, thus the problem is solved.

Alternate Solution (by Titu Andreescu and Cosmin Pohoata): We refer to the same figure as in the first solution. Let X be the midpoint of arc BAC of Ω. A first key step in the problem is to note that D is the orthocenter of triangle XSM. This follows from the fact that $\overline{DK} \perp \overline{KM}$, which implies that line DK must pass through the antipode of M in Ω, which is precisely the point X. This together with the fact that $\overline{MX} \perp \overline{SW}$ implies the claim.

Next, it is essential to notice that I is also the orthocenter of triangle XSI_A, where I_A denotes the A-excenter of triangle ABC. This can be argued as follows: since D is the orthocenter of $\triangle XSM$, we have by Power of a Point that $AX \cdot AS = AD \cdot AM$ (we are implicitly using the fact that the reflection of D across line XS lies on the circumcircle of triangle XSM). However, the 4-tuple (A, I, D, I_A) is a harmonic division and M is the midpoint of $\overline{II_A}$, which easily implies that $AD \cdot AM = AI \cdot AI_A$. By Power of a Point once again, this yields that the reflection of I across line XS lies on the circumcircle of triangle XSI_A, so I must indeed be the orthocenter of triangle XSI_A. This is crucial, since then the circumcircle of triangle MAN is nothing but the nine-point circle of $\triangle XSI_A$, so the foot of altitude L from X on $\overline{SI_A}$ becomes a good candidate for L_1 or L_2. If T denotes the midpoint of segment $\overline{IL}$, then $\overline{TM}$ is a midline in triangle ILI_A, so $\overline{TM} \perp \overline{TX}$; therefore T is on the circle of diameter $\overline{MX}$, which is precisely Ω. It remains to show that L also lies on the circumcircle of triangle KID, but this is clear: $ASKD$ is cyclic, so $XA \cdot XS = XD \cdot XK$; also, $ASLI$ is cyclic, so $XA \cdot XS = XI \cdot XL$; hence $XD \cdot XK = XI \cdot XL$, which by Power of a Point means that $ILKD$ is cyclic, thus completing the proof.

USAMO 4. (Proposed by Maria Monks Gillespie)

We may assume the points have been labeled as $P_1, P_2, \ldots, P_{2n}$ in order, going counterclockwise from $(1, 0)$. Now, write out the color of each point in order, and replace each R with a $+1$ and each B with a -1, to get a list $p_1, \ldots, p_{2n}$ of $+1$'s and -1's. Consider the partial sums $p_1 + \cdots + p_k$ of this sequence, and choose the index k such that the kth partial sum has as small a value as possible; if several partial sums are tied for smallest, let k be the lowest index among them. Now, rotate the circle clockwise so that points $P_1, \ldots, P_k$ are moved past $(1, 0)$; the resulting sequence of $+1$'s and -1's from the new orientation now has all nonnegative partial sums, and the total sum is 0.

Consider any red point in the rotated diagram and label it R_1. The arc $R_1 \to B_1$ does not cross $(1, 0)$, for otherwise the sequence ends with a string of $+1$'s and the partial sums before those $+1$'s would be negative. Furthermore, the sequence of entries from R_1 to B_1 looks like $+1, +1, +1, \ldots, +1, -1$, and so removing R_1 and B_1 is equivalent to removing a consecutive pair of a $+1$ and -1, so the partial sums remain all nonnegative. It follows that the next pairing also doesn't cross $(1, 0)$, and so on, so no matter which way we pick the ordering of the red points in the rotated circle, there are no counterclockwise arcs $R_i \to B_i$ containing $(1, 0)$.

Finally, note that in any ordering of the red points, the blue points among $P_1, \ldots, P_k$ are all paired with red points, and those red points among $P_1, \ldots, P_k$ are paired with blue points in this same subsequence since there are no crossings in the rotated picture. Let m be the difference between the number of blue and red points among $P_1, \ldots, P_k$. Then it follows that exactly m blue points in $P_1, \ldots, P_k$ were matched with red points from $P_{k+1}, \ldots, P_{2n}$. Therefore, when we rotate the circle back to its original position, there are exactly m crossings, no matter which ordering we pick for the red points. Since m is independent of the ordering, the proof is complete.

USAMO 5. (Proposed by Ricky Liu)

The answer is $c < \sqrt{2}$.

First suppose $c < \sqrt{2}$. We can partition $\mathbf{Z}^2$ into two subsets

$$L_1 = \{(x, y) \mid x + y \text{ is odd}\} \quad \text{and} \quad L'_1 = \{(x, y) \mid x + y \text{ is even}\}.$$

Both L_1 and L'_1 are square lattices with unit length $\sqrt{2}$ (that is, they are similar to $\mathbf{Z}^2$ with a scaling factor of $\sqrt{2}$). Hence we can similarly partition L'_1 into two square lattices L_2 and L'_2 with unit length $\sqrt{2}^2$, then partition L'_2 into two square lattices L_3 and L'_3 with unit length $\sqrt{2}^3$, and so forth. Hence for any $N \geq 1$, $\mathbf{Z}^2$ can be partitioned into $N + 1$ square lattices $L_1, L_2, \ldots, L_N, L'_N$ with unit lengths $\sqrt{2}, \sqrt{2}^2, \ldots, \sqrt{2}^N, \sqrt{2}^N$, respectively.

Since $\frac{\sqrt{2}}{c} > 1$, there exists a positive integer N such that $(\frac{\sqrt{2}}{c})^{N+1} \geq \sqrt{2}$, or equivalently, $c^{N+1} \leq \sqrt{2}^N$. For $i = 1, \ldots, N$, label all points in L_i by i, and then label all points in L'_N by $N + 1$. Any two points in L_i lie at least $\sqrt{2}^i > c^i$ apart, while any two points in L'_N lie at least $\sqrt{2}^N \geq c^{N+1}$ apart, so this is a valid labeling.

Suppose instead that $c \geq \sqrt{2}$. For a nonnegative integer m, define

$$R_m = \{(x, y) \mid 1 \leq x \leq 2^a,\, 1 \leq y \leq 2^b\} \subseteq \mathbf{Z}^2, \text{ where } (a, b) = \begin{cases} (\frac{m}{2}, \frac{m}{2}) & \text{if } m \text{ is even,} \\ (\frac{m-1}{2}, \frac{m+1}{2}) & \text{if } m \text{ is odd.} \end{cases}$$

We will show by induction that R_m does not have a valid labeling using only labels at most m, which will prove that $\mathbf{Z}^2$ has no valid labeling. The case $m = 0$ is trivial.

Suppose $m > 0$ is odd and that R_{m-1} does not have a valid labeling using only $1, \ldots, m - 1$ (the inductive hypothesis), but that R_m does have a valid labeling using only $1, \ldots, m$. Consider this labeling of R_m. Since $R_m \supseteq R_{m-1}$, some point (x_0, y_0) with $y_0 \leq 2^{(m-1)/2}$ must be labeled m. But then (x_0, y_0) lies directly below a translate R' of R_{m-1} inside R_m. The distance between (x_0, y_0) and any point in R' is at most

$$\sqrt{(2^{\frac{m-1}{2}} - 1)^2 + (2^{\frac{m-1}{2}})^2} < \sqrt{2}^m \leq c^m,$$

so no points in R' can be labeled m. But by the inductive hypothesis, R' has no valid labeling using only $1, \ldots, m - 1$, which is a contradiction.

Now suppose $m > 0$ is even and that R_{m-1} does not have a valid labeling using only $1, \ldots, m - 1$ (the inductive hypothesis), but R_m does have a valid labeling using only $1, \ldots, m$. By the inductive hypothesis, some point (x_0, y_0) with $\frac{1}{4} \cdot 2^{m/2} < y_0 \leq \frac{3}{4} \cdot 2^{m/2}$ must be labeled m (since the corresponding rows of R_m form a rotated copy of R_{m-1}). But then (x_0, y_0) lies either directly to the left or to the right of a translate R' of R_{m-1} inside R_m. The distance between (x_0, y_0) and any point of R' is less than

$$\sqrt{(\tfrac{3}{4} \cdot 2^{\frac{m}{2}})^2 + (2^{\frac{m-2}{2}})^2} = \frac{\sqrt{13}}{4} \cdot \sqrt{2}^m < \sqrt{2}^m \leq c^m,$$

so no points in R' can be labeled m. But by the inductive hypothesis, R' has no valid labeling using only $1, \ldots, m - 1$, which is a contradiction. This completes the proof.

USAMO 6. (Proposed by Titu Andreescu)

We will show that the minimum is $\frac{2}{3}$. (In particular, the value $\frac{4}{5}$, obtained by making the natural guess $a = b = c = d = 1$, is not the right answer.)

We have

$$\frac{4a}{b^3+4} = a - \frac{ab^3}{b^3+4} \geq a - \frac{ab}{3},$$

since

$$b^3 + 4 = \frac{b^3}{2} + \frac{b^3}{2} + 4 \geq 3b^2,$$

by the Arithmetic Mean-Geometric Mean Inequality.

Then

$$\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4} \geq \frac{a+b+c+d}{4} - \frac{ab+bc+cd+da}{12}.$$

But $a + b + c + d = 4$ and

$$4(ab + bc + cd + da) = 4(a + c)(b + d) \leq (a + b + c + d)^2 = 16.$$

Hence

$$\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4} \geq 1 - \frac{4}{12} = \frac{2}{3}.$$

The minimum is realized when, for example, $a = b = 2$ and $c = d = 0$.