37th United States of America Mathematical Olympiad

1. Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1, \ldots, k_n$, all strictly greater than 1, such that $k_0 k_1 \cdots k_n - 1$ is the product of two consecutive integers.

First solution: We proceed by induction. The case $n = 1$ is clear, since we may pick $k_0 = 3$ and $k_1 = 7$.

Let us assume now that for a certain n there are pairwise relatively prime integers $1 < k_0 < k_1 < \cdots < k_n$ such that $k_0 k_1 \cdots k_n - 1 = a_n(a_n - 1)$, for some positive integer a_n. Then choosing $k_{n+1} = a_n^2 + a_n + 1$ yields

$$k_0 k_1 \cdots k_{n+1} = (a_n^2 - a_n + 1)(a_n^2 + a_n + 1) = a_n^4 + a_n^2 + 1,$$

so $k_0 k_1 \cdots k_{n+1} - 1$ is the product of the two consecutive integers a_n^2 and $a_n^2 + 1$. Moreover,

$$\gcd(k_0 k_1 \cdots k_n, k_{n+1}) = \gcd(a_n^2 - a_n + 1, a_n^2 + a_n + 1) = 1,$$

hence $k_0, k_1, \ldots, k_{n+1}$ are pairwise relatively prime. This completes the proof. $\square$

Second solution: Write the relation to be proved as

$$4k_0 k_1 \cdots k_n = 4a(a+1) + 4 = (2a+1)^2 + 3.$$

There are infinitely many primes for which -3 is a quadratic residue. Let $2 < p_0 < p_1 < \ldots < p_n$ be such primes. Using the Chinese Remainder Theorem to specify a modulo p_n, we can find an integer a such that $(2a+1)^2 + 3 = 4p_0 p_1 \cdots p_n m$ for some positive integer m. Grouping the factors of m appropriately with the p_i's, we obtain $(2a+1)^2 + 3 = 4k_0 k_1 \cdots k_n$ with k_i pairwise relatively prime. We then have $k_0 k_1 \cdots k_n - 1 = a(a+1)$, as desired. $\square$

Third solution: We are supposed to show that for every positive integer n, there is a positive integer x such that $x(x+1) + 1 = x^2 + x + 1$ has at least n distinct prime divisors. We can actually prove a more general statement.

Claim. *Let $P(x) = a_d x^d + \cdots + a_1 x + 1$ be a polynomial of degree $d \geq 1$ with integer coefficients. Then for every positive integer n, there is a positive integer x such that $P(x)$ has at least n distinct prime divisors.*

The proof follows from the following two lemmas.

Lemma 1. *The set*

$$Q = \{p \,|\, p \text{ a prime for which there is an integer } x \text{ such that } p \text{ divides } P(x)\}$$

is infinite.

Proof. The proof is analogous to Euclid's proof that there are infinitely many primes. Namely, if we assume that there are only finitely many primes $p_1, p_2, \ldots, p_k$ in Q, then for each integer m, $P(mp_1p_2\cdots p_k)$ is an integer with no prime factors, which must equal 1 or -1. However, since P has degree $d \geq 1$, it takes each of the values 1 and -1 at most d times, a contradiction. $\square$

Lemma 2. *Let $p_1, p_2, \ldots, p_n$, $n \geq 1$ be primes in Q. Then there is a positive integer x such that $P(x)$ is divisible by $p_1p_2\cdots p_n$.*

Proof. For $i = 1, 2, \ldots, n$, since $p_i \in Q$ we can find an integer c_i such that $P(x)$ is divisible by p_i whenever $x \equiv c_i (\text{mod } p_i)$. By the Chinese Remainder Theorem, the system of n congruences $x \equiv c_i (\text{mod } p_i)$, $i = 1, 2, \ldots, n$ has positive integer solutions. For every positive integer x that solves this system, $P(x)$ is divisible by $p_1p_2\cdots p_n$. $\square$

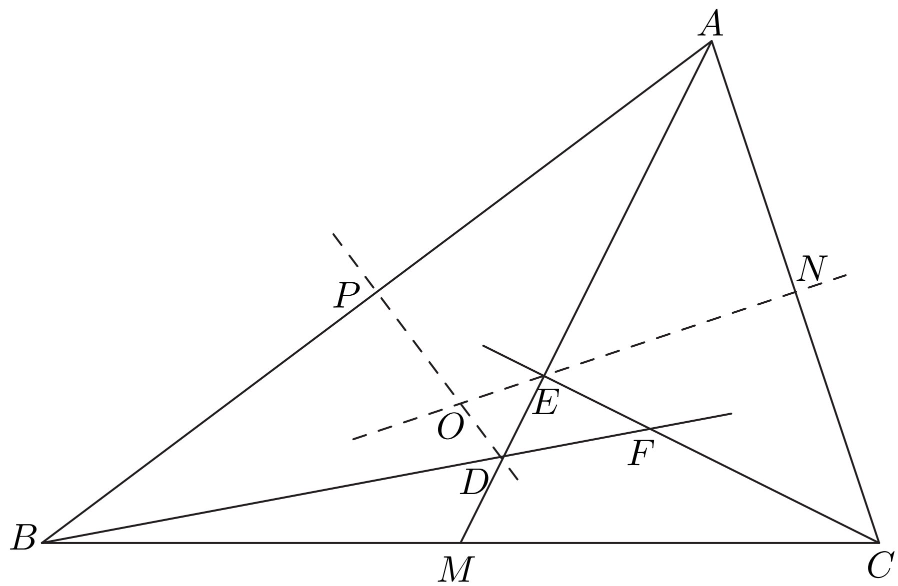
This problem was suggested by Titu Andreescu.

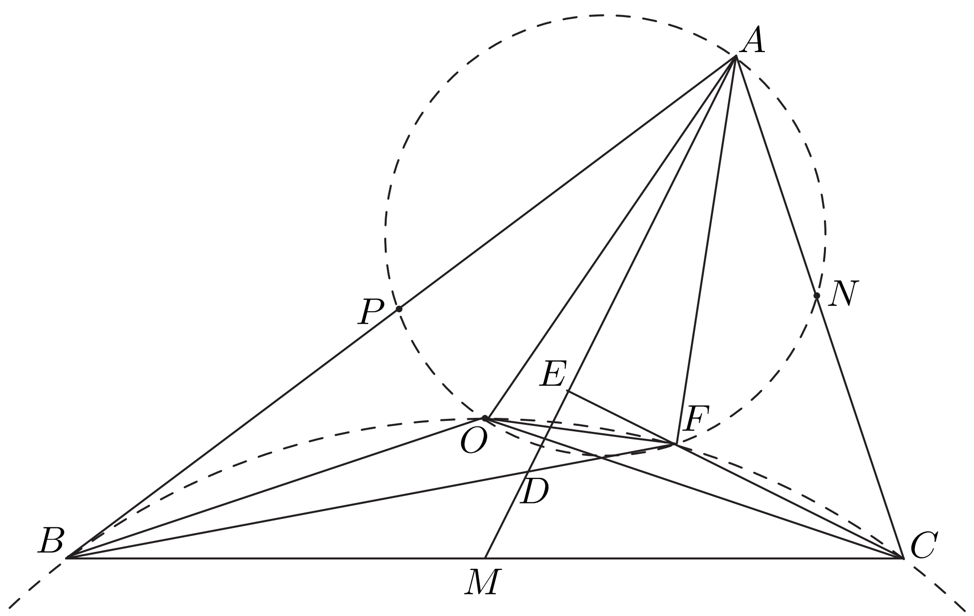
2. Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

First solution: Let O be the circumcenter of triangle ABC. We prove that

$$\angle APO = \angle ANO = \angle AFO = 90^\circ. \tag{1}$$

It will then follow that A, P, O, F, N lie on the circle with diameter $\overline{AO}$. Indeed, the fact that the first two angles in (1) are right is immediate because $\overline{OP}$ and $\overline{ON}$ are the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$, respectively. Thus we need only prove that $\angle AFO = 90^\circ$.





We may assume, without loss of generality, that $AB > AC$. This leads to configurations similar to the ones shown above. The proof can be adapted to other configurations. Because PO is the perpendicular bisector of AB, it follows that triangle ADB is an isosceles triangle with $AD = BD$. Likewise, triangle AEC is isosceles with $AE = CE$. Let $x = \angle ABD = \angle BAD$ and $y = \angle CAE = \angle ACE$, so $x + y = \angle BAC$.

Applying the Law of Sines to triangles ABM and ACM gives

$$\frac{BM}{\sin x} = \frac{AB}{\sin \angle BMA} \qquad \text{and} \qquad \frac{CM}{\sin y} = \frac{AC}{\sin \angle CMA}.$$

Taking the quotient of the two equations and noting that $\sin \angle BMA = \sin \angle CMA$ we find

$$\frac{BM}{CM}\frac{\sin y}{\sin x} = \frac{AB}{AC}\frac{\sin \angle CMA}{\sin \angle BMA} = \frac{AB}{AC}.$$

Because $BM = MC$, we have

$$\frac{\sin x}{\sin y} = \frac{AC}{AB}. \tag{2}$$

Applying the law of sines to triangles ABF and ACF we find

$$\frac{AF}{\sin x} = \frac{AB}{\sin \angle AFB} \qquad \text{and} \qquad \frac{AF}{\sin y} = \frac{AC}{\sin \angle AFC}.$$

Taking the quotient of the two equations yields

$$\frac{\sin x}{\sin y} = \frac{AC}{AB}\frac{\sin \angle AFB}{\sin \angle AFC}, \qquad \text{so by (2),} \qquad \sin \angle AFB = \sin \angle AFC. \tag{3}$$

Because $\angle ADF$ is an exterior angle to triangle ADB, we have $\angle EDF = 2x$. Similarly, $\angle DEF = 2y$. Hence

$$\angle EFD = 180^\circ - 2x - 2y = 180^\circ - 2\angle BAC.$$

Thus $\angle BFC = 2\angle BAC = \angle BOC$, so $BOFC$ is cyclic. In addition,

$$\angle AFB + \angle AFC = 360^\circ - 2\angle BAC > 180^\circ,$$

and hence, from (3), $\angle AFB = \angle AFC = 180^\circ - \angle BAC$. Because $BOFC$ is cyclic and $\triangle BOC$ is isosceles with vertex angle $\angle BOC = 2\angle BAC$, we have $\angle OFB = \angle OCB = 90^\circ - \angle BAC$. Therefore,

$$\angle AFO = \angle AFB - \angle OFB = (180^\circ - \angle BAC) - (90^\circ - \angle BAC) = 90^\circ.$$

This completes the proof. $\square$

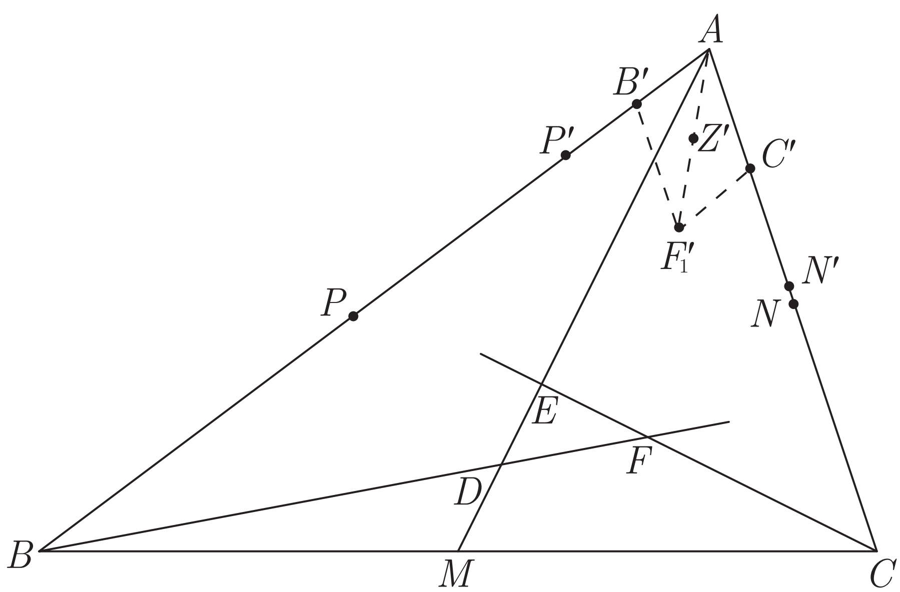
Second solution: Invert the figure about a circle centered at A, and let X' denote the image of the point X under this inversion. Find point F_1' so that $AB'F_1'C'$ is a parallelogram and let Z' denote the center of this parallelogram. Note that $\triangle BAC \sim$

$\triangle C'AB'$ and $\triangle BAD \sim \triangle D'AB'$. Because M is the midpoint of BC and Z' is the midpoint of $B'C'$, we also have $\triangle BAM \sim \triangle C'AZ'$. Thus

$$\angle AF_1'B' = \angle F_1'AC' = \angle Z'AC' = \angle MAB = \angle DAB = \angle DBA = \angle AD'B'.$$

Hence quadrilateral $AB'D'F_1'$ is cyclic and, by a similar argument, quadrilateral $AC'E'F_1'$ is also cyclic. Because the images under the inversion of lines BDF and CFE are circles that intersect in A and F', it follows that $F_1' = F'$.

Next note that B', Z', and C' are collinear and are the images of P', F', and N', respectively, under a homothety centered at A and with ratio $1/2$. It follows that P', F' and N' are collinear, and then that the points A, P, F and N lie on a circle. $\square$



This problem was suggested by Zuming Feng. The second solution was contributed by Gabriel Carroll.

3. Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that
$$|x| + \left|y + \frac{1}{2}\right| < n.$$
A *path* is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \ldots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1 (in other words, the points (x_i, y_i) and (x_{i-1}, y_{i-1}) are neighbors in the lattice of points with integer coordinates).

Prove that the points in S_n cannot be partitioned into fewer than n paths (a partition of S_n into m paths is a set $\mathcal{P}$ of m nonempty paths such that each point in S_n appears in exactly one of the m paths in $\mathcal{P}$).

Solution: Color the points in S_n as follows (see Figure 1):

- if $y \geq 0$, color (x, y) white if $x + y - n$ is even and black if $x + y - n$ is odd;
- if $y < 0$, color (x, y) white if $x + y - n$ is odd and black if $x + y - n$ is even.

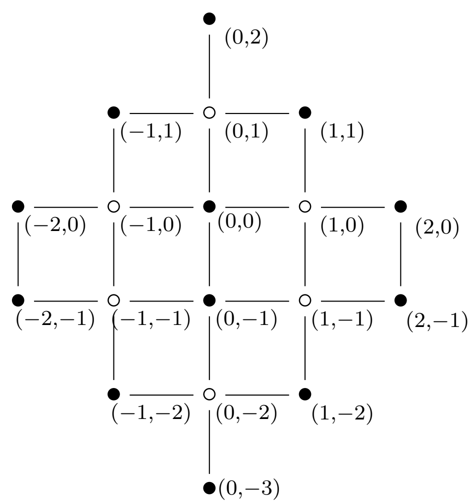


Figure 1: Coloring of S_3

Consider a path $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in S_n. A pair of successive points (x_{i-1}, y_{i-1}) and (x_i, y_i) in the path is called a pair of successive black points if both points in the pair are colored black.

Suppose now that the points of S_n are partitioned into m paths and the total number of successive pairs of black points in all paths is k. By breaking the paths at each pair of successive black points, we obtain $k+m$ paths in each of which the number of black points exceeds the number of white points by at most one. Therefore, the total number of black points in S_n cannot exceed the number of white points by more than $k+m$. On the other hand, the total number of black points in S_n exceeds the total number of white points by exactly $2n$ (there is exactly one more black point in each row of S_n). Therefore,

$$2n \leq k + m.$$

There are exactly n adjacent black points in S_n (call two points in S_n *adjacent* if their distance is 1), namely the pairs

$$(x, 0) \text{ and } (x, -1),$$

for $x = -n+1, -n+3, \ldots, n-3, n-1$. Therefore $k \le n$ (the number of successive pairs of black points in the paths in the partition of S_n cannot exceed the total number of adjacent pairs of black points in S_n) and we have $2n \le k + m \le n + m$, yielding

$$n \le m.$$

□

This problem was suggested by Gabriel Carroll.

4. Let $\mathcal{P}$ be a convex polygon with n sides, $n \ge 3$. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a *triangulation* of $\mathcal{P}$ into $n-2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of n.

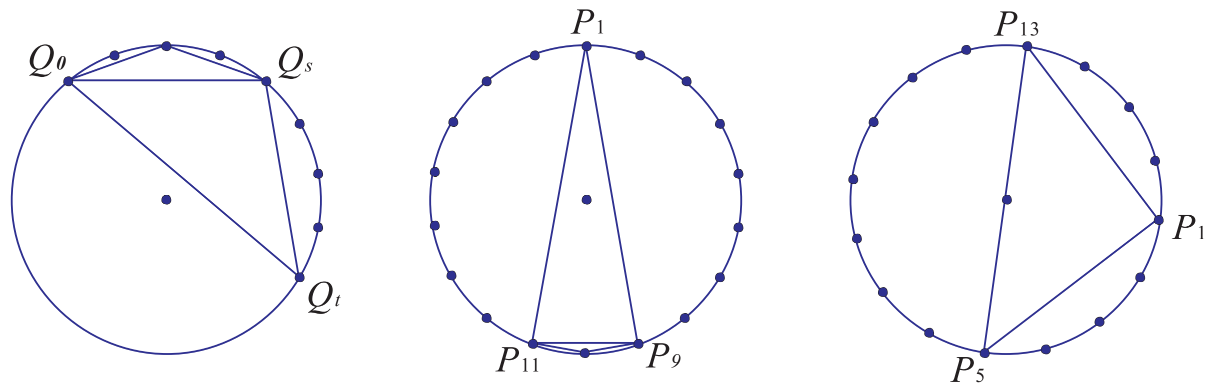
Solution: The answer is $n = 2^{m+1} + 2^k$, where m and k are nonnegative integers. In other words, n is either a power of 2 (when $m + 1 = k$) or the sum of two nonequal powers of 2 (with $1 = 2^0$ being considered as a power of 2).

We start with the following observation.

Lemma. *Let $\mathcal{Q} = Q_0Q_1 \ldots Q_t$ be a convex polygon with $Q_0Q_1 = Q_1Q_2 = \cdots = Q_{t-1}Q_t$. Suppose that $\mathcal{Q}$ is cyclic and its circumcenter does not lie in its interior. If there is a triangulation of $\mathcal{Q}$ consisting only of isosceles triangles, then $t = 2^a$, where a is a positive integer.*

Proof. We call an arc *minor* if its arc measure is less than or equal to 180°. By the given conditions, points $Q_1, \ldots, Q_{t-1}$ lie on the minor arc $\widehat{Q_0Q_t}$ of the circumcircle, so none of the angles $Q_iQ_jQ_k$ $(0 \le i < j < k \le t)$ is acute. (See the left-hand side diagram shown below.) It is not difficult to see that Q_0Q_t is longer than each other side or diagonal of $\mathcal{Q}$. Thus Q_0Q_t must be the base of an isosceles triangle in the triangulation of $\mathcal{Q}$. Therefore, t must be even. We write $t = 2s$. Then $Q_0Q_sQ_t$ is an isosceles triangle in the triangulation. We can apply the same process to polygon $Q_0Q_1 \ldots Q_s$ and show that s is even. Repeating this process leads to the conclusion that $t = 2^a$ for some positive integer a.

The results of the lemma can be generalized by allowing $a = 0$ if we consider the degenerate case $\mathcal{Q} = Q_0Q_1$. $\square$



We are ready to prove our main result. Let $\mathcal{P} = P_1P_2 \dots P_n$ denote the regular polygon. There is an isosceles triangle in the triangulation such that the center of $\mathcal{P}$ lies within the boundary of the triangle. Without loss of generality, we may assume that $P_1P_iP_j$, with $P_1P_i = P_1P_j$ (that is, $P_j = P_{n-i+2}$), is this triangle. Applying the Lemma to the polygons $P_1 \dots P_i$, $P_i \dots P_j$, and $P_j \dots P_1$, we conclude that there are $2^m - 1$, $2^k - 1$, $2^m - 1$ (where m and k are nonnegative integers) vertices in the interiors of the minor arcs $\widehat{P_1P_i}$, $\widehat{P_iP_j}$, $\widehat{P_jP_1}$, respectively. (In other words, $i = 2^m + 1$, $j = 2^k + i$.) Hence

$$n = 2^m - 1 + 2^k - 1 + 2^m - 1 + 3 = 2^{m+1} + 2^k,$$

where m and k are nonnegative integers. The above discussion can easily lead to a triangulation consisting of only isosceles triangles for $n = 2^{m+1} + 2^k$. (The middle diagram shown above illustrates the case $n = 18 = 2^{3+1} + 2^1$. The right-hand side diagram shown above illustrates the case $n = 16 = 2^{2+1} + 2^3$.) $\square$

This problem was suggested by Gregory Galperin.

5. Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1r_1 + a_2r_2 + a_3r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \leq y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

Solution: If two of the a_i vanish, say a_2 and a_3, then r_1 must be zero and we are done. Assume at most one a_i vanishes. If any one a_i vanishes, say a_3, then $r_2/r_1 = -a_1/a_2$

is a nonnegative rational number. Write this number in lowest terms as p/q, and put $r = r_2/p = r_1/q$. We can then write $r_1 = qr$ and $r_2 = pr$. Performing the Euclidean algorithm on r_1 and r_2 will ultimately leave r and 0 on the blackboard. Thus we are done again.

Thus it suffices to consider the case where none of the a_i vanishes. We may also assume none of the r_i vanishes, as otherwise there is nothing to check. In this case we will show that we can perform an operation to obtain r_1', r_2', r_3' for which either one of r_1', r_2', r_3' vanishes, or there exist integers a_1', a_2', a_3', not all zero, with $a_1'r_1' + a_2'r_2' + a_3'r_3' = 0$ and

$$|a_1'| + |a_2'| + |a_3'| < |a_1| + |a_2| + |a_3|.$$

After finitely many steps we must arrive at a case where one of the a_i vanishes, in which case we finish as above.

If two of the r_i are equal, then we are immediately done by choosing them as x and y. Hence we may suppose $0 < r_1, r_2 < r_3$. Since we are free to negate all the a_i, we may assume $a_3 > 0$. Then either $a_1 < -\frac{1}{2}a_3$ or $a_2 < -\frac{1}{2}a_3$ (otherwise $a_1r_1 + a_2r_2 + a_3r_3 > (a_1 + \frac{1}{2}a_3)r_1 + (a_2 + \frac{1}{2}a_3)r_2 > 0$). Without loss of generality, we may assume $a_1 < -\frac{1}{2}a_3$. Then choosing $x = r_1$ and $y = r_3$ gives the triple $(r_1', r_2', r_3') = (r_1, r_2, r_3 - r_1)$ and $(a_1', a_2', a_3') = (a_1 + a_3, a_2, a_3)$. Since $a_1 < a_1 + a_3 < \frac{1}{2}a_3 < -a_1$, we have $|a_1'| = |a_1 + a_3| < |a_1|$ and hence this operation has the desired effect. $\square$

This problem was suggested by Kiran Kedlaya.

6. At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).

Solution: Let n be the number of participants at the conference. We proceed by induction on n.

If $n = 1$, then we have one participant who can eat in either room; that gives us total of $2 = 2^1$ options.

Let $n \geq 2$. The case in which some participant, P, has no friends is trivial. In this case, P can eat in either of the two rooms, so the total number of ways to split n participants is twice as many as the number of ways to split $(n-1)$ participants besides the participant P. By induction, the latter number is a power of two, 2^k, hence the number of ways to split n participants is $2 \times 2^k = 2^{k+1}$, also a power of two. So we assume from here on that every participant has at least one friend.

We consider two different cases separately: the case when some participant has an odd number of friends, and the case when each participant has an even number of friends.

Case 1: *Some participant, Z, has an odd number of friends.*

Remove Z from consideration and for each pair $(X,\ Y)$ of Z's friends, reverse the relationship between X and Y (from friends to strangers or vice versa).

Claim. *The number of possible seatings is unchanged after removing Z and reversing the relationship between X and Y in each pair $(X,\ Y)$ of Z's friends.*

Proof of the claim. Suppose we have an arrangement prior to Z's departure. By assumption, Z has an even number of friends in the room with him.

If this number is 0, the room composition is clearly still valid after Z leaves the room.

If this number is positive, let X be one of Z's friends in the room with him. By assumption, person X also has an even number of friends in the same room. Remove Z from the room; then X will have an odd number of friends left in the room, and there will be an odd number of Z's friends in this room besides X. Reversing the relationship between X and each of Z's friends in this room will therefore restore the parity to even.

The same reasoning applies to any of Z's friends in the other dining room. Indeed, there will be an odd number of them in that room, hence each of them will reverse relationships with an even number of individuals in that room, preserving the parity of the number of friends present.

Moreover, a legitimate seating without Z arises from exactly one arrangement including Z, because in the case under consideration, only one room contains an even number of Z's friends. $\square$

Thus, we have to double the number of seatings for $(n-1)$ participants which is, by the induction hypothesis, a power of 2. Consequently, for n participants we will get again a power of 2 for the number of different arrangements.

Case 2: *Each participant has an even number of friends.*

In this case, each valid split of participants in two rooms gives us an even number of friends in either room.

Let $(A,\ B)$ be any pair of friends. Remove this pair from consideration and for each pair $(C,\ D)$, where C is a friend of A and D is a friend of B, change the relationship between C and D to the opposite; do the same if C is a friend of B and D is a friend of A. Note that if C and D are friends of both A and B, their relationship will be reversed twice, leaving it unchanged.

Consider now an arbitrary participant X different from A and B and choose one of the two dining rooms. [Note that in the case under consideration, the total number of participants is at least 3, so such a triplet $(A,\ B;\ X)$ can be chosen.] Let A have m friends in this room and let B have n friends in this room; both m and n are even. When the pair $(A,\ B)$ is removed, X's relationship will be reversed with either n, or m, or $m+n-2k$ (for k the number of mutual friends of A and B in the chosen room), or 0 people within the chosen room (depending on whether he/she is a friend of only A, only B, both, or neither). Since m and n are both even, the parity of the number of X's friends in that room will be therefore unchanged in any case.

Again, a legitimate seating without A and B will arise from exactly one arrangement that includes the pair $(A,\ B)$: just add each of A and B to the room with an odd number of the other's friends, and then reverse all of the relationships between a friend of A and a friend of B. In this way we create a one-to-one correspondence between all possible seatings before and after the $(A,\ B)$ removal.

Since the number of arrangements for n participants is twice as many as that for $(n-2)$ participants, and that number for $(n-2)$ participants is, by the induction hypothesis, a power of 2, we get in turn a power of 2 for the number of arrangements for n participants. The problem is completely solved. $\square$

This problem was suggested by Sam Vandervelde.