32^{nd} United States of America Mathematical Olympiad

Proposed Solutions

May 1, 2003

Remark: The general philosophy of this marking scheme follows that of IMO 2002. This scheme encourages *complete solutions*. Partial credits will be given under more strict circumstances. Each solution by students shall be graded from one of the two approaches: (1) from 7 going down (a complete solution with possible minor errors); (2) from 0 going up (a solution missing at least one critical idea.) Most partial credits are not additive. Because there are many results need to be proved progressively in problem 3, most partial credits in this problem are accumulative. Many problems have different approaches. Graders are encouraged to choose the approach that most favorable to students. But the partial credits from different approaches are not additive.

1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

   Solution: We proceed by induction. The property is clearly true for $n = 1$. Assume that $N = a_1a_2\dots a_n$ is divisible by 5^n and has only odd digits. Consider the numbers

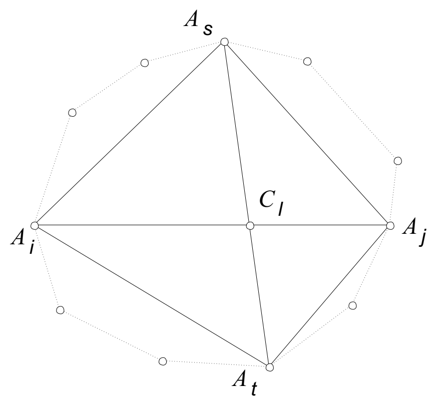
$$\begin{aligned}
N_1 &= 1a_1a_2\dots a_n = 1\cdot 10^n + 5^n M = 5^n(1\cdot 2^n + M),\\
N_2 &= 3a_1a_2\dots a_n = 3\cdot 10^n + 5^n M = 5^n(3\cdot 2^n + M),\\
N_3 &= 5a_1a_2\dots a_n = 5\cdot 10^n + 5^n M = 5^n(5\cdot 2^n + M),\\
N_4 &= 7a_1a_2\dots a_n = 7\cdot 10^n + 5^n M = 5^n(7\cdot 2^n + M),\\
N_5 &= 9a_1a_2\dots a_n = 9\cdot 10^n + 5^n M = 5^n(9\cdot 2^n + M).
\end{aligned}$$

   The numbers $1\cdot 2^n + M, 3\cdot 2^n + M, 5\cdot 2^n + M, 7\cdot 2^n + M, 9\cdot 2^n + M$ give distinct remainders when divided by 5. Otherwise the difference of some two of them would be a multiple of 5, which is impossible, because 2^n is not a multiple of 5, nor is the difference of any two of the numbers $1, 3, 5, 7, 9$. It follows that one of the numbers N_1, N_2, N_3, N_4, N_5 is divisible by $5^n \cdot 5$, and the induction is complete.

2. A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Solution: Let $\mathcal{P} = A_1A_2\dots A_n$, where n is an integer with $n \geq 3$. The problem is trivial for $n = 3$ because there are no diagonals and thus no dissections. We assume that $n \geq 4$. Our proof is based on the following Lemma.

Lemma 1. *Let $ABCD$ be a convex quadrilateral such that all its sides and diagonals have rational lengths. If segments AC and BD meet at P, then segments AP, BP, CP, DP all have rational lengths.*



It is clear by Lemma 1 that the desired result holds when $\mathcal{P}$ is a convex quadrilateral. Let A_iA_j $(1 \leq i < j \leq n)$ be a diagonal of $\mathcal{P}$. Assume that C_1, C_2, $\dots$, C_m are the consecutive division points on diagonal A_iA_j (where point C_1 is the closest to vertex A_i and C_m is the closest to A_j). Then the segments $C_\ell C_{\ell+1}$, $1 \leq \ell \leq m-1$, are the sides of all polygons in the dissection. Let C_ℓ be the point where diagonal A_iA_j meets diagonal A_sA_t. Then quadrilateral $A_iA_sA_jA_t$ satisfies the conditions of Lemma 1. Consequently, segments A_iC_ℓ and $C_\ell A_j$ have rational lengths. Therefore, segments $A_iC_1, A_iC_2, \dots, A_jC_m$ all have rational lengths. Thus, $C_\ell C_{\ell+1} = AC_{\ell+1} - AC_\ell$ is rational. Because i, j, ℓ are arbitrarily chosen, we proved that all sides of all polygons in the dissection are also rational numbers.

Now we present four proofs of Lemma 1 to finish our proof.

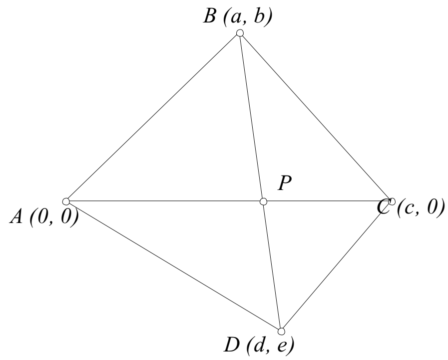
- *First approach* We show only that segment AP is rational, the others being similar. Introduce Cartesian coordinates with $A = (0,0)$ and $C = (c, 0)$. Put $B = (a, b)$ and $D = (d, e)$. Then by hypothesis, the numbers

$$\begin{aligned} &AB = \sqrt{a^2+b^2}, \quad AC = c, \quad AD = \sqrt{d^2+e^2}, \\ &BC = \sqrt{(a-c)^2+b^2}, \quad BD = \sqrt{(a-d)^2+(b-e)^2}, \quad CD = \sqrt{(d-c)^2+e^2}, \end{aligned}$$

are rational. In particular,

$$BC^2 - AB^2 - AC^2 = (a-c)^2 + b^2 - (a^2+b^2) - c^2 = 2ac$$

is rational. Because $c \neq 0$, a is rational. Likewise d is rational.
Now we have that $b^2 = AB^2 - a^2$, $e^2 = AD^2 - d^2$, $(b-e)^2 = BD^2 - (a-d)^2$ are rational, and so that $2be = b^2 + e^2 - (b-e)^2$ is rational. Because quadrilateral $ABCD$ is convex, b and e are nonzero and have opposite sign. Hence $\frac{b}{e} = \frac{2be}{2b^2}$ is rational.



We now calculate

$$P = \left(\frac{bd - ae}{b - e}, 0\right),$$

so

$$AP = \frac{\frac{b}{e} \cdot d - a}{\frac{b}{e} - 1}$$

is rational. ■

- *Second approach*

Note that, for an angle α, if $\cos\alpha$ is rational, then $\sin\alpha = r_\alpha\sqrt{m_\alpha}$ for some rational r and square-free positive integer m (and this expression is unique when r is written in the lowest term). We say two angles α and β with rational cosine are *equivalent* if $m_\alpha = m_\beta$, that is, if $\sin\alpha/\sin\beta$ is rational. We establish the following lemma.

Lemma 2. *Let α and β be two angles.*

(a) If α, β and $\alpha+\beta$ all have rational cosines, then all three are equivalent.

(b) If α and β have rational cosine values and are equivalent, then $\alpha+\beta$ has rational cosine value (and is equivalent to the other two).

(c) If α, β and γ are the angles of a triangle with rational sides, then all three have rational cosine values and are equivalent.

Proof: Assume that $\cos\alpha = s$ and $\cos\beta = t$.

(a) Assume that s and t are rational. By the **Addition formula**, we have

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad (*)$$

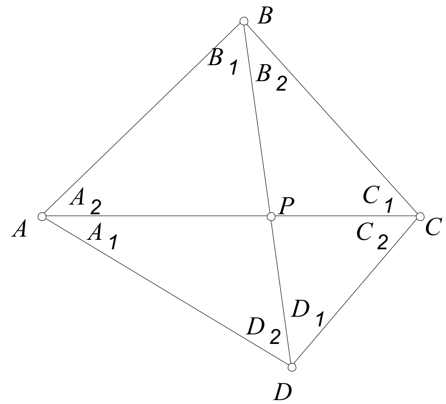
or, $\sin\alpha\sin\beta = st - \cos(\alpha+\beta)$, which is rational by the given conditions. Hence α and β are equivalent. Thus $\sin\alpha = r_a\sqrt{m}$ and $\sin\beta = r_b\sqrt{m}$ for some rational numbers r_a and r_b and some positive square free integer m. By the Addition formula, we have

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = (tr_a + sr_b)\sqrt{m},$$

implying that $\alpha+\beta$ is equivalent to both α and β.

(b) By $(*)$, $\cos(\alpha+\beta)$ is rational if s, t are rational and α and β are equivalent. Then by (a), α, β, $\alpha+\beta$ are equivalent.

(c) Applying the **Law of Cosine** to triangle ABC shows that $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are all rational. Note that $\cos\gamma = \cos(180^\circ - \alpha - \beta) = -\cos(\alpha+\beta)$. The desired conclusions follow from (a). ■



We say a triangle *rational* if all its sides are rational. By Lemma 2 (c), all the angles in a rational triangle have rational cosine values and are equivalent to each other. To prove Lemma 1, we set $\angle DAC = A_1$, $\angle CAB = A_2$, $\angle ABD = B_1$, $\angle DBC = B_2$, $\angle BCA = C_1$, $\angle ACD = C_2$, $\angle CDB = D_1$, $\angle BDA = D_2$. Because triangles ABC, ABD, ADC are rational, angles $A_2, A_1 + A_2, A_1$ all have rational cosine values. By Lemma 2 (a), A_1 and A_2 are equivalent. Similarly, we can show that B_1 and B_2, C_1 and C_2, D_1 and D_2 are equivalent. Because triangle ABC is rational, angles A_2 and C_1 are equivalent. There all angles $A_1, A_2, B_1, \ldots, D_2$ have rational cosine values and are equivalent.

Because angles A_2 and B_1 are equivalent, angle $A_2 + B_1$ has rational values and is equivalent to A_2 and B_1. Thus, $\angle APB = 180^\circ - (A_2 + B_1)$ has rational cosine value and is equivalent to A_2 and B_1. Apply the **Law of Sine** to triangle ABP gives

$$\frac{AB}{\sin \angle APB} = \frac{AP}{\sin \angle B_1} = \frac{BP}{\sin \angle A_2},$$

implying that both AP and BP have rational length. Similarly, we can show that both CP and DP has rational length, proving Lemma 1.

- *Third approach* This approach applies the techniques used in the first approach into the second approach. To prove Lemma 1, we set $\angle DAP = A_1$ and $\angle BAP = A_2$. Applying the Law of Cosine to triangle ABC, ABC, ADC shows that angles $A_1, A_2, A_1 + A_2$ all has rational cosine values. By the Addition formula, we have

$$\sin A_1 \sin A_2 = \cos A_1 \cos A_2 - \cos(A_1 + A_2),$$

implying that $\sin A_1 \sin A_2$ is rational.

Thus,

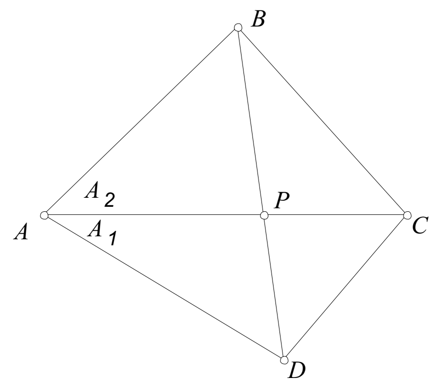
$$\frac{\sin A_2}{\sin A_1} = \frac{\sin A_2 \sin A_1}{\sin^2 A_1} = \frac{\sin A_2 \sin A_1}{1 - \cos^2 A_1}$$

is rational.

Note that the ratio between areas of triangle ADP and ABP is equal to $\frac{PD}{BP}$. Therefore,

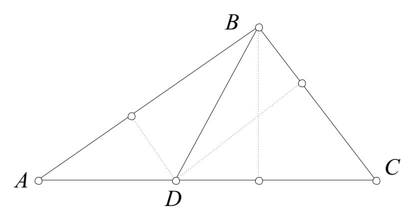
$$\frac{BP}{PD} = \frac{[ABP]}{[ADP]} = \frac{\frac{1}{2}AB \cdot AP \cdot \sin A_2}{\frac{1}{2}AD \cdot AP \cdot \sin A_1} = \frac{AB}{AD} \cdot \frac{\sin A_2}{\sin A_1},$$

implying that $\frac{PD}{BP}$ is rational. Because $BP + PD = BD$ is rational, both BP and PD are rational. Similarly, AP and PC are rational, proving Lemma 1.



- *Fourth approach* This approach is based on the following lemma.

  Lemma 3. *Let ABC be a triangle, D be a point on side AC, $\phi_1 = \angle DAB$, $\phi_2 = \angle DBA$, $\phi_3 = \angle DBC$, $\phi_4 = \angle DCB$, $AB = c$, $BC = a$, $AD = x$, and $DC = y$. If the numbers a, c, and $\cos\phi_i$ $(1 \le i \le 4)$ are all rational, then numbers x and y are also rational.*



Proof: Note that $x + y = AC = c\cos\phi_1 + a\cos\phi_4$ is rational. Hence x is rational if and only if y is rational. Let $BD = z$. Projecting point D onto the lines AB and BC yields

$$\begin{cases} x\cos\phi_1 + z\cos\phi_2 = c, \\ y\cos\phi_4 + z\cos\phi_3 = a, \end{cases}$$

or, denoting $c_i = \cos\phi_i$ for $i = 1, 2, 3, 4$,

$$\begin{cases} c_1 x + c_2 z = c, \\ c_4 y + c_3 z = a. \end{cases}$$

Eliminating z, we get $(c_1c_3)x - (c_2c_4)y = c_3c - c_2a$, which is rational. Hence there exist rational numbers, r_1 and r_2, such that

$$\begin{cases} (c_1c_3)x - (c_2c_4)y = r_1, \\ x + y = r_2. \end{cases}$$

We consider two cases.

- In this case, we assume that the determinant of the above system, $c_1c_3 + c_2c_4$, is not equal to 0, then this system has a unique solution (x, y) in rational numbers.
- In this case, we assume that the determinant $c_1c_3 + c_2c_4 = 0$, or

  $$\cos\phi_1\cos\phi_3 = -\cos\phi_2\cos\phi_4.$$

  Let's denote $\theta = \angle BDC$, then $\phi_2 = \theta - \phi_1$ and $\phi_3 = 180^\circ - (\theta + \phi_4)$. Then the above equation becomes

  $$\cos\phi_1\cos(\theta + \phi_4) = \cos\phi_4\cos(\theta - \phi_1).$$

by the **Product-to-sum formulas**, we have

$$\cos(\theta + \phi_1 + \phi_4) + \cos(\theta + \phi_4 - \phi_1) = \cos(\theta + \phi_4 - \phi_1) + \cos(\theta - \phi_1 - \phi_4),$$

or

$$\cos(\theta + \phi_1 + \phi_4) = \cos(\theta - \phi_1 - \phi_4).$$

It is possible only if $[\theta + \phi_1 + \phi_4] \pm [\theta - \phi_1 - \phi_4] = 360^\circ$, that is, either $\theta = 180^\circ$ or $\phi_1 + \phi_4 = 180^\circ$, which is impossible because they are angles of triangles.

Thus, the determinant $c_1c_3 + c_2c_4$ is not equal to 0 and x and y are both rational numbers. ■

Now we are ready to prove Lemma 1. Applying the Law of Cosine to triangles ABC, ACD, ABD shows that $\cos\angle BAC$, $\cos\angle CAD$, $\cos\angle ABD$, $\cos\angle ADB$ are all rational. Applying Lemma 1 to triangle ABD shows that both of the segments BP and DP have rational lengths. In exactly the same way, we can show that both of the segments AP and CP have rational lengths.

Note: It's interesting how easy it is to get a gap in the proof of the Lemma 1 by using the core idea of the proof of Lemma 3. Here is an example.

Let us project the intersection point of the diagonals, O, onto the four lines of all sides of the quadrilateral. We get the following 4×4 system of linear equations:

$$\begin{cases} \cos\phi_1\ x + \cos\phi_2 y = a, \\ \cos\phi_3 y + \cos\phi_4 z = b, \\ \cos\phi_5 z + \cos\phi_6 t = c, \\ \cos\phi_7 t + \cos\phi_8 x = d. \end{cases}$$

Using the **Kramer's Rule**, we conclude that all x, y, z, and t must be rational numbers, for all the corresponding determinants are rational. However, this logic works only if the determinant of the system is different from 0.

Unfortunately, there are many geometric configurations for which the determinant of the system vanishes (for example, this occurs for rectangles), and you cannot make a conclusion of rationality of the segments x, y, z, and t. That's why Lemma 2 plays the central role in the solution to this problem.

3. Let $n \neq 0$. For every sequence of integers

$$A = a_0, a_1, a_2, \ldots, a_n$$

satisfying $0 \leq a_i \leq i$, for $i = 0, \ldots, n$, define another sequence

$$t(A) = t(a_0), t(a_1), t(a_2), \ldots, t(a_n)$$

by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

Solution: Note first that the transformed sequence $t(A)$ also satisfies the inequalities $0 \leq t(a_i) \leq i$, for $i = 0, \ldots, n$. Call any integer sequence that satisfies these inequalities an *index bounded sequence.*

We prove now that that $a_i \leq t(a_i)$, for $i = 0, \ldots, n$. Indeed, this is clear if $a_i = 0$. Otherwise, let $x = a_i > 0$ and $y = t(a_i)$. None of the first x consecutive terms $a_0, a_1, \ldots, a_{x-1}$ is greater than $x-1$ so they are all different from x and precede x (see the diagram below). Thus $y \geq x$, that is, $a_i \leq t(a_i)$, for $i = 0, \ldots, n$.

index	0	1	...	$x-1$	...	i
A	a_0	a_1	...	a_{x-1}	...	x
$t(A)$	$t(a_0)$	$t(a_1)$	...	$t(a_{x-1})$	...	y

This already shows that the sequences stabilize after finitely many applications of the transformation t, because the value of the index i term in index bounded sequences cannot exceed i. Next we prove that if $a_i = t(a_i)$, for some $i = 0, \ldots, n$, then no further applications of t will ever change the index i term. We consider two cases.

- In this case, we assume that $a_i = t(a_i) = 0$. This means that no term on the left of a_i is different from 0, that is, they are all 0. Therefore the first i terms in $t(A)$ will also be 0 and this repeats (see the diagram below).

index	0	1	...	i
A	0	0	...	0
$t(A)$	0	0	...	0

- In this case, we assume that $a_i = t(a_i) = x > 0$. The first x terms are all different from x. Because $t(a_i) = x$, the terms $a_x, a_{x+1}, \ldots, a_{i-1}$ must then all be equal to x. Consequently, $t(a_j) = x$ for $j = x, \ldots, i-1$ and further applications of t cannot change the index i term (see the diagram below).

index	0	1	...	$x-1$	x	$x+1$	...	i
A	a_0	a_1	...	a_{x-1}	x	x	...	x
$t(A)$	$t(a_0)$	$t(a_1)$	...	$t(a_{x-1})$	x	x	...	x

For $0 \leq i \leq n$, the index i entry satisfies the following properties: (i) it takes integer values; (ii) it is bounded above by i; (iii) its value does not decrease under transformation t; and (iv) once it stabilizes under transformation t, it never changes again. This shows that no more than n applications of t lead to a sequence that is stable under the transformation t.

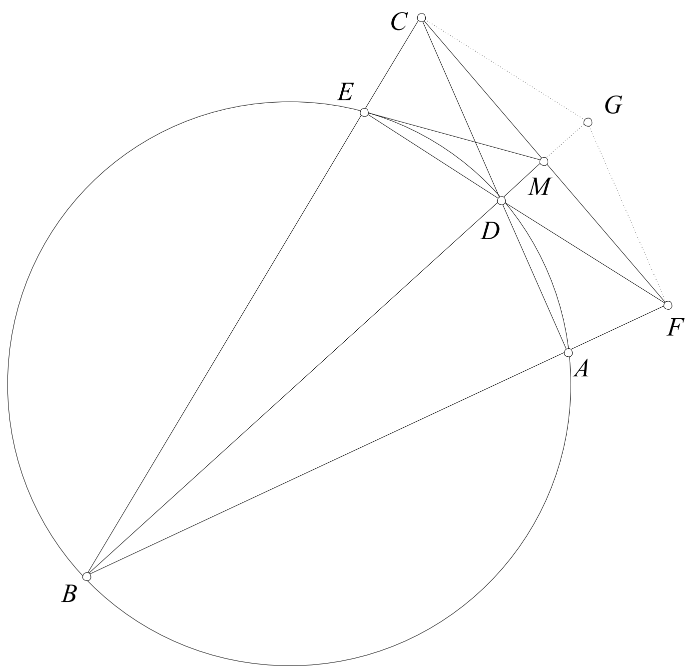
Finally, we need to show that no more than $n-1$ applications of t is needed to obtain a fixed sequence from an initial $n+1$-term index bounded sequence $A = (a_0, a_1, \ldots, a_n)$. We induct on n.

For $n = 1$, the two possible index bounded sequences $(a_0, a_1) = (0, 0)$ and $(a_0, a_1) = (0, 1)$ are already fixed by t so we need zero applications of t.

Assume that any index bounded sequences $(a_0, a_1, \ldots, a_n)$ reach a fixed sequence after no more than $n-1$ applications of t. Consider an index bounded sequence $A = (a_0, a_1, \ldots, a_{n+1})$. It suffices to show that A will be stabilized in no more than n applications of t. We approach indirectly by assume on the contrary that $n+1$ applications of transformations are needed. This can happen only if $a_{n+1} = 0$ and each application of t increased the index $n+1$ term by exactly 1. Under transformation t, the resulting value of index term i will not the effected by index term j for $i < j$. Hence by the induction hypothesis, the subsequence $A' = (a_0, a_1, \ldots, a_n)$ will be stabilized in no more than $n-1$ applications of t. Because index n term is stabilized at value $x \leq n$ after no more than $\min\{x, n-1\}$ applications of t and index $n+1$ term obtains value x after x exactly applications of t under our current assumptions. We conclude that the index $n+1$ term would become equal to the index n term after no more than $n-1$ applications of t. However, once two consecutive terms in a sequence are equal they stay equal and stabilize together. Because the index n term needs no more than $n-1$ transformations to be stabilized, A can be stabilized in no more than $n-1$ applications of t, which contradicts our assumption of $n+1$ applications needed. Thus our assumption was wrong and we need at most n applications of transformation t to stabilize an $(n+1)$-term index bounded sequence. This completes our inductive proof.

4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Rays BA and ED intersect at F while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

First Solution: Extend segment DM through M to G such that $FG \parallel CD$.



Then $MF = MC$ if and only if quadrilateral $CDFG$ is a parallelogram, or, $FD \parallel CG$. Hence $MC = MF$ if and only if $\angle GCD = \angle FDA$, that is, $\angle FDA + \angle CGF = 180^\circ$.

Because quadrilateral $ABED$ is cyclic, $\angle FDA = \angle ABE$. It follows that $MC = MF$ if and only if

$$180^\circ = \angle FDA + \angle CGF = \angle ABE + \angle CGF,$$

that is, quadrilateral $CBFG$ is cyclic, which is equivalent to

$$\angle CBM = \angle CBG = \angle CFG = \angle DCF = \angle DCM.$$

Because $\angle DMC = \angle CMB$, $\angle CBM = \angle DCM$ if and only if triangles BCM and CDM are similar, that is

$$\frac{CM}{BM} = \frac{DM}{CM},$$

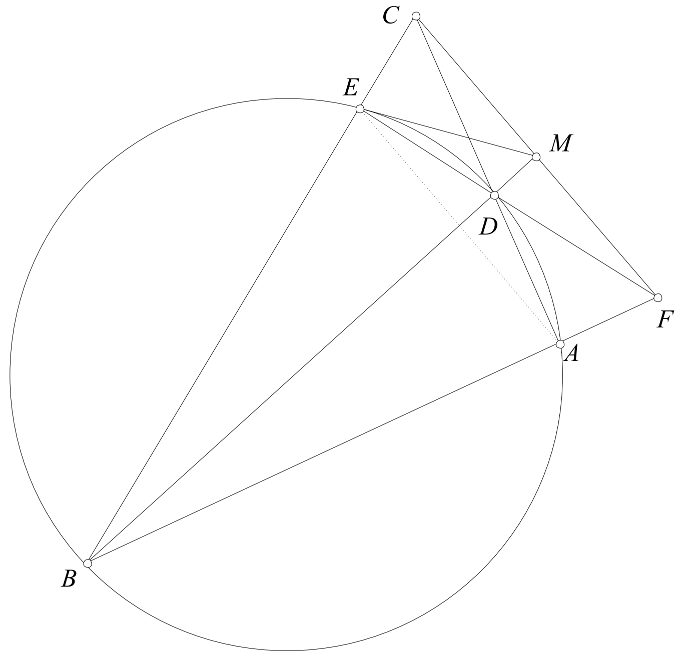
or $MB \cdot MD = MC^2$.

Second Solution:

We first assume that $MB \cdot MD = MC^2$. Because $\frac{MC}{MD} = \frac{MB}{MC}$ and $\angle CMD = \angle BMC$, triangles CMD and BMC are similar. Consequently, $\angle MCD = \angle MBC$.

Because quadrilateral $ABED$ is cyclic, $\angle DAE = \angle DBE$. Hence

$$\angle FCA = \angle MCD = \angle MBC = \angle DBE = \angle DAE = \angle CAE,$$



implying that $AE \parallel CF$, so $\angle AEF = \angle CFE$. Because quadrilateral $ABED$ is cyclic, $\angle ABD = \angle AED$. Hence

$$\angle FBM = \angle ABD = \angle AED = \angle AEF = \angle CFE = \angle MFD.$$

Because $\angle FBM = \angle DFM$ and $\angle FMB = \angle DMF$, triangles BFM and FDM are similar. Consequently, $\frac{FM}{DM} = \frac{BM}{FM}$, or $FM^2 = BM \cdot DM = CM^2$. Therefore $MC^2 = MB \cdot MD$ implies $MC = MF$.

Now we assume that $MC = MF$. Applying **Ceva's Theorem** to triangle BCF and **cevians** BM, CA, FE gives

$$\frac{BA}{AF} \cdot \frac{FM}{MC} \cdot \frac{CE}{EB} = 1,$$

implying that $\frac{BA}{AF} = \frac{BE}{EC}$, so $AE \parallel CF$.

Consequently, $\angle DCM = \angle DAE$. Because quadrilateral $ABED$ is cyclic, $\angle DAE = \angle DBE$. Hence

$$\angle DCM = \angle DAE = \angle DBE = \angle CBM.$$

Because $\angle CBM = \angle DCM$ and $\angle CMB = \angle DMC$, triangles BCM and CDM are similar. Consequently, $\frac{CM}{DM} = \frac{BM}{CM}$, or $CM^2 = BM \cdot DM$.

Combining the above, we conclude that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8.$$

First Solution: By multiplying a, b, and c by a suitable factor, we reduce the problem to the case when $a+b+c=3$. The desired inequality reads

$$\frac{(a+3)^2}{2a^2+(3-a)^2}+\frac{(b+3)^2}{2b^2+(3-b)^2}+\frac{(c+3)^2}{2c^2+(3-c)^2}\le 8.$$

Set

$$f(x)=\frac{(x+3)^2}{2x^2+(3-x)^2}$$

It suffices to prove that $f(a)+f(b)+f(c)\le 8$. Note that

$$\begin{aligned} f(x) &= \frac{x^2+6x+9}{3(x^2-2x+3)}=\frac{1}{3}\cdot\frac{x^2+6x+9}{x^2-2x+3} \\ &= \frac{1}{3}\left(1+\frac{8x+6}{x^2-2x+3}\right)=\frac{1}{3}\left(1+\frac{8x+6}{(x-1)^2+2}\right) \\ &\le \frac{1}{3}\left(1+\frac{8x+6}{2}\right)=\frac{1}{3}(4x+4). \end{aligned}$$

Hence,

$$f(a)+f(b)+f(c)\le\frac{1}{3}(4a+4+4b+4+4c+4)=8,$$

as desired.

Second Solution: Note that

$$\begin{aligned} (2x+y)^2+2(x-y)^2 &= 4x^2+4xy+y^2+2x^2-4xy+2y^2 \\ &= 3(2x^2+y^2). \end{aligned}$$

Setting $x=a$ and $y=b+c$ yields

$$(2a+b+c)^2+2(a-b-c)^2=3(2a^2+(b+c)^2).$$

Thus, we have

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}=\frac{3(2a^2+(b+c)^2)-2(a-b-c)^2}{2a^2+(b+c)^2}=3-\frac{2(a-b-c)^2}{2a^2+(b+c)^2}.$$

and its analogous forms. Thus, the desired inequality is equivalent to

$$\frac{(a-b-c)^2}{2a^2+(b+c)^2}+\frac{(b-a-c)^2}{2b^2+(c+a)^2}+\frac{(c-a-b)^2}{2c^2+(a+b)^2}\ge\frac{1}{2}.$$

Because $(b+c)^2\le 2(b^2+c^2)$, we have $2a^2+(b+c)^2\le 2(a^2+b^2+c^2)$ and its analogous forms. It suffices to show that

$$\frac{(a-b-c)^2}{2(a^2+b^2+c^2)}+\frac{(b-a-c)^2}{2(a^2+b^2+c^2)}+\frac{(c-a-b)^2}{2(a^2+b^2+c^2)}\ge\frac{1}{2},$$

or,

$$(a-b-c)^2+(b-a-c)^2+(c-a-b)^2 \geq a^2+b^2+c^2. \tag{1}$$

Multiplying this out the left-hand side of the last inequality gives $3(a^2+b^2+c^2)-2(ab+bc+ca)$. Therefore the inequality (1) is equivalent to $2[a^2+b^2+c^2-(ab+bc+ca)] \geq 0$, which is evident because

$$2[a^2+b^2+c^2-(ab+bc+ca)] = (a-b)^2+(b-c)^2+(c-a)^2.$$

Equalities hold if $(b+c)^2 = 2(b^2+c^2)$ and $(c+a)^2 = 2(c^2+a^2)$, that is, $a=b=c$.

Third Solution: Given a function f of three variables, define the cyclic sum

$$\sum_{\text{cyc}} f(p,q,r) = f(p,q,r)+f(q,r,p)+f(r,p,q).$$

We first convert the inequality into

$$\frac{2a(a+2b+2c)}{2a^2+(b+c)^2}+\frac{2b(b+2c+2a)}{2b^2+(c+a)^2}+\frac{2c(c+2a+2b)}{2c^2+(a+b)^2} \leq 5.$$

Splitting the 5 among the three terms yields the equivalent form

$$\sum_{\text{cyc}} \frac{4a^2-12a(b+c)+5(b+c)^2}{3[2a^2+(b+c)^2]} \geq 0. \tag{2}$$

The numerator of the term shown factors as $(2a-x)(2a-5x)$, where $x=b+c$. We will show that

$$\frac{(2a-x)(2a-5x)}{3(2a^2+x^2)} \geq -\frac{4(2a-x)}{3(a+x)}. \tag{3}$$

Indeed, (3) is equivalent to

$$(2a-x)[(2a-5x)(a+x)+4(2a^2+x^2)] \geq 0,$$

which reduces to

$$(2a-x)(10a^2-3ax-x^2) = (2a-x)^2(5a+x) \geq 0,$$

evident. We proved that

$$\frac{4a^2-12a(b+c)+5(b+c)^2}{3[2a^2+(b+c)^2]} \geq -\frac{4(2a-b-c)}{3(a+b+c)},$$

hence (2) follows. Equality holds if and only if $2a=b+c$, $2b=c+a$, $2c=a+b$, i.e., when $a=b=c$.

Fourth Solution: Given a function f of three variables, we define the symmetric sum

$$\sum_{\text{sym}} f(x_1,\ldots,x_n) = \sum_{\sigma} f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

where σ runs over all permutations of $1, \ldots, n$ (for a total of $n!$ terms). For example, if $n = 3$, and we write x, y, z for x_1, x_2, x_3,

$$\sum_{\text{sym}} x^3 = 2x^3 + 2y^3 + 2z^3$$
$$\sum_{\text{sym}} x^2y = x^2y + y^2z + z^2x + x^2z + y^2x + z^2y$$
$$\sum_{\text{sym}} xyz = 6xyz.$$

We combine the terms in the desired inequality over a common denominator and use symmetric sum notation to simplify the algebra. The numerator of the difference between the two sides is

$$\sum_{\text{sym}} 8a^6 + 8a^5b + 2a^4b^2 + 10a^4bc + 10a^3b^3 - 52a^3b^2c + 14a^2b^2c^2.$$

Recalling **Schur's Inequality**, we have

$$\begin{aligned} & a^3 + b^3 + c^3 + 3abc - (a^2b + b^2c + c^a + ab^2 + bc^2 + ca^2) \\ = \quad & a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0, \end{aligned}$$

or

$$\sum_{\text{sym}} a^3 - 2a^2b + abc \geq 0.$$

Hence,

$$0 \leq 14abc \sum_{\text{sym}} a^3 - 2a^2b + abc = 14 \sum_{\text{sym}} a^4bc - 28a^3b^2c + 14a^2b^2c^2$$

and by repeated **AM-GM Inequality**,

$$0 \leq \sum_{\text{sym}} 4a^6 - 4a^4bc$$

(because $a^46 + a^6 + a^6 + a^6 + b^6 + c^6 \geq 6a^4bc$ and its analogous forms)

and

$$0 \leq \sum_{\text{sym}} 4a^6 + 8a^5b + 2a^4b^2 + 10a^3b^3 - 24a^3b^2c.$$

Adding these three inequalities yields the desired result.

6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Note: Let

$$A \begin{array}{cc} B & C \\ F & E \end{array} D$$

denote a position, where A, B, C, D, E, F denote the numbers written on the vertices of the hexagon. We write

$$A \begin{array}{cc} B & C \\ F & E \end{array} D \pmod 2$$

if we consider the numbers written modulo 2.

Solution: Define the *sum* and *maximum* of a position to be the sum and maximum of the six numbers at the vertices. We will show that from any position in which the sum is odd, it is possible to reach the all-zero position.

Our strategy alternates between two steps:

(a) from a position with odd sum, move to a position with exactly one odd number;

(b) from a position with exactly one odd number, move to a position with odd sum and strictly smaller maximum, or to the all-zero position.

Note that no move will ever increase the maximum, so this strategy is guaranteed to terminate, because each step of type (b) decreases the maximum by at least one, and it can only terminate at the all-zero position. It suffices to show how each step can be carried out.

First, consider a position

$$A \begin{array}{cc} B & C \\ F & E \end{array} D$$

with odd sum. Then either $A+C+E$ or $B+D+F$ is odd; assume without loss of generality that $A+C+E$ is odd. If exactly one of A, C and E is odd, say A is odd, we can make the sequence of moves

$$1 \begin{array}{cc} B & 0 \\ F & 0 \end{array} D \to 1 \begin{array}{cc} \mathbf{1} & 0 \\ \mathbf{1} & 0 \end{array} \mathbf{0} \to \mathbf{0} \begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} 0 \to 0 \begin{array}{cc} 1 & 0 \\ \mathbf{0} & 0 \end{array} 0 \pmod 2,$$

where a letter or number in boldface represents a move at that vertex, and moves that do not affect each other have been written as a single move for brevity. Hence we can reach a position with exactly one odd number. Similarly, if A, C, E are all odd, then the sequence of moves

$$1 \begin{array}{cc} B & 1 \\ F & 1 \end{array} D \to 1 \begin{array}{cc} \mathbf{0} & 1 \\ \mathbf{0} & 1 \end{array} \mathbf{0} \to 1 \begin{array}{cc} 0 & \mathbf{0} \\ 0 & \mathbf{0} \end{array} 0 \pmod 2,$$

brings us to a position with exactly one odd number. Thus we have shown how to carry out step (a).

Now assume that we have a position

$$A \begin{array}{cc} B & C \\ F & E \end{array} D$$

with A odd and all other numbers even. We want to reach a position with smaller maximum. Let M be the maximum. There are two cases, depending on the parity of M.

- In this case, M is even, so one of B, C, D, E, F is the maximum. In particular, $A < M$. We claim after making moves at B, C, D, E, and F in that order, the sum is odd and the maximum is less than M. Indeed, the following sequence

$$1 \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} 0 \to 1 \begin{matrix} \mathbf{1} & 0 \\ 0 & 0 \end{matrix} 0 \to 1 \begin{matrix} 1 & \mathbf{1} \\ 0 & 0 \end{matrix} 0 \to 1 \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \mathbf{1} \to 1 \begin{matrix} 1 & 1 \\ 0 & \mathbf{1} \end{matrix} 1 \to 1 \begin{matrix} 1 & 1 \\ \mathbf{0} & 1 \end{matrix} 1 \pmod 2.$$

  shows how the numbers change in parity with each move. Call this new position $A' \begin{matrix} B' & C' \\ F' & E' \end{matrix} D'$. The sum is odd, since there are five odd numbers. The numbers A', B', C', D', E' are all less than M, since they are odd and M is even, and the maximum can never increase. Also, $F' = |A' - E'| \le \max\{A', E'\} < M$. So the maximum has been decreased.

- In this case, M is odd, so $M = A$ and the other numbers are all less than M.

  If $C > 0$, then we make moves at B, F, A, and F, in that order. The sequence of positions is

$$1 \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} 0 \to 1 \begin{matrix} \mathbf{1} & 0 \\ 0 & 0 \end{matrix} 0 \to 1 \begin{matrix} 1 & 0 \\ \mathbf{1} & 0 \end{matrix} 0 \to \mathbf{0} \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} 0 \to 0 \begin{matrix} 1 & 0 \\ \mathbf{0} & 0 \end{matrix} 0 \pmod 2.$$

  Call this new position $A' \begin{matrix} B' & C' \\ F' & E' \end{matrix} D'$. The sum is odd, since there is exactly one odd number. As before, the only way the maximum could not decrease is if $B' = A$; but this is impossible, since $B' = |A - C| < A$ because $0 < C < M = A$. Hence we have reached a position with odd sum and lower maximum.

  If $E > 0$, then we apply a similar argument, interchanging B with F and C with E.

  If $C = E = 0$, then we can reach the all-zero position by the following sequence of moves:

$$A \begin{matrix} B & 0 \\ F & 0 \end{matrix} D \to A \begin{matrix} \boldsymbol{A} & 0 \\ \boldsymbol{A} & 0 \end{matrix} \mathbf{0} \to \mathbf{0} \begin{matrix} A & 0 \\ A & 0 \end{matrix} 0 \to 0 \begin{matrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 \end{matrix} 0.$$

  (Here 0 represents zero, not any even number.)

Hence we have shown how to carry out a step of type (b), proving the desired result. The problem statement follows since 2003 is odd.

Note: Observe that from positions of the form

$$0 \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} 0 \qquad \pmod 2 \qquad \text{or rotations}$$

it is impossible to reach the all-zero position, because a move at any vertex leaves the same value modulo 2. Dividing out the greatest common divisor of the six original numbers does not affect whether we can reach the all-zero position, so we may assume that the numbers in the original position are not all even. Then by a more complete analysis in step (a), one can show from any position not of the above form, it is possible to reach a position with exactly one odd number, and thus the all-zero position. This gives a complete characterization of positions from which it is possible to reach the all-zero position.

There are many ways to carry out the case analysis in this problem; the one used here is fairly economical. The important idea is the formulation of a strategy that decreases the maximum value while avoiding the "bad" positions described above.

Second Solution: We will show that if there is a pair of opposite vertices with odd sum (which of course is true if the sum of all the vertices is odd), then we can reduce to a position of all zeros.

Focus on such a pair (a,d) with smallest possible $\max(a,d)$. We will show we can always reduce this smallest maximum of a pair of opposite vertices with odd sum or reduce to the all-zero position. Because the smallest maximum takes nonnegative integer values, we must be able to achieve the all-zero position.

To see this assume without loss of generality that $a \geq d$ and consider an arc (a,x,y,d) of the position

$$a \begin{smallmatrix} x & y \\ * & * \end{smallmatrix} d$$

Consider updating x and y alternately, starting with x. If $\max(x,y) > a$, then in at most two updates we reduce $\max(x,y)$. Thus, we can repeat this *alternate updating* process and we must eventually reach a point when $\max(x,y) \leq a$, and hence this will be true from then on.

Under this alternate updating process, the arc of the hexagon will eventually enter an unique cycle of length four modulo 2 in at most one update. Indeed, we have

$$1 \begin{smallmatrix} 0 & 0 \\ * & * \end{smallmatrix} 0 \to 1 \begin{smallmatrix} \mathbf{1} & 0 \\ * & * \end{smallmatrix} 0 \to 1 \begin{smallmatrix} 1 & \mathbf{1} \\ * & * \end{smallmatrix} 0 \to 1 \begin{smallmatrix} \mathbf{0} & 1 \\ * & * \end{smallmatrix} 0 \to 1 \begin{smallmatrix} 0 & \mathbf{0} \\ * & * \end{smallmatrix} 0 \pmod 2$$

and

$$1 \begin{smallmatrix} 0 & 0 \\ * & * \end{smallmatrix} 0 \to 1 \begin{smallmatrix} 0 & \mathbf{0} \\ * & * \end{smallmatrix} 0 \pmod 2; \quad 1 \begin{smallmatrix} 1 & 0 \\ * & * \end{smallmatrix} 0 \to 1 \begin{smallmatrix} \mathbf{1} & 0 \\ * & * \end{smallmatrix} 0 \pmod 2$$

$$1 \begin{smallmatrix} 1 & 1 \\ * & * \end{smallmatrix} 0 \to 1 \begin{smallmatrix} 1 & \mathbf{1} \\ * & * \end{smallmatrix} 0 \pmod 2; \quad 1 \begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix} 0 \to 1 \begin{smallmatrix} \mathbf{0} & 1 \\ * & * \end{smallmatrix} 0 \pmod 2,$$

or

$$0 \begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix} 1 \to 0 \begin{smallmatrix} \mathbf{1} & 1 \\ * & * \end{smallmatrix} 1 \to 0 \begin{smallmatrix} 1 & \mathbf{0} \\ * & * \end{smallmatrix} 1 \to 0 \begin{smallmatrix} \mathbf{0} & 0 \\ * & * \end{smallmatrix} 1 \to 0 \begin{smallmatrix} 0 & \mathbf{1} \\ * & * \end{smallmatrix} 1 \pmod 2$$

and

$$0 \begin{smallmatrix} 0 & 0 \\ * & * \end{smallmatrix} 1 \to 0 \begin{smallmatrix} \mathbf{0} & 0 \\ * & * \end{smallmatrix} 1 \pmod 2; \quad 0 \begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix} 1 \to 0 \begin{smallmatrix} 0 & \mathbf{1} \\ * & * \end{smallmatrix} 1 \pmod 2$$

$$0 \begin{smallmatrix} 1 & 1 \\ * & * \end{smallmatrix} 1 \to 0 \begin{smallmatrix} \mathbf{1} & 0 \\ * & * \end{smallmatrix} 1 \pmod 2; \quad 0 \begin{smallmatrix} 1 & 0 \\ * & * \end{smallmatrix} 1 \to 0 \begin{smallmatrix} 1 & \mathbf{0} \\ * & * \end{smallmatrix} 1 \pmod 2.$$

Further note that each possible parity for x and y will occur equally often.

Applying this alternate updating process to both arcs (a,b,c,d) and (a,e,f,d) of

$$a \begin{smallmatrix} b & c \\ f & e \end{smallmatrix} d,$$

we can make the other four entries be at most a and control their parity. Thus we can create a position

$$a \begin{smallmatrix} x_1 & x_2 \\ x_5 & x_4 \end{smallmatrix} d$$

with $x_i + x_{i+3}$ $(i = 1,2)$ odd and $M_i = \max(x_i, x_{i+3}) \leq a$. In fact, we can have $m = \min(M_1, M_2) < a$, as claimed, unless both arcs enter a cycle modulo 2 where the values congruent to a modulo 2 are always exactly a. More precisely, because the sum of x_i and x_{i+3} is odd, one of them is not congruent to a and so has its value strictly less than a. Thus both arcs must pass through the state (a, a, a, d) (modulo 2, this is either $(0, 0, 0, 1)$ or $(1, 1, 1, 0)$) in a cycle of length four. It is easy to check that for this to happen, $d = 0$. Therefore, we can achieve the position

$$\begin{array}{cccc} & a & a & \\ a & & & 0. \\ & a & a & \end{array}$$

From this position, the sequence of moves

$$\begin{array}{cccc} & a & a & \\ a & & & 0 \\ & a & a & \end{array} \rightarrow \begin{array}{cccc} & \mathbf{0} & a & \\ a & & & 0 \\ & \mathbf{0} & a & \end{array} \rightarrow \begin{array}{cccc} & 0 & \mathbf{0} & \\ \mathbf{0} & & & 0 \\ & 0 & \mathbf{0} & \end{array}$$

completes the task.