**$43^{\text{rd}}$ United States of America Mathematical Olympiad**

**Day I, II     12:30 PM – 5 PM EDT**

**April 29 - April 30, 2014**

USAMO 1. Using Vieta's identities we have:

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 - x_1x_2x_3x_4 \geq 5$$

and so

$$x_1(x_2 + x_3 + x_4 - x_2x_3x_4) + 1(x_2x_3 + x_2x_4 + x_3x_4 - 1) \geq 4.$$

It follows that

$$4^2 \leq [x_1(x_2 + x_3 + x_4 - x_2x_3x_4) + 1(x_2x_3 + x_2x_4 + x_3x_4 - 1)]^2,$$

and by the Cauchy-Schwarz Inequality,

$$\begin{aligned} 4^2 &\leq (x_1^2 + 1)[(x_2 + x_3 + x_4 - x_2x_3x_4)^2 + (x_2x_3 + x_2x_4 + x_3x_4 - 1)^2] \\ &= (x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1). \end{aligned}$$

The equality holds if and only if

$$x_1(x_2x_3 + x_2x_4 + x_3x_4 - 1) = 1(x_2 + x_3 + x_4 - x_2x_3x_4),$$

which is equivalent to

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = x_1 + x_2 + x_3 + x_4,$$

that is, $a = c$. Taking $x_1 = ... = x_4 = 1$ we obtain $b - d = 5$ and that the smallest value of the product in question is 16.
An alternative, shorter argument runs as follows: we have

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) = P(i)P(-i) =$$

$$((1 - b + d) + i(c - a))(1 - b + d - i(c - a)) = (b - d - 1)^2 + (c - a)^2 \geq 16,$$

with equality if and only if $b - d = 5$ and $a = c$, both attained if $x_1 = ... = x_4 = 1$.

This problem and solutions were suggested by Titu Andreescu.

USAMO 2. Let $f$ be a solution of the problem. Let $p$ be a prime. Since $p$ divides $f(p)^2$, $p$ divides $f(p)$ and so $p$ divides $\frac{f(p)^2}{p}$. Taking $y=0$ and $x=p$, we deduce that $p$ divides $f(0)$. As $p$ is arbitrary, we must have $f(0)=0$. Next, take $y=0$ to obtain $xf(-x)=\frac{f(x)^2}{x}$. Replacing $x$ by $-x$, and combining the two relations yields $f(x)=0$ or $f(x)=x^2$ for all $x$.

Suppose now that there exists $x_0 \neq 0$ such that $f(x_0)=0$. Taking $y=x_0$, we obtain $xf(-x)+x_0^2f(2x)=\frac{f(x)^2}{x}$, yielding $x_0^2f(2x)=0$ for all $x$ and so $f$ vanishes on even numbers. Assume that there exists an odd number $y_0$ such that $f(y_0)\neq 0$, so $f(y_0)=y_0^2$. Taking $y=y_0$, we obtain

$$xf(2y_0^2-x)+y_0^2f(2x-y_0^2)=\frac{f(x)^2}{x}+f(y_0^3).$$

Choosing $x$ even, we deduce that $y_0^2f(2x-y_0^2)=f(y_0^3)$. This forces $f(y_0^3)=0$, as otherwise we would have $f(2x-y_0^2)=(2x-y_0^2)^2$ for all even $x$ and so $y_0^2(2x-y_0^2)^2=f(y_0^3)$ for all such $x$, obviously impossible. Thus $f(2x-y_0^2)=0$ for all even numbers $x$, that is $f$ vanishes on numbers of the form $4k+3$. But since $x^2f(-x)=f(x)^2$, $f$ also vanishes on all $x$ such that $-x\equiv -1 \pmod 4$, that is on $4\mathbb{Z}+1$. Thus $f$ also vanishes on all odd numbers, contradicting the choice of $y_0$. Hence, if $f$ is not the zero map, then $f$ does not vanish outside 0 and so $f(x)=x^2$ for all $x$.

In conclusion, $f(x)=0$ for all $x\in\mathbb{Z}$ and $f(x)=x^2$ for all $x\in\mathbb{Z}$ are the only possible solutions. The first function clearly satisfies the given relation, while the second also satisfies the Sophie Germaine identity

$$x(2y^2-x)^2+y^2(2x-y^2)^2=x^3+y^6$$

for all $x,y\in\mathbb{Z}$.

OR

$f(0)=0$: If $f(0)\neq 0$, set $x=2f(0)$ to obtain

$$2(f(0))^2=\frac{(f(2f(0)))^2}{2f(0)}+f(0)$$

that is

$$2(f(0))^2(2f(0)-1)=f(2f(0))^2.$$

But $2(2f(0)-1)$ cannot be a perfect square since it is of the form $4k+2$. So $f(0)=0$.

This problem and the solutions were suggested by Titu Andreescu and Gabriel Dospinescu.

USAMO 3. We claim that defining $P_n$ to be the point with coordinates $(n, n^3-2014n^2)$ will satisfy the conditions of the problem. Recall that points $(x_1,y_1)$, $(x_2,y_2)$ and $(x_3,y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Therefore we examine the determinant

$$\begin{vmatrix} a & a^3 - 2014a^2 & 1 \\ b & b^3 - 2014b^2 & 1 \\ c & c^3 - 2014c^2 & 1 \end{vmatrix} = \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} - 2014 \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}.$$

The first determinant on the right is a homogenous polynomial of degree four divisible by $(a-b)(b-c)(c-a)$. The remaining factor has degree one, is symmetric, and yields an $ab^3$ term when the product is expanded, hence must be $(a+b+c)$. The second determinant is a homogenous polynomial of degree three divisible by $(a-b)(b-c)(c-a)$, and comparing coefficients of the $ab^2$ term we see that this is the desired polynomial. Thus

$$\begin{vmatrix} a & a^3 - 2014a^2 & 1 \\ b & b^3 - 2014b^2 & 1 \\ c & c^3 - 2014c^2 & 1 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c-2014).$$

It follows that for distinct $a$, $b$ and $c$ this expression will equal zero if and only if $a+b+c = 2014$, as desired.

This solution was suggested by Razvan Gelca.

OR

First, note that the translation $x \mapsto x - 671$ in the indices allows us to replace 2014 in the statement by 1. Now it comes natural to look for a polynomial pattern $(P(x), Q(x))$ in the coordinates of a point. The collinearity condition translates, in coordinates, into

$$P(a)Q(b) + P(b)Q(c) + P(c)Q(a) - P(a)Q(c) - P(b)Q(a) - P(c)Q(b) = 0.$$

This should happen only when $a+b+c-1=0$ or when two of $a, b, c$ are equal. Hence the left-hand side should be of the form $(a+b+c-1)(b-a)(c-b)(a-c)R(a,b,c)$. We can try the simplest case $R=1$ so that the dominant coefficients of both $P(x)$ and $Q(x)$ are 1. $P(x)$ and $Q(x)$ cannot both have even degree because then the 4th degree terms on the left cancel out, while on the right there are clearly 4th degree terms. Hence one of the polynomials $P(x)$ and $Q(x)$ has degree 3, the other has degree 1. By a translation we can turn the degree 1 polynomial into $x$, thus we may assume that $P(x) = x$. Thus we should have

$$\begin{aligned} &(c-b)Q(a) + (a-c)Q(b) + (b-a)Q(c) \\ &\quad = (a+b+c-1)(b-a)(c-b)(a-c). \end{aligned}$$

So we let $Q(x) = x^3 + \alpha x^2 + \beta x + \gamma$. Note that we are free to choose $\beta$ and $\gamma$ any way we want, since they cancel out. So we let $Q(x) = x^3 + \alpha x^2$.

For $a = 0$, $b = -1$, $c = 1$ the above identity yields $-2Q(0) - Q(-1) - Q(1) = 2$, and hence $\alpha = -1$.

Returning to the case of the problem with 2014 instead of 1, we have the points $P_n = (n-671, (n-671)^3 - (n-671)^2)$. But we can simplify this since we can replace $P(x)$ by $x$ and ignore the linear part of $Q(x)$. We thus obtain the simpler infinite family of points

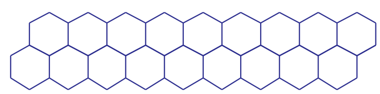
$$P_n = (n, n^3 - 3 \cdot 671n^2 - n^2) = (n, n^3 - 2014n^2)$$

satisfying the conditions of the problem.

This problem and the second solution was suggested by Sam Vandervelde.

USAMO 4. The answer is $k = 6$. First we show that $A$ cannot win for $k \geq 6$. Color the grid in three colors so that no two adjacent spaces have the same color, and arbitrarily pick one color $C$. $B$ will play by always removing a counter from a space colored $C$ that $A$ just played. If there is no such counter, $B$ plays arbitrarily. Because $A$ cannot cover two spaces colored $C$ simultaneously, it is possible for $B$ to play in this fashion. Now note that any line of six consecutive squares contains two spaces colored $C$. For $A$ to win he must cover both, but $B$'s strategy ensures at most one space colored $C$ will have a counter at any time.

Now we show that $A$ can obtain 5 counters in a row. Take a set of cells in the grid forming the shape shown below. We will have $A$ play counters only in this set of grid cells until this is no longer possible. Since $B$ only removes one counter for every two $A$ places, the number of counters in this set will increase each turn, so at some point it will be impossible for $A$ to play in this set anymore. At that point any two adjacent grid spaces in the set have at least one counter between them.



Consider only the top row of cells in the set, and take the lengths of each consecutive run of cells. If there are two adjacent runs that have a combined length of at least 4, then $A$ gets 5 counters in a row by filling the space in between. Otherwise, a bit of case analysis shows that there exists a run of 1 counter which is neither the first nor last run. This single counter has an empty space on either side of it on the first row. As a result, the four spaces of the second row touching these two empty spaces all must have counters. Then $A$ can play in the 5th cell on either side of these 4 to get 5 counters in a row. So in all cases $A$ can win with $k \leq 5$.

This problem and solution was suggested by Palmer Mebane.

USAMO 5. It is well-known that the reflection $H'$ of the orthocenter $H$ in the line $AC$ lies on the circumcircle of triangle $ABC$. Hence, the circumcenter of triangle $CAH'$ coincides with the circumcenter of triangle $ABC$. But since $H'$ is the reflection of $H$ in the line $AC$, the triangles $ACH$ and $CAH'$ are symmetric with respect to $BC$, and the circumcenter $O'$ of triangle $ACH$ must be the reflection of the circumcenter of triangle $CAH'$ in the line $BC$, i. e. the reflection of the circumcenter of triangle $ABC$ in the line $CA$.

Now since the quadrilateral $AHPC$ is cyclic and since $H$, $Y$ are the orthocenters of triangles $ABC$, and $APC$, respectively, we have that

$$\angle ABC = 180^\circ - \angle AHC = 180^\circ - \angle APC = \angle AYC.$$

Hence the point $Y$ lies on the circumcircle of triangle $ABC$, and therefore $OC = OY = R$, where $R$ denotes the circumradius of triangle $ABC$.

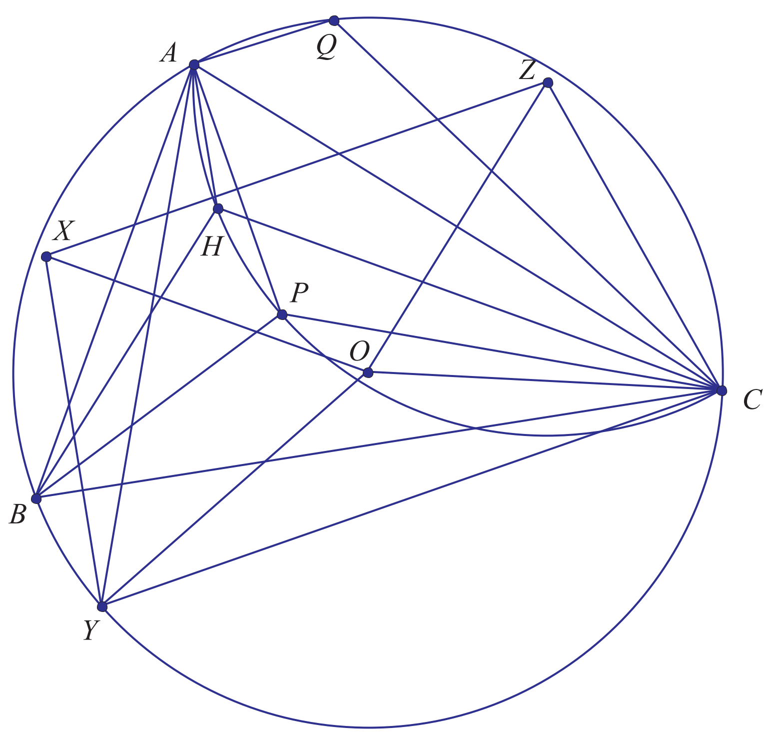
On the other hand, note that the lines $OX$, $XO'$, $O'O$ are the perpendicular bisectors of the segments $AB$, $AP$, and $AC$, respectively, we get

$$\angle OXO' = \angle BAP = \angle PAC = m(\angle XO'O.$$

Thus $OO' = OX$. Combining this with $OC = OY$ and with the parallelism of the lines $XO'$ and $YC$ (note that these two lines are both perpendicular to $AP$), we conclude that the trapezoid $XYCO'$ is isosceles, and therefore $XY = O'C = OC = R$. This completes our proof. $\square$

*Remark.* If $ABC$ is right-angled at $A$, then the statement is trivially true if we convene that the circumcenter of $AB$ is the midpoint of $AB$ and that the orthocenter of $AC$ is the midpoint of $AC$. Then, we have that $XY = \frac{1}{2}BC = R$.

OR




Because $ABC$ is acute, $H$ lies inside the triangle. We consider the configuration show above. (For other possible configurations, it is not difficult to adjust our proof properly.)

Let $O$ and $Z$ denote the circumcenters of triangles $ABC$ and $APC$ respectively. Let $\omega$ and $r$ denote the circumcircle and the circumradius of triangle $ABC$ respectively. We will show that

$$XYCZ \text{ is an isoscelees trapezoid with } XY = CZ = r. \tag{1}$$

Because $X$ and $Z$ are the circumcenters of triangle $APB$ and $APC$, line $XZ$ is the perpendicular bisector of segment $AP$. Because $Y$ is the orthocenter of triangle $APC$, $CY \perp AP$. Hence both lines $XZ$ and $CY$ are perpendicular to line $AP$, implying that $XYZC$ is a trapezoid with $XZ \parallel CY$.

Because $X$ and $O$ are the circumcenters of triangles $APB$ and $ABC$, line $XO$ is the perpendicular bisector of segment $AB$. Because $XO \perp AB$ and $XZ \perp AP$, the acute angles formed by lines $XO$ and $XZ$ is equal to the acute angle formed by lines $AP$ and $AB$; that is, $\angle OXZ = \angle BAP$. Likewise, we can can show that $\angle OZX = \angle CAP$. Therefore, we have $\angle OXZ = \angle BAP = \angle CAP = \angle OZX$, implying that $OX = OZ$; that is, $O$ lies on the perpendicular bisector of segment $XZ$.

Because $H$ is the orthocenter of acute triangle $ABC$, $\angle AHC = 180° - \angle ABC$. Because $APHC$ is cyclic, we have $\angle APC = \angle AHC = 180° - \angle ABC$. Now in obtuse triangle $APC$, $\angle AYC = 180° - \angle APC = \angle ABC$. (This relates to the fact of orthocenter group: if one point is the orthocenter of the triangle formed by the other three points, then any of the four point is the orthocenter of the triangle formed by the other three.) In particular, this means that $Y$ lies on $\omega$; that is, $OY = OC = r$.

Note that in trapezoid $XYCZ$, the perpendicular bisectors of the bases $YC$ and $XZ$ share a common point $O$. Thus, these two bisectors must coincide; that is, $XYCZ$ is an isosceles trapezoid with $XY = CZ$, establishing the first part of (**??**).

To complete our proof, it suffices to show that $CZ = r$. Let $Q$ be the reflection of $H$ across line $AC$. It is well known that $Q$ lies $\omega$ (because $\angle ACQ = \angle ACH = 90° - \angle BAC = \angle ABH = \angle ABQ$.) We note that triangle $AQC$ and its circumcenter $O$ and triangle $AHC$ and its circumcenter $Z$ are respective images of each other across line $AC$. In particular, we conclude that $CZ = CO = r$, completing our proof.

This problem and solutions were suggested by Titu Andreescu and Cosmin Pohoata.

USAMO 6. Let $a, b, n$ be positive integers as in the statement of the problem. Let $P_n$ be the set of prime numbers not exceeding $n$. We will need the following

There is a positive integer $n_0$ such that for all $n \geq n_0$ we have

$$\sum_{p \in P_n} \left(\frac{n}{p} + 1\right)^2 < \frac{2}{3} n^2.$$

*Proof.* Expanding and dividing by $n^2$, and observing that $|P_n| \leq n$, it suffices to prove the inequality

$$\sum_{p \in P_n} \frac{1}{p^2} + \frac{2}{n} \sum_{p \in P_n} \frac{1}{p} + \frac{1}{n} < \frac{2}{3}.$$

Since

$$\frac{2}{n} \sum_{p \in P_n} \frac{1}{p} < \frac{2}{n} \sum_{i=2}^{n} \frac{1}{i} < \frac{2}{n} \log n,$$

it suffices to prove the existence of a constant $r < \frac{2}{3}$ such that $\sum_{p \in P_n} \frac{1}{p^2} < r$. But

$$\sum_{p \in P_n} \frac{1}{p^2} \quad \leq \quad \frac{1}{4} + \frac{1}{9} + \sum_{k=1}^{n} \frac{1}{(2k+1)(2k+3)}$$

$$
\begin{aligned}
&= \frac{1}{4}+\frac{1}{9}+\sum_{k=1}^{n}\frac{1}{2}\left(\frac{1}{2k+1}-\frac{1}{2k+3}\right)\\
&= \frac{1}{4}+\frac{1}{9}+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right) < \frac{1}{4}+\frac{1}{9}+\frac{1}{6} < \frac{1}{3}
\end{aligned}
$$

and we can take $r = \frac{1}{4}+\frac{1}{9}+\frac{1}{6}$. $\square$

From now on we fix such $n_0$, and we prove the statement assuming $n \geq n_0$. Note that for any $p \in P_n$ there are at most $\frac{n}{p}+1$ numbers $i \in \{0,1,\ldots,n-1\}$ such that $p \mid a+i$, and likewise for $j \in \{0,1,\ldots,n-1\}$ such that $p \mid b+j$. Thus there are at most $\left(\frac{n}{p}+1\right)^2$ pairs $(i,j)$ such that $p \mid \gcd(a+i,b+j)$. Using the previous lemma, we deduce that there are less than $\frac{2}{3}n^2$ pairs $(i,j)$ with $i,j \in \{0,1,\ldots,n-1\}$ such that $p \mid \gcd(a+i,b+j)$ for some $p \in P_n$.

Let $N$ be the least integer greater than or equal to $\frac{n^2}{3}$. By the above, there are at least $N$ pairs $(i,j)$ with $i,j \in \{0,1,\ldots,n-1\}$ such that $\gcd(a+i,b+j)$ is not divisible by any prime in $P_n$. Call these pairs $(i_s,j_s)$ for $s=1,2,\ldots,N$. For each pair, choose a prime $p_s$ that divides $\gcd(a+i_s,b+j_s)$ (since, by hypothesis, $\gcd(a+i_s,b+j_s)>1$); thus $p_s > n$. The map $s \mapsto p_s$ is injective, for if $p_s = p_{s'}$, then $p_s \mid i_s - i_{s'}$, implying $i_s = i_{s'}$, and similarly $j_s = j_{s'}$, hence $s = s'$.

We conclude that $\prod_{i=0}^{n-1}(a+i)$ is a multiple of $\prod_{s=1}^{N} p_s$. Since the $p_s$ are distinct prime numbers greater than $n$, then,

$$
(a+n)^n > \prod_{i=0}^{n-1}(a+i) \geq \prod_{s=1}^{N} p_s \geq \prod_{i=1}^{N}(n+2i-1).
$$

Let $X$ be this last product. Then

$$
X^2 = \prod_{i=1}^{N}[(n+2i-1)(n+2(N+1-i)-1)] > \prod_{i=1}^{N}(2Nn) = (2Nn)^N,
$$

where the inequality holds because

$$
(n+2i-1)(n+2(N+1-i)-1) > n(2(N+1-i)-1)+(2i-1)n = 2Nn.
$$

Finally

$$
(a+n)^n > (2Nn)^{\frac{N}{2}} \geq \left(\frac{2n^3}{3}\right)^{\frac{n^2}{6}}.
$$

Thus,

$$
a \geq \left(\frac{2}{3}\right)^{\frac{1}{6}\cdot n} \cdot n^{\frac{n}{2}} - n,
$$

which is larger than $c^n \cdot n^{\frac{n}{2}}$ when $n$ is large enough, for any constant $c < \left(\frac{2}{3}\right)^{\frac{1}{6}}$. Similarly, the same inequality holds for $b$.

This shows that $\min\{a, b\} \geq c^n \cdot n^{\frac{n}{2}}$ as long as $n$ is large enough. By shrinking $c$ sufficiently, we can ensure the inequality holds for all $n$.

One can see that the argument is not sharp, so that the factor $n^{\frac{n}{2}}$ can be improved to $n^{rn}$ for some constant $r$ slightly larger than $\frac{1}{2}$. Consequently, for *any* $c > 0$, the inequality in the problem holds if $n$ is large enough.

This problem and solution was suggested by Titu Andreescu and Gabriel Dospinescu.