

31st United States of America Mathematical Olympiad
Cambridge, Massachusetts
Part I 1 p.m. EDT - 5:30 p.m. EDT
May 3, 2002

1. Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:
 - (a) the union of any two white subsets is white;
 - (b) the union of any two black subsets is black;
 - (c) there are exactly N white subsets.
2. Let ABC be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2 \cot \frac{B}{2}\right)^2 + \left(3 \cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisor and determine these integers.

3. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

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4. Let \mathbb{R} be the set of real numbers. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all real numbers x and y .

5. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence n_1, n_2, \dots, n_k of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i ($1 \leq i < k$).
6. I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of a sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that

$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

for all $n > 0$.