43^{rd} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 30, 2014

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

USAMO 4. Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent spaces in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum exists.

USAMO 5. Let ABC be a triangle with orthocenter H and let P be the second intersection of the circumcircle of triangle AHC with the internal bisector of the angle $\angle BAC$. Let X be the circumcenter of triangle APB and Y the orthocenter of triangle APC. Prove that the length of segment XY is equal to the circumradius of triangle ABC.

USAMO 6. Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a + i, b + j) > 1$ for all $i, j \in \{0, 1, \ldots, n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$